INTERNATIONAL ATOMIC ENERGY AGENCY
UNITED NATIONS EDUCATIONAL, SCIENTIFIC AND CULTURAL ORGANIZATION

INTERNATIONAL CENTRE FOR THEORETICAL PHYSICS
34100 TRIESTE (ITALY) - P.O.B. 586 - MIRAMARE - STRADA COSTIERA 11 - TELEPHONES: 224281/2/3/4/5/6
CABLE: CENTRATOM - TELEX 460392 I

IC/82 - 20

AUTUMN COURSE

ON

VARIATIONAL METHODS IN ANALYSIS AND MATHEMATICAL PHYSICS

20 October - 11 December 1981

VARIATIONAL FORMULATION FOR EVERY NONLINEAR EQUATION

E. TONTI

Facoltà di Ingegneria
Università degli Studi
Trieste
Italy

VARIATIONAL FORMULATION FOR EVERY NONLINEAR EQUATION

E. Tonti (*)

Abstract. Using the operatorial notation we show that for every nonlinear problem does exist a variational formulation and, much more, an infinity of them. The corresponding functionals are easily obtained. It follows that the requirement of the existence of a variational formulation for a given problem as a heuristic criterion to select admissible field equations, equations of motion or equations of interaction is to be abandoned.

(*) This work has been sponsored by the Italian Consiglio Nazionale delle Ricerche.

1 - INTRODUCTION

Roughly speaking the inverse problem of the Calculus of Variations is the problem of finding a variational formulation or a variational principle, for a given set of equations with associated initial or boundary conditions.

The trial and error method is still used in the search for the variational formulation in physics and engineering. On the contrary such a search in mathematical literature is done using the operatorial approach that enables to solve the problem for classes of equations instead of single equations.

The main interest of the mathematician in the variational formulation lies in the fact that it offers a method of proving the existence of the solution of a given problem. One then speaks of variational methods for the existence proofs.

The main interest of the engineer in the variational formulation lies in the possibility of utilizing numerical methods of solution, the so called direct methods of the Calculus of Variations: his aim is the finding of the solution. Furthermore he is often attracted by the synthetic statement that a variational principle offers and by the fact that the functional is often a physically significant quantity.

The main interest of the physicist about the variational formulation is even more different. He is rarely interested in the existence proof, and little interest in direct methods. He is often fascinated by the synthetic statement and by the physical meaning of the functional (it may be a time, an energy, an action, a transition probability, a

phase, an entropy production rate, etc.). But what seems to us the main interest of the physicist, is that he believes that equations obtained from a variational principle constitute a priviledged class among all conceivable equations.

This belief is perhaps due to the fact that most of the field equations come from a variational principle. This, at least, is the case of fields in which reversible phenomena take place, like statics and dynamics of elastic solids, dynamics of perfect fluids, electromagnetic field without conduction currents, gravitational field either classical or relativistic, quantum mechanics, like Schrödinger field, Klein-Gordon field, Dirac field, etc.

Helmholtz, in its classical paper on the principle of minimal action [38], wrote: "Already now one can consider highly probable that such a principle of minimal action represents the general law of all reversible processes in Nature ..." And he continued: "Anyhow the general validity of the principle of minimal action appears to me as a well established one, so that it can assume a high value either as a heuristic principle or as a criterion in the trial on formulating the laws of new classes of phenomena" [38,p.142].

The fact that variational principles are frequently encountered in physics has brought to the construction of a mathematical field theory in which general rules are given to obtain informations about the solutions of an equation starting from a lagrangian. This is the case of Noether theorems, of the automatic relativistic invariance of field equations,

of the automatic construction of the stress-energy-momentum tensor of the field, etc. How can one be not convinced of the priviledge of the equations coming from a variational principle?

Nevertheless there are equations that resist for long time to a variational formulation. This is the case of the Navier-Stokes equations of fluid dynamics that have not yet a variational formulation (see [36]); of Fourier equation of heat conduction and of its companion Fick equation that received a variational formulation twenty years ago, of the equations of irreversible thermodynamics, with the exception of those describing stationary flows for which the principle of minimum entropy production rate has been established by Prigogine.

The existence of equations which will describe physical phenomena, although they are not deducible from a variational principle, have stimulated physicists and engineers to extend the classical Calculus of Variations to give variational formulation to such equations.

After many trials that produced "quasi" variational formulations and "restricted" variational formulations (see [27] for a critical review), some of them quite out of the limits of mathematical correctness, the first rigorous result was obtained by Gurtin [28],[44] who showed how to give variational formulation to linear initial value problems. Gurtin's idea was essentially the introduction of the convolution product of two functions to construct the bilinear form which is necessary for a variational formulation. This

method opened the way to the variational formulation to all linear initial value problems and a large number of papers mainly appeared in the engineering reviews.

The method of Gurtin was later simplified by the present author [26]. The idea of adapting the bilinear form to the given operator was brought to its apex by Magri [6] who showed that every linear equation admits a variational formulation (and really an infinity of them) giving the explicit way to obtain the functional. This result contradicts the common believe that equations admitting a variational formulation form a priviledged class.

Equations of physics are usually of differential kind: integrodifferential equations are rare. The absence of integral or integrodifferential equations has led to overemphasize the role of the lagrangian as a function of the field variables and of its derivatives.

But the use of integrodifferential equations is increasing both in engineering and physics. If a constitutive law is history-dependent the present value of a physical variable depends on the values assumed by another physical variable in all past instants: this leads to an integral constitutive relation. It is the case of the ferromagnetism, of the dispersion of light, of the radiation reaction of fastly accelerate electrons. In engineering of materials we have visco-elastic materials [37,p.20]. In general this is the case of nonlocal or not instantaneous actions. When a variational formulation is given to such physical law the functional does not contain a lagrangian function but an integro-differential expression. One may speak of lagrangian operator.

On the other side the non conventional variational formulations of Gurtin and Magri give rise to integrodifferential lagrangians. Then the possibility of starting a physical theory saying: "given the lagrangian $L(x;\psi(x),\psi_k(x))...$" is restricted to differential equations that admit a classical variational formulation.

One may then think that a priviledged class of equations is the one that admits a variational formulation using the standard bilinear forms, regarding other bilinear forms as "artificial".

But an example taken from physics may be enough to abandon this thought. When Dirac equation system has come out it was written with the matrices α_k and β. De Broglie was troubled with the fact that there was not a variational principle for such equations. Happily the equation was not refused for this lack! Later he discovered that using the matrices γ^μ (introduced by von Neumann) and writing Dirac system in the form $i\hbar\gamma^\mu\partial_\mu\psi - m_o c\psi = 0$ one obtains a variational formulation. This form of the system may be obtained from the previous one through a multiplication by the matrix β. Of course the premultiplication by an invertible matrix changes the form, not the content, of the equation, as the change of representation does. Since the premultiplication by an invertible integrating factor is equivalent to a change of

bilinear form, we see that Dirac equation system admits a variational formulation to the price of abandoning the standard bilinear form.

The historical example tells us that one must discard a field equation if it is not directly derivable from a variational formulation: an integrating factor or the change of bilinear form must be admitted.

For "integrating factor" one usually means a function or a matrix whose elements are functions. In all cases an essential requirement is that the factor is invertible (see section 4).

For example a variational formulation of the problem

$$\begin{cases} mq+h\dot{q}+k\ddot{q}=0 & 0\leqslant t\leqslant T \\ q(0)=a \quad q(0)=b & q(t)\in C^2[0,T] \end{cases} \tag{1.1}$$

is obtained if we multiply the equation for the invertible factor

$$\varphi(t)=\exp\left(\frac{h}{m}\,t\right) \tag{1.2}$$

because the equation assumes the selfadjoint form

$$m\frac{d}{dt}\left[\exp\left(\frac{h}{m}\,t\right)\frac{dq}{dt}\right]+k\,\exp\left(\frac{h}{m}\,t\right)q=0 \tag{1.3}$$

that is necessary for a variational formulation.

In general the inverse problem for newtonian equations of motion

$$f(t;q,\dot{q},\ddot{q})=0 \tag{1.4}$$

is put in the following terms: does it exist an invertible function $\varphi(t;q,\cdot)$ so that the equation

$$\varphi(t;q,\dot{q})f(t;q,\dot{q},\ddot{q})=0 \tag{1.5}$$

is the Euler-Lagrange equation of a functional?

There is no reason to restrict the search for an integrating factor to a function: one may search for an integrating operator like

$$\int_0^T g(t,\tau)\varphi(\tau;q(\tau),\dot{q}(\tau))\;\; f(\tau;q(\tau),\dot{q}(\tau),\ddot{q}(\tau))\,d\tau=0. \tag{1.6}$$

We must require, of course, that the kernel g of this integral transform is such that Eq.(1.6) has the same solutions as (1.4). This integrating operator reduces to the usual integrating factor when $g(t,\tau)$ reduces to the Dirac distribution $\delta(t,\tau)$.

Even if we are open to such an extension (and we cannot find reasons to refuse it) we would be probably disturbed if we were told that every equation of the kind (1.4) admits an integrating operator and then a variational formulation.

This is just a particular case of what we show in the present paper.

The consequence of this is that we have no longer reason to believe that equations coming from a variational principle form a priviledged class.

This statement sounds unpleasant: it seems that we have lost a heuristic principle (compare with Helmholtz quotation) to investigate the admissibility of new equations of physics or to predict possible interactions.

But if instead of insisting on the variational formulation we went deeper to perform an anatomy of the field equations dividing them in their constituents we would realize that there is a simple operatorial structure that lies behind every physical theory (see [29] [33]).

In author's opinion it is just the mathematical structure on which the variational formulation is based, that enables us to find again an heuristic principle that can substitute the previous one.

Stated in other words: the variational formulation is only the emerging part of an iceberg. The deep reason for the "spontaneous" existence of classical variational principles lies in the submerged part of this iceberg and it is there that we can find the heuristic principle to investigate new laws of physics.

2 - VARIATIONAL FORMULATION

The search for a variational formulation of a given problem is the "Inverse Problem" of the Calculus of Variations. We are going to give a precise definition of what the Inverse Problem is.

First of all one must distinguish between Calculus of Variations in the small and in the large. There are correspondingly two forms of the inverse problem: the inverse problem in the small or local inverse problem and the inverse problem in the large or the global inverse problem.

We shall deal here only with local inverse problem (*). Even in this context one can find in the literature at least two different statements. To formalize then we need some background and terminology.

First of all we shall use an operatorial approach: it is both synthetic and conceptually simple. To make it technically simple we shall take an inductive approach.

Talking about equations we mean whatever kind of equation: of differential, integral, or integrodifferential kind; both a single equation and a system of equations; linear or not linear. This paper deals essentialy with nonlinear operators: when the operator is linear it will be expressely said. An equation is usually associated with additional conditions that specify initial, boundary, regularity conditions and the functional class. The functions considered may be real or complex valued, scalar, vector, tensor or spinor-valued. The set formed by an equation and all additional conditions constitutes a problem. Every problem may be written in the general form

---

(*) Those who are interested in the inverse problem in the large may consult [41], [42], [40], [43], [9], [12], [46], [47].

$$N(u)=\theta_v \qquad (2.1)$$

where N denotes an operator (N for "nonlinear"), u is an element of the domain $\mathcal{D}(N)$ that is considered embedded in a vector space U. For generality, the range $R(N)$ is supposed, to be embedded in another vector space V. θ_v denotes the null element of the V-space. The two spaces U and V, that may possibly coincide, are <u>not</u> supposed to be Banach or Hilbert spaces.

After a bilinear form has been introduced one may define a topology that makes continuous the bilinear form both in u and v. Such topology is said <u>consistent</u> with the duality [13,p.130].

A linear problem is suitably written in the form

$$Lu=f \qquad (2.2)$$

where $L:\mathcal{D}(L)\subset M\to V$ denotes a linear operator. The from (2.1) includes this form: we must put N=Lu-f. This operator is nonlinear but of a mild kind: it is an <u>affine</u> operator. It may be characterized as follows: if we denote by

$$N'_u\varphi=N'(u,\varphi)=\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\left[N(u+\varepsilon\varphi)-N(u)\right] \qquad (2.3)$$

the Gateaux differential of the operator N, we may easily see that an affine operator has a Gateaux differential that is independent on u. If we denote by A our affine operator it is

$$A'=\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\left[[L(u+\varepsilon\varphi)-f]-[Lu-f]\right]=L\varphi. \qquad (2.4)$$

The notations $N(u)=\theta_v$ or Lu=f do not exclude that there are functions, different from u and f, that form the operator. So the problem

$$\begin{cases}\frac{d}{dx}\left[p(x)\,\frac{dy(x)}{dx}\right]+q(x)y(x)=f(x)\\ y(0)=0 \quad y(1)=0 \quad 0\le x\le 1\\ y(x)\in C^2[0,1]\end{cases} \qquad (2.5)$$

may be written as follows

$$L\left(p(x),q(x);y(x)\right)=f(x) \qquad (2.6)$$

or simply

$$L\,y(x)=f(x) \qquad (2.7)$$

Even a linear equation with constant coefficients like

$$\begin{cases}a\frac{d^2y}{dx^2}+b\frac{dy}{dx}+cy=f\\ y\in C^2[0,1] \quad 0\le x\le 1 \quad y(0)=0 \quad y(1)=0\end{cases} \qquad (2.8)$$

may be written

$$L(a,b,c;y)=f \quad \text{or} \quad Ly=f. \qquad (2.9)$$

Then an equation that links a "source" function f with a "configuration" function u may be written

$$N(f;u)=\theta_V \quad \text{or} \quad N(u)=\theta_V . \tag{2.10}$$

We remark that the Gateaux differential does not require a norm on the U-space while the Frechet differential requires it: this is a stronger requirement. (*)

We remark that in the case of differential operators the linearity of the operator implies the linearity of the formal differential operator (†) and the linearity of its domain. This in turn requires homogeneous boundary or initial conditions. Nonhomogeneous linear boundary conditions make the domain a convex set and the operator affine.

Now to put the question of the variational formulation of problem (2.1) we need a bilinear form or bilinear functional, we shall denote $\langle v,u\rangle$, i.e. a mapping $B: U\times V \to R$ such that it satisfies these requirements

1) it must be real-valued (even if U and V are vector spaces over the complex number field);
2) it must be bilinear over the real number field (even if U and V are linear over the complex number field;
3) it must be nondegenerate, i.e.
   if $\langle v,u_o\rangle=0$ for every $v\in V$ then $u_o=\theta_U$
   if $\langle v_o,u\rangle=0$ for every $u\in U$ then $v_o=\theta_V$.

The real number $\tau=\langle v,u\rangle$ is then called the scalar product of the two elements $v\in V$ and $u\in U$, the V-space is called dual

(*) In the author's paper [18] the Gateaux derivative was erroneously called "Frechet" derivative.

(†) A "formal" differential operator is any expression that contains symbols of total or partial derivatives.

of the U-space and one writes $V=U^*$. One says also that U and V are put in duality by the canonical bilinear form $\langle v,u\rangle$ (to distinguish this from other possible bilinear forms). One also says that U and V form a dual pair, or a dual system [24,p.123].

The bilinear forms used in practice are of the following kinds. Let us denote $v\cdot u$ the local scalar product of two vector or two tensor or two spinor-valued functions of opposite variance, like

$$v_k u^k \qquad v_{hk} u^{hk} \qquad v_{AB} u^{AB} . \tag{2.11}$$

In the case of tensor or spinors of second rank the two tensors must have the same symmetry: both antisymmetric, as it is the case of electromagnetic tensors $\frac{1}{2}F_{\alpha\beta}G^{\alpha\beta}$, both symmetric, as it is the case of the relativistic gravitational theory, $g_{\mu\upsilon}T^{\mu\upsilon}$.

If we denote with Ω a subset of R^n and with x a point of Ω, i.e. $x=(x^1, x^2, \ldots x^n)\in\Omega$ we shall consider canonical bilinear form the form

$$\langle v,u\rangle_o \doteq \int_\Omega v(x)\cdot u(x)\, d\Omega . \tag{2.12}$$

This is the form usually employed in physical theories. It is not the most general bilinear form. If $A: U\to U$ and $B: V\to V$ are two linear invertible operators whose domain is the whole U and V-space respectively, a more general non degenerate bilinear form is of the kind

$$\langle v,u\rangle = \int_\Omega Bv(x) \cdot Au(x)\, d\Omega. \qquad (2.13)$$

The requirement of nondegeneracy constitutes a restriction on possible pairs of spaces U and V and also on the mappings N candidates for a variational formulation. For example a problem like

$$\begin{cases} \operatorname{div} \vec{D}(\vec{r}) = \rho(\vec{r}) \\ \vec{n} \cdot \vec{D}(\vec{s}) = 0 \end{cases} \quad \text{with} \quad \vec{r} \in \Omega, \ \vec{s} \in \partial\Omega \qquad (2.14)$$

cannot be a candidate for a variational formulation as it is because on the spaces of scalar-valued functions $\rho(\vec{r})$ and of vector-valued functions $D(\vec{r})$ one cannot define a nondegenerate bilinear form.

Of course one may add to problem (2.14) the corresponding adjoint problem

$$\vec{D}(\vec{r}) = -\varepsilon \operatorname{grad} \psi(\vec{r}) \qquad (2.15)$$

The whole problem formed by (2.14) and (2.15) becomes a possible candidate for a variational formulation.

Let us denote with $F : \mathcal{D}(F) \subset U \to R$ a functional. If we consider a variation δu we can write the corresponding variation of F as

$$\delta F = \langle E(u), \delta u \rangle$$

as is usual in the " δ - process" of the Calculus of Variations.

We shall call E the Euler operator of the functional F. The relation between E and F can be expressed sayng that E is the gradient of F and F is the potential of E. An element u_o such that $\delta F = 0$ for whatever δu is called a critical point of the functional; the functional is said stationary at u_o.

Notations. Let us denote T a nonlinear operator: if the following decomposition holds

$$T(u+\varepsilon\psi) - T(u) = L_u \varepsilon\psi + N_u(\varepsilon\psi) \qquad (2.17)$$

where ψ is such that $u+\varepsilon\psi$ belongs to the domain of T for every ε, L_u and N_u are linear and nonlinear operators on $\varepsilon\psi$ respectively and

$$\lim_{\varepsilon \to 0} \frac{N_u(\varepsilon\psi)}{\varepsilon} = 0 \qquad \text{for every } u \in \mathcal{D}(T) \qquad (2.18)$$

then $L_u \varepsilon\psi$ is called the Gateaux differential of T at u and L_u the Gateaux derivative. One write $L_u = T'_u$. It is

$$T'_u \psi = L_u \psi = \lim_{\varepsilon \to 0} \frac{T(u+\varepsilon\psi) - T(u)}{\varepsilon} \qquad (2.19)$$

If one puts

$$\delta u = \varepsilon\psi \qquad \delta T(u) = T'_u \delta u \qquad (2.20)$$

then one can write

$$T(u+\delta u) - T(u) = \delta T(u) + N_u(\delta u) \qquad (2.21)$$

Then the symbol " δ " of the classical Calculus of Variations, that is usually applied to functionals, coincides with the Gateaux differential [4, p.114].

The element $v = E(u)$ is called the variational derivative

of the functional F or also its functional derivative [19]. When E is a differential operator, if we denote by $\mathcal{E}$ the corresponding formal differential operator, the equation

$$\mathcal{E}(u)=0 \tag{2.22}$$

is the usual Euler equation while

$$E(u)=\theta_V \tag{2.23}$$

is the Euler problem. We remark that a variational formulation has sense for a problem not for an equation: boundary or initial conditions play a fundamental role.

Nov we one able to state the kinds of variational formulations mentioned at the beginning of this section.

a) Variational formulation in the strict sense:

given a problem $N(u)=\theta_V$ with $u\in\mathcal{D}(N)\subset U$ and $V=U^*$ find a functional $F[u]$, if any, such that the operator N is the gradient of F, i.e. such that

$$F'[u;\varphi]=<N(u),\varphi>. \tag{2.24}$$

This implies that the solutions of problem $N(u)=\theta_V$ are the critical points of the functional $F[u]$ and viceversa.

This form of the inverse problem is the one given by Hirsch in 1897 [15,p.52]; Atherton and Homsy in 1975 [11,p.35].

In the context of the global inverse problem the "strict" form seems to be the only form considered yet: see Dedecker [40], Anderson and Duchamp [12,p.781]; Olver [13,p.75].

b) Variational formulation in the extended sense:

given a problem $N(u)=\theta_V$ with $u\in\mathcal{D}(N)\subset U$ and $V=U^*$ find a functional $F[u]$, if any, whose critical points are solutions of the problem and viceversa.

This statement is more general of the former because it requires only the coincidence of the critical points with the solutions without the supplementary requirement that N be the gradient of the functional. The gradient of the functional F will be another operator, say $\bar{N}$, that will be linked in some way to the operator N.

This form of the inverse problem is the one given by Davis in 1928 [17] and used by Douglas in 1941 [16,p.71]; Santilli in 1978 [10,p.10].

3 - INVERSE PROBLEM IN THE STRICT SENSE: ITS SOLUTION

Let us consider a very simple example to show the simplicity of the operatorial approach to the inverse problem. Let us consider a system of two equations in two variables

$$\begin{cases} g(x,y) = 0 \\ h(x,y) = 0 \end{cases} \tag{3.1}$$

To this system one may associate a vector field and ask

$$\begin{cases} v_x = g(x,y) \\ v_y = h(x,y) \end{cases} \tag{3.2}$$

ourselves if the vector field is conservative. Let us denote by λ a real parameter that ranges from 0 to 1. A line starting at the point $P_o(x_o,y_o)$ and ending at $P(x,y)$ may be described by two functions $\xi(\lambda)$, $\eta(\lambda)$ with the condition $\xi(0)=x_o$, $\eta(0)=y_o$ and $\xi(1)=x$, $\eta(1)=y$. The circulation of v along this line from P_o to P is given by

$$C(P_o,P) = \int_0^1 \left[g(\zeta(\lambda),\eta(\lambda)) \frac{\partial \zeta(\lambda)}{\partial \lambda} + h(\zeta(\lambda),\eta(\lambda)) \frac{\partial \eta(\lambda)}{\partial \lambda} \right] d\lambda . \tag{3.3}$$

If the circulation of the vector v along every closed line contained in the region Ω vanishes, one may associate a real number to every point with the formula

$$f(P) = f(P_o) + C(P_o,P) . \tag{3.4}$$

The function f(P) is the potential of the vector field. The condition df=0 implies

$$g(x,y) = \partial_x f = 0 \qquad h(x,y) = \partial_y f = 0 . \tag{3.5}$$

If the field is differentiable the condition of path-independence becomes

$$\partial_x v_y = \partial_y v_x . \tag{3.6}$$

This condition may be restated as follows: if we denote by J_u the jacobian of the mapping $u=(x,y) \rightarrow v=(v_x,v_y)$ this matrix

of functions is symmetric for every u. The symmetry may be written as follows

$$J_u \vec{\varphi} \cdot \vec{\psi} = J_u \vec{\psi} \cdot \vec{\varphi} . \tag{3.7}$$

The condition is only local and then it is only necessary for the existence of the potential. To make it sufficient one must require that the region Ω be simply connected. In particular if Ω is a convex set this requirement is satisfied and then condition (3.7) becomes sufficient.

These notions may be easily extended to function spaces. Let us consider a generally nonlinear mapping N from a subset of U to $V = U$ with domain dense in U. To the problem

$$N(u) = \theta_v \tag{3.8}$$

we can associate the "vector field"

$$v = N(u) . \tag{3.9}$$

Now we if consider in the U-space an one-parameter family of elements $\eta(\lambda)$ with the condition $\eta(0)=u_o$, $\eta(1)=u$ (Fig.1a) we can define the "circulation" of the element v by the formula

$$C[u_o,u] = \int_0^1 \left\langle N(\eta(\lambda)), \frac{\partial \eta(\lambda)}{\partial \lambda} \right\rangle d\lambda . \tag{3.10}$$

If this functional is independent on the line in U-space connecting u_o to u for whatever choice of u_o and u in $\mathcal{D}(N)$, one may associate a real number r to every $u \in \mathcal{D}(N)$ by the

formula

$$F[u]=F[u_o] + C[u_o,u] \qquad (3.11)$$

where $r_o=F[u_o]$ is arbitrary. If the circulation between two points u_1 and u_2 chosen arbitrarily, does not depend on the line connecting them, it follows that the circulation along any closed path vanishes. In particular this happens for an infinitesimal loop. Let us take the infinitesimal parallelogram indicated in Fig. 1b. It must be

$$<N(u),\varepsilon\varphi>-<N(u+\varepsilon\varphi),\upsilon\psi>=<N(u),\upsilon\psi>+<N(u+\upsilon\psi),\varepsilon\varphi> \qquad (3.12)$$

from which one obtains

$$<\frac{N(u+\varepsilon\varphi)-N(u)}{\varepsilon},\psi>=<\frac{N(u+\upsilon\psi)-N(u)}{\upsilon},\varphi>. \qquad (3.13)$$

This relation must be valid for every ε and every ν. If the operator N admits a Gateaux derivative N'_u the condition of path-independence reduces to the <u>local</u> condition

$$<N'_u\varphi,\psi>=<N'_u\psi,\varphi>. \qquad (3.14)$$

This means that the Gateaux derivative of the operator must be <u>symmetric</u>; this is an extension to function spaces of Eq. (3.7).

The symmetry condition (3.14) being a local condition is not sufficient for the existence of the potential F[u]. To make it sufficient it is enough to require that the domain $\mathcal{D}(N)$

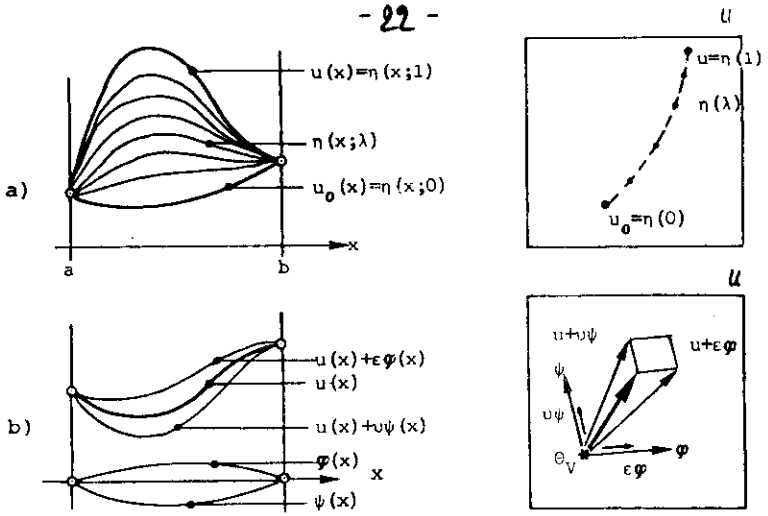

Fig. 1 - a) a line in funtion space; b) an infinitesimal parallelogram in function space.

be simply connected. Since in many cases of practical interest $\mathcal{D}(N)$ is convex, we may conclude that under the hypothesis of convexity of the domain the condition becomes sufficient.

The relation between the functional and the operator is easily obtained from (3.11) and (3.10)

$$F'[u;\varphi]=<N(u),\varphi> \qquad (3.15)$$

so that N is the gradient of F.

We may now solve the inverse problem in its strict form with the

THEOREM I (Volterra, 1913): the necessary and sufficient condition so that an operator $N:\mathcal{D}(N)\subset U\mapsto U$ *with domain* $\mathcal{D}(N)$ *dense in* U *is the gradient of a functional is that the circulation of the element* $v=N(u)$

along every closed line contained in $\mathcal{D}(N)$ vanishes.

If N admits a Gateaux derivative N'_u the necessary condition is that N'_u be symmetric, i.e.

$$\langle N'_u\varphi,\psi\rangle = \langle N'_u\psi,\varphi\rangle. \qquad (3.16)$$

If the domain $\mathcal{D}(N)$ is convex the condition (3.16) becomes sufficient. The functional is given by

$$F[u]=\int_0^T \langle N(\eta(\lambda)),\frac{d\eta(\lambda)}{d\lambda}\rangle d\lambda. \qquad (3.17)$$

Historical remark: 1 - While the symmetry condition (3.16) was established by Volterra in 1887 [19,p.109], the preceeding theorem was established for the first time by Volterra in 1913 [1,p.47]. He wrote: "On peut tirer de même de la formule (12) la condition pour que

$$\int_0^T \cdot |[\overset{b}{x}(t),\eta]|\delta x(\eta)d\eta$$

soit la differentiale totale exacte d'une fonction de ligne que nous saurons alors calculer. Il faut que

$$X'|[\overset{b}{x}(t),\eta,\zeta]|=X'|[\overset{b}{x}(t),\zeta,\eta]|$$

le second paramètre indiquant toujours dans les expressions précédentes, le point où l'on effectue la dérivation.

C'est la condition de symmétrie de la dérivée seconde que nous avons déjà indiquée.

On peut énoncer le résultat précis suivant: la condition

nécessaire et suffisante pour que $X|[\overset{b}{x}(t),\eta]|$ soit la dérivée d'une fonction de ligne $F|[\overset{b}{x}(t)]|$ est que

$$X'|[\overset{b}{x}(t),\eta,\zeta]|=X'|[\overset{b}{x}(t),\zeta,\eta]|$$

On saura calculer cette fonction de ligne.

On aura

$$F|[x(t)]|-F|[x_o(t)]|=\int_o^T ds\int_o^T X|[\overset{b}{x}(t)s,\eta]|\frac{dx(\eta|s)}{ds}\,d\eta$$

As further proof we remark that in 1918 Evans [21] published a book in which at page 23 he wrote: "The condition

$$\phi''[C|MM_1]=\phi''[C|M_1M]$$

... was originally stated by Volterra".

In 1933 Kerner [5,p.550] reported these results and quoted Volterra (its reference 4). In 1964 Vainberg [3,p.56] gives the same results quoting Kerner (its reference 42b). In 1969 Tonti [18] divulgated these results quoting Vainberg (its reference 1). The result of all this is that some authors attribute this theorem to Kerner (see [4], [20,p. 42], [23,p.176]; others to Vainberg (see [11,p.33], [13,p. 75], [22,p.1179], and some others to the present author (see [9]).

Remark 2 - It is often stated that the necessary condition for a variational formulation (in the "strict" sense) is

selfadjointness. This is correct when referred to the equation, (the equation must be selfadjoint) but it is not correct when referred to the operator: the operator must be symmetric. A linear operator S is selfadjoint if $S^*=S$ and is symmetric when $S^* \supset S$. This means that the adjoint operator has a domain $\mathcal{D}(S^*)$ that is larger than that of S and moreover when applied to the elements $u \in \mathcal{D}(S)$ gives the same transform of S. Condition (3.16) means that N' is symmetric, not necessarily selfadjoint.

Remark 3 - The two spaces U and V are requested to be linear over the real or complex number field. In the second case, typical of quantum theory, one may first introduce a sesquilinear form $\langle v|u\rangle$, i.e. a nondegenerate complex-valued form that satisfies the requirements

$$\begin{aligned} &\langle \lambda v|u\rangle = \lambda\langle v|u\rangle \\ &\langle v|\lambda u\rangle = \lambda\langle v|u\rangle . \end{aligned} \qquad (\lambda = \text{complex}) \qquad (3.18)$$

After, to deal with variational formulation, one must construct the form

$$\langle v,u\rangle = \frac{1}{2}\langle v|u\rangle + \langle v|u\rangle \qquad (3.19)$$

that is bilinear on the real number field. An operator that is hermitean with respect to the sesquilinear form (3.18) is automatically symmetric with respect to the bilinear form (3.19).

4 - INVERSE PROBLEM IN THE EXTENDED SENSE: ITS SOLUTION

In matrix theory a system Au=b may always be transformed in another system by premultiplication by another matrix C. If C is invertible the new system CAu=Cb has the same solutions of the old one. In particular if A^* is invertible the system $A^*Au=A^*b$ has the same solutions. In the latter case the vector u that solves the system makes stationary the function

$$f(u_1,u_2,\ldots u_n)=\| Au-b \|^2 \qquad (4.1)$$

and this gives rise to the method of least squares.

The same procedure cannot be performed, in general, on differential operators. To show this reason let us consider the linear differential operator

$$D=\left\{ \frac{d}{dt}, 0\leqslant t\leqslant T, u(0)=0, u(t)\in C^1[0,T] \right\} \qquad (4.2)$$

and the problem

$$Du=f \qquad \text{with} \qquad f\in C[0,T]. \qquad (4.3)$$

The adjoint operator is

$$D^*=\left\{ -\frac{d}{dt}, v(T)=0, v(t)\in AC[0,T] \right\} \qquad (4.4)$$

where AC[O,T] denotes the class of absolutely continuous functions. It is true that D^*D is a symmetric operator but D^* cannot be applied to both members of the problem (4.3) because f does not satisfy the final condition f(T)=0 and then does not belong the field of definition of D^*. When

the method is applied to differential equations it is assumed that $f \in \mathcal{D}(A^*)$ [46,p.496].

Stated in equivalent terms the domain of the operator D^*D is a restriction of the domain of D. If Du=f is an equation representing a physical law all elements $f \in R(D)$ describe possible sources. Then the restriction of the domain necessary for the application of the operator D^* would exclude elements f and then possible source distributions: this cannot be accepted.

We may think of applying to problem (4.3) an integral operator, $K: R(N) \to U$ i.e. to perform an integral transformation, so that f be transformed in a new function $\overline{f}$ that satisfies the final condition $\overline{f}(T)=0$. One may choose an operator as follows

$$\overline{f}(t) = \int_0^T k(t,\tau) f(\tau) d\tau \qquad (4.5)$$

with the condition $k(T,\tau)=0$. In particular one may choose an integral operator of the kind of Volterra

$$\overline{f}(t) = \int_0^t k(t,\tau) f(\tau) d\tau \qquad (4.6)$$

that satisfies automatically the final condition. The problem (4.3) becomes

$$KDu = Kf. \qquad (4.7)$$

We can now apply the operator D^*:

$$D^*KDu = D^*Kf . \qquad (4.8)$$

The new problem (4.8) has the same solutions of the given one if D^* and K are invertible operators. Moreover if the integral operator is symmetric the operator D^*KD is also symmetric and then the Volterra condition (3.15) is satisfied. This implies that the problem (4.8) admits a strict variational formulation and then the problem (4.2) admits an extended variational formulation.

As we see the role of the integral operator K is to modify the range of D making it"digestible" to the operator D^*. This problem does not arise in matrix theory because the domain of a matrix is the whole vector space.

The procedure indicated may be extended to nonlinear operators as it is shown by the following

Theorem II (1981) - Let us consider the problem

$$N(u) = e_v \qquad (4.9)$$

where N *is a generally nonlinear operator that satisfies the following requirements:*

1) $\mathcal{D}(N) \subset U$ *then* $R(N) \subset U^*$

2) $\mathcal{D}(N)$ *is convex (linear nonhomogeneous boundary conditions)*

3) $\mathcal{D}$ *is dense in* U *(i.e.* $\overline{\mathcal{D}}(N) \subset U$*)*

4) $R(N) \ni \Theta_v$ *(the solution exists)*

5) N'_u *exists*

6) N'^*_u *is invertible for every* $u \in \mathcal{D}(N)$.

Then for every choice of the integral operator K *that satisfies the following conditions*

7) $\mathcal{D}(K) \supset R(N)$

8) $R(K) \subset \mathcal{D}(N'^{*}_{u})$

9) is linear

10) is invertible

11) is symmetric

the operator $\bar{N}$ *defined by*

$$\bar{N}(u) = N'^{*}_{u}\, K\, N(u) \tag{4.10}$$

is such that

a) it has the domain of N

b) the problems $\bar{N}(u)=\Theta_v$ *and* $N(u)=\Theta_v$ *have the same solutions*

c) it is a potential operator

It follows that the solutions of the problem (4.9) coincide with the critical points of the functional

$$\bar{F}[u] = \frac{1}{2} \langle KN(u)\ ,\ N(u) \rangle \tag{4.11}$$

whose gradient is the operator $\bar{N}$.

Proof. Fig 2 will help us to make the proof easier. It is easily seen that conditions 7) and 8) assure that the operators N and $\bar{N}$ have the same domain then a) is proved. Condition 6) and the linearity of N'^{*}_{u} implicit in 5) assure that Θ_u is mapped in Θ_v.

Conditions 9) and 10) assure that Θ_v is mapped back in Θ_u by K. Then if

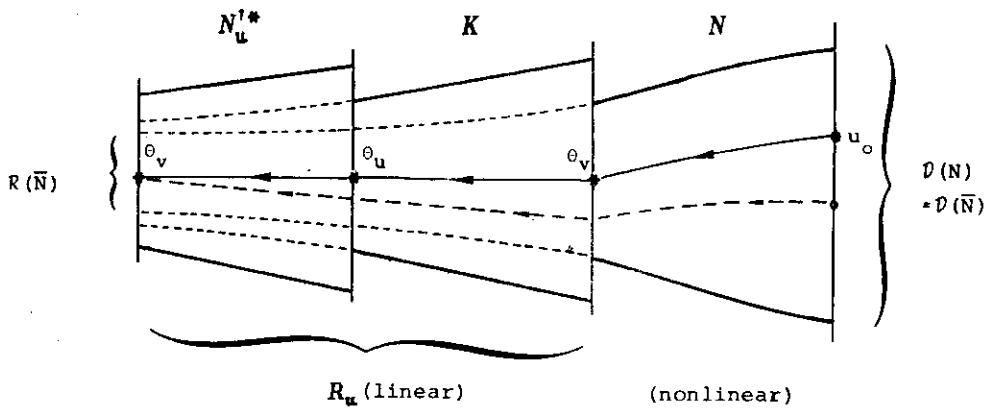

Fig. 2 - Relation between the operators N, K, N'^{*}_{u}.

$$N'^{*}_{u_o} KN(u_o) = \Theta_V \tag{4.12}$$

it follows that

$$N(u_o) = K^{-1} (N'^{*}_{u_o})^{-1} \Theta_V = K^{-1}\, \Theta = \Theta_v \tag{4.13}$$

then u_o is solution of problem (4.9). Viceversa if u_o is solution of (4.9) from (4.10) one sees that u_o is also solution of $\bar{N}(u)=\Theta_v$. This means that a sequence like that represented by a dotted line in Fig. 2 cannot arise. Then b) is proved. Moreover on account of condition 11) we have

$$\langle \delta u, N'^{*}_{u} K N(u) \rangle = \langle \delta N(u), KN(u) \rangle = \\ = \delta \frac{1}{2} \langle N(u), KN(u) \rangle = \delta \bar{F}[u]. \tag{4.14}$$

Condition (4.14) assures only the existence of a local potential F[u]: condition 2) assures that $\mathcal{D}(N)=\mathcal{D}(F)$ is simply connected and then that F[u] is a global potential.

If we put

$$R_u = N_u'^* K \tag{4.15}$$

we see that the domain of $\bar{N}(u)$ coincides with that of $N(u)$. Since the latter is dense in U, as condition 3) says, the relation

$$\delta \bar{F}[u] = \langle \bar{N}(u), \delta u \rangle \tag{4.16}$$

for every δu of a dense subset of U and the nondegenerate nature of the bilinear form implies $\bar{N}(u) = \theta_v$. Then c) is proved: q.e.d.

The relation between the various operators is shown in Fig. 3.

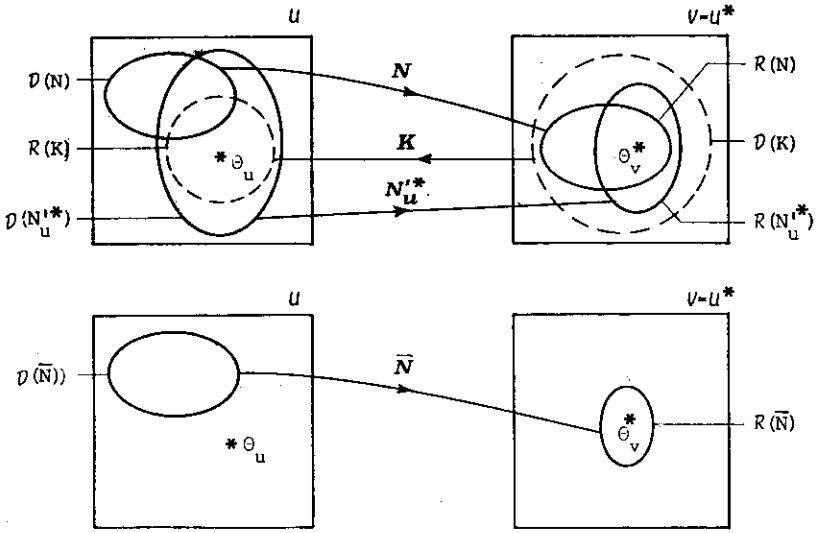

Fig. 3 - The construction of the operator $\bar{N}$

In conclusion we see that the operator R is an integrating operator for the problem $N(u) = \theta_v$. It is a linear operator. The theorem then gives, under mild conditions on the operator N, the conditions from 1) to 6), an infinity of integrating operators. This theorem then provides a general solution to the inverse problem in the extended sense.

Fig. 4 shows the relation between the strict and extended form of the inverse problem.

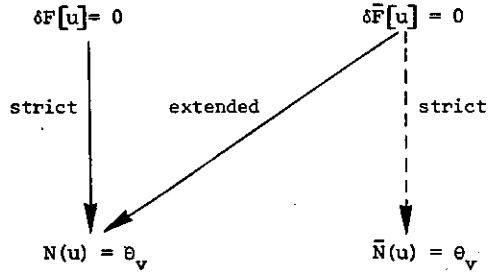

Fig. 4 - The relation between the strict and the extended variational formulation.

Remark 4- To give a variational formulation to a problem one may consider bilinear forms different from the canonical one. So the operator D of Eq. (4.2) that is not symmetric with respect to the canonical bilinear form

$$\langle v,u \rangle = \int_0^T v(t)\, u(t)\, dt \tag{4.17}$$

becomes symmetric with respect to the convolutive bilinear form

$$\langle v,u \rangle_c = \int_0^T v(T-t)u(t)dt = \int_0^T v(t)u(T-t)dt \tag{4.18}$$

as it is easily shown: see [26]. If we denote by C the convolution operator defined by

$$Cv(t) = v(T-t) \tag{4.19}$$

we may write

$$\langle v,u\rangle_c = \langle Cv,u\rangle_o = \langle v,Cu\rangle_o \tag{4.20}$$

because C is a symmetric operator.

The symmetry of D with respect to the convolutive form, i.e.

$$\langle D\varphi,\psi\rangle_c = \langle D\psi,\varphi\rangle_c \tag{4.21}$$

is equivalent to the statement that CD is symmetric:

$$\langle CD\varphi,\psi\rangle_o = \langle D\varphi,\psi\rangle_c = \langle D\psi,\varphi\rangle_c = \langle CD\psi,\varphi\rangle_o. \tag{4.22}$$

This shows that the change of bilinear form is equivalent to the premultiplication by an integrating operator. The statement that the symmetry of an operator is relative to a bilinear form is equivalent to the statement that an operator may be made symmetric by the application (on the left) of an integrating operator. The choice of one or other point of view is a matter of convenience.

Remark 5 - In 1963 Sălov [34] suggested for the linear problem Lu=f the integrating operator SL^{-1} where S is a symmetric, positive definite operator. The problem is then reduced to the equivalent one $Su=SL^{-1}f$. The method can be applied only when L^{-1} is known. This makes the method unpractical because it requires the solution of the problem.

In 1974 Magri [6] obtained the general form of the integrating operator for the problem Lu=f. Its result may be restated as follows: we choose an integral operator K that is symmetric and invertible [6,p.542]. Then starting with the standard bilinear form $\langle v,u\rangle$ we may introduce a new bilinear form

$$\langle v,u\rangle_L = \langle v,KLu\rangle_o = \langle L^*Kv,u\rangle_o \tag{4.23}$$

and the operator L is symmetric with respect to the new bilinear form. The method can be easily applied as is shown in [6].

In 1978 Didenko [35], ignoring paper [6], suggested the integrating operator L^*S where S is an isometric and symmetric operator. Both Sălov and Didenko worked in Hilbert spaces while in [6] the method is given for two spaces in duality.

In 1979 Telega [23] generalized Magri's result to some nonlinear operators.

4.1 - How to choose the operator K

According to theorem II the integral operator K must be invertible, symmetric, with range contained in $\mathcal{D}(N_u'^*)$. A good source of such operators is formed by the inverses of symmetric differential operators: the kernel is then the Green function of the differential operator. So if one considers the operator

$$L = \left\{ -\frac{d^2}{dt^2},\ 0 \le t \le T, u(0)=0, u(T)=0, u(t) \in C^2[0,T] \right\} \tag{4.24}$$

that is symmetric and invertible, its inverse is

$$Kv = \int_0^T [-(t-\tau)H(t-\tau))(T-\tau)]\, v(\tau)\, d\tau \tag{4.25}$$

where H(z) is the unit-step Heaviside function.

In principle every Sturm-Liouville formal differential operator

$$\mathcal{S} = -\frac{d}{dt}\left[p(t)\frac{d}{dt}\right] + q(t)I \tag{4.26}$$

with $p(t) > 0$ and $q(t) \geqslant 0$ in $[0,T]$, associated with boundary conditions that make the operator S symmetric and invertible gives rise to an operator $K = S^{-1}$ that may be utilized to give a normal variational formulation. Table I gives some of these operators.

Often the problem may be decomposed in two pieces

Table I - Integral operators K for functions of one variable defined in [0,T] (*)

	formal differential operator	additional conditions: initial	additional conditions: final	inverse operator $K = \int_0^T g(t,\tau)\ldots d\tau$	positive definite
1	$-\frac{d^2}{dt^2}$	$u(0)=0$	$u(T)=0$	$g(t,\tau) = -(t-\tau)H(t-\tau)+(T-\tau)\frac{t}{T}$	yes
2	$-\frac{d^2}{dt^2}$	$\dot{u}(0)=0$	$u(T)=0$	$g(t,\tau) = -(t-\tau)H(t-\tau)+(T-\tau)$	yes
3	$-C\frac{d}{dt}$	———	$u(T)=0$	$g(t,\tau)=H(T-t-\tau)$	no
4	$-C\frac{d^2}{dt^2}$	———	$u(T)=0$ $\dot{u}(T)=0$	$g(t,\tau) = (T-t-\tau)H(T-t-\tau)$	no
5	$\frac{d^4}{dt^4}$	$u(0)=0$ $\dot{u}(0)=0$	$u(T)=0$ $\dot{u}(T)=0$	$g(t,) = \frac{1}{6}(t-\tau)^3 H(t-\tau) + \frac{1}{2}(T-\tau)^2(T-t) - \frac{1}{6}(T-\tau)^2$	yes

(*) - H(z) denotes the Heaviside (unit step) function; C is the convolutive operator.

$$N(u)=Lu+B(u) \tag{4.27}$$

where L is an easily invertible operator. In this case the problem reduces itself to

$$u+M(u)=\theta_u \tag{4.28}$$

with $M=L^{-1}B$.

In this case the functional (4.11) reduces itself to

$$\overline{F}[u]=\frac{1}{2}\langle v,Kv\rangle+\langle v,KM(u)\rangle+\frac{1}{2}\langle M(u),KM(u)\rangle\ . \tag{4.29}$$

5 - APPLICATIONS

Let us consider some examples.

Example 1: give a variational formulation in the extended sense to the problem

$$\begin{cases} \frac{d}{dt}u(t)=f(t) & 0\le t\le T \\ u(0)=a & u(t)\in C^1[0,T] \end{cases} \tag{5.1}$$

The operator is affine: the adjoint of its Gateaux derivative is

$$N'^{*}_{u}\,\psi=\left\{-\frac{d}{dt}\psi,\psi(T)=0,\psi(t)\in AC[0,T]\right\} \tag{5.2}$$

and is invertible.

If we choose any Green function of Table I (all of them satisfy the final condition $\psi(T)=0$) the integrating operator becomes

$$Rv=-\frac{d}{dt}\int_0^T g(t,\tau)v(\tau)d\tau\equiv\int_0^T-\frac{\partial g(t,\tau)}{\partial t}v(\tau)d\tau \tag{5.3}$$

then in particular choosing the Green functions of the rows 2,3 and 4 of Table I we obtain the integrating operators

$$R^{(1)}v=\int_0^T v(\tau)d\tau \tag{5.4}$$

$$R^{(2)}v=v(T-t)=Cv(t) \tag{5.5}$$

$$R^{(3)}v=\int_0^T v(\tau)d\tau \tag{5.6}$$

and the corresponding functionals become

$$\overline{F}^{(1)}[u]=\frac{1}{2}\int_0^T u^2(t)dt-\int_0^T u(t)\left[\int_0^T f(\tau)d\tau-a\right]dt \tag{5.7}$$

$$\overline{F}^{(2)}[u]=\frac{1}{2}\int_0^T\frac{du(t)}{dt}u(T-t)dt-\int_0^T f(t)u(T-t)dt \tag{5.8}$$

$$\overline{F}^{(3)}[u]=\frac{1}{2}\int_0^T u(t)u(T-t)dt-\int_0^T u(t)\left[\int_0^T f(\tau)d\tau+a\right]dt. \tag{5.9}$$

We have obtained three variational formulations of the initial value problem given thougt its operator is not symmetric.

Example 2 : Let us give a variational formulation to the nonlinear initial value problem

$$\begin{cases} \dot{q}(t)-f(t;q(t))=0 \\ q(0)=a \qquad q(t)\in C^1[0,T] \end{cases} \tag{5.10}$$

where f is an assigned function. It is

$$N(q)=\left\{\frac{d}{dt}\,q-f(t,q),q(0)=a,q(t)\in C^1[0,T]\right\} \tag{5.11}$$

$$N'_q\varphi=\left\{\frac{d}{dt}\varphi-\frac{\partial f}{\partial q}\varphi\ ,\varphi(0)=0,\varphi(t)\in C^1[0,T]\right\} \tag{5.12}$$

$$N'^{*}_q\psi=\left\{-\frac{d}{dt}\psi-\frac{\partial f}{\partial q}\psi\ ,\psi(T)=0,\psi(t)\in AC[0,T]\right\} \tag{5.13}$$

The adjoint homogeneous problem

$$-\frac{d}{dt}\psi-\frac{\partial f}{\partial q}\psi=0 \qquad \psi(T)=0 \tag{5.14}$$

has only the null solution. The domain of N is convex and dense in U with all common topologies.

Any Green function of Table I satisfies the final condition $\Psi(T)=0$. Moreover the operator K so obtained has a domain that may be choosen larger than $C[0,T]$, for ex. $\mathcal{D}(K)=C[0,T]$ and a range that is contained in $AC[0,T]$: all operators K of Table I have $\mathcal{R}(K)\subset C[0,T]\subset AC[0,T]$. Then all conditions of Theorem II are satisfied. One has

$$h(t)=\dot{q}(t)-f(t;q(t)) \tag{5.15}$$

$$\overline{F}[q]=\frac{1}{2}\int_0^T dt\ h(t)\int_0^T g(t,\tau)h(\tau)d\tau. \tag{5.16}$$

We have so obtained at least five variational formulations in the extended sense of the problem.

We remark that with two integrations by parts one can transform the expression of $F[q]$ eliminating the time derivatives of q(t).

Example 3: Let us give a variational formulation in the extended sense to the initial value problem

$$\begin{cases} \ddot{q}(t)-f(t;q(t),\dot{q}(t)) = 0 \\ q(0)=0 \qquad \dot{q}(0)=0 \qquad q(t)\in C^2[0,T] \end{cases} \tag{5.17}$$

where f is an assigned function. We can reduce this problem to the general form (4.28) by two integrations on t: we obtain the nonlinear integro-differential equation

$$q(t)-\int_0^T(t-\tau)f(\tau;q(\tau),\dot{q}(\tau))d\tau=0. \tag{5.18}$$

If we now use a Green function that satisfies the final conditions $\psi(T)=0$, $\dot{\psi}(T)=0$, like those given in rows 4) and 5) of Table I, we obtain the functional of the form (4.29) with

$$M(q)=-\int_0^T (t-\tau) f\left(\tau; q(\tau), \dot{q}(\tau)\right) d\tau \ . \qquad (5.19)$$

The two corresponding functionals give a variational formulation to the given not symmetric initial value problem.

Example 4. Dynamical system.

Let us consider a dynamical system

$$\begin{cases} \frac{d}{dt} q^r(t) - f^r(t; q^1(t), q^2(t), \ldots q^n(t)) = 0 \\ q^r(0) = a^r \qquad q^r(t) \in C^1[0,T] \qquad (r=1,2,\ldots n) \end{cases} \qquad (5.20)$$

where f^r are given functions and a^r given constants. We may utilize the analysis done in Ex. 2: if one selects n^2 Green functions $g_{rs}(t,\tau)$ that satisfy the final condition $g_{rs}(T,\tau)=0$ one obtains a functional. In particular one may choose only one Green function $g(t,\tau)$ and using the metric tensor a_{rs} to put

$$g_{rs}(t,\tau) = a_{rs} g(t,\tau) . \qquad (5.21)$$

Then one obtains

$$\bar{F}[q] = \frac{1}{2} \int_0^T dt \left[\frac{d}{dt} q_r(t) - f_r(t; q(t))\right] \times \times \int_0^T g(t,\tau) \left[\frac{d}{dt} q^r(\tau) - f^r(\tau, q(\tau))\right] d\tau . \qquad (5.22)$$

These functionals (with $g(T,\tau)=0$) give a variational formulation in the extended sense to every dynamical system.

REFERENCES

1. VOLTERRA V., Leçons sur le fonctions de ligne, Gauthier-Villars, 1913.
2. VOLTERRA V., Theory of functionals and integrodifferential equations, London 1929 (reprinted in 1959 by Dover).
3. VAINBERG M.M., Variational Methods for the Study of Nonlinear Operators, Holden-Day, 1964.
4. NASHED M.Z., Differentiability and related properties of non-linear operators: some aspect of the role of differentials in non-linear functional analysis. In Nonlinear Functional Analysis and Applications (Ed. by Rall), Academic Press, 1971.
5. KERNER M., Die Differentiale in der allgemeinen Analysis, Ann. of Math. 34 (1933), pp. 546-572.
6. MAGRI F., Variational Formulation for Every Linear Equation, Int. Journ. Engin. Sci. 12, (1974), pp. 537-549.
7. MAGRI F., An Operator Approach to Poisson Brackets, Ann. Phys. 99, (1976), pp. 334-344.
8. HELMHOLTZ (von) H., Ueber die physikalische Bedeutung des Princips der kleinsten Wirkung, J. Reine Angew. Math. 100, (1887), pp. 137-166.
9. TAKENS F., A Global Version of the Inverse Problem of the Calculus of Variations, Journ. Diff. Geom. 14 (1979), pp. 543-562.
10. SANTILLI R.M., Foundations of Theoretical Mechanics I: The Inverse Problem of Newtonian Mechanics, Springer-Verlag, 1978.
11. ATHERTON R.W. - HOMSY G.M., On the Existence and Formulation of Variational Principles for Nonlinear Differential Equations, Stud. Appl. Math. 54, (1975), pp. 31-60.
12. ANDERSON I.M. DUCHAMP T., On the existence of global variational principles, Journ. of Math., 102, 5 (1980), pp. 781-868.
13. OLVER P.J., On the Hamiltonian structure of evolution equations, Math. Proc. Camb. Phil. Soc. 88 (1980), pp. 71-88.

14. HAVAS P., The Range of Application of the Lagrange Formalism - I, Suppl. Nuovo Cim., Vol. V, Scr. X, (1937), pp. 363-388.
15. HIRSCH A., Ueber eine charakteristische Eigenschaft der Differentialgleichungen der Variationsrechnung, Math. Ann. 49 (1897), pp. 49-72.
16. DOUGLAS J., Solution of the Inverse Problem of the Calculus of Variations, Trans. Am. Math. Soc. 50, (1941), pp. 71-128.
17. DAVIS D.R., The inverse problem of the calculus of variations in high space, Trans. Am. Math. Soc., 30 (1928), pp. 710-736.
18. TONTI E., Variational formulation of nonlinear differential equations, Bull. Acad. Roy. de Belgique, Scr. 5, Vol. LV, (1969), pp. 137-165 (first part), pp. 262-278 (second part).
19. VOLTERRA V., Sopra le funzioni che dipendono da altre funzioni, Rend. Acc. Lincei, 1887, vol. III, pp. 97-105 (I); pp. 141- (II); pp. 153-158 (III).
20. ODEN J.T. and REDDY J.N., Variational methods in theoretical mechanics, Springer Verlag, 1976.
21. EVANS G.C., Functionals and their applications, American Mathematical Society, 1918, (repreinted by Dover in 1964).
22. VANDERBAUWHEDE A.L., Potential operators and the inverse problem of classical mechanics, Hadronic Journ., 1 (1978) pp. 1177-1197.
23. TELEGA J.J., On variational formulation for nonlinear, non-potential operators, J. Inst. Maths. Applics.24, (1979), pp. 175-195.
24. SCHAEFER H., Topological Vector Spaces, Springer Verlag, 1970.
25. MORSE M., FESHBACH H., Methods of theoretical physics, McGraw Hill, 1953.
26. TONTI E., Variational formulation for initial value problems, Annali di Matem. Pura ed Appl., vol. XCV, (1972), pp.331-360.

27. FINLAYSON B.A., SCRIVEN L.E., On the Search for Variational Principles, Journ. Heat and Mass Transfer, vol. 10, (1957), pp. 799-821.
28. GURTIN M.E., Variational principles in the linear theory of viscoelasticity, Arch.Rat.Mech.Anal. 13-3 (1963), pp. 179-197.
29. TONTI E., On the mathematical structure of a large class of physical theories, Rend.Acc.Lincei, vol. LII (1972), pp. 39-56.
30. TONTI E., A mathematical model for physical theories, Rend.Acc.Lincei, vol. LII, (1972), pp. 175-181; 350-356.
31. TONTI E., Gauge invariance and conservation laws, Rend. Acc.Lincei, vol. VL (1968), pp. 293-300.
32. TONTI E., The reason for analogies of hysical theories, Appl.Math.Modelling, vol. I (1976), pp. 37-60.
33. TONTI E., Sulla struttura formale delle teorie fisiche, Rend. Seminario Matematico e Fisico di Milano, vol.XLVI, 1976, pp. 163-257. (Preprint: On the formal structure of Physical theories, 1975, C.N.R.)
34. SALOV V.M., Solution of non-selfadjoint equations by the variational method, Sov. Math. Dokl., IV (1963), pp. 1046-1048.
35. DIDENKO V.P., A variational method of solving boundary value problems whose operator is not symmetric, Dokl. Akad. Nauk. SSSR, vol. 240 (1978), n.6, pp.736-740.
36. FINLAYSON B.A., Existence of Variational Principles for the Navier-Stokes Equation, Phys. of Fluids, Vol. 15, 6, (1972), pp. 963-967.
37. FUNG Y.C., Foundation of Solid Mechanics, Prentice-Hall, 1965.
38. HELMHOLTZ H. (von), Ueber die physikalische Bedeutung des Prinzips der kleinsten Wirkung, Journ. für die reine und angew. Mathem., C., 1886, pp. 137-166; pp. 213-222; (W.A.,III, pp.203-248).
39. HLAVACEK I., Variational Principles for Parabolic Equations, Aplik. Matem., vol. 4, (1969), pp.278-297.

40. DEDECKER P., Sur un problème inverse du calcul des variations, Bull. Acad. R. Belg. Sc. Vol. 36, (1950), pp. 63-70.
41. MORSE M., The critical points of functions and the calculus of variations in the large, Bull. Am. Math. Soc. 1929, vol. 35, pp. 38-54.
42. MORSE M., The calculus of variations in the large, Verhandlungen des Internationalen Mathematiker-Kongress, Züric 1932, pp. 173-183.
43. DEDECKER P. and TULCZYJEW W.M., Spectral Sequences and the Inverse Problem of the Calculus of Variations, Lect. Notes Math. vol. 836, 1979.
44. GURTIN M.E., Variational Principles for Linear Initial--Value Problems, Quart.Appl.Math., 22, (1964), pp.252-256.
45. MIKHLIN S.G., Variational Methods in Mathematical Physics, Pergamon Press, 1964.
46. TULCZYJEW W.M., The Lagrange differential, Bull.Acad.Pol. Sc.Ser.Sc.Math.Astron.Phys. 24, (1976), pp. 1089-1096.
47. TULCZYJEW W.M., The Lagrange complex, Bull.Soc.Math. France 105, (1977), pp. 419-431.

Acknowledgements.The author is grateful to the student F. Beltram, prof. F. Magri, prof. V. Amoia and in particular to prof. G. Tironi for critical reading of the manuscript. The author remains the only responsible of possible flaws.