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ON

VARIATIONAL METHODS IN ANALYSIS AND MATHEMATICAL PHYSICS

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AMANN'S THEOREMS ON THE UNIQUE SOLVABILITY OF OPERATORS EQUATIONS IN HILBERT SPACE

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In this seminar I will speak on some results by Amann, cf. [1], regarding the existence of a unique solution to the operator equation

$$A(u) = F(u)$$

in a given Hilbert space. Their proofs is based on a very beautiful and original application of the theory of monotone operators. So let me start with some introductory remarks on the theory of monotone operators.

Let H be a Hilber space. A mapping $T: K \subseteq H \to H$ is called monotone if

$$(T(u) - T(v) \mid u - v) \geq 0 \qquad (u, v \in K).$$

One of the basic problems about monotone [redacted] operators is the following:

PROBLEM OF VARIATIONAL INEQUALITIES: Find $x_0 \in K$ such that

$$(T(x_0) \mid x - x_0) \geq 0$$

for every $x \in K$.

Observe that when $K = H$, the condition

$$(T(x_0) \mid x - x_0) \geq 0 \qquad (x \in K)$$

is equivalent to say that $T(x_0) = 0$. By this and the fact that

$$T(x) = y \iff T(x) - y = 0$$

and that $T - y$ is monotone whenever T is, we may conclude that the problem of variational inequalities is a problem of surjectivity in case [redacted] $K = H$.

The concept of monotone operators has been introduced independently by KAČUROVSKI [6] and Zarantonello [11]. The first general [redacted] theorems for the [redacted] existence of solutions to the problem of variational inequalities has been proved by Minty [9], cf. also [10] and the references therein. [redacted] Browder [3] was the first who applied the theory of monotone operators to PDE, opening in this way a new field of research. He wrote quite a number of papers on the subject, cf. Browder [4]. Stampacchia developped in a number of papers the theory of regularity of solutions. For a comprehensive and very readable introduction to the theory, cf. KINDERLEHRER – STAMPACCHIA [7]. For applications of monotone operators to [redacted] evolution equations, cf. LIONS [8] and BREZIS [2]. [redacted]

There are various important questions in which the problem of variational equations arises in a natural way. We shall consider three simples examples, in order to have an idea of the importance of this class of operators and of the problem of variational inequalities.

EXAMPLE 1: Consider the fundamental problem of the calculus of variations: to minimize a functional. We have the following result:

<u>Theorem</u>: Let K be a convex subset of the Hilbert space H and $f : K \to \mathbb{R}$ a convex differentiable functional. Then x_0 is a point of minimum for f if and only if x_0 is a solution of the variational inequality:

$$(\text{grad } f(x_0) \mid x - x_0) \geq 0 \qquad (x \in K).$$

<u>Proof</u>: Necessity: Fix $x \in K$ and suppose that x_0 is a minimum for f. Then $x_0 + t(x - x_0) \in K$ and therefore the function

$$\varphi(t) = f(x_0 + t(x - x_0))$$

is well defined on $[0,1]$ and has a minimum at $t = 0$. Consequently we have:

$$0 \leq \varphi'(0) = (\text{grad } f(x_0) \mid x - x_0).$$

Since x is an arbitrary point of K, we have done.

Sufficiency: Suppose that x_0 is a solution to our variational inequality. The convexity of f implies that

$$f(x) \geq f(x_0) + (\text{grad } f(x_0) \mid x - x_0) \qquad (x \in K).$$

But $(\text{grad } f(x_0) \mid x - x_0) \geq 0$, so we have:

$$f(x) \geq f(x_0) \qquad (x \in K).$$

<u>q.e.d.</u>

EXAMPLE 2: The complementarity problem. Now we consider a problem of importance in mathematical programming. Set

$$\mathbb{R}^n_+ = \{ x = (x_1, \ldots, x_n) \in \mathbb{R}^n \mid x_i \geq 0 \text{ for } i = 1, \ldots, n \}.$$

Then $\mathbb{R}^n_+$ is convex. The problem we are interested is the following:

COMPLEMENTARITY PROBLEM: <u>Given</u> $F : \mathbb{R}^n_+ \to \mathbb{R}^n$, <u>find</u> $x_0 \in \mathbb{R}^n_+$ <u>such that</u> $F(x_0) \in \mathbb{R}^n_+$ <u>and</u> $(F(x_0) \mid x_0) = 0$.

We have the following result:

<u>Theorem</u>: x_0 is a solution to the complementarity problem if and only if x_0 is a solution to the variational inequality

$$(F(x_0) \mid x - x_0) \geq 0 \qquad (x \in \mathbb{R}^n_+).$$

<u>Proof</u>: Necessity: Assume that x_0 is a solution to the complementarity problem. Since $F(x_0) \in \mathbb{R}^n_+$, the definition of inner product of $\mathbb{R}^n$ implies that

$$(F(x_0) \mid x) \geq 0 \qquad (x \in \mathbb{R}^n_+).$$

Therefore for every $x \in \mathbb{R}^n_+$ we have:

$$\begin{aligned}(F(x_0) \mid x - x_0) &= (F(x_0) \mid x) - (F(x_0) \mid x_0) \\ &= (F(x_0) \mid x) \\ &\geq 0.\end{aligned}$$

Sufficiency: Now let x_0 be a solution to the variational inequality related to $T = F$. Consider the vector

$$x = x_0 + e_i$$

where $e_i = (0, \ldots, 0, 1, 0, \ldots, 0)$ with the non-null coordinate at the i-th place. Obviously $x \in \mathbb{R}^n_+$. Thus we have:

$$0 \leq (F(x_0) \mid x_0 + e_i - x_0) = (F(x_0) \mid e_i) = F_i(x_0).$$

This shows that $F(x_0) \in \mathbb{R}^n_+$. Selecting $x = 0 \in \mathbb{R}^n_+$, from the variational inequality we have:

$$(F(x_0) \mid x_0) \leq 0.$$

But from the fact that x_0 and $F(x_0)$ belong to $\mathbb{R}^n_+$ it follows, by virtue of the definition of inner product in $\mathbb{R}^n$, that

$$(F(x_0) \mid x_0) \geq 0.$$

Combining with the above we get $(F(x_0) \mid x_0) = 0$. q.e.d.

EXAMPLE 3: Elliptic equations. Let us consider the Dirichlet problem for an elliptic equation:

$$(1) \quad \begin{cases} -\sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(a_{ij}(x) \frac{\partial}{\partial x_j} u\right) + a_0(x)\, u + f(x,u) = 0 \\ u|_{\partial\Omega} = 0 \end{cases}$$

where $a_0 \geq 0$ and $f : \Omega \times \mathbb{R} \to \mathbb{R}$, Ω being a domain of $\mathbb{R}^n$ with smooth boundary. Consider the Dirichlet form:

$$a(u,v) = \sum_{i,j=1}^{n} \int_\Omega a_{ij} \frac{\partial}{\partial x_i} u \frac{\partial}{\partial x_j} v \, dx + \int_\Omega a_0\, u v \, dx.$$

Let $L^2 = L^2(\Omega, \mathbb{R})$ and $H^1_0 = H^1_0(\Omega, \mathbb{R})$. The Dirichlet

form is defined for each $u, v \in H^1_0 \subseteq L^2$. Multiplying the elliptic equation (1) by $\varphi \in H^1_0$ and then integrating, we get that u is a solution to (1) if and only if

$$(2) \quad a(u,\varphi) + \int_\Omega f(x,u)\varphi \, dx = 0 \qquad (\varphi \in H^1_0).$$

For each $u \in H^1_0$, $a(u,\cdot)$ is a continuous linear functional on $H^1_0 \subseteq L^2$. Therefore, by Riezs theorem, $a(u,\cdot)$ can be indentified to a point $A(u) \in H$. For each $u \in L^2$, the map

$$v \mapsto \int_\Omega f(x,u(x))\, v(x) \, dx$$

is a continuous linear functional on L^2, hence can be identify to a point $F(u) \in L^2$. Then (2) is equivalent to the following operator equation

$$(3) \quad A(u) + F(u) = 0.$$

This means that to solve (1) is equivalent to solve (3) in H^1_0. By using Gårding's inequality it is possible to prove the following

Theorem: The operator $A + F$ is monotone in $H^1_0 \subseteq L^2$ whenever $f(x,\cdot) : \mathbb{R} \to \mathbb{R}$ is an increasing function.

By virtue of this result, the solution of (1) is reduced to the surjectivity of a monotone operator (if $A+F$ is surjective, then (3) has a solution).

After this preparation, we want to prove the theorem below, whose statement is inspired by the equivalence (1) $\Leftrightarrow$ (3). We shall see how to apply it to elliptic equations in case $f(x,\cdot)$ is not increasing.

AMANN'S THEOREM: Let H be a Hilbert space, $A : D(A) \subseteq H \longrightarrow H$ a self-adjoint linear operator and $F : H \to H$ a mapping with a symmetric weak Gateaux derivative F'. Moreover, suppose that

(i) there exist two symmetric operators $B^{\pm} \in L(H)$ such that
$$B^- \leq F'(u) \leq B^+ \qquad (u \in H);$$

(ii) there exist two closed subspaces $X^{\pm}$ of H such that $X^+ \cap X^- = 0$, $H = X^+ + X^-$ and

(a) there exists a constant $\gamma > 0$ such that
$$((A - B^-)u|u) \leq -\gamma \|u\|^2 \qquad (u \in D(A) \cap X^-),$$
$$((A - B^+)u|u) \geq \gamma \|u\|^2 \qquad (u \in D(A) \cap X^+);$$

(b) $D(A)$ is invariant under the projections $Q^{\pm} : H \to X^{\pm}$ parallel to $X^{\mp}$.

Then the operator equation
$$A(u) = F(u)$$
has exactly one solution.

Note that the subspaces X^+ and X^- are not required to be orthogonal.

Proof: The proof is divided in two parts: first we have some algebraic considerations, and then we appeal to the theory of monotone operators. Define
$$R = Q^+ - Q^- : H \to H.$$
By showing first that Q^+ and Q^- are closed and that

(4) $$R^2 = Q^+ + Q^- = id_H,$$

it is easily seen that R is continuous and invertible. Thus, setting $v = R^{-1}u$, the given equation
$$A(u) = F(u)$$
is equivalent to
$$AR(v) = FR(v),$$
hence to
$$M(v) = 0$$
where $M = AR - FR$. Set $L = AR$. By some algebraic considerations, it is possible to prove that L is closed and

(5) $$D(L) = D(L^*) = D(A).$$

With this we finish the part related to algebra. Now we start the second part. To begin with, we want to show that F maps bounded sets into bounded sets. To say that F' is the weak Gateaux derivative of F means that $F'(u) \in L(H)$ and

$$\lim_{t\to 0} \frac{1}{t}\left(F(u+th) - F(u) \mid v\right) = \left(F'(u)\cdot h \mid v\right)$$

for all $u, v, h \in H$. Thus for every $u, v, w \in H$, the mean value theorem implies the existence of a number $0<t<1$ such that

$$\left(F(u)-F(v) \mid w\right) = \left(F'(v+t(u-v))\cdot(u-v) \mid w\right).$$

Consequently

$$\|F(u)-F(v)\| \le \sup_{0\le t\le 1} \|F'(v+t(u-v))\|\,\|u-v\|$$

$$\le \max\{\|B^+\|, \|B^-\|\}\cdot\|u-v\|$$

where the last inequality is a consequence of (i). This shows that F maps bounded sets into bounded sets. Now let us state the following inequality:

$$(6) \qquad \left(M(u)-M(v) \mid u-v\right) \ge \frac{\gamma}{2}\|u-v\|^2 \qquad (u,v\in D(A))$$

Fix $u, v \in D(A)$. By the mean value theorem, there is $0<t<1$ such that, setting $B = F'(R(v)+tR(u-v)) \in L(H)$, we have:

$$\left(F(Ru)-F(Rv) \mid u-v\right) = \left(BR(u-v) \mid u-v\right).$$

Consequently, since $A-B$ is self-adjoint, we have:

$$\left(M(u)-M(v) \mid u-v\right) = \left(AR(u-v) \mid u-v\right) - \left(F(R(u))-F(Rv) \mid u-v\right)$$

$$= \left((A-B)\left[Q^+(u-v) - Q^-(u-v)\right] \mid Q^+(u-v)+Q^-(u-v)\right)$$

(by the definition of R and by (4))

$$= \left((A-B)\,Q^+(u-v) \mid Q^+(u-v)\right) - \left((A-B)Q^-(u-v) \mid Q^-(u-v)\right)$$

$$(7) \qquad \ge \left((A-B^+)\,Q^+(u-v) \mid Q^+(u-v)\right) - \left((A-B^-)Q^-(u-v) \mid Q^-(u-v)\right)$$

(by (i))

$$\ge \gamma\left\{\|Q^+(u-v)\|^2 + \|Q^-(u-v)\|^2\right\}$$

(by (a)).

On the other hand, from (4) we have for every $w\in H$:

$$\|w\|^2 = \|Q^+w + Q^-w\|^2 = \|Q^+w\|^2 + 2\left(Q^+w \mid Q^-w\right) + \|Q^-w\|^2$$

$$\le \|Q^+w\|^2 + 2\|Q^+w\|\,\|Q^-w\| + \|Q^-w\|^2$$

$$\le 2\left\{\|Q^+w\|^2 + \|Q^-w\|^2\right\}.$$

Combining this with (7), we get immediately (6). At this point we are in condition to apply the following:

Thm. of Browder: Let H be a Hilbert space, $T: D(T)\subseteq H \to H$ with $D(T)$ a dense linear subspace and $T = L + G$ with L linear and G possibly non-linear. Suppose further that the following assumptions are satisfied:

(α) G is continuous and maps bounded sets into bounded sets;

(β) L is linear, closed and L^* is the closure of its restriction on $D(L) \cap D(L^*)$;

(γ) T is monotone;

(δ) $\lim_{\|u\| \to \infty} \frac{(Tu|u)}{\|u\|} = +\infty$;

Then the range of T is the whole of H.

In fact, all assumptions of this theorem are satisfied for $T = M$, $L = AR$ and $G = FR$, observing that L^* is the closure of its restriction to $D(L) \cap D(L^*)$ by virtue of (5) and that (δ) follows from the following consequence of (6):

$$(M(u)|u) \geq \left\{ \frac{\gamma}{2} \|u\| - \|M(0)\| \right\} \|u\| .$$

q.e.d.

Remark: By taking $u = 0$ in (6) we get that the solution of $M(v) = 0$ is bounded by the following:

$$\|v\| \leq \frac{2}{\gamma} \|M(0)\| .$$

The above theorem implies the following:

COROLLARY: Let $\Omega \subseteq \mathbb{R}^p$ be a bounded open set with suitably smooth boundary. Let $f : \Omega \times \mathbb{R}^q \to \mathbb{R}^q$ be a function satisfying the following assumptions:

(a) $f(x, \cdot) : \mathbb{R}^q \to \mathbb{R}^q$ is continuous for a.a. $x \in \Omega$;

(b) $f(\cdot, u) : \Omega \to \mathbb{R}^q$ is measurable for all $u \in \mathbb{R}^q$;

(c) $\frac{\partial}{\partial u} f(x,u)$ exists and is symmetric for a.a. $x \in \Omega$ and all $u \in \mathbb{R}^q$;

(d) there exist two symmetric matrices b^-, b^+ such that

$$b^- \leq \frac{\partial}{\partial u} f(x,u) \leq b^+$$

for a.a. $x \in \Omega$, all $u \in \mathbb{R}^q$.

Let $L^2 = L^2(\Omega, \mathbb{R}^q)$ and let $B^\pm : L^2 \to L^2$ be the linear operator defined by the formula

$$B^\pm u(x) = b^\pm u(x).$$

Let $A : D(A) \subseteq L^2 \to L^2$ be a self-adjoint linear operator such that B^+ and B^- commute with A and there is no eigenvalue of A in

$$\bigcup_{i=1}^{q} [\lambda_i^-, \lambda_i^+]$$

where $\lambda_i^\pm$ is the i-th eigenvalue of $b^\pm$. Then the operator equation

$$A(u)(x) = f(x, u(x)) \qquad (x \in \Omega)$$

has a unique solution $u \in L^2$.

The proof is a mere application of the above theorem, modulo some technicality due to the heavy use of spectral theory in order to check all the assumptions of the theorem. The spaces $X^\pm$ are defined by using the spectral resolution of the identity for $b^\pm$ and A, while

$$\gamma = \text{dist}\left(\sigma(A), \bigcup_{i=1}^{q} [\lambda_i^-, \lambda_i^+]\right).$$

The interested reader can see the original paper of Amann (see enclosures).

There are many direct applications of the corollary. We outline only two.

APPLICATION 1: The elliptic system

$$-\sum_{i,j} \frac{\partial}{\partial x_i}\left(a_{ij} \frac{\partial}{\partial x_j} u\right) = f(x,u), \quad u/_{\partial\Omega} = 0$$

has a unique solution provided that f satisfies (a), ..., (b) of corollary and $\bigcup_i [\lambda_i^-, \lambda_i^+]$ does not contains any eigenvalue of the linear part.

APPLICATION 2: The system of Schrödinger equations

$$-\Delta u + V(x) u = f(x,u)$$

has a unique periodic solution provided that V and $f(\cdot, u)$ are periodic and moreover f satisfies (a), ..., (b) of corollary and $\bigcup_i [\lambda_i^-, \lambda_i^+]$ does not contain any eigenvalue of $-\Delta + V(x)$.

Open problems: (1) What happens if A and/or $Q^\pm$ are not symmetric?

(2) What happens if the constant $\gamma = 0$?

(3) What happens if in the corollary we replace $f(x, u(x))$ by $f(x-\tau, u(x-\tau))$ in order to treat the periodic boundary value problem of the functional differential equation $u'' = f(x, u(x-\tau))$?

References

1. H. AMANN: On the unique solvability of semi-linear operator equations in Hilbert space, to appear.

2. H. BRÉZIS: Opérateurs maximaux monotones, North Holland, Amsterdam, 1973.

3. F. BROWDER: Variational boundary value problems for quasi-linear elliptic equations of arbitrary order, Proc. Nat. Acad. Sci. USA 50 (1963), 31–37.

4. ———: Nonlinear operators and nonlinear equations of evolution in Banach space, American Mathematical Soc., Providence 1976.

5. G. DUVAUT and J.L. LIONS: Les inequations en Mécanique et en Physique, Dunod, Paris 1972.

6. R.I. KAČUROVSKII: On monotone operators and convex functional (Russian), Uspehi Mat. Nauk 15 (1960), 213–215.

7. D. KINDERLEHRER and G. STAMPACCHIA: An introduction to variational inequalities and their applications, Academic Press, New York 1980.

8. J.L. LIONS: Quelques méthods de résolution des problèmes aux limites non linéaires, Dunod, Paris 1969.

9. G. MINTY: Monotone (nonlinear) operators in Hilbert space, Duke Math. J. 29 (1962), 341–346.

10. ———: On the solvability of nonlinear functional equations of "monotone" type, Pacific J. Math. 14 (1964), 249–255.

11. E.H. ZARANTONELLO: Solving functional equations by contractive averaging, Techn. Rep. 160, Madison 1960.

ON THE UNIQUE SOLVABILITY OF SEMI-LINEAR OPERATOR EQUATIONS IN HILBERT SPACES

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1. *Introduction*

In this paper we study the unique solvability of semi-linear operator equations of the form

$$(1) \qquad Au = F(u)$$

in a real Hilbert space H, where we suppose that $A : \mathrm{dom}(A) \subset H \to H$ is a self-adjoint linear operator with spectrum $\sigma(A)$ and resolvent set $\rho(A)$ and $F : H \to H$ is a Gateaux differentiable gradient operator. (For slightly more general hypotheses we refer to Section 2.)

Equations of this form occur in a variety of situations, in particular in the theory of differential equations. For example, they can describe nonlinear elliptic boundary value problems, or problems concerning periodic solutions of semi-linear wave equations or Hamiltonian systems of ordinary differential equations, to name a few.

Recently much progress has been made towards a better understanding of the solvability properties of equations of this type. In particular it is known that equation (1) possesses multiple solutions if the nonlinearity F interacts suitably with the spectrum of A (cf. [3] and the bibliography given therein).

In this paper we are concerned with the complementary case, where F does not interact with the spectrum of

A at all. In this case, one expects unique solvability, and, in fact, precisely this will be shown in this paper under rather general hypotheses.

First we recall that in a recent paper the author obtained, as a simple corollary to some general considerations on saddle points, the following existence and uniqueness theorem (cf. [2], Theorem (3.4)]), which contains and generalizes most previously known results of this type.

Theorem: Suppose that there exist real numbers $\nu < \mu$ *such that* $[\nu,\mu] \subset \rho(A)$ *and*

$$(2) \qquad \nu \le \frac{\langle F(u)-F(v),u-v\rangle}{\|u-v\|^2} \le \mu \qquad \forall\, u,v \in H,\ u \ne v.$$

Then the equation $Au = F(u)$ *possesses exactly one solution.*

It should be observed that there is no condition whatsoever concerning the nature of the spectrum of A outside the interval $[\nu,\mu]$. Recently J. Mawhin [10] has given a different proof for the above theorem.

Inspite of its generality, the above Theorem is too restrictive for applications to problems, which describe *systems* of equations. To be more precise, suppose that $H=L^2(\Omega,\mathbb{R}^M)$ for some σ-finite measure space Ω, and F is the Nemytskii operator of some function

$$f : \Omega \times \mathbb{R}^M \to \mathbb{R}^M$$

(satisfying the so-called Carathéodory conditions, cf. Section 3 below). Then, if we suppose that the partial derivative D_2f with respect to $\xi \in \mathbb{R}^M$ exists, condition (2) amounts to

$$(3) \qquad \nu \leq D_2f(\omega,\xi) \leq \mu \qquad \forall\ (\omega,\xi) \in \Omega \times \mathbb{R}^M.$$

Observe that $D_2f(\omega,\xi)$ is a symmetric $M \times M$-matrix, since f is a gradient operator. Hence (3) requires all the eigenvalues of $D_2f(\omega,\xi)$ to lie in $[\nu,\mu]$. However, a much more general and more satisfactory "nonresonance" condition would require the eigenvalues of $D_2f(\omega,\xi)$ to lie in possibly distinct gaps of the spectrum of A.

There are already some results in this direction. The first one concerns the existence of 2π-periodic solutions for the system of ordinary differential equations

$$(4) \qquad -u'' = \operatorname{grad}G(u) + p(t),$$

where $G \in C^2(\mathbb{R}^M, \mathbb{R})$, and $p \in C(\mathbb{R}, \mathbb{R})$ is 2π-periodic. It guarantees the existence of exactly one 2π-periodic solution of (4) if

(5)
- there are two constant symmetric $M \times M$-matrices B^- and B^+ such that
- (i) $B^- \leq \operatorname{grad}G(\xi) \leq B^+ \qquad \forall\ \xi \in \mathbb{R}^M$;
- (ii) there exist integers N_k, $k=1,\dots,M$, such that
  $$N_k^2 < \lambda_k^- \leq \lambda_k^+ < (N_k+1)^2,$$
  where $\lambda_1^\pm \dots \leq \lambda_M^\pm$ are the eigenvalues of $B^\pm$

(and the inequalities in (i) are to be understood in the sense that $B^+ \geq B^-$ means that $B^+ - B^-$ is positive semi-definit).

It should be observed that $\{k^2 \mid k \in \mathbb{N}\}$ is precisely the set of eigenvalues of $-u''$, subject to 2π-periodic boundary conditions (i.e. "the spectrum of A").

Under hypothesis (5) the uniqueness assertion has first been proven by Lazer [8], and Ahmad [1] established afterwards the existence of a solution. Very recently Brown and Lin [6] have given a new proof for this existence and uniqueness theorem, based on a global inverse function theorem.

Latly Mawhin [11] has established the unique solvability (in a class of weak solutions) of the system of semilinear wave equations

$$(6) \qquad u_{tt} - u_{xx} = \operatorname{grad}G(u) + h(t,x) \qquad 0 < x < \pi,\ t \in \mathbb{R},$$

under Dirichlet conditions for the x-variable, that is,

$$u(0,t) = u(\pi,t) = 0 \qquad \forall\ t \in \mathbb{R}$$

and a 2π-periodic condition for the t-variable, that is,

$$u(x,t) = u(x,t+2\pi) \qquad \forall\ t \in \mathbb{R},\ x \in [0,\pi].$$

Here h is supposed to be 2π-periodic in t and $G \in C^2(\mathbb{R}^M)$ satisfies condition (5.i). Condition (5.ii) is replaced by the nonresonance condition

$$\bigcup_{j=1}^{M} [\lambda_j^-,\lambda_j^+] \cap \{j^2-k^2 \mid (j,k) \in \mathbb{N}^* \times \mathbb{Z}\} = \emptyset \ .$$

Observe that in each case (that is, for problems (4) and (6)) the nonresonance conditions are of the form

$$\bigcup_{j=1}^{M} [\lambda_j^-,\lambda_j^+] \subset \rho(A),$$

which apparently generalizes conditions (2) and (3).

It is purpose of this paper to prove a much more general result, which contains all the above theorems as special cases and which is also applicable to cases in which the above methods break down (cf. in particular the remarks in Section 4.C). Our main result is Theorem (2.6), although we prove a somewhat more general (and more technical) theorem, namely Theorem (2.10). A direct application of Theorem (2.6) is given in Theorem (4.5), where we prove a vaste generalization of the above mentioned results of Ahmad, Lazer, ans Brown-Lin.

In Section 3 we study particular situations in which the hypotheses of Theorem (2.6) and (2.10) can be verified. These "semi-abstract" results can be applied to systems of partial differential equations, where the differential operator is a diagonal operator with identical entries. Concrete applications of these results are presented in Sections 4A to 4C.

Our proofs are completely different from the proofs given by the above mentioned authors. In particular, our

main result, Theorem (2.6) rests heavily upon an existence theorem for a class of monotone operators, due to F.E. Browder [5]. Although we use in Theorem (2.10) a Galerkin argument, our approach is different from Mawhin's approach in [11]. In particular we do not impose any compactness hypothesis.

Finally, we like to mention that the author reported on a preliminary version of this paper at a meeting on "Nonlinear Boundary Value Problems" in Trieste, Italy, in June 1980.

2. *The Abstract Results*

Throughout this section we denote by H a real Hilbert space with inner product $\langle .,.\rangle$, and we suppose that

$$\begin{cases} X^+ \text{ and } X^- \text{ are closed vector subspaces} \\ \text{of } H \text{ such that } X^+ \cap X^- = \{o\}. \end{cases}$$

Then we denote by $Q^\pm : X^+ + X^- \to X^\pm$ the projection parallel to $X^\mp$. (Of course, here and in the following, in a given context either the lower or the upper index has to be used throughout.) Finally, we let

$$R := Q^+ - Q^- : X^+ + X^- \to H.$$

and prove the following

(2.1) <u>Lemma</u>: (a) *Q^+ and Q^- are closed.*

(b) *R is a closed bijection onto $X^+ + X^-$, and*

$$R^2 = Q^+ + Q^- = id_{X^+ + X^-} .$$

Proof: (a) Let (x_j) be a sequence in $X^+ + X^-$ such that $x_j \to x$ and $Q^+ x_j \to y$. Then $y \in X^+$ since X^+ is closed, and $Q^- x_j = x_j - Q^+ x_j \to x - y$, where $x - y \in X^-$ by the closedness of X^-. Hence $x = y + (x-y) \in X^+ + X^-$ and $Q^+ x = y$, which proves the closedness of Q^+. Similarly one obtains the closedness of Q^-.

(b) Let (x_j) be a sequence in $X^+ + X^-$ such that $x_j \to x$ and $Rx_j \to y$. Then, since $Q^\pm x_j = 2^{-1}(x_j \pm Rx_j)$, it

follows that $Q^\pm x_j \to 2^{-1}(x \pm y)$. Thus, by (a), $x \in X^+ + X^-$ and $Q^\pm x = 2^{-1}(x \pm y)$, which shows that $Rx = y$. Hence R is closed. It is trivial that $R^2 = Q^+ + Q^-$, and this relation implies the bijectivity of R onto $X^+ + X^-$. □

(2.2) <u>Corollary</u>: *If $X^+ + X^- = H$, then $Q^\pm \in L(H)$ and R is a continuous automorphism of H.*

Proof: This follows from Lemma (2.1) and the closed graph theorem. □

We impose now the additional assumption that

$$\begin{cases} A : dom(A) \subset H \to H \text{ is a self-adjoint linear} \\ \text{operator} \end{cases}$$

Then we obtain the following

(2.3) <u>Lemma</u>: *Suppose $Q^\pm(dom(A) \cap (X^+ + X^-)) \subset dom(A)$. Then*

(a) *the restriction of R to $dom(A) \cap (X^+ + X^-)$ is a bijection, and*

(b) *if $X^+ + X^- = H$, then $L := AR$ is closed with $dom(L) = dom(L^*) = dom(A)$.*

Proof: (a) follows easily from Lemma (2.1.b).

(b) Since, by Corollary (2.2), $R \in L(H)$ and A is closed, L is closed and $dom(L) = dom(A)$, by part (a).

Let $y \in dom(A)$. Then

$$\langle Lx, y\rangle = \langle x, R^* Ay\rangle \qquad \forall\, x \in dom(L),$$

which shows that $y \in dom(L^*)$. On the other hand, if $y \in dom(L^*)$, then

$$|\langle Lx,y\rangle| = |\langle ARx,y\rangle| \leq \alpha \|x\| \leq \alpha\|R^{-1}\|\|\mathbb{R}x\|$$

for some constant α and all $x \in dom(L) = dom(A)$. Consequently, since $R|dom(A)$ is a bijection onto $dom(A)$,

$$|\langle Az,y\rangle| \leq \alpha\|R^{-1}\| \|z\| \qquad \forall\ z \in dom(A),$$

which shows that $y \in dom(A) = dom(L)$. Thus $dom(L^*) = dom(L)$. □

We introduce now a further hypothesis.

F : H → H has a symmetric weak Gateaux derivative F'.
There exist symmetric operators $B^{\pm} \in L(H)$ such that

$$(1) \qquad B^- \leq F'(u) \leq B^+ \qquad \forall\ u \in H,$$

and there is a constant $\gamma > 0$ such that

$$\langle (A-B^-)u,u\rangle \leq -\gamma\|u\|^2 \qquad \forall\ u \in X^- \cap dom(A)$$

and

$$\langle (A-B^+)u,u\rangle \geq \gamma\|u\|^2 \qquad \forall\ u \in X^+ \cap dom(A).$$

Clearly, F' is the weak Gateaux derivative of F iff $F'(u) \in L(H)$ and

$$\lim_{t\to 0} t^{-1}\langle F(u+th)-F(u),v\rangle = \langle F'(u)h,v\rangle$$

for all $u,v,h \in H$. Thus for every $u,v,w \in H$, the mean value theorem implies the existence of a number $t \in (0,1)$ such

that

$$\langle F(u)-F(v),w\rangle = \langle F'(v+t(u-v))(u-v),w\rangle.$$

Consequently,

$$\|F(u)-F(v)\| \leq \sup_{0\leq t\leq 1} \|F'(v+t(u-v))\| \, \|u-v\|$$
$$\leq \max\{\|B^+\|,\|B^-\|\}\|u-v\|$$

for all $u,v \in H$, where the last inequality is a consequence of (1). Thus, in particular, F is bounded, that is, maps bounded sets into bounded sets.

(2.5) <u>*Lemma*</u>: *Suppose that $X^+ + X^- = H$ and that $Q^{\pm}(dom(A)) \subset dom(A)$. Let*

$$M := (A-F)\circ R = L-F\circ R : dom(L) \to H.$$

Then

$$\langle M(u)-M(v),u-v\rangle \geq (\gamma/2)\max\{\|u-v\|^2,\|R(u-v)\|\|u-v\|\}$$

for all $u,v \in dom(L)$.

Proof: Let $u,v \in dom(L)$ be fixed. Then, by the mean value theorem, there exists a number $t \in (0,1)$ such that

$$\langle F(Ru)-F(Rv),u-v\rangle = \langle BR(u-v),u-v\rangle,$$

where $B := F'(Rv+tR(u-v)) \in L(H)$.

Consequently, since $A-B$ is self-adjoint, the above assumption implies

$$\langle M(u)-M(v),u-v\rangle = \langle AR(u-v),u-v\rangle - \langle F(Ru)-F(Rv),u-v\rangle$$

$$=\langle (A-B)[Q^+(u-v)-Q^-(u-v)],\ Q^+(u-v)+Q^-(u-v)\rangle$$
$$=\langle (A-B)Q^+(u-v),\ Q^+(u-v)\rangle - \langle (A-B)Q^-(u-v),Q^-(u-v)\rangle$$
$$\geq \langle (A-B^+)Q^+(u-v),\ Q^+(u-v)\rangle - \langle (A-B^-)Q^-(u-v),Q^-(u-v)\rangle$$
$$\geq \gamma(\|Q^+(u-v)\|^2 + \|Q^-(u-v)\|^2).$$

Now, since

$$\|Q^+w\pm Q^-w\|^2 = \|Q^+w\|^2 \pm 2\langle Q^+w,Q^-w\rangle + \|Q^-w\|^2$$
$$\leq \|Q^+w\|^2 + 2\|Q^+w\|\|Q^-w\| + \|Q^-w\|^2$$
$$\leq 2(\|Q^+w\|^2 + \|Q^-w\|^2)$$

for every $w\in H$, the assertion follows. □

After these preparations we can now prove the following *existence and uniqueness result.*

(2.6) <u>*Theorem*</u>: *Let* $A : \mathrm{dom}(A) \subset H \to H$ *be self-adjoint and suppose that* $F : H \to H$ *has a symmetric weak Gateaux derivative* F' . *Moreover, suppose that*

(i) *there exist symmetric operators* $B^{\pm} \in L(H)$ *such that*

$$B^- \leq F'(u) \leq B^+ \qquad \forall\ u\in H;$$

(ii) *there exist closed vector subspaces* $X^{\pm}$ *of* H *such that* $X^+ \cap X^- = \{o\}$ *and* $H = X^+ + X^-$, *and such that*

(a) *there is a constant* $\gamma > o$ *such that*

$$\langle (A-B^-)u,u\rangle \leq -\gamma\|u\|^2 \qquad \forall\ u\in \mathrm{dom}(A)\cap X^-$$

and

$$\langle (A-B^+)u,u\rangle \geq \gamma\|u\|^2 \qquad \forall\ u\in \mathrm{dom}(A)\cap X^+,$$

(b) $\mathrm{dom}(A)$ *is invariant under the projections* $Q^{\pm} : H \to X^{\pm}$ *parallel to* $X^{\mp}$.

Then the equation $Au = F(u)$ *possesses exactly one solution* u^* *and* $\max\{\|u^*\|,\|Ru^*\|\} \leq (2/\gamma)\|F(o)\|$.

Proof: By Lemma (2.3.a), the equation $Au = F(u)$ is equivalent to the equation $M(u) = o$, where $M = L-F\circ R$. By Lemma (2.5), the map M is strongly monotone, hence coercive, namely,

$$\langle M(u),u\rangle \geq [(\gamma/2)\|u\| - \|M(o)\|]\|u\| \qquad \forall\ u\in \mathrm{dom}(L).$$

Since L is closed and $\mathrm{dom}(L) = \mathrm{dom}(L^*)$, by Lemma (2.3), and $F\circ R$ is continuous and bounded, we can apply a result of F.Browder [5, Théorème 16], which guarantees the existence of $v^*\in\mathrm{dom}(L)$ such that $M(v^*) = o$. The uniqueness follows from the strong monotonicity. Finally, by Lemma (2.5),

$$(\gamma/2)\max\{\|Rv^*\|,\|v^*\|\} \leq \|M(o)\| = \|F(o)\|,$$

and the stated estimate follows from the fact that by Lemma (2.1.b) $u^* = Rv^*$ and $v^* = Ru^*$. □

For practical purposes the assumption that $X^+ + X^- = H$ is sometimes too restrictive. For this reason we prove now a more complicated generalization of Theorem (2.6).

Recall that a closed vector subspace X of H is said to *reduce* A iff A commutes with the orthogonal projection P onto X, that is, iff $AP\supset PA$.

In the following we denote by $P^{\pm}$ the *orthogonal projection* of H onto $X^{\pm}$. Then we consider the following assumption.

(A) *There exists a family of closed vector subspaces H_α, $\alpha \in \mathcal{A}$, of H such that:*

(i) $\{H_\alpha \mid \alpha \in \mathcal{A}\}$ *is directed by inclusion* (that is, for each pair $\alpha, \beta \in \mathcal{A}$ there exists a $\gamma \in \mathcal{A}$ such that $H_\alpha \cup H_\beta \subset H_\gamma$);

(ii) *each H_α reduces A and $\bigcup_\alpha H_\alpha$ is dense in H;*

(iii) *the orthogonal projection $P_\alpha : H \to H_\alpha$ commutes with $P^{\pm}$;*

(iv) $X_\alpha^+ + X_\alpha^- = H_\alpha$, *where* $X_\alpha^{\pm} := X^{\pm} \cap H_\alpha$;

(v) $Q^{\pm}(\mathrm{dom}(A) \cap H_\alpha) \subset \mathrm{dom}(A)$ *for each* $\alpha \in \mathcal{A}$.

We derive first some consequences of the above assumptions.

(2.7) <u>*Lemma*</u>: *Let* $D := \mathrm{dom}(A) \cap \bigcup_\alpha H_\alpha$. *Then D is dense in H and a* <u>*core*</u> *for A, that is, A is the closure of its restriction to D.*

Proof: Since $\mathrm{dom}(A)$ and $\bigcup_\alpha H_\alpha$ are dense in H and each H_α reduces A, it is easy to see that D is dense in H.

Let $u \in \mathrm{dom}(A)$ be arbitrary. Then, by the density of $\bigcup_\alpha H_\alpha$ and the directedness of $\{H_\alpha \mid \alpha \in \mathcal{A}\}$, there exists a sequence (α_j) in $\mathcal{A}$ such that $P_{\alpha_j} u \to u$ and $P_{\alpha_j}(Au) = AP_{\alpha_j} u \to Au$ as $j \to \infty$. This proves the assertion. □

(2.8) <u>*Lemma*</u>: *P_α commutes with $Q^{\pm}$ and R.*

Proof: Let $u \in X^+ + X^-$ be arbitrary. Then $u = Q^+u + Q^-u = P^+y + P^-z$ for some $y, z \in H$. Consequently, $P_\alpha u = P_\alpha Q^+ u + P_\alpha Q^- u = P_\alpha P^+ y + P_\alpha P^- z = P^+ P_\alpha y + P^- P_\alpha z \in X^+ + X^-$, by (A .iii). Thus, by the uniqueness of the decomposition,

$$Q^+ P_\alpha u = P^+ P_\alpha y = P_\alpha P^+ y = P_\alpha Q^+ u$$

and

$$Q^- P_\alpha u = P^- P_\alpha z = P_\alpha P^- z = P_\alpha Q^- u,$$

which shows that $P_\alpha Q^{\pm} \subset Q^{\pm} P_\alpha$. Since $R = Q^+ - Q^-$, the second part of the assertion is now obvious. □

(2.9) <u>*Lemma*</u>: *For each $\alpha \in \mathcal{A}$, there exists exactly one $u_\alpha \in H_\alpha \cap \mathrm{dom}(A)$ such that $Au_\alpha = P_\alpha F(u_\alpha)$, and*
$\max\{\|u_\alpha\|, \|Ru_\alpha\|\} \le (2/\gamma)\|F(0)\|$.

Proof: Let $j_\alpha : H_\alpha \to H$ be the natural injection and observe that $j_\alpha^* = P_\alpha$. Moreover, let $F_\alpha := P_\alpha \circ F \circ j_\alpha$ and observe that $F_\alpha : H_\alpha \to H_\alpha$ is weakly Gateaux differentiable with derivative $F_\alpha'(u) = P_\alpha F'(u) j_\alpha$ for $u \in H_\alpha$. Thus $F_\alpha'(u)$ is symmetric and $B_\alpha^- \le F_\alpha'(u) \le B_\alpha^+$ for each $u \in H_\alpha$, where $B_\alpha^{\pm} = P_\alpha B^{\pm} j_\alpha \in \mathcal{L}(H_\alpha)$ is symmetric. Finally, letting $A_\alpha := A \mid H_\alpha \cap \mathrm{dom}(A) = P_\alpha A j_\alpha$, it follows that

$$\langle (A_\alpha - B_\alpha^-)u, u \rangle \le -\gamma \|u\|^2 \qquad \forall\, u \in X_\alpha^- \cap \mathrm{dom}(A)$$

and

$$\langle (A_\alpha - B_\alpha^+)u, u \rangle \ge \gamma \|u\|^2 \qquad \forall\, u \in X_\alpha^+ \cap \mathrm{dom}(A).$$

Now, let $Q_\alpha^\pm$ be the projection of $H_\alpha = X_\alpha^+ + X_\alpha^-$ onto $X_\alpha^\pm$, parallel to $X_\alpha^\mp$, and let $R_\alpha = Q_\alpha^+ - Q_\alpha^-$. Then Lemma (2.7) implies easily that $Q_\alpha^\pm = Q^\pm | H_\alpha$ and, hence, $R_\alpha = R|H_\alpha$. Thus, $Q_\alpha^\pm(\mathrm{dom}(A_\alpha)) \subset \mathrm{dom}(A_\alpha)$ by (A .v). Finally, since $A_\alpha(u) - F_\alpha(u) = Au - P_\alpha F(u)$ for all $u \in H_\alpha$, the assertion follows by applying Theorem (2.6) to the equation $A_\alpha u = F_\alpha(u)$ in H_α. □

After these preparations we can now prove the following more general *existence and uniqueness theorem*.

(2.10) *Theorem: Let* $A : \mathrm{dom}(A) \subset H \to H$ *be self-adjoint and suppose that* $F : H \to H$ *has a symmetric Gateaux derivative* F'. *Moreover, suppose that*

(i) *there exist symmetric operators* $B^\pm \in L(H)$ *such that*

$$B^- \leq F'(u) \leq B^+ \qquad \forall\ u \in H;$$

(ii) *there exist closed vector subspaces* $X^\pm$ *of* H *such that* $X^+ \cap X^- = \{o\}$ *and such that*

(a) *there is a constant* $\gamma > o$ *such that*

$$<(A-B^-)u,u> \leq -\gamma\|u\|^2 \qquad \forall\ u \in \mathrm{dom}(A) \cap X^-,$$

and

$$<(A-B^+)u,u> \geq \gamma\|u\|^2 \qquad \forall\ u \in \mathrm{dom}(A) \cap X^+,$$

(b) *assumption* (A) *is satisfied.*

Then the equation $Au = F(u)$ *possesses exactly one solution.*

Proof: By Lemma (2.9), there exists, for each $\alpha \in \mathcal{A}$, a unique $u_\alpha \in H_\alpha$ such that

$$Au_\alpha = P_\alpha F(u_\alpha)$$

and $\|u_\alpha\|, \|Ru_\alpha\| \leq c_o := (2/\gamma)\|F(o)\|$. For each $\alpha \in \mathcal{A}$, let

$$U_\alpha := \text{w-cl}\{(u_\beta, Ru_\beta) \in H \times H \mid H_\beta \supset H_\alpha\}$$

where w-cl {...} denotes the weak closure, and observe, that the family $\{U_\alpha \mid \alpha \in \mathcal{A}\}$ has the finite intersection property, since the family $\{H_\alpha \mid \alpha \in \mathcal{A}\}$ is directed by inclusion. Since $U_\alpha \subset \overline{\mathbb{B}}(o,c_o) \times \overline{\mathbb{B}}(o,c_o)$, where $\overline{\mathbb{B}}(o,c_o)$ is the closed ball in H about zero with radius c_o, it follows from the weak compactness of closed bounded convex subsets of $H \times H$, that there exists an element $(u,\tilde{u})$ in $\bigcap_\alpha U_\alpha$. Since by Lemma (2.1.a), R is closed, hence weakly closed, we deduce that $u \in X^+ + X^-$ and $\tilde{u} = Ru$.

Now, for each $v \in H_\alpha$ and each $\beta \in \mathcal{A}$ with $H_\beta \supset H_\alpha$,

$$(2) \qquad <Au_\beta, v> = <F(u_\beta), v> .$$

Consequently, by the boundedness of F and of the set $\{u_\alpha \mid \alpha \in A\}$,

$$|<u_\beta, Av>| \leq \|F(u_\beta)\|\|v\| \leq c_1\|v\| \qquad \forall\ v \in H_\alpha \cap \mathrm{dom}(A),$$

where c_1 is an appropriate constant. Hence, by passing to the limit, it follows that

$$|<u,Av>| \leq c_1\|v\| \qquad \forall\ v \in D.$$

This shows that $u \in \mathrm{dom}((A_D)^*)$, where A_D denotes the restriction of A to D. But, by Lemma (2.7), A equal the closure of A_D, that is, $A = \overline{A_D}$, and, consequently since $(A_D)^* = (\overline{A}_D)^*$, $\mathrm{dom}((A_D)^*) = \mathrm{dom}(A)$, that is, $u \in \mathrm{dom}(A)$.

Now it follows from (2) and the fact that $R^2 v = v$ for all $v \in D$, that

$$o = \langle Au_\beta - F(u_\beta), v\rangle = \langle M(Ru_\beta), v\rangle$$

for all $v \in H_\beta \cap \mathrm{dom}(A)$. Thus, in particular,

$$o = \langle M(Ru_\beta), Ru_\beta\rangle \quad \forall\, \beta \in \mathcal{A},$$

since $R(H_\beta \cap \mathrm{dom}(A)) \subset H_\beta \cap \mathrm{dom}(A)$, by Lemmas (2.3.a) and (2.8). Consequently, by the monotonicity of M,

$$\langle M(Rv), Rv - Ru_\beta\rangle = \langle M(Rv) - M(Ru_\beta), Rv - Ru_\beta\rangle \geq o$$

for all $v \in H_\alpha \cap \mathrm{dom}(A)$, all $\beta \in \mathcal{A}$ with $H_\beta \supset H_\alpha$, and all $\alpha \in \mathcal{A}$. Hence, by passing to the limit, we find that

$$\langle Av - F(v), R(v-u)\rangle = \langle M(Rv), R(v-u)\rangle \geq o$$

for all $v \in D$.

As in the proof of Lemma (2.7) we can find a sequence (α_j) in $\mathcal{A}$ such that

$$u_j := P_{\alpha_j} u \to u, \quad Ru_j = P_{\alpha_j} Ru \to Ru, \text{ and}$$

$$Au_j = P_{\alpha_j} Au \to Au \text{ as } j \to \infty$$

(cf. Lemma (2.8)).

Hence, letting $v := u_j + tRh$ in the above inequality, where $t > o$ and $h \in D$, and observing that $R^2 h = h$ by Lemma (2.1),

$$\langle Au_j - F(u_j + th), Ru_j - Ru + th\rangle + t\langle ARh, Ru_j - Ru + th\rangle \geq o$$

for all $j \in \mathbb{N}$. Consequently, letting $j \to \infty$,

$$t\langle Au - F(u+th), h\rangle + t^2\langle ARh, h\rangle \geq o.$$

Finally, by multiplying this inequality by t^{-1} and letting then $t \to o$, we find that

$$\langle Au - F(u), h\rangle \geq o \quad \forall\, h \in D.$$

Thus, since D is dense in H, $Au = F(u)$.

As for the uniqueness, suppose that there is some $v \in \mathrm{dom}(A)$ with $Av = F(v)$. Then there exists a sequence (α_j) in $\mathcal{A}$ such that $u_j := P_{\alpha_j} u \to u$ and $v_j := P_{\alpha_j} v \to v$ as $j \to o$. Then, by the mean-value theorem, and Lemma (2.8),

$$\begin{aligned} o &= \langle Au - F(u) - [Av - F(v)], R(u_j - v_j)\rangle \\ &= \langle A(u_j - v_j), R(u_j - v_j)\rangle - \langle F'(v + t_j(u-v))(u-v), R(u_j - v_j)\rangle \\ &= \langle (A - B_j)(u_j - v_j), R(u_j - v_j)\rangle + \langle B_j[(u_j - v_j) - (u-v)], R(u_j - v_j)\rangle, \end{aligned}$$

where $t_j \in (o,1)$ and $B_j := F'(v_j + t_j(u-v))$. Since

$$\langle (A - B_j)(u_j - v_j), R(u_j - v_j)\rangle = \langle (A - B_j)R(u_j - v_j), u_j - v_j\rangle$$

the proof of Lemma (2.5) shows that

$$\langle (A-B_j)(u_j-v_j), R(u_j-v_j)\rangle \geq (\gamma/2)\|u_j-v_j\|\|R(u_j-v_j)\|.$$

Since, moreover, $\|B_j\| \leq \max\{\|B^+\|,\|B^-\|\} =: c$, we deduce from the above identity the inequality

$$o \geq \|R(u_j-v_j)\|[(\gamma/2)\|u_j-v_j\| - c\|(u_j-v_j)-(u-v)\|]$$

for all $j\in\mathbb{N}$. Hence, either $R(u_j-v_j) = o$ or

$$\|u_j-v_j\| \leq (2c/\gamma)[\|u_j-u\|+\|v_j-v\|],$$

which shows that $u_j-v_j \to o$ as $j \to o$. Thus $u-v = \lim u_j - \lim v_j = \lim (u_j-v_j) = o$, and the theorem is completely proven. □

It should be remarked that the above uniqueness proof has been motivated by an analogous argument of Mawhin [11].

3. *Semi-Abstract Results*

Throughout this section we let Ω be a σ-finite measure space, and $H := L^2(\Omega,\mathbb{R}^M)$ for some $M \geq 1$.

Recall that a map $f : \Omega \times \mathbb{R}^M \to \mathbb{R}^M$ is said to be a *Carathéodory function* if $f(\omega,.) : \mathbb{R}^M \to \mathbb{R}^M$ is continuous for a.a. $\omega\in\Omega$ and $f(.,\xi) : \Omega \to \mathbb{R}^M$ is measurable for all $\xi\in\mathbb{R}^M$.

In the following we denote by $L_s(\mathbb{R}^M)$ the set of all symmetric linear endomorphisms of $\mathbb{R}^M$, which we

identify canonically with the symmetric $M\times M$-matrices.

Then we impose the following hypotheses:

(H1)

(i) $f : \Omega\times\mathbb{R}^M \to \mathbb{R}^M$ *is a Carathéodory function such that* $f(\omega,.)\in C^1(\mathbb{R}^M,\mathbb{R}^M)$, *with a symmetric derivative* $D_2f(\omega,\xi)\in L_s(\mathbb{R}^M)$ *for a.a.* $\omega\in\Omega$ *and all* $\xi\in\mathbb{R}^M$.

(ii) *There exist two matrices* $b^+,b^-\in L_s(\mathbb{R}^M)$ *such that*

$$b^- \leq D_2f(\omega,\xi) \leq b^+$$

for a.a. $\omega\in\Omega$ *and all* $\xi\in\mathbb{R}^M$.

It should be observed that (H1.i) implies that $f(\omega,.)$ is the gradient of some function $\varphi(\omega,.) : \mathbb{R}^M\to \mathbb{R}$ for a.a. $\omega\in\Omega$.

For every $b\in L(\mathbb{R}^M)$, we define $B\in L(H)$ by

$$(Bu)(\omega) = bu(\omega)$$

for all $u\in H$ and a.a. $\omega\in\Omega$. Then B is said to be the *constant multiplication operator* induced by b. It is clear, and that $\sigma(B) = \sigma_p(B) = \sigma(b)$, where $\sigma(.)$ denotes the spectrum and $\sigma_p(.)$ the point spectrum.

Finally we denote by $B^{\pm}$ the constant multiplication operators induced by $b^{\pm}$, and by $\lambda_1^{\pm} \leq\ldots\leq\lambda_M^{\pm}$ the eigenvalues of $b^{\pm}$, where each eigenvalue is repeated according to its multiplicity. Moreover, we let $\nu := \lambda_1^-$ and $\mu := \lambda_M^+$.

Then we consider the following hypothesis:

$$
\text{(H2)}\quad
\begin{cases}
\text{(i)} & A : \mathrm{dom}(A)\subset H \to H \textit{ is a self-adjoint} \\
& \textit{linear operator.} \\
\text{(ii)} & B^+ \textit{ and } B^- \textit{ commute with } A. \\
\text{(iii)} & \bigcup_{j=1}^{M} [\lambda_j^-,\lambda_j^+] \subset \rho(A).
\end{cases}
$$

We are interested in the solvability of the semi-linear equation

$$Au = f(\omega,u) \quad \text{in } \Omega,$$

that is, we are looking for functions $u\in\mathrm{dom}(A)$ such that $Au = F(u)$, where F denotes the Nemytskii operator of f, that is,

$$F(u)(\omega) := f(\omega,u(\omega))$$

for all $u\in H$ and a.a. $\omega\in\Omega$.

It is well known that (H1) implies that F maps all of H into H, and that F has everywhere a Gateaux derivative F', which satisfies

$$B^- \le F'(u) \le B^+ \qquad \forall\, u\in H.$$

Let $\{e_j^\pm \mid j=1,\dots,M\}$ be orthonormal bases of $\mathbb{R}^M$ such that $e_j^\pm$ is an eigenvector to the eigenvalue $\lambda_j^\pm$ of $b^\pm$. Then $b^\pm$ has the spectral resolution

$$b^\pm = \sum_{j=1}^{M} \lambda_j^\pm (e_j^\pm,\cdot)e_j^\pm$$

and, consequently,

$$(b^\pm\xi,\xi) = \sum_{j=1}^{M} \lambda_j^\pm (e_j^\pm,\xi)^2 \qquad \forall\, \xi\in\mathbb{R}^M,$$

where $(\cdot,\cdot)$ is the Euclidean inner product. Hence, by replacing $b^\pm$ by

$$b_\varepsilon^\pm := \sum_{j=1}^{M} (\lambda_j^\pm \pm \varepsilon_j)(e_j^\pm,\cdot)e_j^\pm \in \mathcal{L}_s(\mathbb{R}^M),$$

where $\varepsilon_j \ge 0$ is sufficiently small, we can assume that the eigenvalues $\lambda_j^\pm$ of $b^\pm$ are pairwise distinct. Then $B^\pm$ has the spectral resolution

$$B^\pm = \sum_{j=1}^{M} \lambda_j^\pm P_j^\pm,$$

where $P_j^\pm$ is the orthogonal projection onto the eigenspace $\ker(B^\pm-\lambda_j)$ of $B^\pm$. Of course, $P_j^\pm$ is the constant multiplication operator induced by the projection $p_j^\pm := (e_j^\pm,\cdot)e_j^\pm : \mathbb{R}^M \to \mathbb{R}e_j^\pm$.

Since A is self-adjoint, it possesses a spectral resolution

$$A = \int_{-\infty}^{\infty} \lambda\, dE_\lambda$$

with a right continuous spectral family $\{E_\lambda \mid \lambda\in\mathbb{R}\}$, and we let

$$E(\alpha,\beta) := \int_\alpha^\beta dE$$

for all $\alpha,\beta\in\rho(A)\cup\{\pm\infty\}$ with $\alpha < \beta$.

Since $B^\pm$ commute with A, it is well known that the projections $P_j^\pm$ commute with the resolution of the identity E_λ, $\lambda\in\mathbb{R}$. Consequently, the self-adjoint operators $A-B^\pm$

have the spectral resolution

(1) $$A-B^{\pm} = \sum_{j=1}^{M} \int_{-\infty}^{\infty} (\lambda-\lambda_j^{\pm})\, dE_\lambda P_j^{\pm} .$$

We define now two orthogonal projections $P^{\pm}$ by

$$P^{+} := \sum_{j=1}^{M} E(\lambda_j^{+},\infty)\; P_j^{+}$$

and

$$P^{-} := \sum_{j=1}^{M} E(-\infty,\lambda_j^{-}) P_j^{-} ,$$

and we let

$$X^{\pm} := P^{\pm}(H).$$

Observe that, by (H2.iii),

(2) $$P^{+} = \sum_{j=1}^{M} E(\lambda_j^{-},\infty) P_j^{+} ,$$

and that

$$\gamma := \operatorname{dist}\Big(\bigcup_{j=1}^{M} [\lambda_j^{-},\lambda_j^{+}] \;,\; \sigma(A)\Big) > 0.$$

Moreover, it follows from (1) that

(3) $$\langle (A-B^{-})u,u\rangle \le -\gamma\|u\|^2 \qquad \forall\, u\in X^{-}\cap \operatorname{dom}(A)$$

and

(4) $$\langle (A-B^{+})u,u\rangle \ge \gamma\|u\|^2 \qquad \forall\, u\in X^{+}\cap \operatorname{dom}(A).$$

We can now easily prove the following existence and uniqueness theorem, without any further restriction upon

the linear operator A.

(3.1) <u>*Theorem*</u>: *Let the hypotheses* (H1) *and* (H2) *be satisfied and suppose that the matrices* b^{+} *and* b^{-} *commute. Then the equation*

$$Au = f(\omega,u) \quad \textit{in} \quad \Omega$$

has exactly one solution.

Proof: In this case it is easily seen that (after a possible renumeration of the eigenvalues of $b^{\pm}$) we can assume that $P_j^{+} = P_j^{-}$ for $j=1,\dots,M$. Hence, by (2), $P^{+} = id_H - P^{-}$, that is, $X^{+}=(X^{-})^{\perp}$. Consequently $Q^{\pm} = P^{\pm}$ and the assumptions of Theorem (2.5) are satisfied, which implies the assertion . □

The following existence and uniqueness theorem shows that we can drop the commutativity requirement for b^{+} and b^{-} if we impose some mild restrictions upon the operator A.

For this purpose we recall some facts from spectral theory (e.g. [7,14]). Let

$$Z := E(\nu,\mu)(H),$$

so that Z is a closed vector subspace of H, which reduces A.

For every $\lambda\in\mathbb{R}$, let $P(\lambda) := E_{\lambda}-E_{\lambda-0}$. Then $P(\lambda) \neq 0$ iff $\lambda\in\sigma_p(A)$, in which case $P(\lambda)$ is the orthogonal projection onto the eigenspace $\ker(\lambda-A)$ of the eigenvalue λ. Then A is said to have a <u>pure point spectrum in</u> (ν,μ) if

$$Z = \overline{\operatorname{span}}\{P(\lambda)Z \mid \lambda\in(\nu,\mu)\}.$$

This is the case iff Z *possesses an orthonormal basis of eigenvectors of* A. Moreover, if A has a pure point spectrum in (ν,μ), then $\sigma(A)\cap(\nu,\mu) = \overline{\sigma_p(A)}\cap(\nu,\mu)$. In *particular,* A *has a pure point spectrum in* (ν,μ) *if* $A|Z$ *is compact, or if* $\sigma(A)\cap(\nu,\mu)$ *consists of finitely many eigenvalues of arbitrary multiplicities.* These special cases are particulary important for applications.

After these preparations we can now prove our main result.

(3.2) *Theorem: Let* (H 1) *and* (H2) *be satisfied. Suppose that* A *commutes with every constant multiplication operator, and that* A *has a pure point spectrum in* (ν,μ). *Then the equation*

$$Au = f(\omega,u) \qquad \textit{in } \Omega$$

has exactly one solution.

Proof: Let $H^- := E(-\infty,\nu)(H)$ and $H^+ := E(\mu,\infty)(H)$. Then H has the orthogonal decomposition $H = H^-\oplus Z\oplus H^+$,

and it is clear that $H^{\pm}\subset X^{\pm}$.

By assumption, Z possesses an orthogonal basis $\mathcal{B}$ of eigenvectors A. Let $\mathcal{A}$ be the set of all finite subsets of $\mathcal{B}$ and for each $\alpha\in\mathcal{A}$, let $Z_\alpha := \operatorname{span}(\alpha)$. Hence Z_α is a finite-dimensional subspace of Z, which reduces A. Finally, let H_α be the orthogonal sum.

$$H_\alpha := H^-\oplus Z_\alpha\oplus H^+$$

for each $\alpha\in\mathcal{A}$. Then, each H_α is a closed vector subspace of H, which reduces A, and $\bigcup_\alpha H_\alpha$ is dense in H. Moreover, the family $\{H_\alpha \mid \alpha\in\mathcal{A}\}$ is directed by inclusion, and the orthogonal projection P_α onto H_α commutes with the projections $P^{\pm}$ defined above.

Since $\{e_j^+ \mid j=1,\dots,M\}$ and $\{e_j^- \mid j=1,\dots,M\}$ are orthonormal bases for $\mathbb{R}^M$, there exists a unitary operator $\tilde U\in\mathcal{L}(\mathbb{R}^M)$, such that $\tilde U e_j^+ = e_j^-$, $j=1,\dots,M$. It follows easily that $p_j^- = \tilde U p_j^+ \tilde U^{-1}$, $j=1,\dots,M$. Thus, denoting by U the constant multiplication operator induced by $\tilde U$, we find that $U\in\mathcal{L}(H)$ is unitary and that $P_j^- = UP_j^+U^{-1}$, $j=1,\dots,M$. Since A commutes with U, by assumption, we deduce that

$$P^- = U\Big(\sum_{j=1}^{M} E(-\infty,\lambda_j^-]P_j^+\Big)U^{-1},$$

or, by (2) and the fact, that $E(-\infty,\lambda_j^-) = id_H - E(\lambda_j^-,\infty)$ for $j=1,\dots,M$, that

$$P^- = U[id_H - P^+]U^{-1}. \tag{5}$$

Let $Z_\alpha^\pm := P^\pm(Z_\alpha)$ and observe that $Z_\alpha^\pm \subset Z_\alpha$, since Z_α reduces A and $P^\pm$ commute with A. Moreover,

$$X_\alpha^+ := X_\alpha^+ \cap H_\alpha = Z_\alpha^+ \oplus H^+$$

and

$$X_\alpha^- := X^- \cap H_\alpha = H^- \oplus Z_\alpha^- .$$

Since $Z_\alpha^\pm \subset \mathrm{dom}(A)$, it follows from (3) and (4) that $Z_\alpha^+ \cap Z^-{}_\alpha = \{o\}$, hence $X_\alpha^+ \cap X_\alpha^- = \{o\}$. On the other hand, (5) implies that Z_α^- is isomorphic to the orthogonal complement of Z_α^+ in Z_α. Consequently,

$$\dim Z_\alpha^- = \dim Z_\alpha - \dim Z_\alpha^+$$

and thus, Z_α being finite-dimensional, $Z_\alpha^- + Z_\alpha^+ = Z_\alpha$, which implies that

$$X_\alpha^- + X_\alpha^+ = H_\alpha \qquad \forall\ \alpha \in \mathcal{A}.$$

Next we suppose that $u \in X^+ \cap X^-$. Then there exists a sequence (α_j) in $\mathcal{A}$ such that $u_j := P_{\alpha_j} u \to u$ as $j \to \infty$. Since P_{α_j} commutes with $P^\pm$, it follows that $u_j \in X_{\alpha_j}^+ \cap X_{\alpha_j}^- = \{o\}$ for all $j \in \mathbb{N}$. Hence $u=o$, that is, $X^+ \cap X^- = \{o\}$. Thus assumption (A1) is satisfied.

Finally it is immediate from the definition of $X^\pm$ and the fact that A commutes with the constant multiplication operators $P_j^\pm$, that $Q^\pm(\mathrm{dom}(A) \cap H_\alpha) \subset \mathrm{dom}(A)$. Thus we have verified that assumption (A) is fulfilled. Since the other hypotheses have been established above or have been

postulated, respectively. The assertion follows from Theorem (2.10).

It should be observed that we have proven a slightly more general result. Namely, *Theorem (3.2) remains valid if the condition, that* A *commutes with every constant multiplication operator is being replaced by the following conditions :*

There exists a unitary operator $U \in L(H)$ *such that*

$$P_j^- = U P_j^+ U^{-1} \qquad , \ j=2,..,M,$$

and A *commutes with* U.

4. *Some Applications to Differential Equations*

We begin with a simple technical lemma, which can often be used for verifying commutativity properties.

(4.1) ***Lemma***: *Let* $A : \mathrm{dom}(A) \subset H \to H$ *be a closed linear operator in some Hilbert space* H, *and let* D *be a core of* A. *If* $A|D$ *commutes with* $B \in L(H)$, *then* A *commutes with* B.

Proof: Let $u \in \mathrm{dom}(A)$ be arbitrary, and choose a sequence (u_j) in D such that $u_j \to u$ and $Au_j \to Au$. Then

$Bu_j \to Bu$ and $ABu_j = BAu_j \to BAu$. Thus, by the closedness of A, $Bu \in dom(A)$ and $ABu = BAu$. □

In the remainder of this section we apply now the general results of the preceding section to situations, where A is induced by certain differential operators.

A. *Semilinear Elliptic Boundary Value Problems*

Let Ω be a bounded domain in $\mathbb{R}^N$ with a Lipschitz boundary $\partial\Omega$, and let $a_o, a_{ij} \in L^\infty(\Omega)$ with $a_{ij} = a_{ji}$ for $1 \leq i,j \leq N$. Moreover, for $u,v \in H^1(\Omega) := W_2^1(\Omega)$, let

$$a(u,v) := \sum_{i,j=1}^{N} \int_\Omega a_{ij} D_i u D_j v dx + \int_\Omega a_o uv dx.$$

Then a is a continuous symmetric bilinear form on $H^1(\Omega)$.

Suppose now that V is a closed vector subspace of $H^1(\Omega)$ containing the test functions, that is, $\mathcal{D}(\Omega) \subset V$, such that a is semi-coercive on V, that is, there exist constants $\alpha > o$ and $\lambda \geq o$ such that

$$a(v,v) - \lambda \|v\|_o^2 \geq \alpha \|v\|_1^2 \qquad \forall\ v \in V,$$

where $\|\cdot\|_s$ denotes the norm in $H^s(\Omega)$, s=o,1.

We define now a linear operator $A_o : dom(A_o) \subset L^2(\Omega) \to L^2(\Omega)$ by

$$dom(A_o) := \{u \in V \mid a(u,.) \text{ is continuous on } V \text{ with respect to the } L^2(\Omega)\text{-topology}\}$$

and

$$\langle A_o u, v \rangle := a(u,v) \qquad \forall\ v \in V.$$

Then it can be shown (cf. [9]) that A_o is self-adjoint and (due to the compact imdedding of $H^1(\Omega)$ in $L^2(\Omega)$) that it has a compact resolvent.

Finally we define a self-adjoint linear operator with compact resolvent:

$$A : dom(A) \subset H := L^2(\Omega, \mathbb{R}^M) \to H$$

for some $M \geq 1$, by

$$dom(A) := [dom(A_o)]^M \text{ and } A := diag(A_o, \dots, A_o).$$

It is obvious that A commutes with every constant multiplication operator.

We suppose now that the function

$$f : \Omega \times \mathbb{R}^M \to \mathbb{R}^M$$

satisfies hypothesis (H1), and we denote by F the corresponding Nemystkii operator. Then by a *weak solution of the semi-linear elliptic system*

$$(1) \quad \begin{cases} -\sum_{i,j=1}^{N} D_i(a_{ij}(x) D_j u) + a_o u = f(x,u) & \text{in } \Omega \\ u \in V^M \end{cases}$$

we mean a solution of the operator equation

$$Au = F(u)$$

or equivalently, a function $u \in H^1(\Omega, \mathbb{R}^M)$ such that

$$a(u,v) = \int_\Omega \langle f(x,u(x)), v(x)\rangle_{\mathbb{R}^M} dx \qquad \forall\, v \in H^1(\Omega, \mathbb{R}^M) .$$

As an immediate consequence of the above remarks and Theorem (3.1) we obtain the following

(4.2) <u>*Theorem*</u>: *Suppose the set* $\bigcup_{j=1}^{M} [\lambda_j^-, \lambda_j^+]$ *does not contain an eigenvalue of* A_o. *Then the semi-linear elliptic system* (1) *possesses exactly one weak solution.*

It follows from standard elliptic regularity theory that every weak solution of (1) is a classical solution provided the data , that is, $\partial\Omega$, f and the coefficients a_o, a_{ij}, are sufficiently smooth. Finally we refer to [9] for interpretations of (1) for various choices of V.

8. *Nonlinear Schrödinger Equations*

Suppose that $f : \mathbb{R}^N \times \mathbb{R}^M \to \mathbb{R}^M$, where $N,M \geq 1$, satisfies hypothesis (H1) with $\Omega = \mathbb{R}^N$. Then we consider *systems of semi-linear Schrödinger equations*

$$(2) \qquad -\Delta u + V(x)u = f(x,u) \qquad \text{in } \mathbb{R}^N ,$$

where we suppose that the measurable function $V : \mathbb{R}^N \to \mathbb{R}$, the "potential", has the property that $A := -\Delta + V$ is a self-adjoint linear operator in $L^2(\mathbb{R}^N)$ with core $\mathcal{D}(\mathbb{R}^N)$. There are many sufficient conditions for V guaranteeing this, and guaranteeing also that $\sigma(A)$ has "gaps". In particular, if V is an appropriate periodic potential, then there are conditions guaranteeing gaps in the conditions spectrum (e.g.[13]).

Of course, $-\Delta+V$ operates in (2) as a diagonal operator. It is then clear that $A|\mathcal{D}(\mathbb{R}^N)^M$ commutes with every constant multiplication operator. Hence, by Lemma (4.1), A commutes with every constant multiplication operator, thus, in particular, with $B^\pm$. Consequently the following theorem is an immediate consequence of Theorem (3.1) and (3.2).

(4.3) <u>*Theorem*</u>: *Suppose that either the matrices* b^+ *and* b^- *commute, or that* A *has a pure point spectrum in* (ν,μ). *Then the semilinear system of Schrödinger equations* (2) *possesses exactly one solution, provided*

$$\bigcup_{j=1}^{M} [\lambda_j^-, \lambda_j^+] \subset \rho(-\Delta+V)$$

C. *Periodic Solutions of Semilinear Wave Equations*

Let H be a real separable Hilbert space and suppose that

$A : \mathrm{dom}(A) \subset H \to H$ is self-adjoint and has a compact resolvent.

Moreover, suppose that $F \in C(\mathbb{R} \times H, H)$, such that, for some $T > o$, $F(t+T,.) = F(t,.)$ for all $t \in \mathbb{R}$. Then we are interested in the existence of T-periodic solutions for the semi-linear abstract wave equation

(3) $\ddot{u} + Au = F(t,u), \qquad t \in \mathbb{R},$

where the dot denotes the time-derivative. By a T-periodic solution of (3) we mean a function

$$u \in C^2(\mathbb{R}, H) \cap L^2(\mathbb{R}, D(A))$$

such that

$$\ddot{u}(t) + Au(t) = F(t,u(t))$$

and

$$u(t+T) = u(t)$$

for all $t \in \mathbb{R}$, where $D(A)$ is $\mathrm{dom}(A)$ endowed with the graph norm $\|\cdot\|_A$ that is, $\|u\|_A^2 := \|Au\|^2 + \|u\|^2$.

Let now $\mathbb{H} := L^2((o,T),H)$ and

$$\mathrm{dom}(L_o) := \{u \in C^2([o,T],H) \cap L^2((o,T),D(A)) \mid u(o) = u(T), \dot{u}(0) = \dot{u}(t)\},$$

and define a linear operator

$$L_o := \mathrm{dom}(L_o) \subset \mathbb{H} \to \mathbb{H}$$

by

$$L_o u(t) := \ddot{u}(t) + Au(t) \qquad \forall\, t \in [o,T].$$

Then L_o is obviously densely defined, and partial integration shows that L_o is symmetric, that is, $L_o \subset L_o{}^*$. Finally we let

$$L := L_o{}^*$$

and say $u \in \mathbb{H}$ is a *weak T-periodic solution of* (3) *iff*

$$u \in \mathrm{dom}(L) \text{ and } Lu = F(t,u) \text{ in } [o,T],$$

that is, iff $u \in \mathbb{H}$ and

$$\int_o^T u(t)[\ddot{v}(t) + Av(t)]dt = \int_o^T v(t)F(t,u(t))dt$$

for all $v \in \mathrm{dom}(L_o)$. Clearly, every T-periodic solution of (3) is a weak T-periodic solution.

Since A has a compact resolvent, there exists an orthonormal basis $\{\varphi_j \mid j \in \mathbb{N}^*\}$ in H and a sequence $\lambda_j \in \mathbb{R}$ such that $|\lambda_j| \to \infty$ and $A\varphi_j = \lambda_j \varphi_j$ for all $j \in \mathbb{N}^*$ (if $\mathrm{dom}(H) = \infty$). Let

$$\psi_k(t) := \begin{cases} c_k \cos(k\tau t) & \text{for } k \le o \\ c_k \sin(k\tau t) & \text{for } k \ge 1 \end{cases}$$

and $t \in [o,T]$, where $\tau := 2\pi/T$ and $c_k := \sqrt{2/T}$ if $k \neq o$, and $c_o := 1/\sqrt{T}$. Moreover, let

$$\varphi_{jk}(t) := \psi_k(t)\varphi_j, \qquad o \le t \le T.$$

Then it is easily verified that

(4) $\{\varphi_{jk} \mid (j,k) \in \mathbb{N}^* \times \mathbb{Z}\}$

is an orthonormal basis of $\mathbb{H}$. Moreover, $\varphi_{jk} \in \text{dom}(L_o) \subset \text{dom}(L)$ and

(5) $L\varphi_{jk} = \mu_{jk}\varphi_{jk}$,

where

$$\mu_{jk} := \lambda_j - \tau^2 k^2 \qquad \forall\ (j,k) \in \mathbb{N}^* \times \mathbb{Z}$$

It is not difficult to verify that

$$\text{dom}(L) = \{u \in \mathbb{H} \mid \sum_{j,k} |\mu_{jk} \langle u, \varphi_{jk}\rangle|^2 < \infty\}$$

and that

$$Lu = \sum_{j,k} \mu_{jk} \langle u, \varphi_{jk}\rangle \varphi_{jk}$$

for all $u \in \text{dom}(L)$ (where, of course, the summations extend over all $(j,k) \in \mathbb{N}^* \times \mathbb{Z}$). By means of this representation it follows easily that L is symmetric. Hence

$L^* = L_o^{**} \supset L = L_o^* \supset L_o^{**} = L^*$, since L_o^{**} is the smallest closed extension of L_o. Thus, $L^* = L$, that is, L *is self-adjoint*. Moreover, since, by (5), the orthonormal basis (4) of $\mathbb{H}$ consists of eigenfunctions of L, L *has a pure point spectrum, thus*, in particular, $\sigma(L) = \overline{\sigma_p(L)}$, where

$$\sigma_p(L) = \{\lambda_j - \tau^2 k^2 \mid (j,k) \in \mathbb{N}^* \times \mathbb{Z}\}.$$

Finally, suppose that $B \in \mathcal{L}(H)$ commutes with A, and let

$$(\mathbb{B}u)(t) := Bu(t) \qquad \forall\ t \in [o,T]$$

and all $u \in \mathbb{H}$. Then $\mathbb{B} \in \mathcal{L}(\mathbb{H})$ and it is obvious that $\mathbb{B}$ commutes with L_o. Thus $\mathbb{B}$ commutes with L, by Lemma (4.1).

Suppose now that Ω is a bounded domain in $\mathbb{R}^N$ with a Lipschitz boundary, and let $H := L^2(\Omega, \mathbb{R}^M)$ for some $M \ge 1$. Moreover A is the self-adjoint linear operator in H defined in subsection A above. Then, by the above remarks, L commutes with every constant multiplication operator on $\mathbb{H} = L^2((o,T), L^2(\Omega, \mathbb{R}^M))$, which we identify canonically with $L^1(Q, \mathbb{R}^M)$, where $Q := (o,T) \times \Omega$.

Finally, we suppose that the Carathéodory function

$$f : Q \times \mathbb{R}^M \to \mathbb{R}^M$$

satisfies hypothesis (H1) (with Ω replaced by Q, of course). Then, by a T-*periodic weak solution* of the variational boundary value problem

$$(6)\quad \begin{cases} \dfrac{\partial^2 u}{\partial t^2} - \sum\limits_{i,j=1}^{N} \dfrac{\partial}{\partial x^i}\left(a_{ij}(x)\dfrac{\partial u}{\partial x^j}\right) + a_o(x)u = f(t,x,u) \\[2ex] u(t,.) \in V^M \end{cases}$$

for the semi-linear system of wave equations we mean a solution of the nonlinear operator equation

$$Lu = F(t,u) \text{ in } (o,T),$$

where $F(t,u(t,.)) = f(t,.,u(t,.))$ for all $t \in (o,T)$ and all $u \in \mathbb{H}$.

(4.4) *Theorem: Suppose that*

$$(7)\qquad \left(\bigcup_{j=1}^{M} [\lambda_j^-,\lambda_j^+]\right) \cap \overline{\{\lambda_j - \tau^2 k^2 \mid (j,k) \in \mathbb{N}^* \times \mathbb{Z}\}} = \phi .$$

Then the variational boundary value problem (6) *for the semi-linear system of wave equations possesses exactly one* T-*periodic weak solution.*

Proof: Due to the above considerations, the assertion follows directly from Theorem (3.2) (with A replaced by L, and Ω replaced by Q). □

In general, $\sigma(A)$, hence $\sigma_p(L)$, will not be known explicitly. But, if Ω is the N-dimensional cube $(o,\pi)^N$ and A is induced by the Laplace operator $-\Delta$ under

Dirichlet boundary conditions (that is, $V = H_o^1(\Omega)$), then it is easily seen that each eigenvalue of A is of the form $m_1^2 + \ldots + m_N^2$, where $m_i \in \mathbb{N}^*$, $i=1,\ldots,N$. Consequently,

$$\sigma_p(L) = \{m_1^2 + \ldots + m_N^2 - \tau^2 k^2 \mid m_i \in \mathbb{N}^*, k \in \mathbb{Z}, i=1,\ldots,N\}.$$

Thus, if $\tau = p/q \in \mathbb{Q}$, that is, T is a rational multiple of 2π, then $\sigma_p(L) \subset q^{-2}\mathbb{Z}$, and it follows that $\sigma(L) = \sigma_p(L)$. Hence in this particular case condition (7) can easily be checked.

In the very special case of the standard wave equation in one space dimension under Dirichlet boundary conditions (that is, in the case N = 1 in our particular example) and for $\tau = 1$, Theorem (4.4) has been obtained by Mawhin [11], by a different method, which uses the fact, that $L|[\mathrm{dom}(L) \cap \ker(L)^\perp]$ has a compact inverse in this case. Since, due to the presence of nonzero eigenvalues of infinite multiplicities, this is not true if N > 1, Mawhin's method does not apply to this more general case.

D. Periodic Solutions of Hamiltonian Systems

In this section we study the problem of the existence of periodic solutions for the Hamiltonian system of ordinary differential equations

(8) $\dot{p} = -H_q(t,p,q) \quad , \quad \dot{q} = H_p(t,p,q)$.

Denoting a generic point of $\mathbb{R}^{2N} = \mathbb{R}^N \times \mathbb{R}^N$ by $x := (p,q)$, with $p,q \in \mathbb{R}^N$, we assume that the Hamiltonian function $H : \mathbb{R} \times \mathbb{R}^{2N} \to \mathbb{R}$ satisfies the following conditions:

(i) $H(t+T,\cdot) = H(t,\cdot)$ *for all* $t \in \mathbb{R}$ *and some* $T > 0$.

(ii) H *has a second derivative* H_{xx} *with respect to* x *such that* $H_{xx} \in C(\mathbb{R} \times \mathbb{R}^{2N}, L(\mathbb{R}^{2N}))$.

(iii) *There exist constant symmetric matrices* $b^{\pm} \in L(\mathbb{R}^{2N})$ *such that*

$$b^- \leq H_{xx}(t,y) \leq b^+ \qquad \forall (t,y) \in \mathbb{R} \times \mathbb{R}^{2N}$$

Then, by denoting by $J \in L(\mathbb{R}^{2N})$ the standard symplectic structure of $\mathbb{R}^{2N}$,

$$J = \begin{pmatrix} 0 & -I_N \\ I_N & 0 \end{pmatrix} ,$$

where I_N is the identity in $\mathbb{R}^N$, the Hamiltonian system (8) takes the form

(9) $\dot{y} = J H_x(t,y)$,

and we are looking for T-periodic solutions to (9).

We shall state the solvability criterion in terms of the

purely imaginary eigenvalues of the matrices $Jb^{\pm}$. For this we need some preparation.

Let $b \in L(\mathbb{R}^{2N})$ be a symmetric matrix and let $i\alpha$, with $\alpha \in \mathbb{R}\setminus\{0\}$, be a purely imaginary eigenvalue of Jb . Then it is easily seen that $-i\alpha$ is also an eigenvalue of Jb . In the following we denote by $P_{i\alpha}$ the eigenprojection onto the eigenspaces of the eigenvalues $i\alpha$ and $-i\alpha$. Then $P_{i\alpha}\mathbb{R}^{2N}$ is a symplectic subspace of $\mathbb{R}^{2N}$ which is invariant under Jb . In particular, the dimension of $P_{i\alpha}\mathbb{R}^{2N}$ is even.

Suppose now first that $i\alpha$ is a simple eigenvalue so that $\dim P_{i\alpha}\mathbb{R}^{2N} = 2$. Then there is a linear symplectic coordinate change in $P_{i\alpha}\mathbb{R}^{2N}$ such that the corresponding Hamiltonian has the following normal form on $\mathbb{R}^2$:

$$h(x,y) = \frac{\alpha}{2}(x^2 + y^2) .$$

The number α (which may be positive or negative) is a symplectic invariant, and we call $i\alpha$ the "positively oriented" eigenvalue of the pair $\pm i\alpha$.

Let now the multiplicity of $i\alpha$ be $r > 1$. Then $\dim P_{i\alpha}\mathbb{R}^{2N} = 2r$. If we denote by $E_{i\alpha}$ the complex eigenspace belonging to the eigenvalue $i\alpha$, then

$$(2i)^{-1} \langle v , J\bar{v} \rangle , \qquad v \in E_{i\alpha} ,$$

defines a nondegenerate Hamiltonian form.If this from has an r_+-dimensional positive and an r_--dimensional negative subspace, where $r_+ + r_- = r$, then we set r_+ of the eigenvalues equal to $i|\alpha|$ and r_- of them equal to $-i|\alpha|$. Then

$$\underbrace{i|\alpha|,\ldots,i|\alpha|}_{r_+\text{-times}} \qquad \underbrace{-i|\alpha|,\ldots,-i|\alpha|}_{r_-\text{-times}}$$

are called the "positively oriented eigenvalues" of the restriction of Jb to $P_{i\alpha}\mathbb{R}^{2N}$. If this restriction is symplectically diagonizable, then there is a symplectic coordinate change which puts the corresponding Hamiltonian into the following normal form on $\mathbb{R}^{2N}$:

$$h(x,y) = \frac{|\alpha|}{2}\sum_{j=1}^{r_+}(x_j^2+y_j^2) - \frac{|\alpha|}{2}\sum_{j=1}^{r_-}(x_{r_++j}^2+y_{r_++j}^2) \quad .$$

If the restriction of Jb to $P_{i\alpha}\mathbb{R}^{2N}$ is symplectically diagonizable for every pair of purely imaginary eigenvalues $\pm i\alpha$, then we say "that imaginary part of Jb is symplectically diagonizable". It is known that this is the case if the quadratic Hamiltonian $2^{-1}\langle bx,x\rangle$ is definite or if all the purely imaginary eigenvalues of Jb are simple. For more details we refer to [12].

After these preparations we can now state the following existence and uniqueness result, where by $[M]$ we denote the cardinality of the finite set M.

(4.5) *Theorem: Suppose that the imaginary parts of* $Jb^{\pm}$ *are symplectically diagonizable and that* $\sigma(Jb^{\pm})\cap i\tau\mathbb{Z}=\phi$, *where* $\tau:=2\pi/T$. *Denote by* $S(b^{\pm})$ *the set of all positively oriented purely imaginary eigenvalues of* $Jb^{\pm}$. *Then the Hamiltonian system* (8) *possesses exactly one* T-*periodic solution if*

$$[i\alpha\in S(b^+)\,|\,\tau j<\alpha<\tau(j+1)]-[i\alpha\in S(b^+)\,|\,-\tau(j+1)<\alpha<-\tau j]$$
$$= [i\alpha\in S(b^-)\,|\,\tau j<\alpha<\tau(j+1)]-[i\alpha\in S(b^-)\,|\,-\tau(j+1)<\alpha<-\tau j]$$

for all $j\in\mathbb{N}$.

Proof: We let $H:=L^2((0,T),\mathbb{R}^{2N})$ and define $A:\mathrm{dom}(A)\subset H\to H$ by

$$\mathrm{dom}(A):=\{u\in H^1((0,T),\mathbb{R}^{2N})\,|\,u(0)=u(T)\}$$

and

$$Au:=-J\dot{u} \quad .$$

Then A is self-adjoint and $\sigma(A)=\sigma_p(A)=\tau\mathbb{Z}$, where each $\lambda\in\sigma(A)$ is an eigenvalue of multiplicity $2N$ (cf. [3]).

We define a nonlinear map $F:H\to H$ by

$$F(u)(t):=H_x(t,u(t)) \quad .$$

Then F is G-differentiable and has a symmetric derivative F' such that

$$(10) \qquad B^- \le F'(u) \le B^+ \qquad \forall u\in H,$$

where $B^{\pm}$ are the constant multiplication operators induced by $b^{\pm}$. Writing equation (9) in the form $-J\dot{y}=H_x(t,y)$, we see that every solution of $Au=F(u)$ defines (by T-periodic continuation) a (classical) T-periodic solution of (8). Conversely, every T-periodic solution of (8) defines (by restriction) a solution of $Au=F(u)$. Thus it suffices to show that the equation $Au=F(u)$ is uniquely solvable.

Choose $\beta\in\mathbb{R}_+\setminus\tau\mathbb{Z}$ such that

$$(11) \qquad -\beta\le B^-\le F'(u)\le B^+\le\beta \qquad \forall u\in H.$$

For each $\lambda\in\tau\mathbb{Z}$ let $E(\lambda)$ be the eigenspace of A to the eigenvalue λ , and let

$$H^- := \bigoplus_{\tau j < -\beta} E(\tau j) \quad \text{and} \quad H^+ := \bigoplus_{\tau j > \beta} E(\tau j) .$$

It is not difficult to see (cf. the proof of Lemma (12.3) of [3]) that $E(\tau j) + E(-\tau j)$ is an invariant subspace for $A - B^\pm$, for each $j \in \mathbb{N}$. Thus the restriction of $A - B^\pm$ onto $E(\tau j) + E(-\tau j)$ defines a quadratic form $Q^\pm_{\tau j}$ for all $j \in \mathbb{N}$. Since the imaginary part of $Jb^\pm$ is symplectically diagonizable, it can be shown that the positive Morse index $m(Q^\pm_{\tau j})$ of $Q^\pm_{\tau j}$ (that is, the dimension of a maximal subspace of $E(\tau j) + E(-\tau j)$ on which $Q^\pm_{\tau j}$ is positive definite) is given by

$$m(Q^\pm_{\tau j}) = 2N - 2[i\alpha \in S(b^\pm) \mid \alpha > \tau j] + 2[i\alpha \in S(b^\pm) \mid \alpha < -\tau j]$$

if $j > 0$, and that

$$m(Q^\pm_0) = N + [i\alpha \in S(b^\pm) \mid \alpha > 0] - [i\alpha \in S(b^\pm) \mid \alpha < 0]$$

(cf. Lemma 1 and the proof of Lemma 5 of [4]).

Thus it follows from our hypthesis that

$$(12) \qquad m(Q^+_{\tau j}) = m(Q^-_{\tau j}) \qquad \forall j \in \mathbb{N} .$$

Since $\sigma(Jb^\pm) \cap i\tau\mathbb{Z} = \phi$, it follows that $0 \in \rho(A - B^\pm)$ (cf. [3, Lemma 12.3]). Thus each one of the forms $Q^\pm_{j\tau}$ is nondegenerate.

For each $j \in \mathbb{N}$ with $\tau j < \beta$, we pick a maximal subspace Z^+_j of $E(\tau j) + E(-\tau j)$ on which $Q^+_{\tau j}$ is positive definite, and a maximal subspace Z^-_j on which $Q^-_{\tau j}$ is negative definite. Thus it follows from (12) and the nondegeneracy of the quadratic

forms $Q^\pm_{\tau j}$ that

$$\dim Z^+_j + \dim Z^-_j = \dim(E(\tau j) + E(-\tau j)) .$$

Since, by (10), $A - B^- \geq A - B^+$, it follows that $Z^+_j \cap Z^-_j = \{0\}$. Hence $E(\tau j) + E(-\tau j)$ is the direct sum of Z^+_j and Z^-_j .

Finally, let

$$Z^\pm := H^\pm + \sum_{0 \leq j < \beta} Z^\pm_j .$$

Then $Z^+ \cap Z^- = \{0\}$ and $Z^+ + Z^- = H$. Moreover, there exists a $\gamma > 0$ such that

$$\langle (A-B^-)u,u \rangle \leq -\gamma\|u\|^2 \qquad \forall u \in Z^- \cap \mathrm{dom}(A)$$

and

$$\langle (A-B^+)u,u \rangle \geq \gamma\|u\|^2 \qquad \forall u \in Z^+ \cap \mathrm{dom}(A)$$

Finally it is obvious that the projections $Q^\pm : H \to Z^\pm$, parallel to $Z^\mp$, leave $\mathrm{dom}(A)$ invariant. Hence the assertion follows now from Theorem (2.6) . □

(4.6) *Corollary. Let the hypotheses of Theorem (4.5) be satisfied and suppose, in addition, that H is independent of t , that is, that the Hamiltonian system (8) is autonomous. Then the unique solution of (8) is constant in time.*

Proof: Since the Hamiltonian vector field is time independent, with every solution x the function $t \to x(t+s)$ is also a solution for every $s \in \mathbb{R}$. This implies that the equation $Au = F(u)$ is invariant under a strongly continuous unitary representation $U_\sigma : S^1 \to L(H)$ of the circle group S^1 (cf. the proof of Theorem 4 in [4]).

Consequently, if u is a solution to $Au = F(u)$, then the whole orbit $\{U_\sigma u | \sigma \in S^1\}$ consists of solutions However, we have shown above that $Au = F(u)$ has precisely one solution u. Hence the orbit $\{U_\sigma u | \sigma \in S^1\}$ consists of the point u alone, which means that u is independent of t. □

(4.7) *Remark*: Suppose now that the Hamiltonian $H(t,x)$ is of the special form

$$H(t,x) = \frac{1}{2}|p|^2 + V(t,q) .$$

Then the Hamiltonian system (8) is equivalent to the second order system

$$-\ddot{u} = V_q(t,u) .$$

Suppose also that there are symmetric matrices $c\pm \in L(\mathbb{R}^N)$ such that

$$c^- \leq V_{qq}(t,y) \leq c^+ \qquad \forall (t,y) \in \mathbb{R}\times\mathbb{R}^N .$$

Then, letting

$$b^\pm := \begin{pmatrix} I_N & 0 \\ 0 & c^\pm \end{pmatrix} \in L(\mathbb{R}^{2N}) ,$$

it follows that

$$b^- \leq H_{xx}(t,y) \leq b^+ \qquad \forall (t,y) \in \mathbb{R} \times \mathbb{R}^{2N} .$$

Moreover, if $\lambda_j^\pm$, $1 \leq j \leq N$, are the eigenvalues of $c^\pm$ (which, without loss of generality can be supposed to be simple (cf. the beginning of Section 3)), it is easily verified that the eigenvalues of $Jb^\pm$ are given by $\pm\sqrt{-\lambda_j^\pm}$, $1 \leq j \leq N$. Thus each $\lambda_j^\pm > 0$ corresponds to a pair of purely imaginary eigenvalues of $b^\pm$, and these are the only ones. Consequently, the imaginary parts of $Jb^\pm$ are symplectically diagonizable

and $i\alpha \in S(b^\pm)$ iff $\alpha = \sqrt{\lambda_j^\pm}$ for some $j \in \{1,\dots,N\}$ with $\lambda_j^\pm > 0$. Thus, in particular, $i\alpha \in S(b^\pm)$ implies $\alpha > 0$, and, consequently, the condition of Theorem (4.5) reduces to

$$(13) \qquad [i\alpha \in S(b^+) | \tau j < \alpha < \tau(j+1)] = [i\alpha \in S(b^-) | \tau j < \alpha < \tau(j+1)]$$

for all $j \in \mathbb{N}$.

Now suppose that $\tau = 1$ (that is, we are looking for 2π-periodic solutions) and that there are integers N_j such that

$$N_j^2 < \lambda_j^- \leq \lambda_j^+ < (N_j+1)^2 \qquad j = 1,\dots,N .$$

Then

$$N_j < \sqrt{\lambda_j^-} \leq \sqrt{\lambda_j^+} \leq N_j+1 \qquad j = 1,\dots,N$$

and, consequently, condition (13) is satisfied.

This shows that Theorem (4.5) contains as a very special case the results of Lazer [8] (uniqueness) and Ahmad [1] (existence), and Brown and Lin [6], referred to in the Introduction.

(4.8) *Remark*: It should be observed that Theorem (4.5) is a special case of the following more general result, which has in fact implicitely been proven above:

Suppose that $\sigma(Jb^\pm) \cap i\tau\mathbb{Z} = \phi$, *where* $\tau := 2\pi/T$, *and choose* $\beta \in \mathbb{R}_+ \setminus \tau\mathbb{Z}$ *such that*

$$-\beta \leq B^- \leq F'(u) \leq B^+ \leq \beta.$$

Denote by E *the sum of the eigenspaces of* A *belonging to the eigenvalues in* $(-\beta,\beta)$, *that is,*

$$E = \bigoplus_{-\beta \leq \tau j \leq \beta} E(\tau j).$$

Then the Hamiltonian system (8) *possesses exactly one* T-*periodic solution if*

$$m((A-B^{-})|E) = m((A-B^{+})|E),$$

where $m((A-B^{\pm})|E)$ *denotes the positive Morse index of the quadratic form induced by the restriction of* $A-B^{\pm}$ *to the invariant subspace* E.

However, due to the fact that the interval $(-\beta,\beta)$ contains only finitely many eigenvalues of finite multiplicities, that is, E is finite dimensional, the above formulation is essentially a restatement of Theorem (2.6).

References

[1] S.AHMAD: An existence theorem for periodically pertubed conservative systems. Mich. Math. J., 20 (1973), 385-392.

[2] H. AMANN: Saddle points and multiple solutions of differential equations. Math. Zeitschr. 169 (1979), 127-166.

[3] H. AMANN and E. ZEHNDER: Nontrivial solutions for a class of nonresonance problems and applications to nonlinear differential equations. Annali Scuola Norm. Sup. Pisa, in print.

[4] H. AMANN and E. ZEHNDER: Periodic solutions of asymptotically linear Hamiltonian systems. Manuscripta Math.

[5] F.E. BROWDER: Problèmes Non-Linéaires. Les Presses de l'Université de Montréal, Montréal 1966.

[6] K.J. BROWN and S.S.LIN: Periodically pertubed conservative systems and a global inverse function theorem. Nonlinear Analysis, Theory, Methods Appl., 4 (1980), 193-201.

[7] T. KATO: Pertubation Theory for Linear Operators. Springer Verlag, New York, 1966.

[8] A.C. LAZER: Applications of a lemma on bilinear forms to a problem in nonlinear oscillatons Proc. Amer. Math. Soc., 33 (1972), 89-94.

[9] J.-L. LIONS: Equations différentielles-opérationelles et problèmes aux limites. Springer Verlag, Berlin-Göttingen-Heidelberg, 1961.

[10] J. MAWHIN: Semilinear equations of gradient type in Hilbert spaces and applications to differential equations. Preprint, 1980.

[11] J. MAWHIN: Conservative systems of semi-linear wave equations with periodic-Dirichlet boundary conditions. Preprint, 1980

[12] J. MOSER: New aspects in the theory of stability of Hamiltonian systems. Comm. Pure Appl. Math. 11 (1958), 81-114.

[13] M. REED ans B. SIMON: Methods of Modern Mathematical Physics, Vol.II and IV. Academic Press, New York, 1975 and 1978.

[14] F. RIESZ and B.SZ.-NAGY: Funtional Analysis. Friderich Unger, New York, 1955.