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SMR/92 - 14

AUTUMN COURSE

ON

VARIATIONAL METHODS IN ANALYSIS AND MATHEMATICAL PHYSICS

20 October - 11 December 1981

THE LUSTERNIK-SCHNIERELMAN CATEGORY

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§ 7. The Lusternik-Schnirelman category.

Here we discuss the main topological tool for what follows.

Let M, N be topological spaces. Two ~~ϕ~~ mappings φ_0, $\varphi_1 \in C(M,N)$ are homotopic ($\varphi_0 \sim \varphi_1$) if there exists $\mathcal{H} \in C([0,1]\times M, N)$ such that

$$\mathcal{H}(0,\cdot) = \varphi_0 \qquad \mathcal{H}(1,\cdot) = \varphi_1$$

Let A be a subset of M. A is said contractible in M if the inclusion $i : A \to M$ is homotopic to a constant mapping. In other words, A is contractible if there is $\mathcal{H}(t,x) \in C([0,1]\times A, M)$ such that $\forall x \in A$

$$\mathcal{H}(0,x) = x \qquad \mathcal{H}(1,x) = p$$

for some $p \in X$.

~~The~~

7.1. <u>Definition</u>. <u>We define</u> $\mathrm{cat}(A,M)$ (<u>the L-S category of A with respect to M) as the least integer k such that $A \subset A_1 \cup A_2 \cup \dots \cup A_k$, each A_i being closed in M and contractible in M.</u>

For example, any finite-dimensional sphere S^n has category 2, while ~~for~~ $S = \{u \in E, \|u\| = 1\}$, with E infinite dimensional separable Hilbert space, it results $\mathrm{cat}(S,S) = 1$.
If T is a two-dimensional torus, then $\mathrm{cat}(T,T) = 3$.

We will use the notation $\mathrm{cat}(M)$ for $\mathrm{cat}(M,M)$.

7.2 <u>Proposition</u>. Let $A, A_1, A_2 \subset M$.

(1) $A_1 \subseteq A_2$ implies $\mathrm{cat}(A_1, M) \leq \mathrm{cat}(A_2, M)$

(2) $A = A_1 \cup A_2$ implies $\mathrm{cat}(A,M) \leq \mathrm{cat}(A_1,M) + \mathrm{cat}(A_2,M)$

(3) if A is closed in M and $\alpha \in C(A,M)$, ~~...~~ is homotopic to the inclusion $i : A \to M$, then $\mathrm{cat}(A,M) \leq \mathrm{cat}(\alpha(A),M)$

<u>Proof.</u> We prove only (3); (1) and (2) follow directly from the definition. Let

$$cat(\alpha(A), M) = k$$

so that $\alpha(A) \subset V_1 \cup V_2 \cup \dots \cup V_k$, V_i closed and contractible in M. Setting

$$A_i = \{x \in A : \alpha(x) \in V_i\},$$

each A_i is closed in A and hence in M, because A is closed. Moreover each A_i is contractible because V_i do. Hence $cat(A, M) \le k$, as required. ■

Before to state next ~~the~~ result, we recall that a metric ~~topological~~ space M is a ANR if: for every ~~ϕ~~ metric space T, for every ~~~~ closed subset $S \subset T$ and every $\varphi \in C(S, M)$ there exists a neighborhood U of S and ~~~~ $\hat\varphi \in C(U, M)$ which extends φ (namely $\hat\varphi(x) = \varphi(x)$ $\forall x \in S$).

<u>7.3. Proposition. Let M be a ANR and $A \subset M$ be compact. Then there exists a neighborhood U of A such that $cat(U, M) = cat(A, M)$.</u>

<u>Proof.</u> As in the proof of Proposition 7.2, it is enough to give prove ~~the~~ ~~ϕ~~ in the case $cat(A, M) = 1$. Since A is contractible, there exists $\mathcal{H} \in C([0,1] \times A, M)$ such that

$$\mathcal{H}(0, x) = x \quad \mathcal{H}(1, x) = p \qquad \forall x \in A.$$

~~~~ In the definition of ANR, we take $T = [0,1] \times M$

and $S = ([0,1] \times A) \cup (\{0\} \times M) \cup (\{1\} \times M)$, and

$$\varphi : S \to M$$

defined by:

$$\begin{aligned} \varphi(t, x) &= \mathcal{H}(t, x) && \text{for } x \in A \\ \varphi(0, x) &= x && \text{for } x \in M \\ \varphi(1, x) &= p && \text{for } x \in M \end{aligned}$$

Remark that $\varphi \in C(S, M)$. Hence φ has an extension $\hat\varphi$ defined in a neighborhood V of S. Since A is compact, we can take a neighborhood U of A such that $[0,1] \times \overline{U} \subset V$. Then $\hat\varphi \in C([0,1] \times \overline{U}, M)$ ~~satisfies~~ is a homotopy from the inclusion $i : \overline{U} \to M$ and the constant mapping p. Hence $cat(\overline{U}, M) = 1$, as required. ■

~~It~~ It is possible to ~~~~ show that a Hilbert (or Banach) submanifold of the type described in § 5 are ANR (cf. [Pa 2]).
~~~~

§ 8. Existence of critical points.

In this section we will expose, following J.T. Schwartz [ ] a mini-max procedure to find critical points of a functional on a manifold.

We will always assume $M = h^{-1}(0)$ where $h \in C^{1,1}(E, \mathbb{R})$, E Hilbert space, and (h1) holds (see § 5).

Let $f \in C^{1,1}(E, \mathbb{R})$.

Define

$$A_n = \{ A \subset M : \operatorname{cat}(A, M) \geq n,\ A \text{ compact} \}.$$

We shall assume that for $n \leq \operatorname{cat}(M)$, $A_n \neq \emptyset$, this will be the case in all our applications. Let

$$c_n = \inf_{A \in A_n} \max \{ f(x) : x \in A \}$$

Let us remark that, since A is compact, then $c_n < \infty$. This is the only point where we use this fact. Indeed, we could avoid the assumption $A_n \neq \emptyset$, dropping the compactness requirement in the definition of A_n. A bit more careful analysis would give the same results.

Since $A_n \supset A_{n+1}$, it follows

$$c_1 \leq c_2 \leq c_3 \leq \cdots \leq c_n \leq c_{n+1} \leq \cdots$$

We will use the notations introduced in § 5.

8.1 Theorem. Suppose (f, M) satisfies (P-S). Then

(i) ~~proving~~ for every $c_n > -\infty$, $K_{c_n} \neq \emptyset$

(ii) if $c := c_n = c_{n+1} = \cdots = c_{n+r} \neq -\infty$, then $\operatorname{cat}(K_c, M) \geq r+1$

(iii) if f is bounded from below on M, then f has at least $\operatorname{cat}(M)$ critical points on M.

~~The main tool of the proof will be a deformation lemma. For this, we will follow~~

For the proof of the Theorem, we will need deformations. We will follow the arguments used in Lemma 2.4 and in § 5. If φ denotes the function introduced in Lemma 2.4, we define F by setting

$$F(u) = -\varphi(\| f_M'(u) \|)\, f_M'(u).$$

and $\alpha(t, x)$ by

$$\begin{cases} \dfrac{d\alpha}{dt} = F(\alpha) \\ \alpha(0, x) = x \end{cases}$$

As in § 5, we show that ~~also~~ $\alpha(\cdot, x)$ is defined for every $t \in \mathbb{R}$ (remark that F is bounded), $\alpha(t, x) \in M$ and $t \mapsto f(\alpha(t, x))$ is non-increasing. Moreover, we recall that (cf. formula (5.5))

$$(8.1) \qquad f(\alpha(t_0, x)) - f(\alpha(t_1, x)) = \int_{t_0}^{t_1} \varphi(\| f_M'(\alpha) \|) \| f'(\alpha) \|^2$$

In order to prove theorem 8.1 we need:

8.2 Lemma. For every $\varepsilon > 0$, and, for a given c:

$$W_\varepsilon = \{x \in f^{-1}[c-\varepsilon, c+\varepsilon] : \|f'_M(\alpha(t,x))\| < \varepsilon \quad \text{for some } t \in [0,1]\}.$$

Then given any neighborhood U of K_c, there exists $\bar\varepsilon > 0$ such that $\forall \varepsilon \leq \bar\varepsilon$ it is $W_\varepsilon \subset U$.

Proof. By contradiction, let U be a neighborhood of K_c and $x_n \in M$, $t_n \in [0,1]$, such that

(8.2) $\quad x_n \notin U$

(8.3) $\quad c - \frac{1}{n} \leq f(x_n) \leq c + \frac{1}{n}$

(8.4) $\quad f'_M(y_n) \to 0 \quad$ where $y_n := \alpha(t_n, x_n)$

We can also assume $t \to t^* \in [0,1]$. Since $s^2\varphi(s) \leq 1$ by (8.1) it follows:

$$|f(x) - f(\alpha(t,x))| \leq |t|$$

Thus, for the sequence x_n, we have that $f(x_n) \to c$ (cf. (8.3)) and $t_n \to t^*$ simply $f(y_n)$ is bounded.

Then, by (P-S) it follows $y_n \to \bar y$, and $f'_M(\bar y) = 0$. Moreover, the continuity of $\alpha(\cdot,\cdot)$ gives:

$$x_n := \alpha(t_n, y_n) \to \alpha(t^*, \bar y)$$

Since $\bar y \in K$, then $F(\bar y) = 0$ and $\alpha(t, \bar y) = \bar y$ for all t. Thus we have $x_n \to \bar y \in K$ with $f(x_n) \to c$ and $x_n \in U$, a contradiction. ■

Proof of Theorem 8.1. Let $c := c_n = \dots = c_{n+r} \neq -\infty$, $r > 0$ and suppose

$$\mathrm{cat}(K_c, M) \leq r$$

Remark that, by (P-S), K_c is compact.
We take, according to Prop. 7.3, a neighborhood U of K_c such that

$$\mathrm{cat}(\bar U, M) = \mathrm{cat}(K_c, M) \leq r$$

Using Lemma 8.2 we find ε, $0 < \varepsilon < 1$, such that $W_\varepsilon \subset U$. Let $\alpha(x) := \alpha(1,x)$. We claim that

(8.5) $\quad \alpha(M_{c+\varepsilon^2/2} \setminus U) \subset M_{c-\varepsilon^2/2}$

In fact, let $x \in M_{c+\varepsilon^2/2} \setminus U$. Since $f(\alpha(\cdot,x))$ is non-increasing, if $x \in M_{c-\varepsilon^2/2}$ there is nothing to prove. Hence, we can assume

$$x \in \left(M_{c+\varepsilon^2/2} \setminus M_{c-\varepsilon^2/2}\right) \setminus U \subset$$
$$\subset (M_{c+\varepsilon} \setminus M_{c-\varepsilon}) \setminus U.$$

Since $W_\varepsilon \subset U$, it follows

$$\|f'_M(\alpha(t,x))\| \geq \varepsilon \qquad \forall t \in [0,1]$$

By (8.1) one has

$$f(x) - f(\alpha(x)) = \int_0^1 \varphi(\|f'_M(\alpha)\|) \, \|f'(\alpha)\|^2$$

Since $s^2\varphi(s)$ is increasing, we obtain

$$f(x) - f(\alpha(x)) \geq \varepsilon^2$$

Hence:

$$f(\alpha(x)) \leq f(x) - \varepsilon^2 \leq c - \varepsilon^2/2$$

proving (8.5).
Now, by the definition of $c = c_{n+r}$, given ε as before, there exists $A \in \mathcal{A}_{n+r}$, with $A \subset M_{c+\varepsilon^2/2}$.
Let $B := A \setminus U$. From Prop. 7.2-(2), it follows:

$$cat(B,M) \geq cat(A,M) - cat(U,M) \geq n+r-r = n$$

From Prop. 7.2-(3), we deduce

$$cat(\alpha(B), M) \geq cat(B,M) \geq n$$

Hence $\alpha(B) \in \mathcal{A}_n$. But (8.5) implies $\max_{\alpha(B)} f(x) \leq c - \varepsilon^2/2$ a contradiction with the definition of $c = c_n$. ■
This proves (ii). Similar arguments permit to prove (i).
Lastly, if f is bounded from below on M, then every $c_n \neq -\infty$ and (iii) follows from (i) and (ii). ■

Remark that in the case (ii), f has infinitely many

critical points at ~~the~~ level c. In fact a finite number of points would have category 1 with respect to M (which is connected).

For another approach to find critical points using L-S category, see [Bro].

§ 9. The case of $\mathbb{Z}_2$ symmetry.

By Theorem 8.1 the number of critical points of a functional on a manifold M is bounded from below by cat(M). ~~Unfortunately~~ Now, if we take $M = S = \{x \in E : \|x\| = 1\}$ as in the example in § 8, we have cat$(S) = 1$, ~~~~ and that result is no useful. On the other hand, we know that in the linear case, namely if $f(u) = (Lu, u)$ f has on S infinitely many critical points: the eigenfunctions of the linear problem $Lu = \lambda u$.

We will show, here, that this is true, more in general, for any functional and ~~~~ manifold which have a suitable symmetry. $\boxed{\text{it more precisely we will deal below with the case in which the group } \mathbb{Z}_2 \text{ acts on the Hilbert space } E \text{ through linear isometries } T_i \in \mathcal{L}(E,E)\ i = 0,1\ \ T_0 = Id_E \text{ and } T_1(u) = -u.}$

Let G be a group acting on E through $T_g \in \mathcal{L}(E)$ $g \in G$ with $T_0 = Id_E$ (0 = null element in G)

A set S is G-invariant if $T_g x \in S$ for all $x \in S$, $g \in G$.

T_g is free on S if $T_g x \neq x$ $\forall g \in G, g \neq 0$, $\forall x \in S$. A mapping $\varphi : E \to E$ is G-equivariant if $\varphi \circ T_g = T_g \circ \varphi$, $\forall g \in G$. Denoted by Σ the class of all (closed) subsets of $E - \{0\}$ which are G-invariants and such that T_g is free, we can define $\gamma_G : \Sigma \to \mathbb{N} \cup \{+\infty\}$ (index or G-genus) satisfying the following:

(1) $S \subset S'$ implies $\gamma_G(S) \leq \gamma_G(S')$

(2) $\gamma_G(S \cup S') \leq \gamma_G(S) + \gamma_G(S')$

(4) if $\varphi \in C(E,E)$ is G-equivariant, then $\gamma_G(\overline{\varphi(S)}) \geq \gamma_G(S)$

(3) if S is compact, then $\gamma_G(S) < \infty$ and there exists a neighborhood U of S such that $\overline{U} \in \Sigma$ and $\gamma_G(\overline{U}) = \gamma_G(S)$.

We will consider here the case in which $G = \mathbb{Z}_2 = \{0, 1\}$ and T_g are: $T_0 = Id$, ~~~~ $T_1 = -Id$.
~~~~ Now S is $\mathbb{Z}_2$-invariant provided it is symmetric with respect to $0 \in E$, i.e. if $-x \in S$ $\forall x \in S$; φ is equivariant if $\varphi(-x) = -\varphi(x)$, i.e. if φ is odd; The action is free on S provided ~~~~ $0 \notin S$ and Σ is the class of all closed subsets $S \subset E \setminus \{0\}$, symmetric with respect to 0.

When we will deal with periodic solvability of autonomous systems, we shall

The mapping ~~~~ γ (we will not write the subscript $\mathbb{Z}_2$ in this case) is defined ~~in~~ in the following way:

Definition: Let $S \in \Sigma$; $\gamma(S)$ is the least integer n such that $\exists\ \varphi \in C(S, \mathbb{R}^n)$, φ odd, $\varphi(x) \neq 0$ $\forall x \in S$. We define $\gamma(\emptyset) = 0$ and $\gamma(S) = +\infty$ if there are no integers with the above property.

In the definition we could take $\varphi \in C(E, \mathbb{R}^n)$, φ odd and $\varphi(x) \neq 0$ $\forall x \in S$. In fact every $\varphi \in C(S, \mathbb{R}^n)$ can be extended (by the Dugundji theorem) to a $\tilde{\varphi} \in C(E, \mathbb{R}^n)$ and then $\frac{1}{2}(\tilde{\varphi}(x) - \tilde{\varphi}(-x))$ is odd and $\neq 0$ $\forall x \in S$.

For example, if $S = A \cup (-A)$, with $A \subset E \setminus \{0\}$, closed and $A \cap (-A) = \emptyset$, then $\gamma(S) = 1$. In fact, we can define $\varphi : S \to \mathbb{R} - \{0\}$ by setting $\varphi(x) = a \neq 0$ for $x \in A$ and $\varphi(-x) = -a$.

The following is less trivial:
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9.1. Proposition. Let $E = \mathbb{R}^n$ and $A = \partial\Omega$, where Ω is an open bounded symmetric neighborhood of $0 \in \mathbb{R}^n$. Then $\gamma(A) = n$.

Proof. Evidently $\gamma(A) \leq n$, because the identity is admissible for the definition of genus. Suppose, by contradiction, that $\gamma(A) = n - k < n$. Then, in view of the definition and the subsequent remark, we can assume $\exists\, \varphi \in C(\mathbb{R}^n, \mathbb{R}^{n-k})$, φ odd and $\varphi(x) \neq 0$ $\forall x \in A$. Therefore the topological degree $d(\varphi, \Omega, 0)$ is well defined. Moreover, since φ is odd and Ω is symmetric with respect to the origin, then the Borsuk-Ulam theorem (cf., e.g., [Schw. 2, Chap. 3]) implies $\deg(\varphi, \Omega, 0) =$ odd integer $\neq 0$. Since $\deg(\varphi, \Omega, \cdot)$ is continuous, then also $\deg(\varphi, \Omega, \xi) \neq 0$ for ξ in some ball B_ε around the origin in $\mathbb{R}^n$. But then a well known property of the topological degree implies $\varphi(\Omega) \supset B_\varepsilon$, in contradiction with the fact that $\varphi(\Omega) \subset \subset \mathbb{R}^{n-k}$. ■

As consequence, we have

9.2. Corollary. If E is a separable infinite-dimensional Hilbert space, and $S = \{x \in E : \|x\| = 1\}$, then $\gamma(S) = +\infty$.

We prove now the properties (P_4) of γ.

~~8.1. The Properties~~

(1) trivial.

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(2) Let $\gamma(S) = n$ and $\gamma(S') = m$ (otherwise it is trivial) then $\exists\, \varphi \in C(E, \mathbb{R}^n)$ (resp. $\psi \in C(E, \mathbb{R}^m)$) both odd and such that $\varphi(x) \neq 0$ (resp. $\psi(x) \neq 0$) for all $x \in S$ ($\forall x \in S'$)

Define:

$$\chi(x) = (\varphi(x), \psi(x)) \in \mathbb{R}^{n+m}$$

Evidently $\chi \in C(E, \mathbb{R}^{n+m})$, χ is odd and $\chi(x) \neq 0$ $\forall x \in S \cup S'$ because at least one between $\varphi(x)$ and $\psi(x)$ are $\neq 0$.

(3) For all $x \in S$, we take $\varepsilon > 0$ such that $B_\varepsilon(x) \cap B_\varepsilon(-x) \neq \emptyset$, where $B_\varepsilon(x) = \{u \in E : \|u - x\| \leq \varepsilon\}$. We know that $\gamma(B_\varepsilon(x) \cup B_\varepsilon(-x)) = 1$. Since S is compact, it can be covered by a finite number of such balls, and by (2) we then deduce $\gamma(S) < \infty$. Let $\gamma(S) = n$. Then $\exists\, \varphi \in C(E, \mathbb{R}^n)$, φ odd and $\varphi(x) \neq 0$ $\forall x \in S$. For all $x \in S$, let $\delta > 0$ be such that $\varphi(y) \neq 0$ $\forall y \in B_\delta(x)$. Since S is compact $\exists\, \bar{\delta} > 0$ such that for every $y \in N_{\bar{\delta}}(S) = \{y \in E : \text{distance of } y \text{ to } S \text{ is} \leq \bar{\delta}\}$ $\varphi(y) \neq 0$. Thus, taking $\bar{U} = N_{\bar{\delta}}(S)$ we have $\gamma(\bar{U}) \leq n$. Since $\bar{U} \supset S$ we can conclude $\gamma(\bar{U}) = \gamma(S) = n$.

(4) Let α be a continuous odd map and $\gamma(\alpha(S)) = n$ (otherwise it is trivial). There is $\varphi \in C(E, \mathbb{R}^n)$, odd $\varphi(y) \neq 0$ $\forall y \in \alpha(S)$. Define $\psi = \varphi \circ \alpha$

$$\begin{tikzcd} S \arrow[r, "\alpha"] \arrow[dr, "\psi = \varphi\circ\alpha"'] & \alpha(S) \arrow[d, "\varphi"] \\ & \mathbb{R}^n - \{0\} \end{tikzcd}$$

The mapping $\psi \in C(E, \mathbb{R}^n)$ is odd and $\forall x \in S$ one has $\psi(x) = \varphi(\alpha(x)) \neq 0$, in view of the property of φ. Hence ψ is admissible for the definition of γ and it follows: $\gamma(S) \leq n$. ■

~~The relationship between the genus and the L-S category is the following~~

To sketch the relationships between genus and L-S category we consider, for example, the unit sphere S in an infinite-dimensional separable Hilbert space E. The projective space $\mathbb{P}^\infty$ is defined as the space of pairs $(u,-u)$ $u \in S$. For $A \subset S$, $A \in \Sigma$, we set $A^* = \{(u,-u) ; u \in A\}$. Then it is possible to show that

$$\gamma(A) = cat(A^*; \mathbb{P}^\infty)$$

We can use the genus to obtain existence of critical points in presence of symmetries: for example, theorem 8.1 can now be stated in the following way: let

(9.1) f is even and $M \in \Sigma$

For every $n \leq \gamma(M)$, set

$$A_n^* = \{A \subset M,\ A \text{ compact},\ A \in \Sigma,\ \gamma(A) \geq n\}$$

$$c_n^* = \inf_{A \in A_n^*} \max_{x \in A} f(x)$$

and suppose $A_n^* \neq \emptyset$.

9.3 Theorem. Suppose (9.1) holds and (f, M) satisfies (P-S). Then every $c_n^* \neq -\infty$ is a critical level and if $c^* = c_n^* = \dots = c_{n+r}^* \neq -\infty$ then $\gamma(K_{c^*}) \geq r+1$. If f is bounded from below on M, then f has on M at least $\gamma(M)$ pairs of critical points.

Proof. The argument is the same used in theorem 8.1 taking into account that, since f and h are even, then $F(u) = -\varphi(\|f_M'(u)\|) f_M'(u)$ is odd, then $\alpha(t, \cdot)$ is odd and property 4 of γ is applicable. ■

The analogous of Theorem 5.6 is:

9.4. Theorem. Let $f \in C^{1,1}(E, \mathbb{R})$ be even and satisfy (f5.1 – 5.2). Then the eigenvalue problem

(9.2) $$f'(u) = \lambda u \qquad \|u\| = 1$$

has infinitely-many solutions (λ, u). Moreover there exists a sequence λ_n of such λ such that $\lambda_n \to 0$.

Proof. We apply theorem 9.3, with $M = S = \{x \in E : \|x\| = 1\}$. Recall that, as in the proof of theorem 5.6, the (P-S) condition is satisfied in $S_a = \{x \in S : f(s) \leq a\}$ for any $a < 0$. However it is clear that $c_n^* < 0$ $\forall n$, and hence this is enough. Since $\gamma(S) = +\infty$ (cf. Corollary

9.2, we deduce (9.2) has infinitely many solutions (λ, u). It remains to prove that there exists a sequence of critical points u_n such that for the corresponding λ_n

$$\lambda_n = (f'(u_n), u_n)$$

one has $\lambda_n \to 0$. Now, for any $a < 0$, f is bounded on S_a and satisfies (P-S) there (cfr. remark before). Then it is known that $\gamma(S_a) < +\infty$. Hence, if we take $n > \gamma(S_a)$, for the corresponding c_n^* we must have $c_n^* \geq a$. Hence $\exists\, u_n \in K_{c_n^*}$ i.e. with $f(u_n) \to 0$. We can also assume that $u_n \to \bar{u}$ and one has $f(\bar{u}) = 0$ namely $\bar{u} = 0$. Hence $\lambda_n = (f'(u_n), u_n) \to (f'(\bar{u}), \bar{u}) = 0$, as required. ∎

Before we have used a result we state explicitly:

9.5 Theorem. *If (f, M) satisfies (P-S) and f is bounded on M, then $\mathrm{cat}(M) < \infty$. If f is even and $M \in \Sigma$, then $\gamma(M) < \infty$.*

As for theorem 5.7 we have here:

9.6 Theorem. *Suppose (g 5.1 - 3) hold and $g(x, -t) = -g(x, t)$: then the eigenvalue problem*

$$\begin{cases} -\Delta u = \lambda g(x, u) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

has infinitely many solutions satisfying $\|u\|_{1,2} = 1$. Moreover there exists a sequence (λ_n, u_n) of such solutions with $\lambda_n \to \infty$.

Proof. The oddness of $g(x, \cdot)$ implies $f(u) = \int G(x, u)$ is even. The remainder is the same. Remark that actually $\lambda_n = 1/((f(u_n), u_n))$ and hence $\lambda_n \to \infty$. ∎