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AUTUMN COURSE

ON

VARIATIONAL METHODS IN ANALYSIS AND MATHEMATICAL PHYSICS

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# "CLASSICAL" OPTIMAL CONTROL PROBLEMS

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"CLASSICAL" OPTIMAL CONTROL PROBLEMS

STATE EQUATION

$f(y, v) = 0 \longrightarrow y = y(v), \ v \in \mathcal{U}_{ad}$

COST FUNCTION

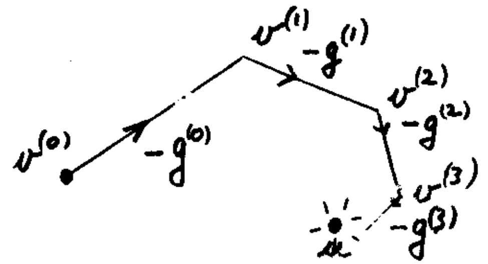
$J(y, v) = J(y(v), v) = J_1(v)$

PROBLEM

$$\inf_{v \in \mathcal{U}_{ad}} J_1(v)$$

SOLUTION BY A GRADIENT METHOD

$$J_1'(v) = \begin{bmatrix} \frac{\partial J_1}{\partial v_1} \\ \vdots \\ \frac{\partial J_1}{\partial v_N} \end{bmatrix} = g(v)$$






## How to find the gradient?

$J_1(v) = J(y(v), v), \quad f(y(v), v) = 0.$

Define Lagrangian $\mathcal{L}(y, v, p)$ independent variables!

$$\mathcal{L}(y, v, p) = J(y, v) + (p, f(y, v))$$

(i) Remark $\mathcal{L}(y(v), v, p) = J(y(v), v) = J_1(v)$

(ii) Thus $J_1'(v) = \frac{\partial \mathcal{L}}{\partial y} y'(v) + \frac{\partial \mathcal{L}}{\partial v} \quad \forall p$

$= \frac{\partial \mathcal{L}}{\partial v}$ if $\frac{\partial \mathcal{L}}{\partial y} = 0$

(iii) But $\frac{\partial \mathcal{L}}{\partial v} = \frac{\partial J}{\partial v} + \left(p, \frac{\partial f}{\partial v}\right) =$

$$\boxed{\frac{\partial J}{\partial v} + \left(\frac{\partial f}{\partial v}\right)^* p = J_1'(v)}$$

gradient

and

(iv) $\left(\frac{\partial \mathcal{L}}{\partial y}, \psi\right) = \left(\frac{\partial J}{\partial y}, \psi\right) + \left(p, \frac{\partial f}{\partial y}\psi\right) =$

$\left(\frac{\partial J}{\partial y} + \left(\frac{\partial f}{\partial y}\right)^* p, \psi\right) = 0 \quad \forall \psi \quad \Leftrightarrow$

$$\boxed{\frac{\partial J}{\partial y} + \left(\frac{\partial f}{\partial y}\right)^* p = 0}$$

adjoint state $p$.

Design of minimization of $J(y,v)$ subject to $f(y,v)=0$.

$v$ given

↓

calculate state $y(v)$
$f(y,v)=0$

↓

calculate adjoint state $p$
$\left[\frac{\partial f}{\partial y}(y,v)\right]^* p = -\frac{\partial J}{\partial y}(y,v)$

↓

calculate gradient $g = \frac{d}{dv} J(y(v),v)$
$g = \frac{\partial J}{\partial v}(y,v) + \left[\frac{\partial f}{\partial v}(y,v)\right]^* p$

Plug into any gradient method for minimizing a functional!



State equations

$$\frac{\partial s}{\partial t} - \Delta s + R(s,i) = 0 \quad , \quad \frac{\partial i}{\partial t} - \Delta i = 0,$$

$$s|_\Sigma = g(x), \; s(x,0) = 0 \quad , \quad i|_\Sigma = v(x,t), \; i(x,0) = 0.$$

Cost function

$$J(s,i,v) = \frac{1}{2}\int_\Sigma \left(\frac{\partial s}{\partial n} - z_d\right)^2 d\Sigma$$

Lagrangian

$$\mathcal{L}(s,i,v,p,q,r) = J(s,i,v) + \int_Q p\left(\frac{\partial s}{\partial t} - \Delta s + R\right) dQ + \int_Q q\left(\frac{\partial i}{\partial t} - \Delta i\right) dQ + \int_\Sigma r(i-v)\, d\Sigma$$

gradient

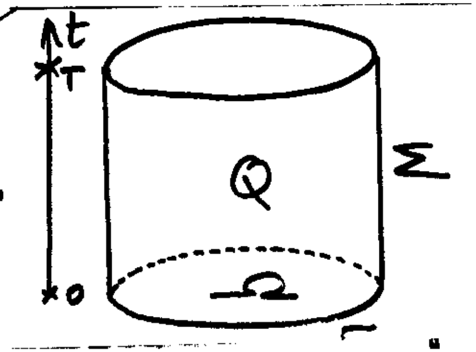
$$(g,\varphi) = \left(\frac{\partial \mathcal{L}}{\partial v}, \varphi\right) = -\int_\Sigma r\varphi\, d\Sigma \Rightarrow \boxed{g = -r}$$

provided we can achieve

$$\frac{\partial \mathcal{L}}{\partial s} = 0 \quad \left(\Leftrightarrow \left(\frac{\partial \mathcal{L}}{\partial s}, \psi\right) = 0 \quad \forall \psi \; \begin{cases} \psi(x,t) = 0 & t = 0 \\ \psi(x,t) = 0 & x \in \Gamma \end{cases}\right)$$

$$\frac{\partial \mathcal{L}}{\partial i} = 0 \quad \left(\Leftrightarrow \left(\frac{\partial \mathcal{L}}{\partial i}, \psi\right) = 0 \quad \forall \psi \quad \psi(x,0) = 0\right)$$

$Q = \Omega \times ]0,T[$
$\Sigma = \Gamma \times ]0,T[$
$dQ = dx\, dt, \; dx = dx_1 \ldots dx_n$
$d\Sigma = d\Gamma\, dt$



$$\frac{\partial \mathcal{L}}{\partial s} = 0 \quad ?$$

$$\mathcal{L}(s,i,v,p,q,r) = \frac{1}{2}\int_\Sigma \left(\frac{\partial s}{\partial n} - z_d\right)^2 d\Sigma + \int_Q p\left(\frac{\partial s}{\partial t} - \Delta s + R(s,i)\right) dx\,dt + \int_Q q\left(\frac{\partial i}{\partial t} - \Delta i\right) dx\,dt + \int_\Sigma r(i - v)\, d\Gamma\, dt.$$

$$\left(\frac{\partial \mathcal{L}}{\partial s}, \psi\right) = \int_\Sigma \left(\frac{\partial s}{\partial n} - z_d\right)\frac{\partial \psi}{\partial n}\, d\Sigma + \int_Q p\left(\frac{\partial \psi}{\partial t} - \Delta\psi + \frac{\partial R}{\partial s}\psi\right) dQ.$$

$$= \int_\Sigma \left(\frac{\partial s}{\partial n} - z_d\right)\frac{\partial \psi}{\partial n}\, d\Sigma + \int_\Omega \Big(\cancel{p(x,T)\overset{\text{o.k}}{\psi(x,T)}} - p\cancel{(x,0)\psi(x,0)}\Big) dx$$

$$- \int_Q \frac{\partial p}{\partial t}\psi\, dx\,dt - \int_\Sigma p\frac{\partial \psi}{\partial n}\, d\Sigma + \cancel{\int_\Sigma \frac{\partial p}{\partial n}\psi\, d\Sigma}$$

$$- \int_Q (\Delta p)\psi\, dx\,dt + \int_Q \frac{\partial R}{\partial s} p\psi\, dx\,dt = 0 \quad \forall \psi,$$

$\psi = 0$ for $t = 0$ and $\psi = 0$ on $\Sigma$, $\Longleftrightarrow$

$$\boxed{\begin{array}{ll} -\dfrac{\partial p}{\partial t} - \Delta p + \dfrac{\partial R}{\partial s} p = 0 & \text{in } \Omega \times ]0,T[ \\ p = 0 & \text{on } \Sigma \\ p(T) = 0 & \end{array}}$$

$p$ = adjoint state for $s$.



$$\frac{\partial \mathcal{L}}{\partial i} = 0 \quad ?$$

$$\mathcal{L}(s,i,v,p,q,r) = \frac{1}{2}\int_\Sigma \left(\frac{\partial s}{\partial n} - z_d\right)^2 d\Sigma + \int_Q p\left(\frac{\partial s}{\partial t} - \Delta s + R(s,i)\right) dx\,dt + \int_Q q\left(\frac{\partial i}{\partial t} - \Delta i\right) dx\,dt + \int_\Sigma r(i - v)\, d\Gamma\, dt$$

$$\left(\frac{\partial \mathcal{L}}{\partial i}, \psi\right) = \int_Q p\frac{\partial R}{\partial i}\psi\, dx\,dt + \int_Q q\left(\frac{\partial \psi}{\partial t} - \Delta\psi\right) dx\,dt + \int_\Sigma r\psi\, d\Sigma = 0 \quad \forall \psi,$$

$\psi = 0$ at $t = 0$, $\Longleftrightarrow$

$$\int_Q \frac{\partial R}{\partial i} p\psi\, dx\,dt + \int_\Omega \big(q(x,T)\psi(x,T) - q(x,0)\psi(x,0)\big) dx$$

$$- \int_Q \frac{\partial q}{\partial t}\psi\, dx\,dt - \int_\Sigma q\frac{\partial \psi}{\partial n}\, d\Sigma + \int_\Sigma \frac{\partial q}{\partial n}\psi\, d\Sigma$$

$$- \int_Q (\Delta q)\psi\, dx\,dt + \int_\Sigma r\psi\, d\Sigma = 0 \quad \forall \psi,\ \psi(0) = 0$$

$\Longleftrightarrow$

$$\boxed{\begin{array}{l} -\dfrac{\partial q}{\partial t} - \Delta q = -\dfrac{\partial R}{\partial i} p \\ q_{|\Sigma} = 0 \\ q(T) = 0 \end{array}} \quad \boxed{r = -\frac{\partial q}{\partial n}}$$

$q$, state adjoint to $i$

$r$, state adjoint to the constraint $i_{|\Sigma} = v$.

$$\boxed{g = \frac{\partial q}{\partial n}}.$$

**Difficulty with optimal control of systems with multiple steady states.**

state equation $f(y,v)=0 \not\Rightarrow y=y(v)$

NO MORE UNIQUELY DEFINED

cost function

$J(y,v)$

**Possible solution to this difficulty** (J.L. LIONS)

Consider the problem for $x=(y,v)$:

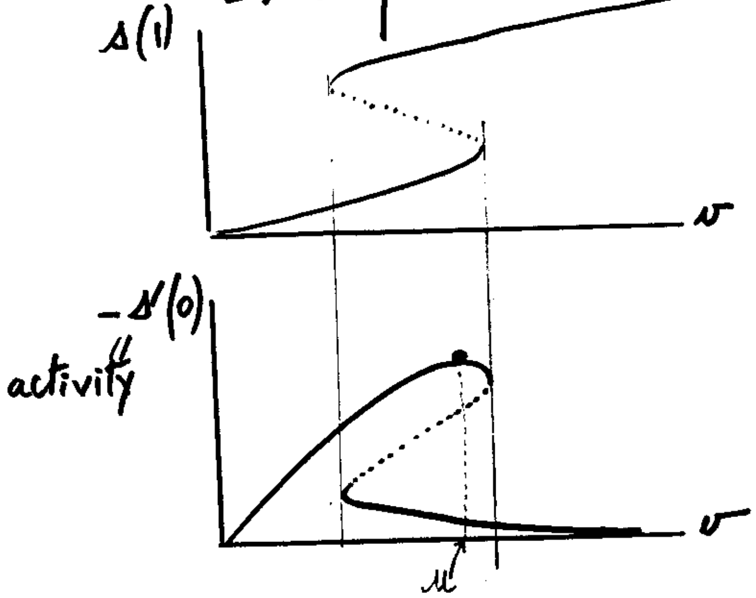
$$\inf J(x)$$

subject to the constraint

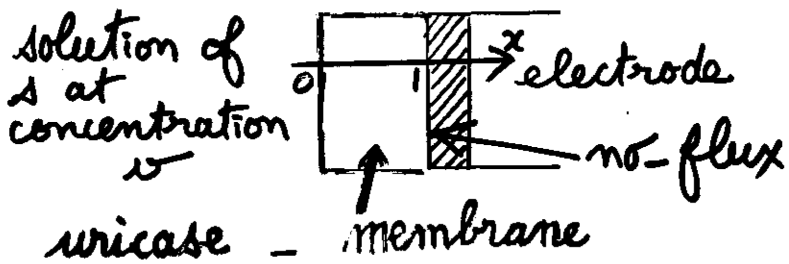
$$f(x)=0 .$$



Example

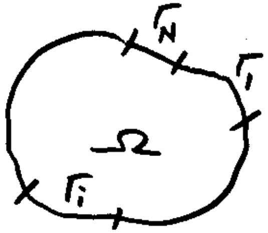



$$\boxed{\begin{aligned} &-\frac{d^2 s}{dx^2} + F(s) = 0 \ , \ s(0) = v \ , \ s'(1) = 0 \\ &F(s) = \sigma\, s \,/\, (1 + s + k s^2) \end{aligned}}$$



# Optimal control problem

## State (s!)




$$(1)\quad \begin{cases} -\Delta s + F(s) = 0 & \text{in } \Omega \\ s|_{\Gamma_i} = v_i \ , \ i = 1, 2, \dots, N \ . \end{cases}$$

**Control** $v = (v_1, \dots, v_N) \in \mathbb{R}_+^N$ .

**Cost function**

$$J(s,v) = -\int_\Gamma \frac{\partial s}{\partial n}\, d\Gamma = -\int_\Omega F(s(x))\, dx \ .$$

**Pb.**

$$\boxed{\begin{array}{l} \inf J(s,v) \\ \begin{cases} (s,v) \text{ constrained by } (1) \\ v \in \mathbb{R}_+^N \end{cases} \end{array}}$$

**Assumptions** : $s, \Delta s \in L^2(\Omega)$

($\Rightarrow s|_\Gamma = v$ makes sense )



## Transformation of the problem . .

(i) Define $y(v) \in L^2(\Omega)$ by

$$\boxed{\begin{cases} -\Delta y = 0 & \text{in } \Omega \\ y|_\Gamma = v \end{cases}}$$

(ii) Define $z = s - y$ .

The new problem is

$$(2)\quad \boxed{\begin{array}{l} \text{Minimize } J(z,v) = -\int_\Omega F(y(v)+z)\, dx \\ z \text{ and } v \text{ being related by} \\ \quad -\Delta z + F(y(v)+z) = 0, \ z|_\Gamma = 0 \end{array}}$$

**Remark.** If $\Gamma$ is smooth enough,

$(2) \Rightarrow \qquad z \in H^2(\Omega) \cap H_0^1(\Omega)$

and even, since

$$F(u) = \sigma \frac{u}{1+u+ku^2} < \sigma ,$$

$$\boxed{z \in \text{bounded set of } H^2(\Omega) \cap H_0^1(\Omega)} \ .$$

## Existence of an optimal pair $(\tilde{z}, u)$.

Let $(z_m, v_m)$ be a minimizing sequence

i.e. $J(z_m, v_m) \to \inf J(z, v) = \mu$.

Remark that $-\infty < \mu < 0$ since

$$J(z, v) = -\int_\Omega F(\dots)\,dx \geq -\sigma\, \mathrm{mes}(\Omega).$$

Assume (see below) we know that $\left\| \begin{array}{l} v_m \in \text{bounded set of } \mathbb{R}_+^N \\ z_m \in \quad \text{—} \quad H^2 \cap H_0^1. \end{array} \right.$

Thus, after eventually extraction of a subsequence

$$\left[ \begin{array}{l} v_m \to u \text{ in } \mathbb{R}_+^N,\ z_m \rightharpoonup \tilde{z} \text{ in } H^2 \cap H_0^1 \text{ weak}, \\ z_m \to \tilde{z} \text{ in } H^1 \text{ strong}, L^2 \text{ strong and a.e.}, \\ y(v_m) \to y(u) \text{ in } L^2(\Omega) \text{ strong} \\ F(y(v_m) + z_m) \to F(y(u) + \tilde{z}) \text{ in } L^2 \text{ strong} \\ -\Delta z_m \rightharpoonup -\Delta \tilde{z} \text{ in } L^2 \text{ weak}, \text{ so that} \end{array} \right.$$

$$\boxed{\begin{array}{l} -\Delta \tilde{z} + F(y(u) + \tilde{z}) = 0,\ \tilde{z}|_\Gamma = 0 \\ J(\tilde{z}, u) = \inf J(z, v) \end{array}}$$

$$-\Delta \hat{s} + F(\hat{s}) = 0, \quad \hat{s}|_\Gamma = u$$



## Boundedness of $v_m$ for a minimizing sequence.

Define $w_i$ $(i = 1, 2, \dots, N)$ by

$$\left\{ \begin{array}{ll} -\Delta w_i + \sigma w_i = 0 & \text{in } \Omega \\ w_i|_{\Gamma_j} = \delta_{ij} & , j = 1, \dots, N \end{array} \right.$$

Define $w^m$ as $w^m = \sum_{i=1}^N (v_m)_i\, w_i$, so that

$$\left\{ \begin{array}{ll} -\Delta w^m + \sigma w^m = 0 & \text{in } \Omega \\ w^m|_\Gamma = v_m \end{array} \right.$$

Consider the difference $s_m - w^m = p_m$:

$-\Delta p_m + \sigma p_m = \sigma s_m - F(s_m) \geq 0$, $p_m|_\Gamma = 0$,

whence $p_m \geq 0$, or $s_m \geq w^m$.

Suppose $v_m \to \infty$ in $\mathbb{R}^N$. Then

$w^m(x) \to \infty$ a.e., $s_m(x) \geq w^m(x) \to \infty$,

$F(s_m(x)) \to 0$ a.e., $\int_\Omega F(s_m(x)) \to 0$.

But this contradicts the fact that

$J(s_m, v_m) \to \mu < 0$. ∎

# Optimality conditions

In the classical case

$$\boxed{\begin{aligned} & f(\tilde{y},u)=0 \\ & \left[\frac{\partial f}{\partial y}(\tilde{y},u)\right]^* p = -\frac{\partial J}{\partial y}(\tilde{y},u) \\ & g = \left[\frac{\partial f}{\partial v}(\tilde{y},u)\right]^* p + \frac{\partial J}{\partial v}(\tilde{y},u) \\ & (g, v-u) \geq 0 \quad \forall v \in K . \end{aligned}}$$

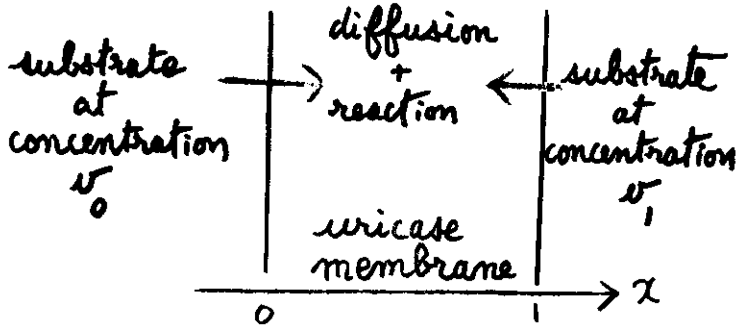
Here we cannot obtain directly these conditions.

The method for obtaining them will be through <u>penalty method</u> (J.L. LIONS).



# Optimal control problem

<u>The physical system</u>




<u>Relation between state $z$ and control $v$.</u>

$$(C) \quad \boxed{\begin{aligned} & -z''(x) + F(z(x)) = 0 \qquad 0<x<1 \\ & z(0)=v_0 \quad , \; z(1)=v_1 \end{aligned}}$$

$$F(z) = \sigma z / (1+z+z^2)$$

<u>Constraints</u>

$$z, z'' \in L^2(0,1)$$

$$z(0)=v_0 \geq 0 \;, \; z(1)=v_1 \geq 0$$

<u>Cost function</u>

$$J(z,v) = -\int_0^1 z(x)\,dx$$

$z, v$ related by $(C)$

## Penalty method

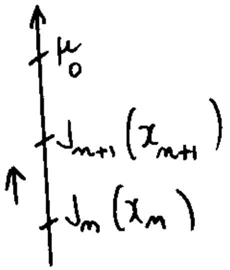
Suppose $\mu_0 = \min \{ J(x) : f(x) = 0 \}$ (1)

Define $J_m(x) = J(x) + K_m f(x)$ with $K_m \uparrow$, $K_m \to \infty$

Suppose $J_m(x)$ is minimum for $x_m$ :

$$J_m(x_m) \leq J_m(x) \quad \forall x$$

Then we have :




1. $J_{m+1}(x_{m+1}) \geqslant J_m(x_m)$
2. $\mu_0 \geqslant J_m(x_m)$
3. $\lim K_m f(x_m) = 0$
4. if $J$ and $f$ are lower semi continuous and $x_m \to x_0$, then $x_0$ is solution of problem (1)



## Penalized problem

$$\boxed{J_m(z, v) = -\int_0^1 F(z(x))\,dx + K_m \left[ \int_0^1 \left(-z''(x) + F(z(x))\right)^2 dx + (z(0) - v_0)^2 + (z(1) - v_1)^2 \right]}$$

$z, z'' \in L^2(0,1)$ , $v_0, v_1 \geq 0$ .

Remark : we can restrict to those $(z, v)$ such that

$$J_m(z, v) \leq J_m(\tilde{z}, \tilde{v}) = \mu_0$$

Thus $\int_0^1 (-z'' + F(z))^2 dx + (z(0) - v_0)^2 + (z(1) - v_1)^2 \leq$ constant

whence

| | | |
|---|---|---|
| $z'' + F(z)$ | $\in$ | bounded set of $L^2(0,1)$ |
| $z''$ | $\in$ | — |
| $z$ | $\in$ | Bounded set of $H^2(0,1)$ |
| $z(0), z(1)$ | $\in$ | — $\mathbb{R}$ |
| $v_0, v_1$ | $\in$ | — $\mathbb{R}_+^2$ . |

## Consequence for a fixed value of $n$. 

Consider a minimizing sequence $x^k = (z^k, v^k)$:

$J_n(z^k, v^k) \to \inf J_n(z, v) = \mu_n$.

One can extract a subsequence such that

$$\left|\begin{array}{llll} z^k & \rightharpoonup z_n & \text{in } H^2 \\ z^k & \to z_n & \text{in } L^2 \\ v^k & \to v_n & \text{in } \mathbb{R}_+^2 \end{array}\right.$$

$$\begin{array}{l} \mu_0 \\ J_n(x^k) \downarrow \\ \mu_n \; J_n(x_n) \end{array}$$

Recall

$$J_n(z^k, v^k) = -\int_0^1 F(z^k)\,dx + K_n\Big[\int_0^1 (-z^{k\,\prime\prime} + F(z^k))^2\,dx + (z^k(0) - v_0^k)^2 + (z^k(1) - v_1^k)^2\Big].$$

Then

$$\mu_n = \lim\inf J_n(z^k, v^k) \geq -\int_0^1 F(z_n)\,dx + K_n\Big[\int_0^1 (-z_n'' + F(z_n))^2\,dx + (z_n(0) - v_{n0})^2 + (z_n(1) - v_{n1})^2\Big] = J(z_n, v_n)$$

Thus we have <u>existence of $x_n$</u> $(z_n, v_n)$ minimizing $J_n(x)$. Also:

$$\left|\begin{array}{l} -z_n'' + F(z_n) \to 0 \quad L^2 \\ z_n(0) - v_{n0} \to 0\,, \quad z_n(1) - v_{n1} \to 0. \end{array}\right.$$

## <u>Consequence as $n \to \infty$</u> 

After eventually extracting a subsequence $(z_n, v_n)$, we have

$$\begin{array}{lll} z_n \longrightarrow \tilde{\tilde{z}} & & H^2 \\ z_n \longrightarrow \tilde{\tilde{z}} & & L^2 \\ v_n \longrightarrow \tilde{\tilde{v}} = (\tilde{\tilde{v}}_0, \tilde{\tilde{v}}_1) & & \mathbb{R}_+^2 \end{array}$$

$$\begin{cases} -\tilde{\tilde{z}}'' + F(\tilde{\tilde{z}}) = 0 \\ \tilde{\tilde{z}}(0) = \tilde{\tilde{v}}_0\,, \quad \tilde{\tilde{z}}(1) = \tilde{\tilde{v}}_1 \end{cases}$$

$\tilde{\tilde{x}} = (\tilde{\tilde{z}}, \tilde{\tilde{v}})$ sol. of original problem.

$$\left( \begin{array}{l} \mu_0 \leq J(\tilde{\tilde{x}}) \\ J(x_n) \leq J_n(x_n) \leq \mu_0 \\ \downarrow \\ J(\tilde{\tilde{x}}) \end{array} \right).$$

Optimality condition for $z_m, \chi_m$.

$$\frac{\partial J_m}{\partial z}(z_m, \chi_m) = 0$$

$$\frac{\partial J_m}{\partial v_0}(z_m, \chi_m)(v_0 - v_{m0}) \geq$$

$$\frac{\partial J_m}{\partial v_1}(z_m, v_m)(v_1 - v_{m1}) \geq$$

$$- z_m'' + F(z_m) = \ldots$$

$$\begin{cases} - p_m'' + F'(z_m)\, p_m = F(z_m) \\ p_m(0) = p_m(1) = 0 \end{cases}$$

$$\begin{cases} - p_m'(0) + 2K_m(z_m(0) - v_{m0}) = 0 \\ p_m'(1) + 2K_m(z_m(1) - v_{m1}) = 0 \end{cases}$$

$$\begin{cases} - p_m'(0)(v_0 - v_{m0}) \geq 0 & \forall v_0 \geq 0 \\ p_m'(1)(v_1 - v_{m1}) \geq 0 & \forall v_1 \geq 0 \end{cases}$$



A priori estimate on $p_m$.

$p_m$ is bounded in $L^2(\Omega)$

if not, let $q_m = p_m / \|p_m\|_{L^2}$.

$$\begin{cases} - q_m'' + F'(z_m)\, q_m = \frac{1}{\|p_m\|} F'(z_m) \\ q_m(0) = q_m(1) = 0 \\ - q_m'(0)(v_0 - v_{m0}) \geq 0 & \forall v_0 \geq 0 \\ q_m'(1)(v_1 - v_{m1}) \geq 0 & \forall v_1 \geq 0 \end{cases}$$

$q_m, q_m'' \in$ bounded set of $L^2(0,1)$

$q_m \quad \in$ bounded set of $H^2 \cap H^1$

$q_m \to q \quad H^2 \cap H_0^1$ weak

$q_m \to q \quad L^2$ strong

$$\begin{cases} - q'' + F'(\tilde{\tilde{z}})\, q = 0 \\ q(0) = q(1) = 0 \\ - q'(0)(v_0 - \tilde{\tilde{v}}_0) \geq 0 & \forall v_0 \\ q'(1)(v_1 - \tilde{\tilde{v}}_1) \geq 0 & \forall v_1 \end{cases}$$

Since $(\tilde{\tilde{v}}_0, \tilde{\tilde{v}}_1) \neq (0,0)$, this implies $q'(0) = 0$ for example $\Rightarrow q \equiv 0$, and $\|q\| = 1$ !

System of optimality

$$\boxed{\begin{aligned} & -z'' + F(z) = 0 \\ & z(0) = v_0 \ , \ z(1) = v_1 \\ & -p'' + F'(z)\,p = F'(z) \\ & p(0) = p(1) = 0 \\ & -p'(0)\,(v - v_0) \geq 0 \qquad \forall v \geq 0 \\ & p'(1)\,(v - v_1) \geq 0 \qquad \forall v \geq 0 \end{aligned}}$$

$$\boxed{\begin{aligned} & -z'' + F(z) = 0 \\ & -p'' + F'(z)\,p = F'(z)\ ,\ p(0) = p(1) = 0 \\ & p'(0) = p'(1) = 0 \end{aligned}}$$

$$\boxed{\begin{aligned} & -z'' + F(z) = 0 \ , \ z(0) = 0 \\ & -p'' + F'(z)\,p = F'(z)\ ,\ p(0) = p(1) = 0 \\ & p'(1) = 0 \end{aligned}}$$