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34100 TRIESTE (ITALY) - P.O.B. 586 - MIRAMARE - STRADA COSTIERA 11 - TELEPHONES: 224281/2/3/4/5/6
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VARIATIONAL METHODS IN ANALYSIS AND MATHEMATICAL PHYSICS

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ON BIFURCATION AND STABILITY OF POSITIVE SOLUTIONS OF NONLINEAR ELLIPTIC EIGENVALUE PROBLEMS

P. HESS
Mathematisches Institut
Universität Zurich
Freiestrasse 36
8032 Zürich
Switzerland

On bifurcation and stability of positive solutions of nonlinear elliptic eigenvalue problems.

Peter Hess

On the bounded domain $\Omega \subset \mathbb{R}^N (N\geq 1)$ having smooth boundary $\partial\Omega$, let ℓ:

$$\ell u = - \sum_{j,k=1}^{N} a_{jk} \frac{\partial^2 u}{\partial x_j \partial x_k} + \sum_{j=1}^{N} a_j \frac{\partial u}{\partial x_j} + a_0 u$$

be a strongly uniformly elliptic differential expression of second order having real-valued coefficient functions $a_{jk} = a_{kj}$, a_j, $a_0 \geq 0$ belonging to $C^\theta(\bar{\Omega})$ $(0<\theta<1)$. Let further $g : \bar{\Omega} \times \mathbb{R} \to \mathbb{R}$ be a sufficiently smooth function with $g(.,0) = 0$. We investigate the bifurcation of positive solutions (λ,u) of the nonlinear elliptic eigenvalue problem

(NEVP) $\qquad \ell u = \lambda g(.,u)$ in Ω, $u=0$ on $\partial\Omega$

from the line $\mathbb{R} \times \{0\}$ of trivial solutions, and the stability of u as steady-state solution of the autonomous diffusion equation $(u = u(t,x))$

$$(0.1) \qquad \begin{cases} \frac{\partial u}{\partial t} + \ell u - \lambda g(.,u) = 0 & \text{in } \mathbb{R}^+ \times \Omega, \\ u(t,.) = 0 & \text{on } \mathbb{R}^+ \times \partial\Omega, \\ u(0,.) = u_0 & \text{given.} \end{cases}$$

The paper is organized as follows. In Section I we consider the linear elliptic eigenvalue problem

(LEVP) $\qquad \ell u = \lambda m u$ in Ω, $u=0$ on $\partial\Omega$,

where $m \in C(\bar{\Omega})$ is a given weight function which may change sign in Ω. We recall the main results of [5] and prove some supplementary facts which are needed in Sections II and III. While a necessary and sufficient condition for bifurcation of positive solutions from the line of trivial solutions is proved in [5], we give a detailed study of the set of (positive) solutions in the neighbourhood of the bifurcation point, as well as a discussion of their stability, in Section II. In Section III a global discussion follows in the special cases where $g(x,s)$ is either convex or concave in $s\geq 0$. We generalize results which are known for positive solutions g (e.g. [1, Sections 25 and 26]).

I wish to thank my collegue H. Amann for constructive discussions which led to the remarkably short and simple proofs in Section III.

I. The linear eigenvalue problem.

We look at the linear eigenvalue problem

(LEVP) $\qquad \ell u = \lambda m u$ in Ω, $u=0$ on $\partial\Omega$,

where $m \in C(\bar{\Omega})$ is a given real-valued function.

Let L_0 denote the differential operator induced by ℓ and the Dirichlet boundary conditions, with domain $D(L_0)=\{v \in C^{2+\theta}(\bar{\Omega}) : v=0 \text{ on } \partial\Omega\}$. Then L_0 is closable in the (real) Banach space $Y := C(\bar{\Omega})$ (L_0 admits a closed extension in $L^p(\Omega)$, $1<p<\infty$, having domain $W_0^{1,p}(\Omega) \cap W^{2,p}(\Omega)$). Set $L :=$ closure of L_0 in Y. Denoting by X the (real) vector space D(L), equipped with the graph norm $\|v\|_X := \|Lv\|_Y + \|v\|_Y$, it is a consequence of the L^p- theory for linear elliptic boundary value problems that $X \subset C_0^1(\bar{\Omega}) := \{v \in C^1(\bar{\Omega}) : v=0 \text{ on } \partial\Omega\}$, and that L is an isomorphism of X onto Y.

Let the spaces X,Y be provided with the natural ordering given by the positive cones P_X, P_Y of pointwise nonnegative functions. The standard notations of ordered Banach spaces are employed: $v \geq 0$ if $v \in P$, $v > 0$ if $v \in P \setminus \{0\}$ ($P = P_X$ or P_Y). Note that P_X has nonempty interior $\mathrm{Int}(P_X)$, and that (by the strong maximum principle) L^{-1} is strongly positive : $L^{-1}(P_Y \setminus \{0\}) \subset \mathrm{Int}(P_X)$. We write $v \gg 0$ if $v \in \mathrm{Int}(P_X)$.

Let $M : X \to Y$ be the (compact) multiplication operator by the function m. We say that λ is eigenvalue of the (LEVP) and u associated eigenfunction if $u \in X$, $u \neq 0$, and

(1.1) $$Lu = \lambda Mu.$$

Note that (1.1) is equivalent to the equation

(1.1') $$u = \lambda L^{-1} Mu$$

in X.

<u>Theorem 1.2</u> ([5]) The (LEVP) admits a positive eigenvalue with a positive eigenfunction if and only if $m(x) > 0$ for some $x \in \Omega$. If m is positive somewhere in Ω, there exists a unique positive eigenvalue $\lambda_1(m)$ having a positive eigenfunction u_1. Moreover $u_1 \in \mathrm{Int}(P_X)$, and

(i) if $\hat{\lambda} \in \mathbb{C}$ is eigenvalue (of the problem obtained by complexification) with $\mathrm{Re}\,\hat{\lambda} > 0$, then $\mathrm{Re}\,\hat{\lambda} \geq \lambda_1(m)$;

(ii) $\mu_1(m) := {}^1/\lambda_1(m)$ is eigenvalue of the compact operator $L^{-1}M : X \to X$ with algebraic multiplicity 1.

<u>Remark 1.3</u>. There is no eigenvalue $\hat{\lambda} \in \mathbb{C}$ with $\mathrm{Re}\,\hat{\lambda} = 0$ (cf. [5]).

<u>Remark 1.4</u>. Gossez-Lami Dozo [4] proved that $\lambda_1(m)$ is the only eigenvalue $\hat{\lambda} \in \mathbb{C}$ with $\mathrm{Re}\,\hat{\lambda} = \lambda_1(m)$.

By rescaling, if necessary, we may assume $|m| < 1$ on $\bar{\Omega}$.

Introducing

$$J := \text{imbedding mapping } X \hookrightarrow Y,$$

for $\lambda \geq 0$ we have the following equivalence

$$Lu = \lambda Mu \Leftrightarrow u = \lambda (L + \lambda J)^{-1}(M+J)u.$$

Note that $M + J : X \to Y$ is the multiplication operator by the positive function $(m+1)$. Set

$$K_\lambda := (L+\lambda J)^{-1}(M+J).$$

Then $K_\lambda : X \to X$ is compact and strongly positive, and $\lambda \geq 0$ is eigenvalue of the (LEVP) with eigenfunction u iff $u = \lambda K_\lambda u$.

Let m^+ denote the positive part of the function m.

<u>Lemma 1.5</u>. Let $\tau \in \mathbb{R}$, $\tau < \bar{\tau} := \|m^+\|_Y$. Then $\tau \mapsto \lambda_1(m-\tau)$ is a strictly increasing, analytic function of τ, and

$$\lim_{\tau \nearrow \bar{\tau}} \lambda_1(m-\tau) = +\infty \ , \ \lim_{\tau \searrow -\infty} \lambda_1(m-\tau) = 0.$$

The assertions of Lemma 1.5 have been proved in [5] in the course of the proof of Theorem 1.2, except for the last (easy) limit relation and the analyticity of $\lambda_1(m-\tau)$. In order to prove the analyticity, let us recall the following

<u>Definition 1.6</u> ([3]). Let $T, K \in \mathcal{B}(X,Y)$. Then $\nu \in \mathbb{R}$ is a K-simple eigenvalue of T if $\dim N(T-\nu K) = \mathrm{codim}\, R(T-\nu K) = 1$ and, if $N(T-\nu K) = \mathrm{span}\,\{x\}$, $Kx \notin R(T-\nu K)$.

<u>Lemma 1.7</u> ([3]). Let $T_0, K \in \mathcal{B}(X,Y)$, and let $\nu_0 \in \mathbb{R}$ be a K-simple eigenvalue of T_0. Then there exists $\delta > 0$ such that if $T \in \mathcal{B}(X,Y)$ with $\|T-T_0\| < \delta$, there is a unique $\nu(T) \in \mathbb{R}$ with $|\nu(T)-\nu_0| < \delta$ for which $T-\nu(T)K$ is singular. The map $T \mapsto \nu(T)$ is analytic, and $\nu(T)$ is K-simple eigenvalue of T. If $N(T_0-\nu_0 K) = \mathrm{span}\,\{x_0\}$ and Z is a complement of

span $\{x_0\}$ in X, there is a unique null vector $x(T)$ of $T-\nu(T)K$ satisfying $x(T)-x_0 \in Z$. The map $T \mapsto x(T)$ is also analytic.

Denoting by I the identity mapping in X, Theorem 1.2(ii) implies that for $\tau < \bar{\tau}$, $\mu_1(m-\tau) := {}^1/\lambda_1(m-\tau)$ is an I-simple eigenvalue of $L^{-1}(M-\tau J) : X \to X$. The analyticity of $\mu_1(m-\tau)$ in τ is thus an immediate consequence of Lemma 1.7.

We turn to the inhomogeneous problem

(1.8) $\qquad (L - \lambda M)u = h,$

$h \in Y$ given. (1.8) is of course equivalent to the equation

(1.8') $\qquad (I - \lambda L^{-1}M)u = L^{-1}h$

in X. By the Riesz-Schauder theory, (1.8') is solvable for arbitrary $h \in Y$ iff λ is not an eigenvalue of (1.1).

Proposition 1.9 ([5]). Suppose $m(x)>0$ for some $x \in \Omega$.

(i) Let $0 \le \lambda < \lambda_1(m)$, $h \ge 0$, and let u be the solution of (1.8). Then $u \ge 0$.

(ii) Let $\lambda \ge \lambda_1(m)$, $h>0$, and let u be the solution of (1.8). Then $u \not\ge 0$.

Proposition 1.9(ii) can be sharpened as follows.

Proposition 1.10 (Anti-maximum-principle, [6]). Let $h>0$ be given. Then there exists a number $\delta = \delta(h)>0$ such that if $\lambda_1(m)<\lambda<\lambda_1(m)+\delta$ and u is solution of (1.8), then $u<0$.

Set $\lambda_1 := \lambda_1(m)$, and let $u_1 \in \mathrm{Int}(P_X)$ be associated eigenfunction. As a consequence of Theorem 1.2(ii), the space X admits the direct topological decomposition

(1.11) $\qquad X = \mathrm{span}\,\{u_1\} \oplus Z,$

$Z = R(I-\lambda_1 L^{-1}M)$, which plays an important rôle e.g. in the proof of Proposition 1.10.

Let $h \in Y$, and decompose $L^{-1}h$ as

$$L^{-1}h = \alpha u_1 + b$$

($\alpha \in \mathbb{R}$, $b \in Z$) according to (1.11). Then $b = (I-\lambda_1 L^{-1}M)w$ for some $w \in X$, and we have

$$h = \alpha L u_1 + Lw - \lambda_1 M w$$
$$= \alpha L u_1 + (L+\lambda_1 J)w - \lambda_1 (M+J)w.$$

Hence

(1.12) $\qquad (L+\lambda_1 J)^{-1}h = \alpha(L+\lambda_1 J)^{-1}Lu_1 + (I-\lambda_1 K_{\lambda_1})w.$

Since K_{λ_1} is a strongly positive, compact operator in X and $\mu_1 u_1 = K_{\lambda_1} u_1$ ($\mu_1 = {}^1/\lambda_1 > 0$), the Krein-Rutman theorem implies that μ_1 is also eigenvalue of the (Banach space) adjoint operator $K^*_{\lambda_1} : X^* \to X^*$ with positive eigenfunction u_1^*. Let $\langle .,. \rangle$ denote the duality pairing between X^* and X. From (1.12),

$$\langle u_1^*, (L+\lambda_1 J)^{-1}h \rangle = \alpha \langle u_1^*, (L+\lambda_1 J)^{-1}Lu_1 \rangle$$

follows.

Lemma 1.13. $\chi := \langle u_1^*, (L+\lambda_1 J)^{-1}Lu_1 \rangle > 0$.

We conclude that

$$\alpha = \chi^{-1} \langle u_1^*, (L+\lambda_1 J)^{-1}h \rangle .$$

In particular, $\alpha>0$ if $h>0$, since then $(L+\lambda_1 J)^{-1}h \in \mathrm{Int}\,(P_X)$.

We remark that Lemma 1.13 is not immediate, since $\chi = \lambda_1 \langle u_1^*, (L+\lambda_1 J)^{-1}Mu_1 \rangle$, and Mu_1 may change sign in Ω. An indirect proof of Lemma 1.13 is given in [6], a direct one in [7]. We include a new proof here, based on Lemma 1.7.

Proof of Lemma 1.13. First we note that for $\lambda \ge 0$ there exists a unique eigenvalue $\gamma = \gamma(\lambda) \in \mathbb{R}$ of the problem

(1.14) $\qquad (L-\lambda M)v = \gamma J v$

having a positive eigenfunction $v=v(\lambda) \in X$ ((1.14) is equivalent to the "standard" eigenvalue problem

$$\tilde{L}v := (L+\lambda(J-M))v = (\gamma+\lambda)Jv =: \tilde{\gamma}Jv,$$

where the 0^{th}-order term of $\tilde{L}$ has a nonnegative coefficient function).

Since we may write (1.14) in the form

$$Lv(\lambda) = \lambda(M + \frac{\gamma(\lambda)}{\lambda} J)v(\lambda)$$

with $v(\lambda)>0$, we conclude that the function $m + \frac{\gamma(\lambda)}{\lambda}$ is positive somewhere in Ω, and that

$$\lambda = \lambda_1(m + \frac{\gamma(\lambda)}{\lambda}) .$$

Hence by Lemma 1.5,

(1.15) $\gamma(\lambda_1)=0$, $\gamma(\lambda)>0$ for $\lambda<\lambda_1$, $\gamma(\lambda)<0$ for $\lambda>\lambda_1$.

For a finer study of the function γ in a neighbourhood of $\lambda=\lambda_1$, observe that

(1.16) 0 is J-simple eigenvalue of $L-\lambda_1 M : X \to Y$.

In fact, $\dim N(L-\lambda_1 M) = \operatorname{codim} R(L-\lambda_1 M) = 1$ and $N(L-\lambda_1 M) = \operatorname{span}\{u_1\}$. Suppose $Ju_1 = (L-\lambda_1 M)w = ((L+\lambda_1 J) - \lambda_1(M+J))w$ for some $w \in X$. Then $(L+\lambda_1 J)^{-1}Ju_1 = (I-\lambda_1 K_{\lambda_1})w$ and hence $\langle u_1^*, (L+\lambda_1 J)^{-1}Ju_1\rangle = 0$, which is impossible since $(L+\lambda_1 J)^{-1}Ju_1 \in \operatorname{Int}(P_X)$.

By Lemma 1.7, $\gamma(\lambda)$ is thus an analytic function of λ (in a neighbourhood of λ_1), and we can choose $v(.)$ with $v(\lambda_1) = u_1$ to depend analytically on λ. (1.15) implies that $\gamma'(\lambda_1) \le 0$. Differentiating (1.14) with respect to λ, at $\lambda=\lambda_1$ we obtain

(1.17) $$-Mu_1 + (L-\lambda_1 M)v'(\lambda_1) = \gamma'(\lambda_1)Ju_1 .$$

Since $Mu_1 \notin R(L-\lambda_1 M)$ by Theorem 1.2(ii), it follows that $\gamma'(\lambda_1) \neq 0$ and hence

(1.18) $$\gamma'(\lambda_1) < 0.$$

From (1.17) we infer that

$$-(L+\lambda_1 J)^{-1}Mu_1 + (I-\lambda_1 K_{\lambda_1})v'(\lambda_1) = \gamma'(\lambda_1)(L+\lambda_1 J)^{-1}Ju_1$$

and consequently

$$-\langle u_1^*, (L+\lambda_1 J)^{-1}Mu_1\rangle = \gamma'(\lambda_1)\langle u_1^*, (L+\lambda_1 J)^{-1}Ju_1\rangle .$$

Since $\gamma'(\lambda_1)<0$ and $\langle u_1^*, (L+\lambda_1 J)^{-1}Ju_1\rangle > 0$, the assertion of Lemma 1.13 follows. □

II. The nonlinear eigenvalue problem.

Let $g : (x,s) \in \overline{\Omega} \times \mathbb{R} \mapsto g(x,s) \in \mathbb{R}$ be a continuous function with $g(.,0) = 0$, having continuous partial derivatives g_s and g_{ss}, and let G denote the Nemytskii operator associated with g. Since the imbedding $X \hookrightarrow Y$ is compact, $G : X \to Y$ is a compact mapping. The pair $(\lambda,u) \in \mathbb{R} \times X$ is a positive solution of

(NEVP) $$Lu = \lambda g(.,u) \text{ in } \Omega, \quad u=0 \text{ on } \partial\Omega,$$

if $\lambda>0$, $u>0$, and

(2.1) $$Lu = \lambda G(u).$$

Of course (2.1) is equivalent to the equation

(2.1') $$u = \lambda L^{-1}G(u)$$

in the space X. Note that if (λ,u) is positive solution, then $u \in \operatorname{Int}(P_X)$.

Let the function $m \in Y$ be defined by

(2.2) $$m(x) := g_s(x,0),$$

and let $M : X \to Y$ denote the multiplication operator by m. Then $M = G'(0)$, the Fréchet derivative of G at $u=0$. It is well-known that if $(\lambda,0)$ is bifurcation point for positive solutions, then λ is characteristic value of the linear operator $L^{-1}M = (L^{-1}G)'(0) : X \to X$ having a positive eigenfunction.

Let Σ denote the closure (in $\mathbb{R} \times X$) of the set of positive solutions of (2.1). The following result is an easy consequence of Theorem 1.2 and Rabinowitz' global bifurcation theorem.

<u>Theorem 2.3</u> ([5]). There is bifurcation for positive solutions of the (NEVP) from the line of trivial solutions if and only if $m(x)>0$ for some $x \in \Omega$. If m is positive somewhere in Ω, Σ contains an unbounded connected component Σ_0 in $\mathbb{R} \times X$ with $(\lambda_1(m),0) \in \Sigma_0$. Moreover $(\lambda_1(m),0)$ is the only bifurcation point for positive solutions from the line of trivial solutions.

For the rest of the paper we assume that $m(x)>0$ at some point $x \in \Omega$, and write $\lambda_1 := \lambda_1(m)$. A more detailed description of Σ_0 in the neighbourhood of $(\lambda_1,0)$ can be given. Consider the C^2-mapping $F : \mathbb{R} \times X \to Y$ defined by

$$F(\lambda,u) := Lu - \lambda G(u).$$

Theorem 1.2(ii) implies (with the partial derivatives $F_u(\lambda,0)=L-\lambda M$, $F_{\lambda,u}(\lambda,0) = -M$) that 0 is $F_{\lambda,u}(\lambda_1,0)$ - simple eigenvalue of the operator $F_u(\lambda_1,0)$. The following result is a consequence of [2, Theorem 1.7].

<u>Proposition 2.4</u>. In a neighbourhood $\mathcal{U}$ of $(\lambda_1,0)$ in $\mathbb{R} \times X$, the set of solutions of (2.1) consists precisely of the line $(\mathbb{R}\times\{0\})\cap\mathcal{U}$ and a C^1-curve $\{(\lambda(s),u(s)),\ s \in (-\alpha,\alpha)\}$, where $\lambda(0) = \lambda_1$. With respect to the decomposition (1.11), $u(s)$ admits the representation $u(s)=s(u_1+\psi(s))$, where the map $s\mapsto\psi(s) \in Z$ is continuously differentiable and $\psi(0)=0$. Hence $\Sigma_0 \cap \mathcal{U} = \{(\lambda(s),u(s)),\ 0\le s<\alpha\}$.

It is well-known that the <u>principle of linearized stability</u> holds for the steady-state solutions of problem (0.1): if u is a steady-state solution and $\mu \in \mathbb{R}$ denotes the smallest eigenvalue of the linearized (elliptic) problem

(2.5) $$(L - \lambda G'(u))v = \mu J v,$$

then u is (Lyapunow) asymptotically stable if $\mu>0$, and unstable if $\mu<0$. (In fact, if $\mu>0$, let $v_1 \in \mathrm{Int}(P_X)$ be eigenfunction : $(L-\lambda G'(u))v_1=\mu J v_1$, and observe that for fixed $\varepsilon \in (0,\mu)$ and $0<\delta$ sufficiently small,

$$\bar{u}(t,x) := u(x) + \delta e^{-\varepsilon t} v_1(x)$$

is supersolution of problem (0.1), while

$$\underline{u}(t,x) := u(x) - \delta e^{-\varepsilon t} v_1(x)$$

is subsolution. Hence, if $\hat{u}$ is a solution of (0.1) with $\underline{u}(0,.)\le\hat{u}(0,.)\le\bar{u}(0,.)$, it follows that for $t\ge0$, $\underline{u}(t,.)\le\hat{u}(t,.)\le\bar{u}(t,.)$. Since both $\underline{u}$ and $\bar{u}$ tend to u (exponentially) as $t\to+\infty$, the Lyapunow asymptotic stability

of u follows).

Thus the sign of the smallest eigenvalue of (2.5) is crucial. For the stability of the trivial solution $(\lambda,0)$ we have

<u>Proposition 2.6</u>. Let $(\lambda,0)$ be a trivial solution. Then 0 is stable for $0 \le \lambda < \lambda_1$ and unstable for $\lambda > \lambda_1$.

Recalling (1.14), this follows immediately from (1.15).

We now investigate the stability of the positive solutions $(\lambda(s),u(s))$ in a neighbourhood of $(\lambda_1,0)$. Since (by (1.16)) 0 is J-simple eigenvalue of $F_u(\lambda_1,0)$, Lemma 1.7 implies the existence of C^1-functions $\mu : (-\alpha,\alpha) \to \mathbb{R}$ and $w : (-\alpha,\alpha) \to X$ with $\mu(0)=0$, $w(0)=u_1$, such that

$$(L - \lambda(s)\ G'(u(s)))\ w(s) = \mu(s)\ J\ w(s)$$

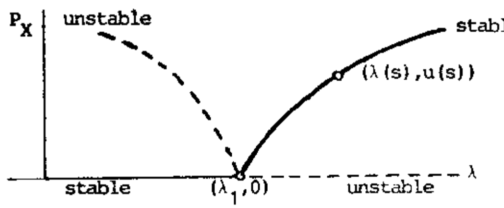
for $s \in (-\alpha,\alpha)$, $\alpha>0$ sufficiently small. Moreover by [3, Theorem 1.16], $\mu(s)$ and $s\lambda'(s)$ have the same zeroes near $s=0$, and

$$\lim_{\substack{s\to0\\ \mu(s)\neq0}} \frac{-s\lambda'(s)\gamma'(\lambda_1)}{\mu(s)} = 1.$$

By (1.18) we thus obtain

<u>Proposition 2.7</u>. Let $(\lambda(s),u(s)) \in \Sigma_0$ and $0<s$ sufficiently small. Then $u(s)$ is stable if $\lambda'(s) > 0$ and unstable if $\lambda'(s) < 0$.

We have the following picture of "<u>exchange of stability</u>" for the nonnegative solutions of the (NEVP) in a neighborhood of $(\lambda_1,0)$ in $\mathbb{R} \times X$:



If $m>0$ on $\bar{\Omega}$, Propositions 2.6 and 2.7 are well-known, e.g. [3, Example 2.3], [8].

III. Convex and concave nonlinearities.

We consider now the special cases where the function $g(x,s)$ is either convex or concave in $s\geq 0$.

III.A. Let $g(x,s)$ be convex in $s\geq 0$ for all $x \in \bar{\Omega}$, strictly convex for at least one $x \in \Omega$.

Proposition 3.1. Let $(\lambda,u) \in \mathbb{R}\times X$ be a positive solution of the (NEVP). Then $\lambda<\lambda_1$, and u is unstable steady-state solution of (0.1).

Proof. Let (λ,u) be positive solution: $Lu=\lambda G(u)$. We know that $u \in Int(P_X)$.

(i) By convexity of g, $G(u)>Mu$. Thus $Lu = \lambda Mu + h$ with $h>0$, and Proposition 1.9(ii) implies $\lambda<\lambda_1$.

(ii) By rescaling, we may assume that the function $s \mapsto g(x,s)+s$ has positive partial derivative with respect to s, for all $x \in \bar{\Omega}$, $0\leq s\leq \|u\|_Y + 1$. Consider the mapping $H : \mathbb{R}^+ \times X \to X$ given by

(3.2) $$H(\lambda,v) := \lambda(L+\lambda J)^{-1}(G(v)+Jv).$$

Then $u = H(\lambda,u)$, and the partial derivative

$$H_v(\lambda,u) = \lambda(L+\lambda J)^{-1}(G'(u)+J)$$

at (λ,u) is a strongly positive, compact linear operator in X. We show that $r^* := spr(H_v(\lambda,u)) > 1$.

Suppose $0<r^*\leq 1$ ($r^*=0$ is impossible since $H_v(\lambda,u)$ is strongly positive). By the Krein - Rutman theorem there exists $w \in Int(P_X)$ such that

(3.3) $$H_v(\lambda,u)w = r^*w.$$

Let $\|w\|_X$ be so small that $u-w$, and hence $u-r^*w$, is contained in $Int(P_X)$. By order convexity

$$H(\lambda,u-w)>>H(\lambda,u)-H_v(\lambda,u)w = u-r^*w.$$

There exists τ : $0<\tau<1$, such that $\tau u-(u- r^*w)\in \partial P_X$. Since $H(\lambda,.)$ is increasing and convex in the order interval $[0,u]$ of X and $0<u-w\leq u-r^*w\leq \tau u<u$, we obtain

$$u-r^*w<<H(\lambda,u-w) \leq H(\lambda,\tau u) \leq \tau H(\lambda,u) = \tau u.$$

Thus $\tau u-(u-r^*w) \in Int(P_X)$, a contradiction.

(3.3) implies that

$$(L+\lambda J)w = \frac{\lambda}{r^*} (G'(u)+J)w, \qquad w>0.$$

We conclude that $\lambda/r^* = \tilde{\lambda}_1(g_s(.,u)+1)$, the principal eigenvalue of the operator $\tilde{L} := L+\lambda J$ with respect to the weight function $\tilde{m} := g_s(.,u)+1$. Since $\lambda>(\lambda/r^*)$, by Lemma 1.5 there exists $\varepsilon>0$ such that $\lambda=\tilde{\lambda}_1(\tilde{m}-\varepsilon)$. Hence for some function $v>0$,

$$(L+\lambda J)v = \lambda(G'(u)+(1-\varepsilon)J)v,$$

i.e.

$$(L-\lambda G'(u))v = (-\lambda\varepsilon)Jv.$$

Since $\mu := -\lambda\varepsilon < 0$, the instability follows. □

III.B. Let $g(x,s)$ be concave in $s\geq 0$ for all $x \in \bar{\Omega}$, strictly concave for at least one $x \in \Omega$.

Proposition 3.4. Let $(\lambda,u) \in \mathbb{R}\times X$ be positive solution of the (NEVP). Then $\lambda>\lambda_1$. For each $\lambda>\lambda_1$, there is at most one positive solution (λ,u), and u is stable. There exists a number $\bar{\lambda} \in (\lambda_1,+\infty]$ and a continuous map $\tilde{u}(.) : [\lambda_1,\bar{\lambda}) \to P_X$ with $\tilde{u}(\lambda_1) = 0$, such that $\Sigma_0 = \{(\lambda,\tilde{u}(\lambda)) : \lambda_1\leq\lambda<\bar{\lambda}\}$. Moreover $\tilde{u}(.)$ is continuously differentiable on $(\lambda_1,\bar{\lambda})$, and if $\bar{\lambda}<+\infty$, then $\lim_{\lambda\nearrow\bar{\lambda}} \|\tilde{u}(\lambda)\|_X = +\infty$.

Proof. (i) Let (λ,u) be positive solution. We may assume again that for all $x \in \bar{\Omega}$, $0\leq s\leq\|u\|_Y+1$, the function $s\mapsto g(x,s)+s$ has positive partial derivative with respect to s. Since $u \in Int(P_X)$ and g is concave,

$$(L+\lambda J)u = \lambda(G(u)+Ju) < \lambda(M+J)u$$

and hence $u<<\lambda K_\lambda u$. Now [5, Lemma 1] implies that $\lambda_1<\lambda$.

(ii) Next we show that if (λ,u) is positive solution, then u is stable. Let $H : \mathbb{R}^+\times X\to X$ be the mapping introduced in (3.2). We claim that $r^* := \mathrm{spr}(H_v(\lambda,u)) < 1$.

Suppose, to the contrary, that $r^* \geq 1$, and let $w \in \mathrm{Int}(P_X)$ be such that (3.3) holds. Since $u \in \mathrm{Int}(P_X)$ and g is concave,

$$H(\lambda,u-tw) \ll H(\lambda,u) - tH_v(\lambda,u)w = u-tr^*w$$

for $t>0$ such that $u-tr^*w \in P_X$. There exists $t_0>0$ such that $u-t_0r^*w \in \partial P_X$. Since $H(\lambda,.)$ is increasing on $[0,u]$ and $H(\lambda,0)=0$,

(3.5) $$H(\lambda,u-t_0r^*w) \in P_X.$$

On the other hand,

(3.6) $$H(\lambda,u-t_0r^*w) \leq H(\lambda,u-t_0w) \ll u-t_0r^*w \in \partial P_X.$$

But (3.5) and (3.6) are incompatible. Thus $0<r^*<1$.

In a similar way as in the proof of Proposition 3.1 we obtain now $\varepsilon>0$ and $v>0$ such that

$$(L-\lambda G'(u))v = (\lambda\varepsilon)Jv.$$

Here $\mu := \lambda\varepsilon > 0$ is the smallest eigenvalue of the linearized problem, and u is stable.

(iii) Let $\lambda>\lambda_1$, and suppose (λ,u_1), (λ,u_2) are positive solutions with $u_1 \neq u_2$ ($u_1,u_2 \in \mathrm{Int}(P_X)$). We may again assume that for all $x \in \bar{\Omega}$ the function $s \mapsto g(x,s)+s$ is strictly increasing in s, $0 \leq s \leq \bar{s} := \mathrm{Max}\,\{\|u_1\|_Y, \|u_2\|_Y\} + 1$, and that $u_2 \not\leq u_1$. There exists $\tau : 0 < \tau < 1$, such that $u_1 - \tau u_2 \in \partial P_X$. We conclude that

$$u_1 = H(\lambda,u_1) \geq H(\lambda,\tau u_2) \gg \tau H(\lambda,u_2) = \tau u_2,$$

contradicting the choice of τ.

(iv) Set $\bar{\lambda} := \sup\{\lambda : (\lambda,u) \in \Sigma_0\}$. It is a simple consequence of the implicit function theorem that $\Sigma_0 \setminus \{(\lambda_1,0)\}$ is a C^1-curve, parametrized by $\lambda \in (\lambda_1,\bar{\lambda})$. In fact, a positive solution (λ,u) of the (NEVP) solves $u-H(\lambda,u) = 0$, and since $r^* = \mathrm{spr}(H_v(\lambda,u)) < 1$, the operator $I-H_v(\lambda,u)$ is invertible in X.

(v) Let $\bar{\lambda}<+\infty$, and suppose there is a sequence $\{(\lambda_n,u_n) : n \in \mathbb{N}\}$ in Σ_0 with $\lambda_n \to \bar{\lambda}$, $\|u_n\|_X \leq C$. By compactness we infer that (for a subsequence) $u_n \to \bar{u}$ in X, where $(\bar{\lambda},\bar{u})$ is positive solution ($(\lambda_1,0)$ is the only bifurcation point for positive solutions from the trivial solutions). The implicit function theorem asserts that Σ_0 can be continued beyond $(\bar{\lambda},\bar{u})$, contradicting the definition of $\bar{\lambda}$. □

We note that the proof of stability in Proposition 3.4 is drastically simpler than that given in [1, Section 25] for a special case. If g is not positive, we do not know whether the map $\lambda \mapsto \|\bar{u}(\lambda)\|_X$ is increasing.

We conclude with two results guaranteeing that $\bar{\lambda} < +\infty$ or $\bar{\lambda} = +\infty$, respectively (cf. [7]).

Proposition 3.7. Suppose g is asymptotically linear, i.e. that there exists $m_\infty(x) := \lim_{s\to+\infty} s^{-1}g(x,s)$, uniformly in $x \in \bar{\Omega}$, and that m_∞ is positive somewhere in Ω. Then $\bar{\lambda} = \lambda_1(m_\infty)$.

Proposition 3.8. Suppose there exists a function $w \in C_0^1(\bar{\Omega}) \cap C^2(\Omega)$, $w>0$, such that $Lw \geq 0$, $G(w) \leq 0$. Then $u<w$ for all $(\lambda,u) \in \Sigma_0$. Hence $\bar{\lambda} = +\infty$.

References.

[1] H. Amann: Fixed point equations and nonlinear eigenvalue problems in ordered Banach spaces. SIAM Review 18 (1976), 620-709.

[2] M.G. Crandall and P.H. Rabinowitz: Bifurcation from simple eigenvalues. J. Functional Analysis 8(1971), 321-340.

[3] ----: Bifurcation, perturbation of simple eigenvalues, and linearized stability. Arch. Rat. Mech. Anal. 52(1973), 161-180.

[4] J.-P. Gossez and E. Lami Dozo: On the principal eigenvalue of a second order linear elliptic problem. (to appear).

[5] P. Hess and T. Kato: On some liner and nonlienar eigenvalue problems with an indefinite weight function. Comm. Partial Differential Equations 5(1980), 999-1030.

[6] P. Hess: An anti-maximum principle for linear elliptic equations with an indefinite weight function. J. Differential Equations. (to appear).

[7] ----: On bifurcation from infinity for positive solutions of second order elliptic eigenvalue problems. In: Proceedings of the internat. Conference on "Nonlinear phenomena in mathematical sciences", Academic Press. (to appear).

[8] D.H. Sattinger: Stability of bifurcating solutions by Leray - Schauder degree. Arch. Rat. Mech. Anal. 43(1971), 154-166.

Mathematics Institute
University of Zürich
Freiestr. 36
8032 Zürich, Switzerland