INTERNATIONAL ATOMIC ENERGY AGENCY
UNITED NATIONS EDUCATIONAL, SCIENTIFIC AND CULTURAL ORGANIZATION

INTERNATIONAL CENTRE FOR THEORETICAL PHYSICS
34100 TRIESTE (ITALY) - P.O.B. 586 - MIRAMARE - STRADA COSTIERA 11 - TELEPHONES: 224281/2/3/4/5/6
CABLE: CENTRATOM - TELEX 460392 I

SMR/92 - 17

AUTUMN COURSE

ON

VARIATIONAL METHODS IN ANALYSIS AND MATHEMATICAL PHYSICS

20 October - 11 December 1981

OPTIMAL CONTROL OF GAS TRANSPORTATION NETWORK

C. SAGUEZ

Institut National de Recherche
en Informatique et en Automatique
Domaine de Voluceau - Rocquencourt
B.P. 105
78150 Le Chesnay
France

OPTIMAL CONTROL OF GAS TRANSPORTATION NETWORK

The problem of optimal control of transportation network is very important from an industrial point of view. Here we consider the case of a gas transportation network. Mathematically, we have an optimal control problem of a system of non linear transport equations,

The lecture is divided in three parts :

- Physical problem and mathematical formulation,

- Theoritical results
  - . State equations
  - . Optimal control problem

- Algorithm and numerical results.

This work has been done at INRIA by A. SORINE - A. BAMBERGER - J.P. YVON in collaboration with G.D.F. (Gaz de France).

For details we refer to M. SORINE - M. SOULAS [3].

I - PHYSICAL PROBLEM and MATHEMATICAL FORMULATION.

A gas transportation network is constitued of pipes connecting by nodes (See figure 1.1). We have thre types of nodes :

- the feeding nodes
- the consumption nodes
- the pumping stations.

As example, we consider for the numerical experiences, we present figure 1.1. an important part of the French network, feeding the Paris area. It is about 192 km long.

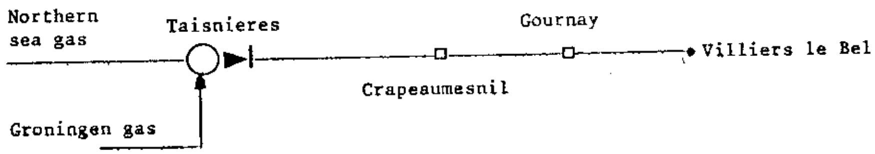

Figure 1.1.

The problem is to determine the optimal power , function of time, of the pumping station, such that the flow entering the feeding nodes remains as closed as possible to given functions.

To present the mathematical formulation, in a simple way, we consider the following network with a pumping station.

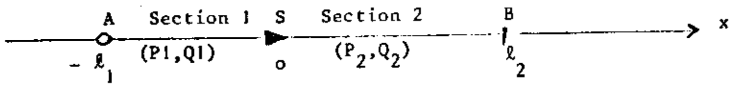

Figure 1.2.

S is a pumping station, A a feeding node and B a consumption node.

I-1) State equations.

For each pipe, the state of the gas is characterized by the pressure P and the gas velocity Q, which verify the following equations.

$$\begin{cases} \dfrac{\partial P_i}{\partial t} + c_{1i}\dfrac{\partial Q_i}{\partial x} = 0 & (1.1) \\[2ex] \dfrac{\partial P_i}{\partial x} + c_{2i}\dfrac{Q_i|Q_i|}{P_i} = 0 \qquad i=1,2 & (1.2) \end{cases}$$

The boundary conditions are :

- for feeding node A :

$$P_1(-\ell_1, t) = P_A(t) \tag{1.3}$$

- for the pumping station S :

$$\begin{cases} P_2(0,t) = P_1(0,t)\, \exp(r\, u(t)/q(t)) \\[1ex] Q_2(0,t) = Q_1(0,t) = q(t) \end{cases} \tag{1.4}$$

- for the consumption node B :

$$Q_2(\ell_2, t) = Q_B(t) \tag{1.5}$$

where $P_A(t)$ the pressure at point A and $Q_B(t)$ the consumption at point B are given,

and we have the initial condition :

$$P_i(x,0) = P_{i0}(x) \qquad i=1,2 \tag{1.6}$$

I-2) Optimal control problem.

The control variable is u(t), the power of the pumping station. The objective is to remain the flow entering the feeding node A as closed as possible to a given function $Q_d(t)$. So we introduce the cost function :

$$J(u) = \int_0^T |Q_1(-\ell_1, t) - Q_d(t)|^2\, dt \tag{1.7}$$

We have the following constraints on the control and on the state (to respect the security norms) :

- at the consumption node B :

$$P_2(\ell_2, t) \geq P_{min} \tag{1.8}$$

- at the pumping station :

$$P_2(0,t) \leq P_{Max} \tag{1.9}$$

$$Q_{min} \leq q(t) \leq Q_{Max} \tag{1.10}$$

$$\frac{P_2(0,t)}{P_1(0,t)} \leq \tau_{Max} \tag{1.11}$$

$$u_{min} \leq u(t) \leq u_{Max} \tag{1.12}$$

Then, if we denote by $\mathcal{U}_{ad}$ the admissible set of control

$$\mathcal{U}_{ad} = \{ u(t) \in L^2(0,T) \mid u(t) \text{ s.t. (1-8)} - \text{(1-12) are verified} \}$$

we have the optimal control problem.

$$
(1.13)\quad \begin{cases} \text{To find } \bar{u} \in \mathcal{U}_{ad} \text{ s.t} \\ J(\bar{u}) \leqslant J(u) \quad \forall u \in \mathcal{U}_{ad}. \end{cases}
$$

II – THEORICAL RESULTS.

In this chapter, we present some theoritical results for the problem (1.13).

II-1) The state equation.

To study the state equation for a pipe, an efficient method is to eliminate the variable Q. Then, if we denote by y(x,t), the variable:

$$
(2.1)\quad y_i(x,t) = P_i^2(x,t)
$$

$y_i(x,t)$ is solution of :

$$
(2.2)\quad \frac{\partial}{\partial t}\beta(y_i) - \sigma\frac{\partial}{\partial x}\left(\beta\left(\frac{\partial y_i}{\partial x}\right)\right) = 0 \quad ; \quad \text{with } \beta(\lambda) = |\lambda|^{-1/2}\lambda
$$

$$
\sigma = c_1/\sqrt{2c_2}
$$

with B.C. and I.C. $\left(y_i(x,0) = y_0(x)\right)$

This equation is equivalent to (1.1)-(1.2).

Remark : (2.2) is a non linear degenerated parabolic equation. The study of such equations has been done by A. BAMBERGER[1]. For differents boundary conditions, he proves the existence of a solution and for the system associated to the network figure 1.2.

We have the results (A. BAMBERGER [1]) : (we suppose $\Omega =]0,1[$)

- Neumann boundary conditions, i.e.

$$
\beta\left(\frac{\partial y}{\partial x}\right)(0) = d_0(t) \quad ; \quad \beta\left(\frac{\partial y}{\partial x}\right)(1) = d_1(t)
$$

under the assumptions

$$
y_0 \in L^\infty(\Omega) \quad ; \quad d_0(t) \text{ and } d_1(t) \text{ continuous on }]0,T[
$$

there exists an unique solution :

$$
y \in L^{3/2}(0,T;\, H^1(\Omega)) \cap L^\infty(\Omega \times]0,T[)
$$

$$
\frac{d}{dt}\beta(y) \in L^3(0,T;\, H^{-1}(\Omega))
$$

- Dirichlet boundary conditions, i.e.

$$
y(0,t) = P_0(t) \quad ; \quad y(1,t) = P_1(t)
$$

under the assumptions $y_0 \in C^1(\Omega)$; $P_0 \in C^1(0,T)$; $P_1 \in C^1(0,T)$

and $P_0(0) = y_0(0)$; $P_1(0) = y_0(1)$

there exists an unique solution :

$$
y \in L^{3/2}(0,T;\, H^1(\Omega)) \cap L^\infty(\Omega \times]0,T[)
$$

$$
\frac{d}{dt}\beta(y) \in L^3(0,T;\, H^{-1}(\Omega))
$$

II-2) Optimal control problem.

To solve the optimal control problem, we introduce the penalized functionnal:

$$
\begin{aligned}
J_\varepsilon(u) = J(u) &+ \frac{1}{\varepsilon_1}\Bigg[\int_0^T \left\{\left[P_2(\ell_2,t) - P_{min}\right]^-\right\}^2 dt \\
(2.3)\qquad &+ \int_0^T \left\{\left[P_2(0,t) - P_{max}\right]^+\right\}^2 dt + \int_0^T \left\{\left[P_2(0,t) - \tau_{max} P_1(0,t)\right]^+\right\}^2 dt\Bigg] \\
&+ \frac{1}{\varepsilon_2}\left[\int_0^T \left\{\left[q(t) - Q_{min}\right]^-\right\}^2 dt + \int_0^T \left\{\left[q(t) - Q_{max}\right]^+\right\}^2 dt\right]
\end{aligned}
$$

with $\varepsilon_1 \in \mathbb{R}^+$, $\varepsilon_2 \in \mathbb{R}^+$

So we have the penalized optimal control problem :

$$(2.4)\quad \begin{cases} \text{To find } \bar{u}_\varepsilon \in \tilde{\mathcal{U}}_{ad} \text{ s.t} \\ J_\varepsilon(\bar{u}_\varepsilon) \leq J_\varepsilon(u) \quad \forall u \in \tilde{\mathcal{U}}_{ad} \end{cases}$$

with $\tilde{\mathcal{U}}_{ad} = \{ u \in L^2(0,T) \mid u_{min} \leq u(t) \leq u_{max} \}$

Then we have when

$$\varepsilon_1, \varepsilon_2 \longrightarrow +\infty \; ; \; \bar{u}_\varepsilon \longrightarrow \bar{u}$$

To solve the optimal control problem (2.4), we want to use a gradient. For this, we introduce the following adjoint state equations. (ϕ_i, ψ_i are the adjoint states on the pipe i).

$$\begin{cases} (2.5) & -\dfrac{\partial \varphi_i}{\partial t} + C_{1i}\,|P_i|\,\dfrac{\partial \psi_i}{\partial t} = 0 \\ (2.6) & \dfrac{\partial \varphi_i}{\partial x} + 2\,C_{2i}\,|Q_i|\,\psi_i = 0 \qquad i = 1,2 \end{cases}$$

With the boundary conditions :

$$(2.7)\quad \psi_1(-l_1,t) = 2\left(Q_1(-l_1,t) - Q_d(t)\right)$$

$$(2.8)\quad \psi_2(l_2,t) = -\frac{2}{\varepsilon_1}\left(P_2(l_2,t) - P_{min}\right)^- / P_2(l_2,t)$$

$$(2.9)\quad \psi_1(0,t)\left[P_1(0,t)\right]^2 - \psi_2(0,t)\left[P_2(0,t)\right]^2 = \frac{2}{\varepsilon_1}\Big[P_2(0,t)\left(P_2(0,t) - P_{max}\right)^+ + \left(|P_2(0,t)| - \tau_{max}|P_1(0,t)|\right)\left(P_2(0,t) - \tau_{max}P_1(0,t)\right)^+ \Big]$$

$$(2.10)\quad \varphi_1(0,t) - \varphi(0,t) - \psi_1(0,t)\,|P_1(0,t)|\,\frac{2u(t)}{(q(t))^2}\,P_1(0,t) = \frac{2}{\varepsilon_2}\Big[\left(q(t) - Q_{min}\right)^- - \left(q(t) - Q_{max}\right)^+ + \tau_{max}\frac{\varepsilon_2}{\varepsilon_1}\frac{2u(t)}{(q(t))^2}P_1(0,t)\left(P_2(0,t) - \tau_{max}P_1(0,t)\right)^+ \Big]$$

and the final condition

$$(2.11)\quad \varphi_i(x,T) = 0 \qquad i = 1,2$$

Then the gradient of $J_\varepsilon(u)$ is given by :

$$(2.12)\quad 2\,|P_1(0,t)|\,P_1(0,t)\,\psi_1(0,t)/q(t)$$

III - <u>ALGORITHM AND NUMERICAL RESULTS.</u>

In fact to solve the initial problem (1.13) we consider the problem (2.4). We use a conjugate gratient algorithm with projection on the admissible set of control $\tilde{\mathcal{U}}_{ad}$.

To solve the state equations, we use an implicit finite difference method. At each time step, the discretized nonlinear system is solved by a Newton procedure.

The adjoint state equations is solved using the implicit finite difference scheme fit to the one chosen for the state equations.

Figures 3.1 - 3.2, we present results obtained for the network figure 1.1. The control variable is the pumping station Taisnieres.

For Trials 1-2-3, we take $Q_d(t)$ = 487 750 m^3/h.

- for Trial 1, we see that the period is too short (we begin to central at 12 h15). It is impossible to obtain the desired flow level ;

- for Trial 2, state constraints are relaxed. In this case we obtain the desired flow level, but the pressure constraint at Taisnieres is not verified ;

- for Trial 3, we consider all the constraints. Then we don't obtain the desired flow level.

For Trial 4, we take $Q_d(t)$ = 425 000 m^3/h. Then it is possible to obtain the desired flow level, when all constraints are taken into account.

Then we see that with this method, it is possible to determined a feasible flow level at the beginning of each period (on day for example). This fact is the main problem of the dispatcher.

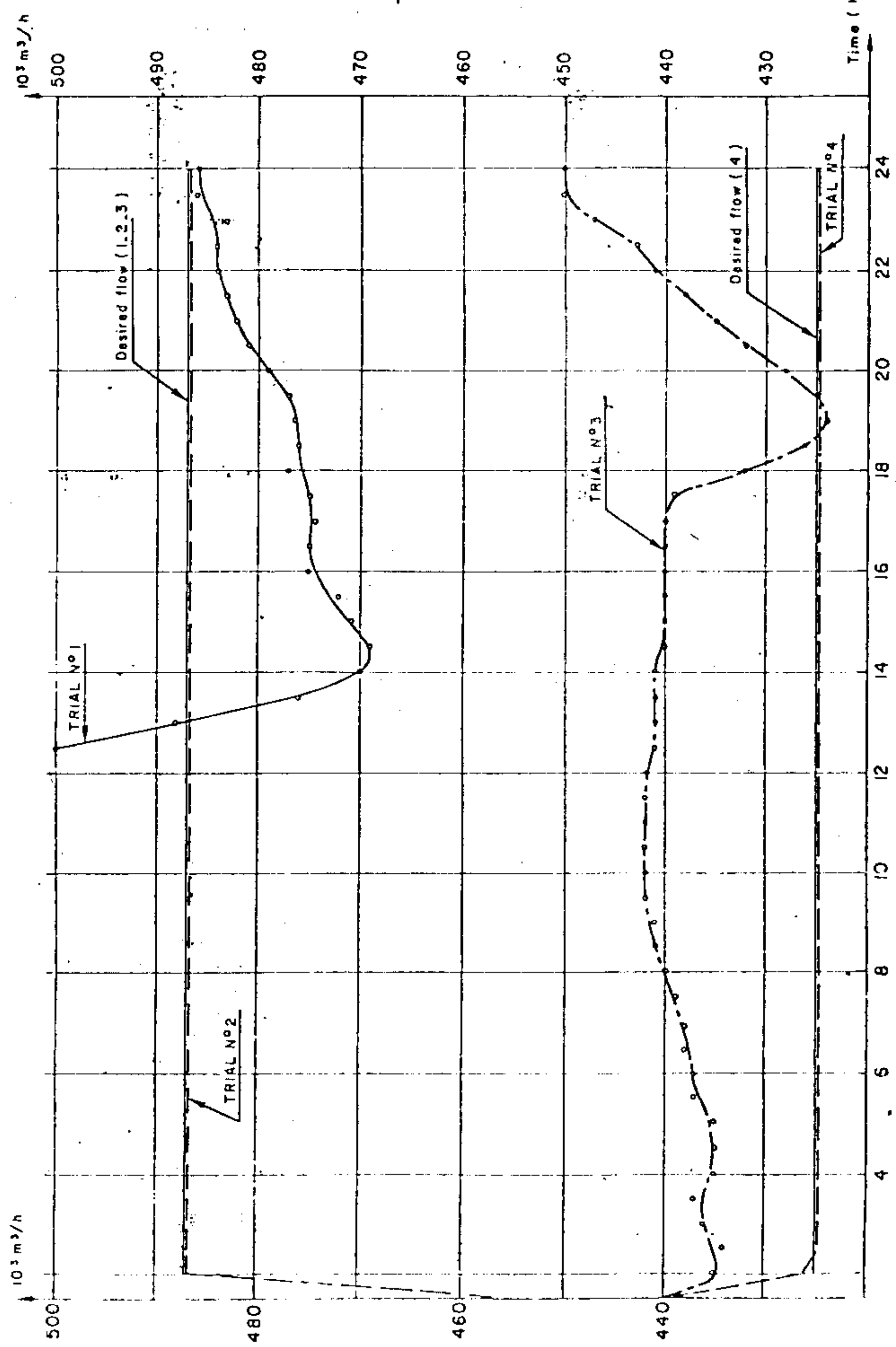

Figure 3.1. : FLOWS AT TAISNIERES

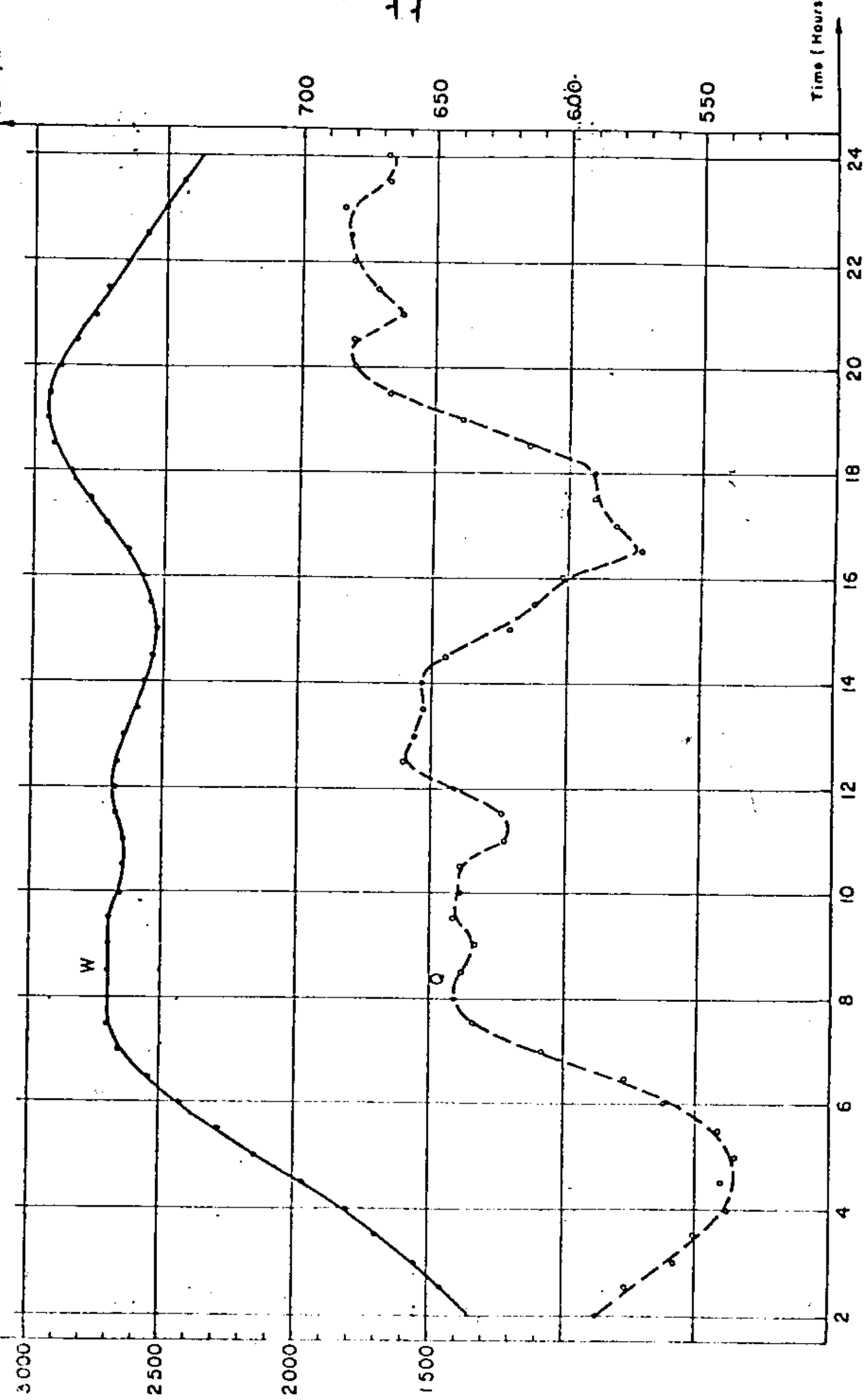

Figure 3.2. : TRIAL N° 4

Q : Flow at Villiers-le-Bel
W : Power at Taisnières

REFERENCES

[1] A. BAMBERGER. Analyse, contrôle et identification de certains systèmes. (Thesis, Paris VI, 1978).

[2] J.L. LIONS. Quelques méthodes de résolution des problèmes aux limites non linéaires. (Dunod, 1969).

[3] M. SORINE - M. SOULAS. Simulation and control of unsteady flows in gas transportation network. (Joint National Meeting TIMS/ORSA, New-Orleans, 1979).

ooooooooooooooo