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ON

VARIATIONAL METHODS IN ANALYSIS AND MATHEMATICAL PHYSICS

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OPTIMAL CONTROL OF WATER SOLIDIFICATION
OBSERVATION OF THE FREE-BOUNDARY

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OPTIMAL CONTROL OF WATER SOLIDIFICATION

OBSERVATION OF THE FREE-BOUNDARY

The solidification of water is one of the simplest free-boundary problem, the state (the temperature) of which can be modelized with a one phase Stefan problem. Here the optimal control of a such system is studied when :

- the state system is given by the variational inequality (V.I) associated with the one phase Stefan problem ;

- the observation is the free-boundary (interface between water and ice), more precisely the domain of ice.

The main difficulties of this problem are :

- the differentiability of the solution of a variational inequality ;

- the dependance of the domain of ice, with respect to the control.

The lecture is divided in three parts :

1./ Mathematical formulation of the problem.
2./ Theoritical results.
3./ Numerical results.

For details we refer to C. SAGUEZ [5].

I - MATHEMATICAL FORMULATION.

We consider the two dimensional case. Let Ω be a bounded open set of $\mathbb{R}^2$, with $\Gamma = \Gamma_1 \cup \Gamma_2$ the boundary (see figure 1.1).

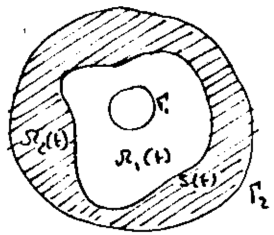


Figure 1.1.

. $\Omega_1(t)$ (resp. $\Omega_2(t)$) denotes the liquid domain (resp. the solid domain) and $S(t)$ the free-boundary. We suppose that $\Omega_1(t) \subset \Omega \quad \forall t \in]0,T[$

I-1) State system.

$\theta_1(x,t)$ the temperature of water and θ_2 the temperature of ice, verify the equations :

In the liquid

$$
\begin{array}{ll}
(1\text{-}1) & \left\{ \begin{array}{ll}
\dfrac{\partial \theta_1}{\partial t} - \Delta \theta_1 = 0 & \text{in } Q_1 = \bigcup_{t \in]0,T[} \Omega_1(t) \times]0,T[\\
\theta_1|_{\Sigma_1} = w_0(x,t) & (\Sigma_1 = \Gamma_1 \times]0,T[) \\
\theta_1|_{S(t)} = 0 & \\
\theta_1(x,0) = \theta_0(x) & (\text{we suppose } \theta_0(x) \geq 0)
\end{array} \right. \\
(1\text{-}2) & \\
(1\text{-}3) & \\
(1\text{-}4) &
\end{array}
$$

In the solid

(1.5) $$\theta_2(x,t) = 0$$

Along the free-boundary $S(t)$

(1.6) $$\frac{\partial \theta_1}{\partial n}\Big|_{S(t)} = -L\,.\,\vec{v}.\vec{n}$$

(where L is the latent heat, and $\vec{V}.\vec{n}$ the normal speed of the free-boundary).

If $\tilde{\theta}_1(x,t)$ is the extension of θ_1 by zero in Ω, we define the new variable $y(x,t)$:

(1.7) $$y(x,t) = \int_0^t \tilde{\theta}_1(x,\tau)\,d\tau$$

Then $y(x,t)$ is solution of the following variational inequality (G. DUVAUT [2], J.L. LIONS [3]) :

To find $y \in L^2(0,T;V)$; $\frac{\partial y}{\partial t} \in L^2(0,T;L^2(\Omega))$ s.t :

$$
\begin{cases}
\left(\frac{\partial y}{\partial t}, \xi - y\right) + a(y, \xi - y) \geq (f, \xi - y) \quad \forall \xi \in K_1(t) & (1.8)\\
y(\cdot,t) \in K_1(t) & (1.9)\\
y(x,0) = 0 & (1.10)
\end{cases}
$$

with
$$V = \{\xi \mid \xi \in H^1(\Omega)\ ;\ \xi|_{\Gamma_2} = 0\}$$
$$K_1(t) = \{\xi \mid \xi \in V\ ,\ \xi|_{\Gamma_1} = v\ ,\ \xi \geq 0 \quad \text{a.e}\}$$
$$a(\varphi,\psi) = \int_\Omega \operatorname{grad}\varphi\, \operatorname{grad}\psi\, d\Omega$$
$$v(x,t) = \int_0^t w_0(x,\tau)\,d\tau.$$
$$f(x,t) = \tilde{\theta}_0(x) - L\,(1 - \chi_{\Omega_1(0)})$$

Remark 1.1. To study the V.I (1.8)-(1.10), it is classical to introduce the penalized problem:

$$
\begin{cases}
\frac{\partial y_\varepsilon}{\partial t} - \Delta y_\varepsilon - \frac{1}{\varepsilon} y_\varepsilon^- = f & (1.11)\\
y_\varepsilon|_{\Sigma_1} = v & (1.12)\\
y_\varepsilon|_{\Sigma_2} = 0 & (1.13)\\
y_\varepsilon(x,0) = 0 & (1.14)
\end{cases}
$$

□

I-2) Optimal control problem.

We define the domain F (domain of ice) by :

(1.15) $$F = \{(x,t) \mid y(x,t) = 0\}$$

Then the following optimal control problem is considered :

- the state y is solution of the V.I (1.8) - (1.10),
- the control variable is v, boundary value along Γ_1,
- the set of admissible control $\mathcal{U}_{ad}$ is a convex, bounded, closed subset of $\mathcal{U} = H^{3/2,3/4}(\Sigma_1)$, such that $v(x,0) = 0$; $v(x,t) \geq 0$ a.e.
- the functional is defined by

(1.16) $$J(v) = \|\chi_F - \chi_d\|^2_{L^2(Q)} + \nu\|v\|^2_{\mathcal{U}} \qquad \nu \geq 0$$

(χ_F characteristic function of F, χ_d characteristic function of $F_d \subset Q$).

The optimal control problem is :

$$(1.17) \qquad \begin{cases} \text{To find } \bar{v} \in \mathcal{U}_{ad} \text{ s.t.} \\ J(\bar{v}) \leq J(v) \quad \forall v \in \mathcal{U}_{ad}. \end{cases}$$

Remark 1.2. We define the penalized optimal control problem :

- the state y_ε is solution of (1.11)-(1.14) ;
- the control variable v , and the admissible set of control $\mathcal{U}_{ad}$ as above ;
- the functional

$$J_\varepsilon(v) = \| \chi_{F_\varepsilon} - \chi_d \|^2_{L^2(Q)} + \nu \|v\|^2_{\mathcal{U}}$$

with

$$F_\varepsilon = \{(x,t) \mid y_\varepsilon(x,t) \leq 0\} \qquad \square$$

II - THEORITICAL RESULTS.

New we assume that :

$$(2.1) \qquad f \in L^2(Q) \ ; \quad \operatorname{meas}\{(x,t) \mid f(x,t) = 0\} = 0$$

In this chapter, we present some theoritical results. For the details of the demonstration we refer to C. SAGUEZ [5].

II-1) Variational inequality.

Proposition 2.1. : The variational inequality (1.8)-(1.10) has an unique solution $y \in H^{2,1}(Q)$, with

$$\|y\|_{H^{2,1}(Q)} \leq c \left(\|f\|_{L^2(Q)} + \|v\|_{\mathcal{U}} \right)$$

Proposition 2.2. : The penalized problem (1.11)-(1.14) has an unique solution $y_\varepsilon \in H^{2,1}(Q)$

and $y_\varepsilon \longrightarrow y$ in $H^{2,1}(Q)$ weakly

with $y_\varepsilon \leq y$ a.e.

Proposition 2.3. : The application $v \longmapsto y(v)$, solution of the V.I, is continuous from $\mathcal{U} = H^{3/2,3/4}(\Sigma_1) \times L^2(Q)$ weak into $H^{2,1}(Q)$ weak, and we have the same property for $y_\varepsilon(v)$ solution of the penalized problem.

II-2) Optimal control problem.

Due to the assumption (2.1), the characteristic function of F is characterized by :

$$(2.2) \qquad f\chi_F = f - \frac{\partial y}{\partial t} - \Delta y$$

From (2.2) we deduce :

Proposition 2.4. : The application $v \mapsto \chi_F$ is continuous from $H^{3/2,3/4}(\Sigma_1)$ weak into $L^2(Q)$ strong :

Proof : Let $\{v_n\}$ be a sequence of elements of $\mathcal{U}$ such that $v_n \longrightarrow v$ in $H^{3/2,3/4}(\Sigma_1)$ weakly.

If F_n is the contact set associated with v_n and $y_n = y(v_n)$ we have

$$f\chi_{F_n} = f - \frac{\partial y_n}{\partial t} - \Delta y_n.$$

Because X_{F_n} is bounded in $L^2(Q)$, there exists a subsequence (still denoted X_{F_n}) and a function $\ell \in L^2(Q)$ such that :

$$X_{F_n} \longrightarrow \ell \text{ in } L^2(Q) \text{ weakly.}$$

At the limit $(n \longrightarrow +\infty)$, with the property 2.3, we obtain :

$$g\,\ell = f - \frac{\partial y}{\partial t} - \Delta y$$

then

$$\ell = X_F$$

The strong convergence is deduced because we consider characteristic functions.

□

For the penalized problem we have :

Proposition 2.5. : The application $v \longrightarrow X_{F_\varepsilon}$ is continuous from $\mathcal{U} = H^{3/2,3/4}(\Sigma_1)$ weak into $L^2(Q)$ strong and $X_{F_\varepsilon} \longrightarrow X_F$ in $L^2(Q)$ strongly.

□

Remark 2.1. We don't detail here how to obtain necessary optimality conditions. This problem is very difficult and many questions are always open.

The principle is the following:

We consider the regularized penalized problem

$$\begin{cases} \dfrac{\partial y_\varepsilon^\eta}{\partial t} - \Delta y_\varepsilon^\eta + \dfrac{1}{\varepsilon}\varphi_\eta(y_\varepsilon^\eta) = f \\ y_\varepsilon^\eta|_{\Sigma_1} = v \quad ; \quad y_\varepsilon^\eta|_{\Sigma_2} = 0 \\ y_\varepsilon^\eta(x,0) = 0 \end{cases}$$

where $\varphi_\eta(x)$ is a very regular function, which regularizes $-x^-$.

For this problem, necessary optimality conditions are defined and at the limit $(\eta \rightarrow 0, \varepsilon \rightarrow 0)$, with technical assumptions, we obtain necessary optimality conditions. (See V. BARBU[1], F. MIGNOT[4], C.SAGUEZ[5])

□

III - NUMERICAL RESULTS.

- Numerically, we have solve the penalized problem with a gradient method.

- The problem has been discretized by finite element method. The triangulation is given Figure 3.1.

- Figure 3.2, we give the results for

$$\begin{cases} f = -80 \quad ; \quad v = 0 \\ F_d = \{(\rho,\theta) \mid \rho \leq R_1 + t(R_2 - R_1)(\cos\theta + 1)/2\} \end{cases}$$

and Figure 3.3, for

$$\begin{cases} f = -80 \quad ; \quad v = 0 \\ F_d = \{(\rho,\theta) \mid \rho \leq R_1 + t\cdot(R_2 - R_1)|\cos\theta|\} \end{cases}$$

The time of computation is 2 mn 20 s in IBM 370/168, for 10 steps of time and 8 iterations of gradient.

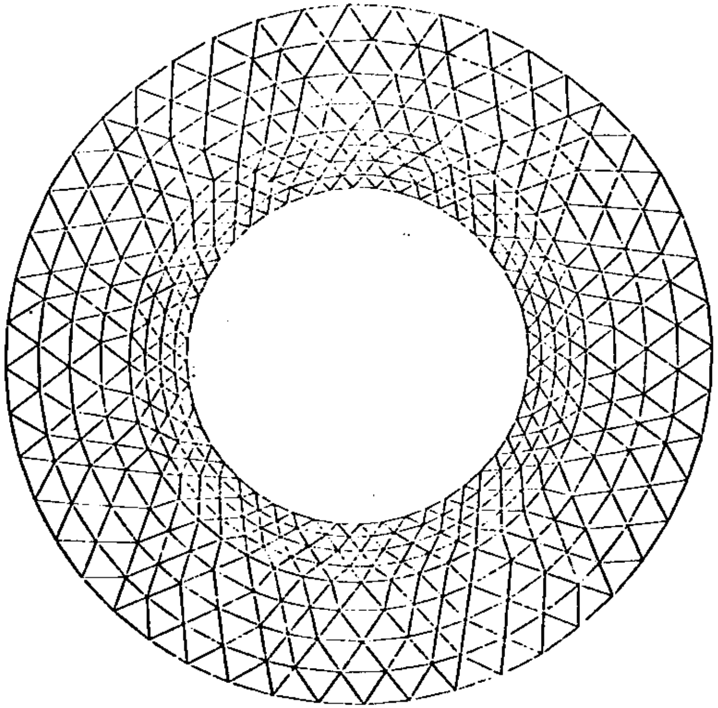
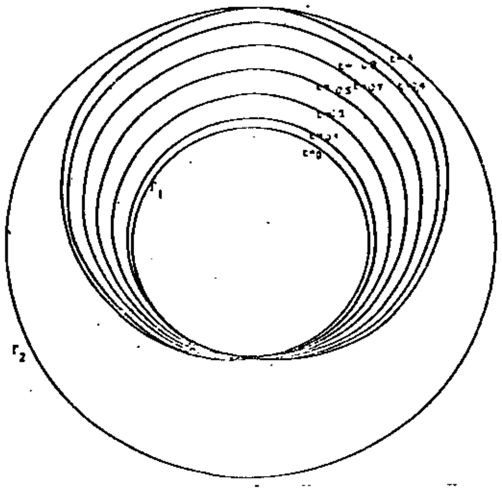
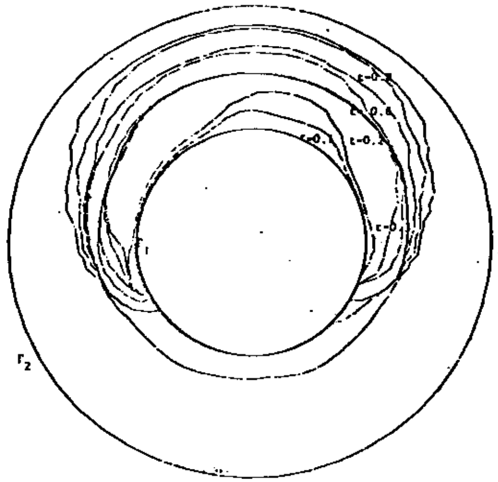


Figure 3.1.

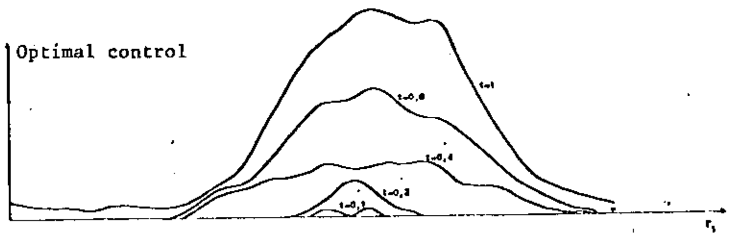
Triangulation of the domain : 728 triangles
482 nodes



Desired free-boundary $\rho_d = R1 + t(R2 - R1)(\cos\theta + 1)/2$

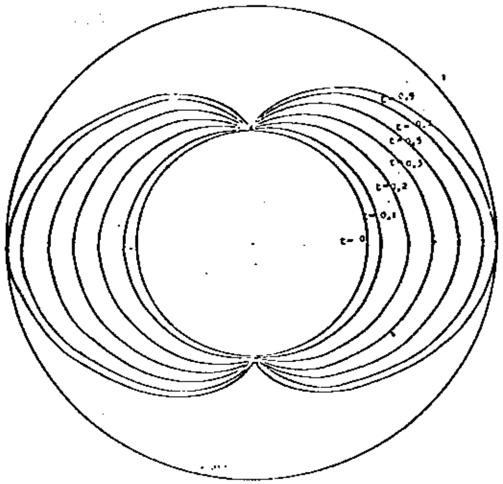


Obtained free-boundary

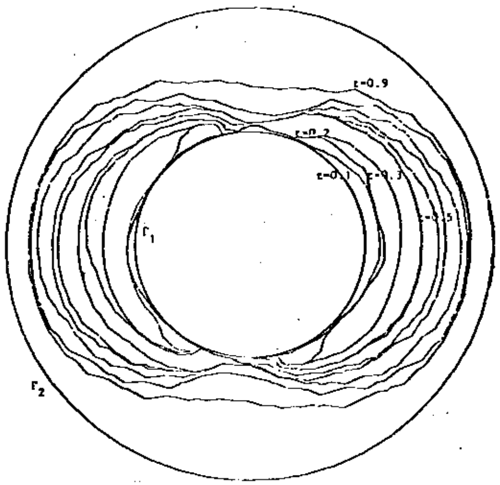


Optimal control along Γ_1

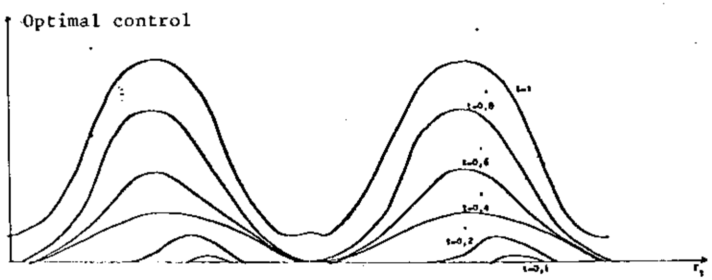
Figure 3.2.



Desired free-boundary $\rho_d = R1 + t(R2-R1)|\cos\theta|$



Computed free-boundary



Optimal control along Γ_1

Figure 3.3.

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