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AUTUMN COURSE

ON

VARIATIONAL METHODS IN ANALYSIS AND MATHEMATICAL PHYSICS

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TOPICS IN CRITICAL POINTS THEORY

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These are preliminary lecture notes, intended only for distribution to participants.
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§0. Preliminaries

a) The Frechét derivative.

Let X, Y be Banach spaces, S an open subset of X and $x \in S$. A function $f : S \to Y$ is Frechét-differentiable at x if there exists $A \in \mathcal{L}(X,Y)$, depending on x, such that:

$$(0.1) \qquad f(x+h) - f(x) = A(h) + o(h) \qquad , h \in E$$

The linear continuous mapping A is uniquely determined by (0.1) and depends only on the topology of X. In the following we will denote by df_x or $df(x)$ or $f'(x)$ the Frechét-derivative of f at x.

0.1. Examples. (i) If $A \in \mathcal{L}(X,Y)$ then A is F-differentiable at every $x \in X$ and $dA_x = A$;

(ii) the norm $\|\cdot\| : X \to \mathbb{R}$ is not F-differentiable at $x = 0$;

(iii) if E is a Hilbert space, with scalar product $(\cdot,\cdot)$ and norm $\|\cdot\| = (\cdot,\cdot)^{1/2}$, then the mapping $\|\cdot\|^2 : E \to \mathbb{R}$ is F-differentiable at every $x \in E$, with derivative given by the (linear) mapping:

$$h \longmapsto 2(x,h);$$

(iv) let $\mathcal{J}(X,Y)$ be the open subset of $\mathcal{L}(X,Y)$ consisting of those $A \in \mathcal{L}(X,Y)$ invertible, with continuous inverse $A^{-1} \in \mathcal{L}(Y,X)$. The mapping $f : \mathcal{J}(X,Y) \to \mathcal{L}(Y,X)$

defined by $f(A) = A^{-1}$, is F-differentiable at every A and it results

$$df_A : T \longrightarrow -A^{-1} \circ T \circ A.$$

The usual rules of differentiation hold: for example, if $f : X \to Y$ is F-differentiable at x and $g : Y \to Z$ is F-differentiable at $y = f(x)$, then $g \circ f$ is F-differentiable at x and one has

$$d(g \circ f)_x = dg_y \circ df_x$$

We will say that $f : S \to Y$ is F-differentiable in S if f is F-differentiable at every point $x \in S$. By $C^1(S,Y)$ we mean the space of all functions $f : S \to Y$ which are F-differentiable in S and such that the mapping

$$x \longrightarrow df(x)$$

from S to $\mathcal{L}(X,Y)$ is continuous.

A mapping $f \in C(S,Y)$ is said <u>locally invertible</u> at $x \in S$ if there exist:

i) a neighborhood U of x, $U \subset S$;

ii) a neighborhood V of $y := f(x)$;

iii) a continuous function $g : V \to U$;

such that

$$(0.2)\quad \begin{cases} g(f(u)) = u & \forall u \in U \\ f(g(v)) = v & \forall v \in V \end{cases}$$

With a not very precise notation we will set $g = f^{-1}$. The following theorem gives a condition for a C^1 mapping to be locally invertible:

0.2 Local Inversion theorem. Let $f \in C^1(S,Y)$ and $x \in S$. Suppose that $df(x) \in \mathcal{J}(X,Y)$ (cf. the notation introduced in 0.1-(iv)). Then f is locally invertible at x. Moreover f^{-1} is of class C^1 and it results

$$d f^{-1}_y = \left(d f_x\right)^{-1} , \quad y = f(x)$$

It is also possible to define higher order derivatives: if the mapping $x \mapsto df_x$ is differentiable at x we define the second Frechet derivative of f at x by setting

$$d^2 f_x = d\,(d f_x)_x$$

Clearly $d^2 f_x \in \mathcal{L}(X, \mathcal{L}(X,Y)) \approx \mathcal{L}_2(X,Y)$ (the space of continuous bilinear forms from $X \times X$ to Y). In the same way we can define the third, fourth, ... derivative.

Proof of Local Inv. thm. Assume w.l. of g. $x = y = 0$.

Set $A = (d f_0)^{-1}$, and consider $g = A \circ f$

$$S \xrightarrow{\quad g \quad} X$$

$$S \xrightarrow{f} Y \xrightarrow{A} X$$

$g \in C^1(S,X)$, $g(0) = 0$ and $dg_0 = A \circ df_0 = Id_X$

Set

$$\Phi(x) = g(x) - x$$

Now

$$d\Phi_0 = dg_0 - Id_X = 0$$

Since Φ is C^1 (indeed g is so), given $0 < \alpha < 1$ $\exists\, \rho > 0$ such that

$$\| d\Phi_x \| \le \alpha \qquad \text{for all } x \in \overline{B}_\rho$$

We claim that Φ is a contraction of modulus α in $\overline{B}_\rho$. In fact, if $u, v \in \overline{B}_\rho$ then

$$(1) \qquad \| \Phi(u) - \Phi(v) \| \le \text{(by the mean-value thm)} \le$$

$$\le \sup_{x \in B_\rho} \| d\Phi_x \| \cdot \| u - v \| \le$$

$$\le \alpha \, \| u - v \|$$

In order to solve the equation $g(x) = y$, we consider

$$T_y(x) \equiv y - \Phi(x) \qquad \text{for } x \in \overline{B}_\rho \text{ and } y \in \overline{B}_{(1-\alpha)\rho}$$

From (1) we get, for $u, v \in \bar{B}_\rho$ and $y \in \bar{B}_{(1-\alpha)\rho}$

$$\| T_y(u) - T_y(v) \| = \| \Phi(u) - \Phi(v) \| \leq$$

$$\leq \alpha \| u - v \|$$

Thus T_y is a contraction of ~~...~~ mod. α in $\bar{B}_\rho$, as well.

Moreover, from

$$\| T_y(u) \| \leq \| y \| + \| \Phi(u) \| \leq (1-\alpha)\rho + \| \Phi(u) \|$$

$$\leq (\text{from } (1), \text{ putting } v = 0) \leq (1-\alpha)\rho + \alpha\rho = \rho$$

it follows that $T_y(\bar{B}_\rho) \subset \bar{B}_\rho$. Hence, by the Banach contraction principle, we conclude that T_y has a unique fixed point in $\bar{B}_\rho$, i.e. $\exists\, x \in \bar{B}_\rho$ such that $T_y(x) = x$ or, in other words, $y - \Phi(x) = x$. Therefore

~~$y = g(x)$~~

for every $y \in \bar{B}_{(1-\alpha)\rho}$ there is a unique $x \in \bar{B}_\rho$ such that $g(x) = y$. We denote ~~the~~ by g^{-1} the inverse of g, defined on $\bar{B}_{(1-\alpha)\rho}$.

We prove that:

1) g^{-1} is continuous (indeed Lipschitz-cont, with Lipschitz constant $= \frac{1}{1-\alpha}$): let $x_i = g^{-1}(y_i)$ $i = 1, 2$; one has:

$$x_i + (g(x_i) - x_i) = y_i \quad \text{or}$$

$$x_i + \Phi(x_i) = y_i$$

and hence

$$\| x_2 - x_1 \| - \| \Phi(x_2) - \Phi(x_1) \| \leq \| y_2 - y_1 \|$$

(by (1))

$$(1-\alpha) \| x_2 - x_1 \| \leq \| y_2 - y_1 \|.$$

2) g^{-1} is differentiable in $y \in \bar{B}_{(1-\alpha)\rho}$.

Let $x = g^{-1}(y)$ and, for $\tilde{y} \in B_{(1-\alpha)\rho}$, $\tilde{x} = g^{-1}(\tilde{y})$.

$$\| \tilde{x} - x - (Dg_x)^{-1}(\tilde{y} - y) \| =$$

$$= \| (Dg_x)^{-1} [Dg_x(\tilde{x} - x) - (\tilde{y} - y)] \|$$

$$\leq \| (Dg_x)^{-1} \| \, \| Dg_x(\tilde{x} - x) - \underbrace{(\tilde{y} - y)}_{= (g(\tilde{x}) - g(x))} \|$$

Since g is diff. at x, we get:

$$\| g(\tilde{x}) - g(x) - dg_x(\tilde{x} - x) \| = o(\tilde{x} - x)$$

and since g^{-1} is Lipschitz we ~~conclude~~ have

$$o(\tilde{x} - x) = o(g^{-1}(\tilde{y}) - g^{-1}(y)) = o(\tilde{y} - y).$$

Lastly we can show that g^{-1} is C^1, because the mapping $y \longmapsto dg^{-1}_y$ can be thought as:

$$y \xrightarrow[g^{-1}]{} x \xrightarrow[dg]{} dg_x \longrightarrow (dg_x)^{-1}$$