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REDUCTION METHOD VIA MINIMAX

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This notes are the material presented by the author in the ICTP Autumn Course on Variational Methods (1981). They are intended for the [redacted] reader who is acquainted with the basics of Hilbert space theory. Lemmas 4 and 5 explain the title.

This material is divided in three sections. In the first we state various minimax principles proving only relevant aspects. In the [redacted] second we apply the minimax principles to obtain a partial extension of a result due to A. Ambrosetti and G. Prodi [2]. In the last section we sketch alternatives for further study.

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1. Minimax principles.

The following two results are well known facts from optimization theory. We include them in order to point out that this method takes us to recent results in Nonlinear Functional Analysis starting from very elementary facts of Hilbert space theory.

Throughout this section H denotes a real Hilbert

space with inner product $\langle\ ,\ \rangle$.

Lemma 1 (Banach-Sachs) *If $\{u_n\}$ is a sequence in H converging weakly to u, then there exists a subsequence $\{u_{n_j}\}$ such that $\{z_K = (\sum_{j=1}^{n} u_{n_j})/K\}$ converges to u.*

Proof: Without loss of generality we can assume that $u=0$. Hence the sequence $\{\langle u_n, u_1\rangle\}$ converges to $0 \in \mathbb{R}$. Thus, there exists an integer n_1 such that $|\langle u_n, u_1\rangle| < (1/2)$ for $n \geq n_1$. Since also $\langle u_n, u_{n_1}\rangle \to 0$, there exists $n_2 > n_1$ such that $|\langle u_n, u_{n_1}\rangle| < (1/2)^2$ for $n \geq n_2$. Iterating this process we obtain a sequence $n_1 < n_2 < n_3 < \dots$ such that $|\langle u_n, u_{n_{K-1}}\rangle| < (1/2)^K$ for $n \geq n_K$, $K = 1, 2, \dots$. We claim that the sequence $\{z_K = (\sum_{j=1}^{K} u_{n_j})/K\}$ converges to 0. In fact

$$(1)\qquad \|z_K\|^2 = \langle z_K, z_K\rangle = \frac{1}{K^2}\Big(\sum_{j=1}^{K}\langle u_{n_j}, u_{n_j}\rangle + 2\sum_{j=2}^{K}\sum_{i<j}\langle u_{n_i}, u_{n_j}\rangle\Big)$$

$$\leq \Big(\sum_{j=1}^{K}\|u_{n_j}\|^2 + 2\sum_{i=1}^{K-1}(K-i)(1/2)^i\Big)/K^2$$

Since $\{u_n\}$ converges weakly, $\{u_n\}$ is bounded (see [10, pp. 28]). Let $M \in \mathbb{R}$ be such that $\|u_n\|^2 \leq M$ for all n. Hence from (1) we have $\|z_K\|^2 \leq (M+2)/K$ which proves the Lemma.

Lemma 2. *Let $J : H \to \mathbb{R}$ be convex and continuous. If*

$J(u)\to\infty$ as $\|u\|\to\infty$ then J has a minimum.

Proof: Let $\{u_n\}$ be a sequence with $\lim_n J(u_n) = \inf\{J(u); u\in H\}$. Since $J(u)\to\infty$ as $\|u\|\to\infty$, the sequence $\{u_n\}$ is bounded. Hence (see [10, Th. 1.8.1]) we can assume that $\{u_n\}$ converges weakly to $\bar u\in H$. Let $\{z_k\}$ be as in Lemma 1. Therefore, by the continuity and convexity of J we have

(2) $$J(\bar u) = \lim_k J(z_k) \le \liminf_k \Big(\Big(\sum_{j=1}^{k} J(u_{n_j})\Big)/k\Big)$$

$$= \lim_n J(u_n) = \inf\{J(u); u\in H\}.$$

This implies that J attains its minimum value at $\bar u$ and the Lemma is proved.

From now on we assume that $J:H\to\mathbb{R}$ is a function of class C^1, that is, we assume that there exists a continuous function $\nabla J: H\to H$ such that

(3) $$\lim_{t\to 0+} \frac{J(u+tv)-J(u)}{t} = \langle\nabla J(u), v\rangle$$

for all $u,v\in H$.

Lemma 3. Let X and Y be closed subspaces of H with $H = X\oplus Y$. If there exit $m>0$ and $\alpha>1$ such that

(4) $$\langle \nabla J(x+y) - \nabla J(x+y_1), y-y_1\rangle \ge m\|y-y_1\|^\alpha$$

for all $x\in X$, $y, y_1\in Y$, then:

i) There exists a continuous function $r: X\to Y$ such that $J(x+r(x)) = \min\{J(x+y); y\in Y\}$. For each $x\in X$, $r(x)$ is the only critical point of the functional $J_x: Y\to\mathbb{R}$, $y\mapsto J(x+y)$.

ii) The function $\tilde J: X\to\mathbb{R}$, $x\mapsto J(x+r(x))$ is of class C^1 and $\langle\nabla\tilde J(x), x_1\rangle = \langle\nabla J(x+r(x)), x_1\rangle$ for $x, x_1\in X$.

iii) An element $x\in X$ is a critical point of $\tilde J$ iff $x+r(x)$ is a critical point of J.

Proof: For $y_1, y_2\in Y$ given, let $\rho(s) \equiv J(x+y_1+s(y_2-y_1))$. Since $(d\rho/ds)(t) = \langle\nabla J(x+y_1+t(y_2-y_1)), y_2-y_1\rangle$, by (4) we have for $\bar t > t$

$$\frac{d\rho}{ds}(\bar t) - \frac{d\rho}{ds}(t) = \langle \nabla J(x+y_1+\bar t(y_2-y_1)) - \nabla J(x+y_1+t(y_2-y_1)),$$

$$y_2-y_1\rangle \ge m\|y_2-y_1\|^\alpha(\bar t - t)^{\alpha-1} > 0.$$

Hence $(d\rho/ds)$ is increasing which proves that J_x is convex. Since $J(x+y) = J(x) + \int_0^1\langle\nabla J(x+sy), y\rangle ds \ge J(x) - \|\nabla J(x)\|\|y\| + \int_0^1 \langle\nabla J(x+sy) - \nabla J(x), y\rangle ds$, by (4) we have

(5) $$J_x(y) \ge J(x) - \|\nabla J(x)\|\|y\| + m\|y\|^\alpha/\alpha.$$

Since $\alpha>1$ and $m>0$, the above expresion implies that

$J_x(y) \to \infty$ as $\|y\| \to \infty$. Thus, by Lemma 2, J_x attains its minimum value at some point $r(x)$. Since (4) implies that J_x can not have two critical points we have proved the existence of the function r. The continuity of r follows routinary arguments and we leave it as an <u>exercise</u> (see [3, Lemma 2]).

Since J_x takes its minimum value at $r(x)$ we have for $t>0$

$$(6) \quad \frac{\tilde{J}(x+tx_1) - J(x)}{t} = \frac{J(x+tx_1+r(x+tx_1)) - J(x+r(x))}{t}$$

$$\leq \frac{J(x+tx_1+r(x)) - J(x+r(x))}{t}$$

$$= \int_0^1 \langle \nabla J(x+stx_1+r(x)), x_1 \rangle ds.$$

Hence, by the continuity of ∇J $\limsup_{t\to 0^+} ((\tilde{J}(u+tv) - J(u))/t) \leq \langle \nabla J(x+r(x)), x_1 \rangle$. In the same manner it is proved that $\liminf_{t\to 0^+} ((\tilde{J}(x+tx_1) - \tilde{J}(x))/t) \geq \langle \nabla J(x+r(x)), x_1 \rangle$. Therefore, by the continuity of r and ∇J we see that $\tilde{J}$ is of class C^1 and the conclusion ii) has been demonstrated.

Now we prove claim iii). Since any $u \in H$ can be

written in the form $u = x_1 + y$, with $x \in X$ and $y \in Y$; if x is a critical point of $\tilde{J}$ then

$$(7) \quad \langle \nabla J(x+r(x)), u \rangle = \langle \nabla J(x+r(x)), x_1 \rangle + \langle \nabla J(x+r(x)), y \rangle = \langle \nabla \tilde{J}(x), x_1 \rangle = 0,$$

where we have also used that $r(x)$ is a critical point of J_x. From (7) clearly follows that $x+r(x)$ is a critical point of J if x is a critical point of $\tilde{J}$. The proof of the converse is equally easy and we leave it as an <u>exercise</u>. This completes the proof of Lemma 3.

The former Lemma can be extended to situations in which r arises from <u>minimax points</u> or <u>saddle points</u>. The purpose of the next Lemma is to express such a case.

Lemma 4. <u>Let</u> X, Y <u>and</u> Z <u>be closed subspaces of</u> H <u>with</u> $H = X \oplus Y \oplus Z$. <u>If there exist</u> $m > 0$ <u>and</u> $\alpha > 1$ <u>such that</u>

$$(8) \quad \langle \nabla J(x+y+z) - \nabla J(x_1+y+z), x - x_1 \rangle \leq -m\|x-x_1\|^\alpha$$

$$(9) \quad \langle \nabla J(x+y+z) - \nabla J(x+y_1+z), y - y_1 \rangle \geq m\|y-y_1\|^\alpha$$

<u>then</u>:

i) <u>There exists a continuous function</u> $r: Z \longrightarrow X \oplus Y$ <u>such that</u> $J(z+r(z)) = \max\{\min\{J(x+y+z); y \in Y\}; x \in X\} = \min\{\max\{J(x+y+z); x \in X\}; y \in Y\}$. <u>Moreover, for each</u> $z \in Z$,

$r(z)$ is the only critical point of $J_z : X \oplus Y \to \mathbb{R}$, $x+y \mapsto J(x+y+z)$.

ii) The function $\tilde{J} : Z \to \mathbb{R}$, $z \mapsto J(z+r(z))$ is of class C^1 and $\langle \nabla \tilde{J}(z), z_1 \rangle = \langle \nabla J(z+r(z)), z_1 \rangle$ for $z, z_1 \in Z$.

iii) An element $z \in Z$ is a critical point of $\tilde{J}$ iff $z+r(z)$ is a critical point of $\tilde{J}$.

Proof: From (9) and Lemma 3 we see that there exists a continuous function $\alpha(z, \ ) : X \to Y$ such that $\tilde{J}_z(x) \equiv J_z(x + \alpha(z,x)) = \min\{J(x+y+z);\ y \in Y\}$. Let $\{x_n\}$ be a sequence such that $\tilde{J}_z(x_n) \to \sup\{\tilde{J}_z(x);\ x \in X\}$. From (8), arguing as in the proof of Lemma 3, we see that $J_z(x) \to -\infty$ as $\|x\| \to \infty$. Hence, since $\tilde{J}_z(x) \le J_z(x)$, $\tilde{J}_z(x) \to -\infty$ as $\|x\| \to \infty$. Therefore, x_n is a bounded sequence. Thus, without loss of generality we can assume that converges weakly to $\bar{x} \in X$. From (8) follows that the function $x \to J(z + x + \alpha(z, \bar{x}))$ is concave. Hence, imitating the proof of Lemma 2 we see that $J(z + \bar{x} + \alpha(z, \bar{x})) \ge \limsup J(x + z_n + \alpha(z, \bar{x}))$. All these facts together give

$$\lim_n \tilde{J}_z(x_n) = \lim_n J(z + x_n + \alpha(z, x_n))$$
$$\le \limsup J(z + x_n + \alpha(z, \bar{x})) \le J(z + \bar{x} + \alpha(z, \bar{x})).$$

Hence $\tilde{J}_z$ attains its maximum value at $\bar{x}$. Thus, if we denote $r(z) = \bar{x} + \alpha(z, \bar{x})$ then we have $J(z + r(z)) = J(z + \bar{x} + \alpha(z, \bar{x})) = \max\{J(z + x + \alpha(z,x));\ x \in X\} = \max\{\min\{J(z+x+y);\ y \in Y\};\ x \in X\}$, and this proves the existence of $r(z)$. Another way to obtain a critical point for J_z is first to maximize the function $x \to J(x+y+z)$, for y fixed, and then minimize. Since, from (8) and (9), follows immediatelly that J_z has at most one critical point then

$$\min\{\max\{J_z(x+y);\ x \in X\};\ y \in Y\} = \max\{\min\{J_z(x+y);\ y \in Y\};\ x \in X\}.$$

This equality is crucial in the proof of the differentiability of $\tilde{J}$. We encourage the reader to finish the proof of this Lemma as an exercise.

2. Application to a problem on jumping nonlinearities.

Let Ω be a bounded region in $\mathbb{R}^n$, $g : \mathbb{R} \to \mathbb{R}$ a continuous function and L a second order selfadjoint uniformly elliptic operator in Ω of the form

$$Lu \equiv \sum_{i,j=1}^{n} \frac{\partial}{\partial x_j}\left(a_{ij}(\xi) \frac{\partial u}{\partial x}\right) + c(\xi) u,$$

with $a_{ij} = a_{ji} \in L^\infty(\Omega)$, $c \in L^\infty(\Omega)$. We consider the existence of weak solutions to the Dirichlet problem

$$(10) \qquad Lu + g(u) = q(\xi) + t\varphi_1(\xi) \quad \xi \in \Omega, \quad u \equiv 0 \text{ on } \partial\Omega,$$

where $q \in L^2(\Omega)$, $\int_\Omega q\varphi_1 = 0$ and φ_1 is an eigenfunction corresponding to the smallest eigenvalue of the problem $L\varphi + \lambda\varphi = 0$ in Ω, $u = 0$ on $\partial\Omega$. The existence of such a smallest eigenvalue follows from standar spectral theory of elliptic boundary value problems (see [6]). In fact, it is well known that the above spectral problem has a sequence of eigenvalues $\lambda_1 < \lambda_2 \leq \cdots \to \infty$ and a corresponding sequence of eigenfunctions $\{\varphi_i\}$ which form a complete orthonormal set in $L^2(\Omega)$. Moreover, φ_1 can be assumed to satisfy

(11) $$\varphi_1(\xi) \geq 0 \quad \text{for all } \xi \in \Omega.$$

Let $H \equiv \overset{\circ}{H}{}^1(\Omega)$ be the Sobolev space of real valued square integrable functions defined in Ω vanishing on $\partial\Omega$ and having first order generalized partial derivatives in $L^2(\Omega)$ (see [1]). The inner product in H is given by $\langle u, v\rangle = \int_\Omega \sum_{i=1}^n \frac{\partial u}{\partial x_i}\cdot\frac{\partial v}{\partial x_i}$.

We define $u \in H$ to be a <u>weak solution</u> of (10) iff u is a critical point of $J(\ ,q,t): H \to \mathbb{R}$ given by $J(u,q,t) = B(u,u) - 2\int_\Omega G(u(\xi))d\xi + 2\int_\Omega (q(\xi) + t\varphi_1(\xi))u(\xi)d\xi$. Here

(12) $$B(u,v) \equiv \int_\Omega \sum_{i,j} a_{ij}(\xi)\frac{\partial v}{\partial x_j}\cdot\frac{\partial u}{\partial x_i} - \int_\Omega c(\xi)u(\xi)v(\xi)d\xi,$$

$G(u) = \int_0^u g(s)ds$.

The next theorem, which is a partial extension of the main result of [2], states the solvability of (10) in term of the parameter t when the following hypotheses are assumed:

(13) There exists $\gamma \in (\lambda_1, \lambda_2)$ such that $(g(u) - g(v))(u-v) \leq \gamma(u-v)^2$ for $u, v \in \mathbb{R}$.

(14) $\lim_{u\to-\infty} (g(u)/u) \equiv g'(-\infty) \in (-\infty, \lambda_1)$ and $\lim_{u\to\infty} (g(u)/u) \in (\lambda_1, \lambda_2)$.

We remark that without loss of generality we can assume that

(15) $$g'(-\infty) < 0 < \lambda_1 < g'(\infty).$$

This is because if (15) does not hold we just replace L by $L+\eta$ and g by $g-\eta$ with $\eta \in (g'(-\infty), \lambda_1)$ obtaining a problem equivalent to (10) which satisfies (15). Since $\lambda_1 > 0$ implies that the bilinear form B is equivalent to the inner product in H we will take the norm in H as $\|u\| = (B(u,u))^{1/2}$. Now we state our main theorem.

Theorem 5. <u>If Ω, L, φ_1 and g are as above, then for each $q \in L^2(\Omega)$ with $\int_\Omega q\varphi_1 = 0$ there exists</u>

a real number $d(q)$ such that (10) has a weak solution iff $t \geq d(q)$. If $t > d(q)$ then (10) has at least two weak solutions. Moreover, if $\{q^n\}$ converges weakly to q in $L^2(\Omega)$ then $d(q^n) \longrightarrow d(q)$.

Remark: Using the maximum principle for second order elliptic operators it can be proven that if, in addition, g is convex then for $t = d(q)$ (10) has precisely one solution and for $t > d(q)$ (10) has precisely two solutions.

Proof of Theorem 5. Let X be the subspace generated by the element φ_1 and Y the closed subspace of H generated by the set $\{\varphi_2, \varphi_3, ..\}$. Hence $H = X \oplus Y$. The orthogonality of the set $\{\varphi_i ; i = 1,2,\dots\}$ and its spectral properties imply that for $i \neq j$ $B(\varphi_i, \varphi_j) = 0$ and $B(\varphi_i, \varphi_i) = \lambda_i$ (see (11)). Thus, for $y \in Y$,

$$(16) \qquad B(y,y) \geq \lambda_2 \int_\Omega y^2(\xi) d\xi .$$

From the definition of J follows that

$$(17) \quad \langle \nabla J(u,q,t), v \rangle = 2\Big(B(u,v) - \int g(u) v + t\int (q + t\varphi_1) v\Big)$$

for $u, v \in H$. In particular, if $x \in X$, $y_1, y_2 \in Y$, by (13) and (16), we have

$$(18) \quad \langle \nabla J(x+y_1,q,t) - \nabla J(x+y_2,q,t), y_1 - y_2 \rangle = 2\Big(B(y_1-y_2, y_1-y_2) - \int_\Omega (g(x+y_1) - g(x+y_2))(y_1-y_2)\Big)$$

$$\geq 2\Big(B(y_1-y_2, y_1-y_2) - \gamma \int_\Omega (y_1-y_2)^2\Big)$$

$$\geq 2(1-(\gamma/\lambda_2))B(y_1-y_2, y_1-y_2) = 2(1-(\gamma/\lambda_2))\|y_1-y_2\|^2.$$

Since by (14) $(1-(\gamma/\lambda_2)) > 0$, Lemma 3 implies that there exists a continuous function $r(\ ,q,t) : X \longrightarrow Y$ such that $J(x + r(x,q,t), q, t) = \min\{J(x+y,q,t) ; y \in Y\}$ and $r(x,q,t)$ is the only critical point of $y \longmapsto J(x+y,q,t)$.

Now we prove that r is independent of the third variable. From (17) we have

$$(19) \quad 0 = \langle \nabla J(x + r(x,q,t), q, t), y \rangle$$

$$= 2\Big(B(x + r(x,q,t), y) - \int_\Omega (g(x + r(x,q,t)) - q + t\varphi_1)\varphi_1\Big)$$

$$= 2\Big(B(r(x,q,t), y) - \int_\Omega (g(x+r(x,q,t)) - q)\varphi_1\Big)$$

for all $y \in Y$. In (19) we have used that for $x \in X$ and $y \in Y$ $B(x,y) = 0 = \int xy$. From (19) we see that $r(x,q,t)$ is a critical point of $y \longmapsto J(y,q,0)$. Consequently, by the uniqueness of the critical point of the latter function, we $r(x,q,t) = r(x,q,0)$ for all q and all x. Thus we write $r(x,q) \equiv r(x,q,t)$.

From the conclusions ii) and iii) of Lemma 3 we know that in order to prove that (10) has a weak solution, i.e. that $J(\ ,q,t)$ has a critical point, it is necessary and sufficient to prove that the equation

$$(20)\qquad 0 = \langle \nabla \tilde{J}(x,q,t), \varphi_1 \rangle$$

$$= \langle \nabla J(x + r(x,q,t), q, t), \varphi_1 \rangle$$

$$= (B(x,\varphi_1) - \int_\Omega g(x + r(x,q))\varphi_1 + t\int_\Omega \varphi_1^2)2$$

has a solution $x \in X$. Since $\int_\Omega \varphi_1^2 = 1$, solving this equation is equivalent to prove that $-t$ belongs to the range of the function $\tau(\ ,q): X \longrightarrow \mathbb{R}, x \longmapsto B(x,\varphi_1) - \int g(x + r(x,q))\varphi_1$. We claim that

$$(21)\qquad \tau(x,q) \longrightarrow -\infty \quad \text{as } |x| \to \infty, \quad x \in X.$$

From (15) follows that g is bounded below, say $g(u) \geq \bar{d}$ for all $u \in \mathbb{R}$. Let us write $x = s\varphi_1$, with $s \in \mathbb{R}$. Hence we have

$$(22)\qquad \tau(s\varphi_1, q) \leq s\,B(\varphi_1,\varphi_1) - \bar{d}\int\varphi_1$$

$$= s\lambda_1 - \bar{d}\int\varphi_1 \longrightarrow -\infty \quad \text{as } s \to -\infty$$

From (13) it follows that if $0 < \varepsilon < g'(\infty) - \lambda_1$ then there exists $d_1 \in \mathbb{R}$ such that $g(u) - (\lambda_1 + \varepsilon)u \geq d_1$, for all $u \in \mathbb{R}$. Hence by (11) we have

$$(2\ )\qquad \tau(s\varphi_1, q) = s\lambda_1 - \int_\Omega g(s\varphi_1 + r(s\varphi_1,q))\varphi_1$$

$$\leq s\lambda_1 - \int_\Omega (\lambda_1 + \varepsilon)[s\varphi_1 + r(s\varphi_1,q)]\varphi_1 - d_1\int_\Omega \varphi_1$$

$$= -\varepsilon s - d_1\int_\Omega \varphi_1 \longrightarrow -\infty$$

as $s \to \infty$. Thus from (22) and (23) we see that $\tau(x,q) \to -\infty$ as $\|x\| \to \infty$. Therefore (10) has a weak solution iff

$$t \geq \min\{-\tau(x,q);\, x \in X\} \equiv d(q) > -\infty.$$

From (22) and (23) it is clear that for $t > d(q)$ (10) has at least two weak solutions. The continuity of is left as an exercise.

Remark: From (22) and (23) it follows that d is bounded below, in fact $d(q) \geq \min\{d_1\int\varphi_1, \bar{d}\int\varphi_1\}$.

3. Suggestions for further study.

Problem 1. It was proved in [4] that if $q: \mathbb{R} \longrightarrow \mathbb{R}$ is 2π-pe riodic function with $\int_0^{2\pi} q = 0$ and $\int_0^{2\pi} q^2 < \infty$, then there exists real numbers $d(q) \le 0 \le D(q)$ such that if $t \in \mathbb{R}$ then the <u>pendulum equation</u> $u'' + \sin(u) = q(t) + t$ has a 2π-periodic solution iff $t \in [d(q), D(q)]$. Open question: "Is there q with $d(q) = 0 = D(q)$". From the methods of [4] follows that $d(q) \neq 0$ iff $D(q) \neq 0$.

<u>Problem 2.</u> Give a version of Theorem 5 for the hyperbolic problem

$$\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial t^2} + g(u) = q(x,t) + \alpha \sin(x)$$

$$u(x,0) = u(x,\pi) = 0$$

$$u(x,t) = u(x, t+2\pi).$$

For details about the critical point theory concerning this problem see [9].

<u>Problem 3.</u> Let X and Y be as in Lemma 1. Suppose $J: H \to \mathbb{R}$ is of class C^1 satisfies (P-S) and

$$J(x+y) \longrightarrow \infty \qquad \text{as } \|y\| \to \infty$$

$$J(x) \longrightarrow -\infty \qquad \text{as } \|x\| \to \infty,$$

$x \in X$, $y \in Y$. It is known these hypotheses imply the

existence of a critical value β (see [8,]).

Open question: Does J have a critical point $x_0 + y_0$ with $\beta = J(x_0+y_0) = \min_y J(x_0+y)$?

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