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ON POSITIVE SOLUTIONS OF NONLINEAR ELLIPTIC EIGENVALUE PROBLEMS

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The purpose of this lecture is to give a summary of recent results on the existence of positive solutions of nonlinear elliptic eigenvalue problems. Let Ω be a bounded domain in $\mathbb{R}^N$ $(N \geq 1)$ having smooth boundary $\partial\Omega$, and let L:

$$Lu = -\sum_{j,k=1}^{N} a_{jk} \frac{\partial^2 u}{\partial x_j \partial x_k} + \sum_{j=1}^{N} a_j \frac{\partial u}{\partial x_j} + a_0 u$$

be a strongly uniformly elliptic differential expression of second order having real-valued coefficient functions $a_{jk} = a_{kj}$, a_j, $a_0 \geq 0$ belonging to $C^{\theta}(\bar{\Omega})$ $(0 < \theta \leq 1)$. Let further $g : \bar{\Omega} \times \mathbb{R} \to \mathbb{R}$ be a sufficiently smooth function with $g(.,0) = 0$. Our results concern the bifurcation of positive solutions (λ,u) of the nonlinear eigenvalue problem

(NEVP) $\qquad Lu = \lambda g(.,u)$ in Ω, $\quad u = 0$ on $\partial\Omega$

from the line $\mathbb{R} \times \{0\}$ of trivial solutions, and the stability of u considered as steady-state solution of the associated autonomous diffusion equation.

Basic for our investigations are the results on the linear eigenvalue problem

(LEVP) $\qquad Lu = \lambda mu$ in Ω, $\quad u = 0$ on $\partial\Omega$

obtained to a large extent by T. Kato and the author in [5] and stated in Section I. Here $m \in C(\bar{\Omega})$ is a real-valued weight function which may change sign in Ω.

Section II contains results on the nonlinear problem. In Section III we mention related research as well as some open problems.

I. The linear eigenvalue problem.

In the real Banach space $Y := C(\bar{\Omega})$, let $L : Y \supset D(L) \to Y$ denote the realization of L, subject to zero Dirichlet boundary conditions. It is a consequence of the L^p-theory for linear elliptic boundary value problems that $X := D(L) \subset C_0^1(\bar{\Omega}) := \{v \in C^1(\bar{\Omega}) : v = 0$ on $\partial\Omega\}$, and that L is an isomorphism of X onto Y.

Let X and Y be provided with the natural ordering given by the positive cones P_X and P_Y of pointwise nonnegative functions. Note that P_X has nonempty interior $\mathrm{Int}(P_X)$ in X, and that by the strong maximum principle $L^{-1}(P_Y \setminus \{0\}) \subset \mathrm{Int}(P_X)$. The standard notations of ordered Banach spaces are employed in the sequel.

Let $M : Y \to Y$ be the multiplication operator by the continuous function m. We say that λ is eigenvalue of the (LEVP) and u associated eigenfunction, if $u \in X$, $u \neq 0$, and

(1.1) $\qquad Lu = \lambda Mu$.

(1.1) is of course equivalent to asking that

(1.1') $\qquad u = \lambda L^{-1} Mu$.

By the maximum principle, $m \not\leq 0$ is necessary for the (LEVP) to have a positive eigenvalue with a positive eigenfunction. It turns out that this condition is also sufficient.

Theorem 1.2 [5]. *The (LEVP) admits a positive eigenvalue with a positive eigenfunction if and only if*

$m(x) > 0$ for some $x \in \Omega$. If m is positive somewhere in Ω, there exists a unique positive eigenvalue $\lambda_1(m)$ having a positive eigenfunction u_1. Moreover $u_1 \in \mathrm{Int}(P_X)$, and

(i) if $\hat{\lambda} \in \mathbb{C}$ is eigenvalue (of the problem obtained by complexification) with $\mathrm{Re}\,\hat{\lambda} > 0$, then $\mathrm{Re}\,\hat{\lambda} \geq \lambda_1(m)$;

(ii) $\mu_1(m) := 1/\lambda_1(m)$ is eigenvalue of the compact operator $L^{-1}M : Y \to Y$ with algebraic multiplicity 1.

Remark 1.3 There is no eigenvalue $\hat{\lambda} \in \mathbb{C}$ with $\mathrm{Re}\,\hat{\lambda} = 0$ (cf. [5]).

For $m > 0$ on $\overline{\Omega}$, Theorem 1.2 is a well-known consequence of the Krein-Rutman theorem [12] and a result of Protter-Weinberger [15]. Three steps, which we want to single out now, are crucial for the proof of our extension. First we note that we may assume $|m| < 1$ on $\overline{\Omega}$, if necessary by rescaling. For $\lambda \geq 0$, we then have the following equivalence

$$Lu = \lambda Mu \iff u = \lambda(L+\lambda)^{-1}(M+1)u.$$

Here $M+1 : Y \to Y$ is the multiplication operator by the (positive) function $m+1$. Set

$$K_\lambda := (L+\lambda)^{-1}(M+1).$$

Then $K_\lambda : Y \to Y$ is compact and positive, and $\lambda > 0$ is eigenvalue of the (LEVP) with eigenfunction u iff $u = \lambda K_\lambda u$.

Lemma 1.4 *Suppose we know a number $\alpha > 0$ and a function $w \in Y$, $w > 0$, such that*

$$w \leq \alpha K_\alpha w.$$

Then there exist $\lambda : 0 < \lambda \leq \alpha$, and $u \in Y$, $u > 0$, with

$u = \lambda K_\lambda u.$

Lemma 1.5 *If m is positive somewhere in Ω, we can construct a number α and a function w satisfying the hypotheses of Lemma 1.4.*

Lemmata 1.4 and 1.5 guarantee the existence of a desired eigenvalue. Set $\lambda_1(m) := \inf\{\lambda > 0 : \lambda$ is eigenvalue having a positive eigenfunction$\}$. Assertion (i) of Theorem 1.2 is now a consequence of Lemma 1.4 and

Lemma 1.6 *Let $\hat{\lambda} \in \mathbb{C}$ be eigenvalue of the (LEVP) with $\mathrm{Re}\,\hat{\lambda} \geq 0$, and u associated eigenfunction. Then*

$$(1.7) \qquad |u| \leq (\mathrm{Re}\,\hat{\lambda})\, K_{\mathrm{Re}\,\hat{\lambda}} |u|.$$

(1.7) is an extension of what is sometimes called the "Kato inequality", introduced in [11] for the study of the essential selfadjointness of Schrödinger operators.

The proofs of uniqueness of a positive eigenvalue having a positive eigenfunction, and of the assertion about the algebraic mulitplicity, are more subtle and use analytic perturbation theory.

Theorem 1.2(i) can be sharpened.

Proposition 1.8 *$\lambda_1(m)$ is the only eigenvalue $\hat{\lambda} \in \mathbb{C}$ of the (LEVP) with $\mathrm{Re}\,\hat{\lambda} = \lambda_1(m)$.*

This result has been obtained by Gossez-Lami Dozo [4] under additional regularity assumptions on L and m. The proof of the Proposition in the present generality is given in [9], and is based on the following observation regarding inequality (1.7).

Lemma 1.9 *Suppose u is eigenfunction of the (LEVP) to the eigenvalue $\hat{\lambda} \in \mathbb{C}$ with $\mathrm{Re}\,\hat{\lambda} > 0$, and suppose*

$$|u| = (\mathrm{Re}\,\hat{\lambda})K_{\mathrm{Re}\,\hat{\lambda}}|u|.$$

Then $\hat{\lambda} = \lambda_1(m)$ *and* $u \in \mathrm{span}[u_1]$.

We now turn to the inhomogeneous problem

$$(1.10) \qquad (L-\lambda M)u = h,$$

$h \in Y$ given. (1.10) is of course equivalent to the equation

$$(1.10') \qquad (I-\lambda L^{-1}M)u = L^{-1}h$$

(in either the space Y or X). By the Riesz-Schauder theory for compact linear operators, (1.10') is uniquely solvable for arbitrary $h \in Y$ iff λ is not a characteristic value of (1.1').

Proposition 1.11 [5]. *Suppose* $m(x) > 0$ *for some* $x \in \Omega$.

(i) *Let* $0 \le \lambda < \lambda_1(m)$, $h \ge 0$, *and let* u *be the solution of* (1.10). *Then* $u \ge 0$.

(ii) *Let* $\lambda \ge \lambda_1(m)$, $h > 0$, *and let* u *be solution of* (1.10). *Then* $u \not> 0$.

The last statement can be sharpened.

Proposition 1.12 (*Anti-maximum principle,* [6]). *Let* $h > 0$ *be given. Then there exists a number* $\delta = \delta(h) > 0$ *such that if* $\lambda_1(m) < \lambda < \lambda_1(m) + \delta$ *and* u *is the solution of* (1.10), *then* $u < 0$.

If m admits both positive and negative values in Ω, we can apply all the above results also to the problem

$$Lu = (-\lambda)(-m)u \quad \text{in} \quad \Omega, \qquad u = 0 \quad \text{on} \quad \partial\Omega,$$

and obtain in addition an eigenvalue $\lambda_{-1}(m) < 0$ having a positive eigenfunction.

II. The nonlinear eigenvalue problem.

For technical reasons it is advantageous to work now in the space $X := D(L)$. Let $g : (x,s) \in \overline{\Omega} \times \mathbb{R} \to g(x,s) \in \mathbb{R}$ be a continuous function with $g(.,0) = 0$, having continuous partial derivatives g_s and g_{ss}, and let G denote the Nemytskii operator associated with g. The pair $(\lambda,u) \in \mathbb{R} \times X$ is called a positive solution of the (NEVP) if $\lambda > 0$, $u > 0$, and

$$(2.1) \qquad Lu = \lambda G(u).$$

Of course (2.1) is equivalent to the equation

$$(2.1') \qquad u = \lambda L^{-1}G(u)$$

in the space X. Note that if (λ,u) is a positive solution, then $u \in \mathrm{Int}(P_X)$.

Let the function $m_0 \in Y$ be defined by

$$m_0(x) := g_s(x,0),$$

and let M_0 be the multiplication operator by m_0. Then $L^{-1}M_0 = (L^{-1}G)'(0)$, the Fréchet derivative (in X) of the mapping $L^{-1}G : X \to X$ at $u = 0$. It is well-known that if $(\lambda,0)$ is bifurcation point for positive solutions, then λ is characteristic value of the linear operator $L^{-1}M_0$ having a positive eigenfunction.

Let Σ denote the closure (in $\mathbb{R} \times X$) of the set of positive solutions of (2.1). The following result is an immediate consequence of Theorem 1.2 and Rabinowitz' global bifurcation theorem [16].

Theorem 2.2 [5]. *There is bifurcation for positive solutions of the* (NEVP) *from the line of trivial solutions if and only if* $m_0(x) > 0$ *for some* $x \in \Omega$. *If* m_0

is positive somewhere in Ω, Σ contains an unbounded connected component Σ_0 in $\mathbb{R} \times X$ with $(\lambda_1(m_0),0) \in \Sigma_0$. Moreover $(\lambda_1(m_0),0)$ is the only bifurcation point for positive solutions from the line of trivial solutions.

In the following we assume that $m_0(x) > 0$ at some point $x \in \Omega$ and set $\lambda_1 := \lambda_1(m_0)$. Employing results of [3], a more detailed description of Σ_0 in the neighborhood of $(\lambda_1,0)$ can be given.

__*Proposition 2.3*__ [8]. *In a sufficiently small neighborhood U of $(\lambda_1,0)$ in $\mathbb{R} \times X$, the set of solutions of (2.1) consists precisely of the line $(\mathbb{R} \times \{0\}) \cap U$ and a C^1-curve $\{(\lambda(s),u(s)) : s \in (-\alpha,\alpha)\}$, where $\lambda(0) = \lambda_1$. Hence $\Sigma_0 \cap U = \{(\lambda(s),u(s)) : 0 \le s < \alpha\}$.*

We now turn to the question of stability of positive solutions of the (NEVP), considered as steady-state solutions of the associated autonomous diffusion equation $(v = v(t,x))$

$$(2.4)\qquad \begin{cases} \dfrac{\partial v}{\partial t} + Lv - \lambda g(.,v) = 0, & (t,x) \in \mathbb{R}^+ \times \Omega \\ v(t,x) = 0, & (t,x) \in \mathbb{R}^+ \times \partial\Omega \\ v(0,x) = v_0(x) & \text{given } (x \in \Omega). \end{cases}$$

According to the *principle of linearized stability* (e.g. [17]), if u is a steady-state solution of (2.4) and $\mu \in \mathbb{R}$ denotes the smallest eigenvalue of the linearized (elliptic) problem

$$(L - \lambda G'(u))w = \mu w,$$

then u is Lyapunow asymptotically stable provided $\mu > 0$, and unstable provided $\mu < 0$. In this context, stability of u means that if v_0 is sufficiently near to u (in Y),

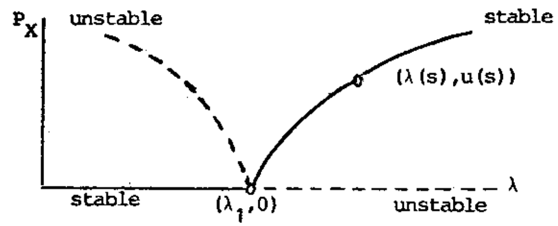
then for the solution v of (2.4) we have $\|v(t,.)-u\|_Y \to 0$ exponentially as $t \to +\infty$.

A first stability result is

__*Proposition 2.5*__ [8]. (i) *Let $(\lambda,0)$ be a trivial solution of the (NEVP). Then 0 is stable for $0 \le \lambda < \lambda_1$ and unstable for $\lambda > \lambda_1$.*

(ii) *Let $(\lambda(s),u(s)) \in \Sigma_0$ and $s > 0$ sufficiently small. Then $u(s)$ is stabe if $\lambda'(s) > 0$, and unstable if $\lambda'(s) < 0$.*

We thus have the following picture of "exchange of stability" for the nonnegative solutions of the (NEVP) in a neighborhood of $(\lambda_1,0)$ in $\mathbb{R} \times X$:



We add two global stability results for positive solutions, for special classes of nonlinearities g.

__*Proposition 2.6*__ [8]. *Let $g(x,s)$ be __convex__ in $s \ge 0$ for all $x \in \bar{\Omega}$, strictly convex for at least one $x \in \Omega$, and let $(\lambda,u) \in \mathbb{R} \times X$ be a positive solution of the (NEVP). Then $\lambda < \lambda_1$, and u is unstable.*

__*Proposition 2.7*__ [8]. *Let $g(x,s)$ be __concave__ in $s \ge 0$ for all $x \in \bar{\Omega}$, strictly concave for at least one $x \in \Omega$, and let $(\lambda,u) \in \mathbb{R} \times X$ be a positive solution of the (NEVP). Then $\lambda > \lambda_1$. For each $\lambda > \lambda_1$, there is at most* *one positive solution* (λ,u), *and* u *is stable. There exists a number* $\bar\lambda \in (\lambda_1, +\infty]$ *and a continuous map* $\tilde u(.) : [\lambda_1,\bar\lambda) \to P_X$ *with* $\tilde u(\lambda_1) = 0$, *such that* $\Sigma_0 = \{(\lambda_1,\tilde u(\lambda)) : \lambda_1 \le \lambda < \bar\lambda\}$. *(This means that* Σ_0 *can be parametrized by* λ.*) Moreover* $\tilde u(.)$ *is continuously differentiable on* $(\lambda_1,\bar\lambda)$, *and if* $\bar\lambda < +\infty$, *then* $\lim_{\lambda\nearrow\bar\lambda} \|\tilde u(\lambda)\|_X = +\infty$.

We note that these results are well-known for *positive* functions g ([1, Chapter V]).

We conclude this Section with some results on *bifurcation from infinity* for positive solutions of the (NEVP). Suppose g is asymptotically linear for $s \to +\infty$, i.e. that there exists

$$m_\infty(x) := \lim_{s\to+\infty} s^{-1} g(x,s),$$

uniformly in $x \in \bar\Omega$. Note that $m_\infty \in Y$.

Theorem 2.8 [7]. *There is bifurcation from infinity for positive solutions of the* (NEVP) *if and only if* $m_\infty(x) > 0$ *for some* $x \in \Omega$. *If* m_∞ *is positive somewhere in* Ω, Σ *contains a connected component* Σ_∞ *in* $\mathbb{R} \times X$ *that meets* $(\lambda_1(m_\infty),\infty)$. *Moreover* $(\lambda_1(m_\infty),\infty)$ *is the only bifurcation point from infinity for positive solutions.*

Combining Theorems 2.2 and 2.8, we obtain existence and multiplicity results for positive solutions of the (NEVP) provided g is asymptotically linear.

Proposition 2.9 *Let* g *be asymptotically linear, and suppose there exists* $m \in Y$ *with* $m(x) > 0$ *at some* $x \in \Omega$, *such that* $g(x,s) \ge m(x)s$ *for all* $x \in \bar\Omega$, $s \ge 0$. *Then* $0 < \lambda \le \lambda_1(m)$ *for all* $(\lambda,u) \in \Sigma$. *Hence* $\Sigma_0 = \Sigma_\infty$, *and for each* λ *between* $\lambda_1(m_0)$ *and* $\lambda_1(m_\infty)$ *there is at least one positive solution* (λ,u) *of the* (NEVP).

Proposition 2.10 *Let again* g *be asymptotically linear, and suppose there is both bifurcation from the trivial solutions and from infinity. Suppose further that there exists a function* $w \in C^1(\bar\Omega) \cap C^2(\Omega)$, $w > 0$, *such that* $Lw \ge 0$ *and* $G(w) \le 0$. *Then* $u < w$ *for all* $(\lambda,u) \in \Sigma_0$. *Hence* $\Sigma_0 \ne \Sigma_\infty$, *and for each* $\lambda > \mathrm{Max}\{\lambda_1(m_0),\lambda_1(m_\infty)\}$, *the* (NEVP) *admits at least two positive solutions.*

III. Additional remarks and open problems.

(i) Nothing seems to be known in general about the existence of a principal eigenvalue of the (LEVP) if only $m \in L^\infty(\Omega)$, with $m > 0$ on a set of positive measure. For formally selfadjoint L, this condition is sufficient for the existence of (infinitely many) positive eigenvalues; cf. the pioneering work of Manes-Micheletti [14], and [2].

(ii) Senn and the author [19] investigate the interesting Neuman problem

$$(3.1) \qquad Lu = \lambda m u \quad \text{in} \quad \Omega, \quad \frac{\partial u}{\partial n} = 0 \quad \text{on} \quad \partial\Omega,$$

where

$$Lu = -\sum_{j,k=1}^{N} a_{jk} \frac{\partial^2 u}{\partial x_j \partial x_k} + \sum_{j=1}^{N} a_j \frac{\partial u}{\partial x_j},$$

assuming that the continuous function m changes sign in Ω. Here the operator L associated with L and the Neumann boundary conditions is not invertible, and 0 is an eigenvalue of (3.1) (eigenfunction = constant). Let v^* be the (positive) eigenfunction of the adjoint operator L^* to the eigenvalue 0; one shows readily that $v^* \in L^p(\Omega)$. Then there exists a positive (negative) eigenvalue having a positive eigenfunction provided $\int_\Omega m v^* < 0$ ($\int_\Omega m v^* > 0$).

The limit case $\int_\Omega mv^* = 0$ is particularly subtle. For $L = -\Delta$, problem (3.1) has been studied by variational methods in [2]; cf. also [18].

(iii) The eigenvalue problems for the weakly coupled linear system

$$(3.2)\qquad \begin{cases} L_k u_k = \lambda \sum_{l=1}^{r} m_{kl}\, u_l & \text{in } \Omega \\ u_k = 0 & \text{on } \partial\Omega \end{cases}$$

and its nonlinear generalization

$$(3.3)\qquad \begin{cases} L_k u_k = \lambda g_k(x,u_1,\ldots,u_r) & \text{in } \Omega \\ u_k = 0 & \text{on } \partial\Omega \end{cases}$$

$(k=1,\ldots,r)$ are discussed in [10]. Under the assumption that $m_{kl} \geq 0$ for all $k,l=1,\ldots,r$, $k \neq l$ (such a condition is necessary, in a certain sense), it is proved that (3.2) admits a positive eigenvalue λ_1 having a positive eigenfunction $\underline{u} = (u_1,\ldots,u_r)$ provided at least one of the functions $m_{kk} \in C(\bar{\Omega})$ $(k=1,\ldots,r)$ is positive somewhere in Ω. Results of Turner [20] are generalized.

(iv) Lazer [13] has recently introduced the concept of "principal eigenvalue" for the operator L obtained from the parabolic differential expression L:

$$Lu = \frac{\partial u}{\partial t} - \sum_{j,k=1}^{N} a_{jk}(t,x) \frac{\partial^2 u}{\partial x_j \partial x_k} + \sum_{j=1}^{N} a_j(t,x) \frac{\partial u}{\partial x_j} + a_0(t,x)u,$$

subject to periodic-Dirichlet boundary conditions. Here the coefficient functions of L are assumed to be periodic in t, with the some period T as imposed on the solutions u.

It is natural to ask whether Lazer's result can be extended to the more general eigenvalue problem

$$(3.4)\qquad \begin{cases} Lu = \lambda m(t,x)u & (t,x) \in \mathbb{R} \times \Omega, \\ u(t+T,x) = u(t,x) & \\ u(t,x) = 0 & (t,x) \in \mathbb{R} \times \partial\Omega, \end{cases}$$

where also the continuous function m is T-periodic in t. By the maximum principle for parabolic equations, $m \not\leq 0$ in $(0,T) \times \Omega$ is a necessary condition for the existence of an eigenvalue $\lambda > 0$ having a positive eigenfunction. Using similar arguments as in the proof of Theorem 1.2, we are able to prove its existence only provided at some $x \in \Omega$, $m(t,x) > 0$ for *all* $t \in \mathbb{R}$ (the difficulty lying in the construction of a number $\alpha > 0$ and a T-periodic function $w > 0$ as in Lemma 1.5).

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