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STABILITY QUESTIONS ON THE STUDY OF CONSTANT MEAN CURVATURE SURFACES

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There are many problems in Differential Geometry where calculus of variations tecniques have been applied sucessifully. In this lecture I intend to present one of them, namely the study of constant mean curvature stable surfaces.

Let's start considering an immersion $X: M \rightarrow R^3$ of a two dimensional, oriented, connected manifold M into the 3-dimensional Euclidean space. (All manifolds and mappings will be assumed to be C^∞). Let D represent a simply connected domain on M with compact ~~snap~~ closure $\bar{D}$ and piecewise smooth

boundary ∂D. A variation of X is a differentiable one-parameter family of immersions $X_t : M \rightarrow R^3$ such that $X_0 = X$. We say that X_t ~~preserves~~ keeps ∂D fixed if $X_t(p) = X(p)$ for all p on ∂D. Given a variation X_t of X we represent by $A(t,D)$ the area of $X_t(D)$ and by $V(t,D)$ the volume of the cone determined by $X_t(D)$ and the origin of R^3. The following is then a well known result:

Theorem 1: The immersion $X: M \rightarrow R^3$ has constant mean curvature if and only if for any given domain $D \subset M$ one has

$$\frac{d}{dt} A(t,D) \Big|_{t=0} = 0$$

for all variations of X keeping $V(t,D)$ and ∂D fixed.

<u>Definition</u> 1 : Let $X : M \to R^3$ be an immersion with constant mean curvature. We say that a domain $D \subset M$ is <u>stable</u> if

$$\frac{d^2}{dt^2} A(t,D)\Big|_{t=0} \geq 0$$

for all variations of X keeping $V(t,D)$ and ∂D fixed. We also say that X is stable if all $D \subset M$ are stable.

We can then formulate two problem

<u>Problem</u> 1 : Given an immersion $X : M \to R^3$ wi constant mean curvature, find simple conditions in terms of the metric of M such that if $D \subset M$ satisfies such conditions then D is stable.

<u>Problem</u> 2 : Find all stable immersions $X : M \to R^3$ under the hypothesis that th

metric induced in M by X is a complete metric.

The study of critical points of $A(t,D)$ for volume preserving variations is a so-called isoperimetric problem in Calculus of Variations. The standard procedure is to ~~take~~ use Lagrange multipliers rule, namely to consider the new function

$$J(t,D) = A(t,D) + \lambda V(t,D) \quad ,$$

where λ is a constant to be determined, and determine the critical points of J for all variations keeping ∂D fixed. A simple computation shows that

$$\frac{d}{dt} J(t,D)\Big|_{t=0} = \int_D (-2H+\lambda) f \, dM \, .$$

where H is the mean curvature function of X, dM is the element of area of the metric induced in M by X and $f = \langle N, E \rangle$, being E the variational vector field

$$E(p) = \frac{d}{dt} X_t(p) \Big|_{t=0}$$

and being N a unit normal vector field along M.

The function f satisfies the condition "$f = 0$ on ∂D" since X_t keeps ∂D fixed and, under this condition, is quite general as one can see by observing that, for any given f, $X_t = X + tfN$ is a variation of X for which $\langle N, E \rangle = f$.

If $H \equiv 0$ ~~one shall observe that $\lambda = 0$ and therefore~~ the critical points of J are just the critical points

of A without further conditions. A surface for which $H \equiv 0$ is called a minimal surface and, for them, problems (1) and (2) have been answered as follows:

<u>Theorem</u> 2: (J. L. Barbosa and M. do Carmo [2])
<u>Let</u> $X : M \to R^3$ <u>be a minimal immersion and</u> $N : M \to S^2$ <u>be its Gauss-mapping</u>. <u>Assume</u> $D \subset M$ <u>is such that</u> $\mathrm{Area}(N(D)) < 2\pi$. <u>Then</u> D <u>is stable</u>. <u>This result is sharp</u>.

"Sharp" means that, given $\varepsilon > 0$, there exists nonstable domain D_ε with $\mathrm{Area}\ N(D_\varepsilon) = 2\pi + \varepsilon$.

Theorem 2 has been used in a number of situations (see Nitsch [12], Nitsch [13] and Meeks [11]

A condition weaker than the one in theorem 2, but sufficient for most applications, is obtained by noticing that

$$\int_D |K|\, dM > \text{Area}\,(N(D))$$

where K denotes the Gauss Curvature of M in the induced metric. Thus theorem 2 implies that,

"if $\int_D |K|\, dM < 2\pi$ then D is stable"

Theorem 3: Let $X : M \longrightarrow R^3$ be a complete stable minimal immersion. Then $X(M) \subset R^3$ is a plane.

This result was proved by M.P. do Carmo and C.K. Peng [8] and independently

by D. Fisher-Colbrie and R. Schoen [9] and completely answers problem 2 for the case of minimal ~~submani~~ surfaces.

Problems 1 and 2 make sense, in ~~much~~ general, for minimal submanifolds of a Riemannian manifold Some partial answers have been given for both and I shall refer the reader to [7] for a quite complete report on the subject.

~~Let's now~~ Let me now give an idea of the proof of theorem 2 before getting into the case when $H \not\equiv 0$.

First of all one computes the

second variation of A obtaining

$$I_D(f) := \frac{1}{2} \frac{d^2 A}{dt^2}\Big|_{t=0} = \int_D (-f\Delta f - |B|^2 f^2)\, dm$$

where Δf denotes the Laplacian of f in the induced metric and $|B|^2$ stands for the square of the norm of the second quadratic form of X. One shows that $|B| = 0$ only at isolated points. ~~For proving theorem 2~~ One now replaces the metric ds^2 of M by the pseudo-metric $d\sigma^2 = \frac{1}{2}|B|^2 ds^2$. It turns out the curvature of this new metric is exactly one. The mapping $N: M \to S^2$ is then a branched covering projection onto $N(M)$, which is an isometry outside

the set of branch points. We can also rewrite $I_D(f)$ as:

$$I_D(f) = \int_D (-f\, \Delta_\sigma f - 2 f^2)\, dm_\sigma$$

where $\Delta_\sigma f$ stands for the Laplacian of f on M endowed with the new metric. Then one proves the following proposition.

<u>Proposition 1</u>: <u>Let $\pi: M \to S$ be a branched covering projection onto S that is also an isometry outside of the branch points; let $D \subset M$ be a domain with compact closure and smooth boundary and let $L_S = \Delta_S + m$ and $L_M = \Delta_M + m(\pi)$ be operators on S and M respectively. Then $\lambda_1(\pi(D)) \leq \lambda_1(D)$ where λ_1 stands for the first eigen value of each operator in the</u>

respective domain.

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~~Massag~~ ~~b~~.
Applying the above proposition to our problem with $N = \Pi$, $L_S = \Delta_S$ and $L_m = \Delta_m$ we obtain that $\lambda_1(D) \geq \lambda_1(N(D))$. By the well known Faber-Krahn inequality we have $\lambda_1(N(D)) \geq \lambda_1(D^*)$ where D^* is a geodesic ball in S^2 of the same area as $N(D)$. Since by hypothesis $\mathrm{Area}(N(D)) < 2\pi$ then D^* is contained in a hemisphere and hence $\lambda_1(D^*) > 2$. Therefore $\lambda_1(D) > 2$ and so $I_D(f) \geq 0$.

As this proof points out it is an interesting ~~question~~ ~~problem~~ question to obtain an estimation of the first eigen value

of a problem of the type

$$(*) \begin{cases} \Delta u + \lambda m u = 0 & \text{on } D \\ u = 0 & \text{on } \partial D \end{cases}$$

(with suitable assumptions on $D, \partial D$ an m) by some function of $\mathcal{M} := \int_D m \, dM$. Using an isoperimetric inequality, Barbosa and do Carmo [3] were able to prove the following proposition relating the first eigen value of the above problem in a surface M with the first eigen-value of the problem

$$(**) \begin{cases} \Delta u + \lambda u = 0 & \text{on } D^* \\ u = 0 & \text{on } \partial D^* \end{cases}$$

on a simply connected, complete surface $M^*(a)$ of constant curvature a.

<u>Proposition 2</u> : <u>If</u> $m > 0$ <u>in</u> D <u>and</u> ~~if~~ <u>the metric</u> $d\sigma^2 = m\, ds^2$ <u>has curvature</u> $K_\sigma \leq a$, <u>then the first eigen-value of</u> (*) <u>is greater than or equal to the first eigen-value of</u> (* *) <u>in</u> $D^* \subset M^*(a)$ <u>where</u> D^* <u>is a geodesic disk such that</u>

$$\text{Area}(D^*) = \int_D m\, dM$$

It would be ~~of~~ interesting to know if one can obtain further information about λ_1 of (*) under weaker conditions, or ~~Now~~ ~~let's~~ even the same information when dimension of M is greater than 2.

Let's ~~now~~ return to the case when $H \neq 0$. Now, the second variation

formula must be computed with respect to that variations that keep $V(t, D)$ infinitesimally constant. This means that we must restrict ourselves to the variations X_t for which

$$\frac{d}{dt} V(t, D) = 0$$

besides the condition of keeping ∂D fixed. A straightforward computation shows this to be equivalent to:

$$\int_D f\, dM = 0 \quad , \quad f|_{\partial D} = 0$$

Therefore, D stable means that

$$I_D(f) = \int_D (-f\Delta f - |B|^2 f^2)\, dm \geq 0$$

for all functions satisfying the above conditions.

If M is compact and $D = M$ then this is equivalent to say that the first eigen-value of $\Delta + |B|^2$ on M is positive. Barbosa and do Carmo [5] recently pointed out that the estimate obtained by Reilly [14] for the first eigen-value of the Laplatian of M can be used to prove the following theorem.

<u>Theorem 4</u>: <u>If M is compact and $X : M \longrightarrow R^3$ is a stable immersion with constant mean curvature H, then M is the sphere S^2 and X is a standard immersion of S^2 with curvature H^2.</u>

I shall observe that Hopf conjecture states that the sole hypothesis that the mean curvature of x is constant implies that x is an isometry to a standard sphere. This conjecture was proved to be true if M is homeomorphic to a sphere (Hopf [10]) or if x is an embedding (Alexandroff [1]). The general case ~~remain~~ remains open and is a challenging problem. Theorem 4 shows that if there is a counter-example of Hopf conjecture, it ~~will be~~ is unstable and so it can not be easily observed in the nature.

For stability of domains most of the questions are still open. To state the known results let's ~~make~~ first make a definition

<u>Definition</u> 2: Let $x: M \longrightarrow R^3$ be an immersion with constant mean curvature. If $D \subset M$ is a domain such that $I_D(u) \geq 0$ for all $u: \bar{D} \longrightarrow R$ with $u = 0$ on ∂D, then we say that D is <u>strongly</u> <u>stable</u>

It is clear that D strongly stable implies D stable. I must observe that in the literature strong stability has been sometimes called stability.

<u>Theorem 5</u>: <u>If</u> $X: M \to R^3$ <u>is an immersion with constant mean curvature and</u> $D \subset M$ <u>is such that</u>

$$\int_D |B|^2 dm \leq 4\pi$$

<u>then</u> D <u>is strongly stable</u>. <u>This result is sharp</u>

This was ~~proved~~ obtained by Ruchert [15]. ~~The and generalized by Barbosa proof~~ and it can be proved following ~~following~~ the lines of the proof of theorem 2, only that now ~~one takes~~ after taking $d\sigma^2 = \frac{1}{2}|B|^2 ds^2$, one proves that the curvature of this new metric is bounded above by 1 and concludes the result using proposition 2. To prove sharpness, ~~one~~ just consider domains in the sphere S^2 that ~~sb~~ contain a hemisphere and have area $2\pi+\varepsilon$.

Let's observe that if $D\subset M$ is not stable it is possible to find a function $u: D \to R$ with compact suport such that

$$\begin{aligned} &\Delta u + |B|^2 u + c = 0 &&\text{on } D \\ &u = 0 &&\text{on } \partial D \\ &\int_D u\, dM = 0 \end{aligned}$$

when c is a constant. The function u changes signs and determines two domains $D^+ = \{p\,;\, u(p)>0\}$ and $D^- = \{p\,;\, u(p)<0\}$. At least one of then, say D^+, satifies

$$\int_{D^+} |B|^2 dM \leq \frac{1}{2} \int_D |B|^2 dM$$

~~and~~ Since we have $I_{D^+}(u) = 0$ and u is not an eigen-function for $\Delta + |B|^2$

on D^+, then we conclude that D^+ is not strongly stable. This observation togheter with Theorem 5 gives the following result

<u>Teorem</u> 6 : <u>If</u> $X: M \to R^3$ <u>is</u> <u>an</u> <u>immersion</u> <u>with</u> <u>constant</u> <u>non-zero</u> <u>mean</u> <u>curvature</u>, <u>and</u> $D \subset M$ <u>is</u> <u>such</u> <u>that</u>

$$\int_D |B|^2 dM \leq 8\pi$$

<u>then</u> D <u>is</u> <u>stable</u>

I do not know if this result is sharp or not.

The proof of theorem 3 can be ~~sketch~~ modified to give a proof of the following theorem, as was first observed by Prof. H. Mori.

<u>Theorem 7</u>: <u>Let</u> $X : M \longrightarrow R^3$ <u>be a complete immersion with constant mean curvature. If</u> X <u>is strongly stable then</u> $X(M)$ <u>is a plane</u>.

Would be interesting to know if it is possible to replace the hypothesis of strong stability by stability in this theorem.