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WEAK AND STRONG DERIVATIVES IN ORLICZ SPACES

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1. INTRODUCTION.

One classical result in partial differential equations is the equality of weak derivatives (i.e.derivatives in the distribution sense) and strong derivatives (i.e. derivatives obtained from smooth functions by a limiting process). Such an equality holds in the setting of L^p spaces, as was proved locally by Friedrichs [9] in 1944 and later globally by Meyers-Serrin [18] in 1964.

Orlicz spaces are generalizations of L^p spaces where, roughly speaking, the defining function $t \to |t|^p$ is replaced by a function of the same shape but whose growth is not necessarily of polynomial type. These spaces have been successfully used in recent years in the study of several questions from partial differential equations, as the limiting case of the Sobolev imbedding theorem (cf.[21]) or the existence theory for strongly nonlinear boundary value problems (cf.[4, 10,11, 5, 8]). In the latter theory, problems related to the equality of weak and strong derivatives in the setting of Orlicz spaces play an important role. Such an equality however does not hold for general Orlicz spaces. In [13] a substitute to this equality was introduced by slightly modifying the limiting process involved in the definition of strong derivatives.

It is our purpose here to describe some of our results of [13] and to show how questions of this sort enter the theory of strongly nonlinear elliptic boundary value problems.

2. WEAK AND STRONG DERIVATIVES.

Let $W^{m,p}(\Omega)$, $1 \leq p < \infty$, Ω open in $\mathbb{R}^N$, be the (Sobolev) space of functions u such that u and its distributional derivatives up to order m belong to $L^p(\Omega)$. Denote by $H^{m,p}(\Omega)$ the closure in $W^{m,p}(\Omega)$ (with respect to the usual norm) of $C^\infty(\Omega) \cap W^{m,p}(\Omega)$. Derivatives of a function in $W^{m,p}(\Omega)$ (resp.$H^{m,p}(\Omega)$) have sometimes been called weak (resp.strong) derivatives. Friedrichs [9], introducing on this occasion his mollifiers, showed that if $u \in W^{m,p}(\Omega)$, then $u \in H^{m,p}(\Omega')$ for any open set Ω' with $\Omega' \subset\subset \Omega$. Meyers-Serrin [18] proved later that actually $W^{m,p}(\Omega)=H^{m,p}(\Omega)$; the decisive point here is that no smoothness condition is imposed on the boundary of Ω.

Let us now briefly recall some definitions and basic facts about Orlicz spaces (cf.e.g.[15,16,1]). Let M be a N-function, i.e. a continuous convex even function from $\mathbb{R}$ to $\mathbb{R}$ with $M(t)=0$ iff $t=0$ and $M(t)/t \to 0$ (resp.$+\infty$) as $t \to 0$ (resp.$+\infty$). The Orlicz space $L_M(\Omega)$ is defined as the set of all (measurable) functions $u: \Omega \to \mathbb{R}$ such that

$$\|u\|_M = \inf \{\lambda > 0; \int_\Omega M(u/\lambda) < 1 \} < +\infty.$$

Let $E_M(\Omega)$ be the closure in $L_M(\Omega)$ of the $L^\infty(\Omega)$ functions with compact support in $\bar{\Omega}$; in general $E_M(\Omega) \subsetneq L_M(\Omega)$. If $\bar{M}$ denotes the conjugate convex fonction of M, i.e. $\bar{M}(t)=\sup\{ts-M(s);s \in \mathbb{R}\}$, then $\bar{M}$ is a N-function, $\bar{\bar{M}} = M$, $\int_\Omega uv$ is a well defined pairing on $L_M(\Omega) \times L_{\bar{M}}(\Omega)$, and the dual of $E_M(\Omega)$ (resp. $E_{\bar{M}}(\Omega)$) is $L_{\bar{M}}(\Omega)$ (resp.$L_M(\Omega)$).

We first consider the analogue of the Meyers-Serrin theorem in the setting of Orlicz spaces, i.e. the question whether $C^\infty(\Omega) \cap W^mL_M(\Omega)$ is norm dense in $W^mL_M(\Omega)$, where $W^mL_M(\Omega)$ is the (Sobolev) space of functions u such that u and its distributional derivatives up to order m belong to $L_M(\Omega)$, with the norm $\sum_{|\alpha|\leq m} \|D^\alpha u\|_M$. In general the answer is negative, even when m=0, i.e. $C^\infty(\Omega) \cap L_M(\Omega)$ may not be dense in $L_M(\Omega)$. Indeed this would imply that each $u \in L_M(\Omega)$ belongs to $E_M(\Omega')$ for any open set Ω' with $\Omega' \subset\subset \Omega$, which is not true as is seen for instance by considering $\Omega =]-1,+1[$, $\Omega'=]-1/2,+1/2[$, $M(t)=e^{t^2}-1$ and $u(x)=(\log|x|^{-1})^{1/2}$.

Starting from this observation, we can first ask whether $C^\infty(\Omega) \cap W^m E_M(\Omega)$ is norm dense in $W^mE_M(\Omega)$; this is true, as was proved by Donaldson-Trudinger [6]. We can also ask whether $C^\infty(\Omega) \cap W^mL_M(\Omega)$ is dense in $W^mL_M(\Omega)$ with respect to a weaker topology. Two such topologies arise in a natural way : $\sigma(\pi L_M,\pi L_{\bar{M}})$ and $\sigma(\pi L_M,\pi E_{\bar{M}})$; here $u_k \to u$ for $\sigma(\pi L_M,\pi L_{\bar{M}})$ (resp.$\sigma(\pi L_M,\pi E_{\bar{M}})$) if for each $|\alpha|\leq m$ and each $v \in L_{\bar{M}}(\Omega)$ (resp.$E_{\bar{M}}(\Omega)$), $\int_\Omega(D^\alpha u_k-D^\alpha u)\, v \to 0$. We will see in theorem 1 below that the Meyers-Serrin property holds with respect to any of these two topologies. Thus, playing upon words, we could say that in a general Orlicz space, weak derivatives are not necessarily strong derivatives, but they are weakly strong !

3. STRONGLY NONLINEAR ELLIPTIC PROBLEMS.

Related density questions involving the two weak topologies above enter the study of the so-called strongly nonlinear elliptic boundary value problems.

Consider the problem of finding $u(x)$, $x \in \Omega$, such

$$(1)\qquad \begin{cases} -\sum_{i=1}^{N} \frac{\partial}{\partial x_i}\left[\varphi\left(\frac{\partial u}{\partial x_i}\right)\right] + \varphi(u) = f \text{ in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega, \end{cases}$$

where $\varphi: \mathbb{R} \to \mathbb{R}$ is continuous, odd and strictly increasing from $-\infty$ to $+\infty$; note that no condition is imposed on the nature of the growth of φ. Typical examples are $\varphi(t)=t$ (linear case, Laplacian operator), $\varphi(t)=|t|^{p-2}t$ with $1 < p < \infty$ (polynomial growth), $\varphi(t)=t|e|^{t}$ (rapid growth), $\varphi(t)=\text{sgnt} \log(1+|t|)$ (slow growth). Existence and unicity results for (1) (and for much more general equations and boundary conditions) have been obtained since 1971 by developping a theory of mappings of monotone type in nonreflexive Banach spaces, along the lines of the work of Browder [3], Leray-Lions [17], Brézis [2],... (cf.[4,10,11,12]). More recently techniques from optimization theory have been applied to (1) (cf. [8]), and regularity has also been considered (cf. [20]).

Let $W_0^1 L_M(\Omega)$ be the closure of $\mathcal{D}(\Omega)$ in $W^1 L_M(\Omega)$ with respect to $\sigma(\pi L_M, \pi E_{\bar{M}})$. When $\partial\Omega$ is sufficiently smooth, one can show that the functions in $W_0^1 L_M(\Omega)$ are precisely those in $W^1 L_M(\Omega)$ which vanish on $\Omega\partial$ in some suitable generalized sense (cf.[7,12]). The fact that $W_0^1 L_M(\Omega)$ is defined here by means of the $\sigma(\pi L_M, \pi E_{\bar{M}})$ topology (for which bounded sets are relatively compact) is important in the existence part of the following proposition.

PROPOSITION (cf.[10]). Assume that Ω has the segment property (i.e. that there exists an open covering $\{U_i\}$ of $\bar{\Omega}$ and corresponding vectors $\{y_i \in \mathbb{R}^N\}$ such that for $x \in \bar{\Omega} \cap U_i$ and $0 < t < 1$, $x + ty_i \in \Omega$). Let $f \in E_{\bar{M}}(\Omega)$. Then there exists an unique $u \in W_0^1 L_M(\Omega)$, with $\varphi(\frac{\partial u}{\partial x_i}) \in L_{\bar{M}}(\Omega)$, $i=1,\dots,N$, and $\varphi(u) \in L_{\bar{M}}(\Omega)$, satifying the equation in (1) in the distribution sense in Ω.

Let us consider the unicity statement. Let u and v be two solutions.

Then

$$(2)\qquad \int_\Omega \sum_{i=1}^{N} \left[\varphi\left(\frac{\partial u}{\partial x_i}\right) - \varphi\left(\frac{\partial v}{\partial x_i}\right)\right] \frac{\partial w}{\partial x_i} + [\varphi(u) - \varphi(v)]\, w = 0$$

for all $w \in \mathcal{D}(\Omega)$, and so, for all w in the $\sigma(\pi L_M, \pi L_{\bar{M}})$ closure of $\mathcal{D}(\Omega)$ in $W^1 L_M(\Omega)$. This however does not allow us to put $w = u - v$ in (2), unless the closures of $\mathcal{D}(\Omega)$ in $W^1 L_M(\Omega)$ with respect to $\sigma(\pi L_M, \pi L_{\bar{M}})$ and $\sigma(\pi L_M, \pi E_{\bar{M}})$ coincide. This equality, which enters also the existence part of the above proposition in a crucial way, will follow from theorem 2 below.

4. RESULTS.

A sequence $u_k \in L_M(\Omega)$ is said to be modular convergent (cf.[19]) to $u \in L_M(\Omega)$ if for some $\lambda > 0$, $\int_\Omega M((u_k - u)/\lambda) \to 0$. This convergence is (in general strictly) intermediate between the norm convergence and the $\sigma(\pi L_M, \pi L_{\bar{M}})$ convergence. A corresponding convergence can be defined on $W^m L_M(\Omega)$ in an obvious way, by requiring the above for the function and each of its derivatives up to order m.

THEOREM 1 (cf.[13]). Each function in $W^m L_M(\Omega)$ can be approximated with respect to the modular convergence by a sequence in $C^\infty(\Omega) \cap W^m L_M(\Omega)$.

THEOREM 2 (cf.[13]). Assume that Ω has the segment property. Then each function in $W_0^m L_M(\Omega)$, the $\sigma(\pi L_M, \pi E_{\bar{M}})$ closure of $\mathcal{D}(\Omega)$ in $W^m L_M(\Omega)$, can be approximated with respect to the modular convergence by a sequence in $\mathcal{D}(\Omega)$.

We suspect that the smoothness assumption on $\partial\Omega$ in theorem 2 can be removed. If $M(t)=|t|^p$, then theorem 1 reduces to the original Meyers-Serrin result, while theorem 2 has no counterpart; indeed $W_0^{m,p}(\Omega)$ is the closure of $\mathcal{D}(\Omega)$ in $W^{m,p}(\Omega)$ with respect to the norm topology as well as the $\sigma(\pi L_p, \pi L_{p'})$ topology since these two topologies on $W^{m,p}(\Omega)$ lead to the same dual space.

The general idea for constructing the approximate functions in the proofs of theorems 1 and 2 is inspired from standard L^p techniques of approximation. The main difficulties arise in the estimates when the modular functional $u \to \int_\Omega M(u)$ is involved. We will briefly illustrate this point by two examples, one relative to the regularization procedure, the other relative to the use of partitions of unity.

Let $u \in L_M(\mathbb{R}^N)$ and let $u_\varepsilon = u * \rho_\varepsilon$ be its regularization, with ρ_ε an approximate identity. In contrast with the L^p situation, it is not generally true here that u_ε converges in norm to u as $\varepsilon \to 0$. However one can show that if $\lambda > 0$ verifies $\int_{\mathbb{R}^N} M(2u/\lambda) < \infty$, then $\int_{\mathbb{R}^N} M((u_\varepsilon - u)/\lambda) \to 0$. Thus u_ε converges to u with respect to the modular convergence.

Let us consider now a covering $\{U_i\}$ of $\bar{\Omega}$, $\{\psi_i\}$ an associated partition of unity, and suppose that one wishes to approximate a function u by a function v of the form $v = \sum_i \psi_i v_i$. If the approximation is intended with respect to a norm, then

$$\|u-v\| = \|\sum_i \psi_i(u-v_i)\| \leq \sum_i \|\psi_i(u-v_i)\|$$

and we are reduced to construct the v_i's by working inside each piece of the covering at a time. If on the other hand we deal with the modular $\int_\Omega M(u)$, we only get

$$\int_\Omega M(u-v) = \int_\Omega M(\sum_i \psi_i(u-v_i)),$$

where the right hand side cannot be estimated by $\sum_i \int_\Omega M(\psi_i(u-v_i))$ (actually $M(t+s) \geq M(t)+M(s)$ for $t,s \in \mathbb{R}^+$). However if the covering has a finite number R of pieces, then from the convexity of M follows that

$$\int_\Omega M(u-v) \leq \frac{1}{R} \sum_{i=1}^{R} \int_\Omega M(R\psi_i(u-v_i)),$$

and we are again reduced to working inside each U_i at a time. This argument can be adapted to the case of an arbitrary covering by using the theory of the Lebesgue dimension (cf. e.g. [14]).

REFERENCES.

[1] R.ADAMS, Sobolev spaces, Ac.Press, 1975.

[2] H.BREZIS, Equations et inéquations non linéaires dans les espaces vectoriels en dualité, Ann.Inst.Fourier, 18 (1968), 115-175.

[3] F.E.BROWDER, Nonlinear elliptic boundary value problems, Bull.Amer. Math.Soc., 69 (1963), 862-874.

[4] T.DONALDSON, Nonlinear elliptic boundary value problems in Orlicz-Sobolev spaces, J.Diff.Equat., 10 (1971), 507-528.

[5] T.DONALDSON, Inhomogeneous Orlicz-Sobolev spaces and nonlinear parabolic initial value problems, J.Diff.Eq., 16 (1974), 201-256.

[6] T.DONALDSON and N.S.TRUDINGER, Orlicz-Sobolev spaces and imbedding theorems, J.Funct.Anal., 8 (1971), 52-75.

[7] A.FOUGERES, Théorèmes de trace et de prolongement dans les espaces de Sobolev et de Sobolev-Orlicz, C.R.Ac.Sc.Paris, Série A., 274 (1972) 181-184.

[8] A.FOUGERES, Optimisation convexe dans les Banachs non réflexifs: méthode de relaxation-projection, Seminaire Choquet, 1978-79, 1-30.

[9] K.FRIEDRICHS, The identity of weak and strong extensions of differential operators, Trans.Amer.Math.Soc., 55 (1944), 132-151.

[10] J.P.GOSSEZ, Nonlinear elliptic boundary value problems for equations with rapidly (or slowly) increasing coefficients, Trans.Amer.Math. Soc., 190 (1974), 163-205.

[11] J.P.GOSSEZ, Surjectivity results for pseudo-monotone mappings in complementary systems, J.Math.Anal.Appl., 53 (1976), 484-494.

[12] J.P.GOSSEZ, A remark on strongly nonlinear elliptic boundary value problems, Bol.Soc.Brasiliera Mat., 8 (1977), 53-63.

[13] J.P.GOSSEZ, Some approximation properties in Orlicz-Sobolev spaces, Studia Mat., to appear.

[14] W.HUREWICZ and WALLMAN, Dimension Theory, Princeton Univ.press, 1941

[15] M.KRASNOSEL'SKII and J.RUTICKII, Convex functions and Orlicz spaces, Noordhoff, 1961.

[16] A.KUFNER, O.JOHN and S.FUCIK, Function spaces, Academia Praha, 1977.

[17] J.LERAY and J.L.LIONS, Quelques résultats de Višik sur les problèmes elliptiques non linéaires par les méthodes de Minty-Browder, Bull. Soc.Math.France, 93 (1965), 97-107.

[18] N.MEYERS and J.SERRIN, H=W, Proc.Nat.Ac.Sc.U.S.A., 51 (1964) 1055-1056.

[19] J.MUSIELAK and W.ORLICZ, On modular spaces Studia Mat., 18 (1959),49-65.

[20] F. de THELIN, Régularité de la solution d'un problème de Dirichlet fortement non linéaire, Thèse, Toulouse, 1981.

[21] N.S.TRUDINGER, On imbeddings into Orlicz spaces and some applications, J.Math.Mech., 17 (1967), 473-483.

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