INTERNATIONAL ATOMIC ENERGY AGENCY
UNITED NATIONS EDUCATIONAL SCIENTIFIC AND CULTURAL ORGANIZATION

INTERNATIONAL CENTRE FOR THEORETICAL PHYSICS
34100 TRIESTE (ITALY) - P.O.B. 586 - MIRAMARE - STRADA COSTIERA 11 - TELEPHONES: 224281/2/3/4/5/6
CABLE: CENTRATOM - TELEX 460392-I

SMR/92 - 31

AUTUMN COURSE

ON

VARIATIONAL METHODS IN ANALYSIS AND MATHEMATICAL PHYSICS

20 October - 11 December 1981

MORSE THEORY

H. BERESTYCKI

Analyse Numerique (Tour 55)
Universite Paris VI
4 Place Jussieu
75230 Paris Cedex 05
France

Chapter I. Introduction to Morse Theory - A few examples

In this chapter we start by giving some important examples of equations that appear in applied mathematics to motivate the study of Morse Theory. Then, we present, in a geometrical and heuristic form, its basic ideas.

1.1 - Very basic definitions

We begin by recalling some basic definitions

ⓐ Let E be a Banach space, with norm $\|\cdot\|$. We define:

$\mathcal{L}(E;\mathbb{R}) = \{\varphi : E \to \mathbb{R},\ \varphi \text{ Linear and continuous}\}$

$E^* = \mathcal{L}(E;\mathbb{R})$ provided with the norm

$$\|\varphi\|_* = \sup_{\substack{x\in E\\ \|x\|=1}} \langle \varphi, x\rangle \quad , \varphi \in E^*$$

where $\langle\varphi, x\rangle = \varphi(x)$

E^* is a Banach space and it's called the dual space of E.

We say that $f : E \to \mathbb{R}$ has a derivative $f'(x) \in E^*$ at $x \in E$ if

$$f(x+h) - f(x) = f'(x)\,h + R_x(h) \qquad \forall h \in E.$$

with $\lim\limits_{h\to 0} \dfrac{\|R_x(h)\|}{\|h\|} = 0.$

Exercise 1 -

(i) Show that the derivative of f at x, if it exists, is unique.

(ii) Show that, if $E = \mathbb{R}^N$ and $f'(x)$ exists, then

$$f'(x) = \nabla f(x) = \left(\frac{\partial f}{\partial x_1}(x), \frac{\partial f}{\partial x_2}(x), \dots, \frac{\partial f}{\partial x_N}(x)\right)$$

We have the following classical definitions

Definition.

(i) $f : E \to \mathbb{R}$ is a C^1-function or a function of class C^1 if f has a derivative at all $x \in E$ and the application

$f' : E \to E^*$, $x \mapsto f'(x)$ is continuous.

In this case, we write $f \in C^1(E;\mathbb{R})$ and we define

(ii) Critical point of f - $x \in E$ is a critical point of f if $f'(x) = 0$.

(iii) Critical value of f - c is a critical value of f if there exists a critical point of f such that $f(x) = c$.

(iv) Regular value of f. $a \in \mathbb{R}$ is a regular value of f if it is not a critical value.

Exercise 2 - Let $A : E \to E^*$ be a continuous linear and symmetric operator (ie, $\langle Ax, y \rangle = \langle x, Ay \rangle \quad \forall x, y \in E$). Show that $f(x) = \frac{1}{2}\langle Ax, x \rangle$ is a $C^1(E;\mathbb{R})$ fonction with $f'(x) = Ax \quad \forall x \in E$.

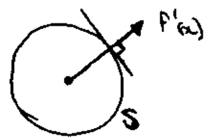
(b) We consider now a situation where f is a real fonction defined on a manifold of the whole space. More precisely, let H be a hilbert space of norm $\|.\|$ and scalar product $(.)$. It is a straightforward consequence of the Riesz's representation theorem that we can identify $H^* \equiv H$.

Let $f \in C^1(H;\mathbb{R})$ and S be the unit sphere dans H,

$S = \{ x \in H, \|x\| = 1 \}$

We consider $\tilde{f} = f|_S$, the restriction of f to S. In chapter IV we will give a precise definition of the derivative of $\tilde{f}$ at $x \in S$.

For the moment, we just say that $\tilde{f}'(x) = 0$ means that $f'(x)$ is orthogonal to S at x. (see figure below)



We express this fact by writing

(1.1) $\tilde{f}'(x) = 0 \iff \exists \lambda \in \mathbb{R}, \; f'(x) = \lambda x$.

(λ is called the Lagrange multiplier of f at x.)

that is, the critical points and critical values of $\tilde{f}$ are, respective eigenvectors and eigenvalues of A.

1.2 - Some examples of variational problems

1.2.1 - A non-linear Dirichlet problem

Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with boundary $\partial\Omega$ "smoo

We want to find a solution of the problem

$$(D) \quad \begin{cases} -\Delta u = g(u) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

where $\Delta u = \sum_{i=1}^{N} \frac{\partial^2 u}{\partial x_i^2}$ is the Laplacian operator.

$g : \mathbb{R} \to \mathbb{R}$ is a (possibly non-linear) fonction.

We consider $G(s) = \int_0^s g(\xi)\, d\xi$ and

$$f(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 - \int_\Omega G(u)$$

where u is a fonction such that $u = 0$ on $\partial\Omega$. We will show that, formally, we have $f'(u) = -\Delta u - g(u)$ and then,

$$f'(u) = 0 \iff (D). \tag{1.2}$$

In fact, set $f(u) = f_1(u) + f_2(u)$, where $f_1(u) = \frac{1}{2}\int_\Omega |\nabla u|^2$

$$f_2(u) = \int_\Omega G(u)$$

Using green's formula we get

$$f_1(u) = \frac{1}{2}\int_\Omega (-\Delta u, u)$$

with $-\Delta$ a linear, symetric operator.

If we suppose that $-\Delta$ is also continuous, from exercise 2 we get $f_1'(u) = -\Delta u$.

For $f_2(u)$ we have

$$f(u+v) - f(u) = \int_\Omega g(u)v + R_u(v) \qquad \text{with}$$

$$R_u(v) = \int G(u+v) - G(u) - g(u)v$$

If $R_u(v) = o(\|v\|)$ then we arrive at (1.2).

Remarks:

(i) This reasoning can be formalized if we take the right space, with the right norm. (and appropriate hypothesis on g)

A careful look at the problem indicates that a reasonable

choice to the norm is $\|u\| = \left(\int_\Omega u(x)^2 + |\nabla u(x)|^2 dx\right)$ and the consequent choice of E can be the completion of $C_0^1(\Omega)$ to this norm. This is precisely the Sobolev space

$$H_0^1(\Omega) \text{"="} \{u \in L^2(\Omega),\ |\nabla u| \in L^2(\Omega),\ u=0 \text{ on } \partial\Omega\}$$

(ii) (D) is called the Euler-Lagrange equation of f, which is known as the energy associated to the problem.

1.2.2 - The Hartree equation

Hartree equation is an opproximate probabilistic model describing the behavior of electrons in an atom.

In the case of the Helium atom $(z=2)$ we have the following equation.

Find $(u,\lambda) \in H^1(\mathbb{R}^3) \times \mathbb{R}$ such that

$$(H)\begin{cases} -\frac{1}{2}\Delta u(x) - \frac{z}{|x|}u(x) + \left(\int_{\mathbb{R}^3} \frac{u^2(y)}{|x-y|}dy\right)u(x) = \lambda u(x) \\ \int_{\mathbb{R}^3} u^2(x)\,dx = 1 \end{cases}$$

We consider $H = H^1(\mathbb{R}^3)$ and $\Sigma = \{u \in H, \int_{\mathbb{R}^3} u^2 = 1\}$. L

$$f(u) = \frac{1}{4}\int_{\mathbb{R}^3} |\nabla u|^2 - \frac{z}{2}\int_{\mathbb{R}^3}\frac{u^2(x)}{|x|}dx + \frac{1}{4}\iint_{\mathbb{R}^3\times\mathbb{R}^3}\frac{u^2(x)\,u^2(y)}{|x-y|}dx\,d$$

$\tilde{f} = f|_{\Sigma}$

Then, by (1.1), we have

$$\begin{cases} (f|_\Sigma)'(u) = 0 \\ u \in \Sigma \end{cases} \Longrightarrow \exists\, \lambda \in \mathbb{R} \text{ such that } \begin{cases} f'(u) = \lambda u \\ u \in \Sigma \end{cases}$$

We leave to the reader the care to verify that

$$f(u+v) - f(u) = \frac{1}{2}\int_{\mathbb{R}^3} \nabla u \nabla v - Z\int_{\mathbb{R}^3} \frac{uv}{|x|}\,dx + \iint_{\mathbb{R}^3\times\mathbb{R}^3} \frac{u^2(y)}{|x-y|}u(x)v(x)$$

$$+ R_u(v)$$

where $R_u(v) = \frac{1}{4}\int_{\mathbb{R}^3} |\nabla v|^2 - \frac{Z}{2}\int_{\mathbb{R}^3} \frac{v^2(x)}{|x|}\,dx + \frac{1}{2}\iint_{\mathbb{R}^3\times\mathbb{R}^3} \frac{u^2(x)\,v^2(y)}{|x-y|}$

$$+ \iint_{\mathbb{R}^3\times\mathbb{R}^3} u(x)u(y)v(x)v(y) + \iint_{\mathbb{R}^3\times\mathbb{R}^3} u(x)v(x) + \frac{1}{4}\iint_{\mathbb{R}^3\times\mathbb{R}^3} \frac{v^2(x)v^2(y)}{|x-y|}$$

Again, we can show that $\lim_{v\to 0} \frac{|R_u(v)|}{\|v\|_H} = 0$ and we use Green's lemma to get

$$f'(u)v = \frac{1}{2}\int_{\mathbb{R}^3} \nabla u \nabla v - Z\int_{\mathbb{R}^3} \frac{uv}{|x|}\,dx + \iint_{\mathbb{R}^3\times\mathbb{R}^3} \frac{u^2(y)}{|x-y|}u(x)v(x)$$

$$= \int_{\mathbb{R}^3} \left(-\frac{1}{2}\Delta u - \frac{Z}{|x|}u + \left(\int_{\mathbb{R}^3} \frac{u^2(y)}{|x-y|}\right)u\right) v\,dx$$

Thus, if (u,λ) is a solution of (H) $\Longleftrightarrow f'(u) = \lambda u \Longleftrightarrow \tilde{f}'(u)$

1.2.3 - Vibrations of Hamiltonian systems

1.2.3.1 - Free vibrations

We consider the following equation, describing the movement of N particles in a mechanical system

$$(1.3) \qquad \ddot{x} + V'(x) = 0 \quad , \; V \in C^1(\mathbb{R}^N, \mathbb{R})$$

where $\ddot{x} = \frac{d^2x}{dt^2}$ is the second derivative of $x(t) : \mathbb{R} \to \mathbb{R}^N$.

We look for periodic solutions of (1.3), ie, solutions satisfy

$$(V) \quad \begin{cases} \ddot{x} + V'(x) = 0 \\ x(0) = x(T) \\ \dot{x}(0) = \dot{x}(T) \end{cases}$$

We consider

$$E = \{x : \mathbb{R} \to \mathbb{R}^N,\ x(t+T) = x(t)\ \forall t \in \mathbb{R},\ \int_0^T |x|^2 + \int_0^T |\dot{x}|^2 < +\infty$$

with $\|x\|_E = \left(\int_0^T |x|^2 + \int_0^T |\dot{x}|^2\right)^{1/2}$

$(E \equiv (H^1(\mathbb{R}/T\mathbb{Z}))^N)$

$$f(x) = \frac{1}{2}\int_0^T |\dot{x}|^2\,dt - \int_0^T V(x(t))\,dt \quad , x \in E.$$

f is the sum of the kinectic energie of the system $\frac{1}{2}\int_0^T |\dot{x}|^2 dt$ with the potential energie $-\int_0^T V(x(t))\,dt$ and is, then, the energie fonction associated to (1.3)

We look for critical points of f in E. We have, as before,

$$(1.4)\quad f'(x)v = \int_0^T \dot{x}\dot{v} - \int_0^T V'(x)v = -\int_0^T (-\ddot{x} - V'(x))v + \dot{x}(T)v(T) - \dot{x}(0)v(0)$$

Therefore, if x is a solution of (V),

$$f'(x)v = 0 \quad \forall v \in E$$

and $f'(x) = 0$.

Conversely, $f'(x)=0$ implies $f'(x)v = 0$ $\forall v \in E$.

In particular if we take $v \in C_0^\infty(\Omega)$ then

$$(1.5)\quad \int_0^T (-\ddot{x} - V'(x))v\,dt = 0$$

and we have (1.3)

It remains to show that $\dot{x}(T) = \dot{x}(0)$. It suffices to take $v \in E$ such that $v(T) = v(0) \neq 0$ and, from (1.4), (1.5),

$$f'(x)v = \dot{x}(T)v(T) - \dot{x}(0)v(0) = 0 \implies \dot{x}(T) = \dot{x}(0).$$

1.2.3.2 _ Forced vibrations

We can also study T-periodic vibrations of a hamiltonian

system submit to an external T-periodic force $F(t)$. We have the problem

$$(V_f)\quad \begin{cases} \ddot{x} + V'(x) = F(t) \\ x(0) = x(T) \\ \dot{x}(0) = \dot{x}(T) \end{cases}$$

and the associated energie

$$f(x) = \frac{1}{2}\int_0^T |\dot{x}|^2 - \int_0^T V(x) + \int_0^T F(t)x(t)\,dt \quad , x \in E$$

(where E is defined as before and $(\,,\,)$ is the scalar product of $\mathbb{R}^N$. So, $\int_0^T (F(t), x(t))$ represents the work contribution of the external forces.)

Exercise 3 - Verify, as before, that x is a critical point of f in E if and only if x is a solution of (V_f).

1.2.3.3 - General Hamiltonian Systems

Let $H \in C^1(\mathbb{R}^N \times \mathbb{R}^N, \mathbb{R})$. We consider the following problem.

Find $(p,q) \in C^1(\mathbb{R}, \mathbb{R}^N \times \mathbb{R}^N)$ solution of

$$(HS)\quad \begin{cases} \dot{p} = -\dfrac{\partial H}{\partial q}(p,q) \\[2mm] \dot{q} = \dfrac{\partial H}{\partial p}(p,q) \\ p(0) = p(T) \\ q(0) = q(T) \end{cases}$$

Exercise 4 - Show that the preceding free vibration example 1.2.3.1 is a particular case of this one, for $H(p,q) = \frac{1}{2}|p|^2 + V(q)$.

We remark that, if (p,q) satisfies (HS), then

$$\frac{d}{dt} H(p(t),q(t)) = \frac{\partial H}{\partial p}(p,q)\,\dot p + \frac{\partial H}{\partial q}(p,q)\,\dot q = 0$$

and there exists $h \in \mathbb{R}$ with

$$(1.6) \qquad H(p(t),q(t)) = h \qquad \forall\, t \in [0,T].$$

H is the energie function associated with (HS) and (1.6) says that any solution vibrates in a constant level energy h.

We have, then, two parameters related to (HS). The period T and the energie h. We will indicated how we can obtain a solution by fixing one of them. Again, we will use variational principles.

Solutions with a given period T.

We will proceed as before. First, we introduce some notations. Let

$z(t) = (p(t), q(t))$

$H'(z) = \left(\frac{\partial H}{\partial p}, \frac{\partial H}{\partial q}\right)$

$\sigma = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}$, ie, $\sigma(p,q) = (-q, p)$

It is easy to see that $\sigma^* = \sigma^{-1} = -\sigma$ and so, (HS) is equivalent to

$$(1.7) \qquad \begin{cases} -\sigma \dot z = H'(z) \\ z(0) = z(T) \end{cases}$$

We consider again

$$E = \left(H^1(\mathbb{R}/_{T\mathbb{Z}})\right)^{2N} = \left\{ x \in C(\mathbb{R}, \mathbb{R}^{2N}),\ x(t+T) = x(t)\ \forall t \in \mathbb{R},\ \int_0^T |x|^2 + \int_0^T |\dot x|^2 < +\infty \right\}$$

and we define

$S(z) = -\sigma \dot z$, $z \in E$

$f(z) = \frac{1}{2} \int_0^T (Sz, z) - \int_0^T H(z)$

We leave to the reader to verify that S is a symetric operator of E and that $f'(z) = Sz - H'(z)$. We have, then,

$$\begin{cases} f'(z) = 0 \\ z \in E \end{cases} \iff z \text{ satisfies } (1.7)$$

Solutions with a given constant energie h.

We utilise here the Maupertius' principle of least action, ie, we search for a critical point of the integral action

$$A(z) = \frac{1}{2}\int_0^1 (Sz, z)\, dt \ , \quad z \in E$$

over $\Sigma_1 = \{ z \in E \ , \ \int_0^1 H(z)\, dt = h \}$

E is defined as before but we take $T = 1$ and Σ_1 is supposed to be non-void.

We know that

$$(A|_{\Sigma_1})'(z) = 0 \iff \exists\ \lambda \in \mathbb{R} \text{ such that } A'(z) = \lambda H'(z)$$

which is equivalent to

$$\dot{z}(t) = \lambda\, \sigma\, H'(z)(t) \qquad \forall\, t \in [$$
$$z(0) = z(T)$$

Let us suppose also that we can garantee that $\lambda \neq 0$. Then we make a time-variable change in order to get (1.7). We consider

$u(t) = z(t/\lambda)$

then, $\dot{u}(t) = \frac{1}{\lambda}\,\dot{\ }(t/\lambda)$ and $\dot{u}(t) = \sigma H'(u(t))$

This implies also that $H(u(t)) = \bar{h} \quad \forall\, t \in [0, T]$. It remains us to prove that $\bar{h} = h$. But $z \in \Sigma_1$ and so,

$$h = \int_0^1 H(z(t))\, dt = \frac{1}{\lambda}\int_0^\lambda H(u(t))\, dt = \bar{h}$$

1.3 - The subject of Morse Theory.

The examples given in § 1.2 should convince the reader of the importance of the study of critical points. An observation of these examples shows us that

(i) We have to study critical points of functionals f defined in a whole space E or on a manifold Σ_1 of it.

(ii) In many situations local minimum (or maximum) does not exist (or gives us trivial solutions) and we want to search for critical points of "minimax" type.

Then, natural questions arise:

1 - How many critical point does f have? How do they look like?

2 - Can the knowledge of some critical points of f lead us to some others?

(For instance, if $f: \mathbb{R} \to \mathbb{R}$, we have the following situation

two local minimum $\Rightarrow$ 1 local maximum

Can it be generalized?)

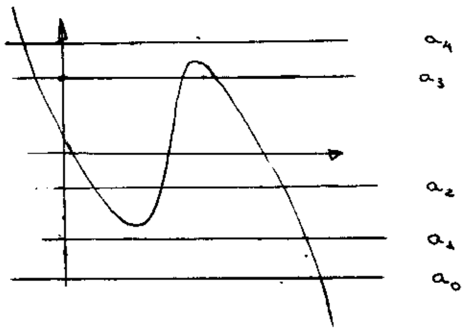
We're going to show how Morse theory answers these questions (and some other ones). This is done essentially by studying the changes of topology occurring in the set $M_a = \{x \in \mathrm{Dom}(f),\ f(x) \leq a\}$ as a varies.

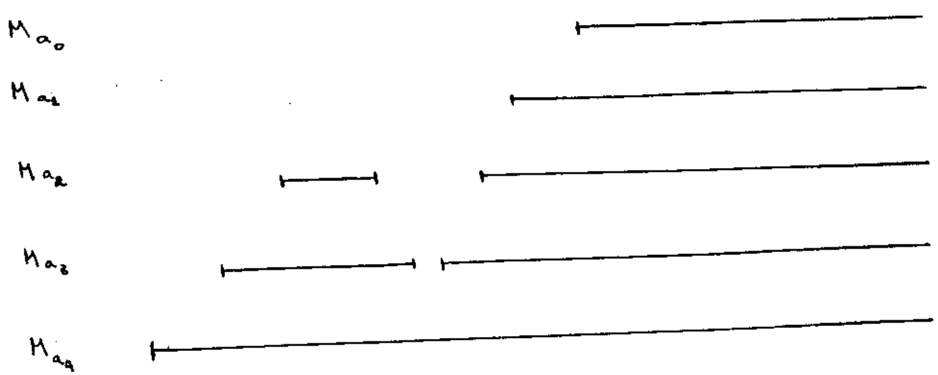
1.3.1 - Morse theory from a geometrical point of view

Let us take some simple cases of functions defined in $\mathbb{R}^1$ or $\mathbb{R}^2$ to see geometrically how do the sets M_a look like.

Example 1 -

$f: \mathbb{R} \to \mathbb{R}$





We see that, from a topological point of view, changes of M_a occur only when a crosses a local minimum or local maximum critical value of f. We can describe these changes by

$M_{a_1} \mapsto M_{a_2}$ attaching of a disjoint 1-cell

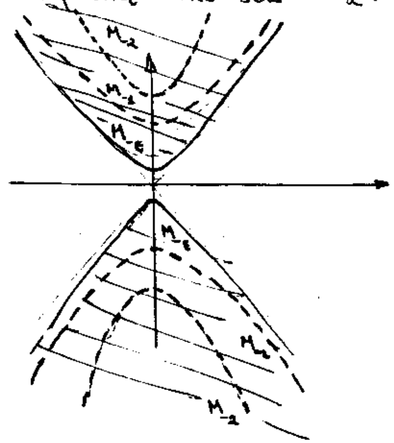
$M_{a_3} \mapsto M_{a_4}$ attaching of a disjoint 1-cell by gluing it to M_{a_3}

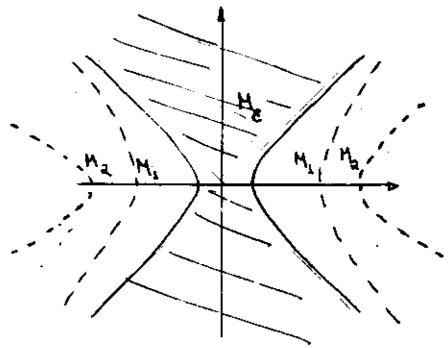
Example 2 -

$f: \mathbb{R}^2 \to \mathbb{R}$, $f(x,y) = x^2 - y^2$.

$(0,0)$ is the only critical point of f, with critical level equal t

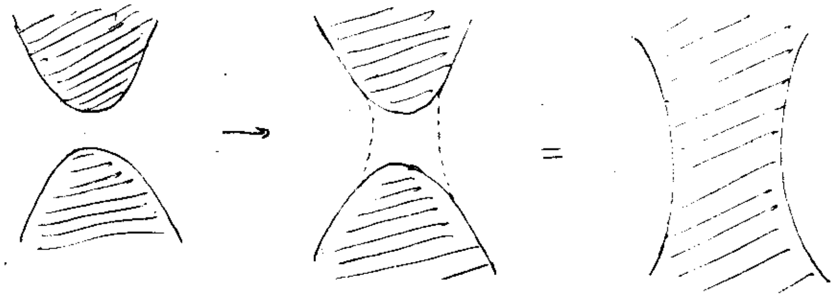
We represent the sets M_a.





As one sees easily, is when we cross the critical value 0 (and only there!) that a topological changes happens. That's the way u going to proceed to determinate critical values of f.

Moreover, we have a way to obtain M_{ε} from $M_{-\varepsilon}$:



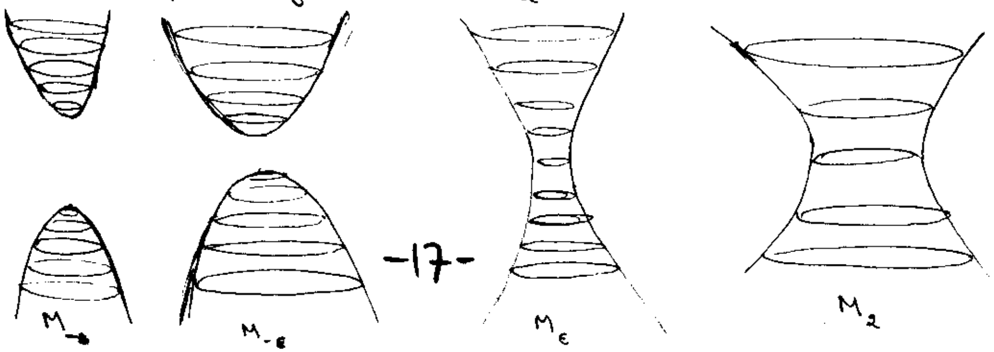
by attaching a $(1,1)$ cell ($(1-1)$ handle) to $M_{-\varepsilon}$, that is, a topological deformation of the square $D^1 \times D^1$  D^1, such that $\partial D^1 \times D^1$ is mapped into the boundary of $M_{-\varepsilon}$.

Example 3.

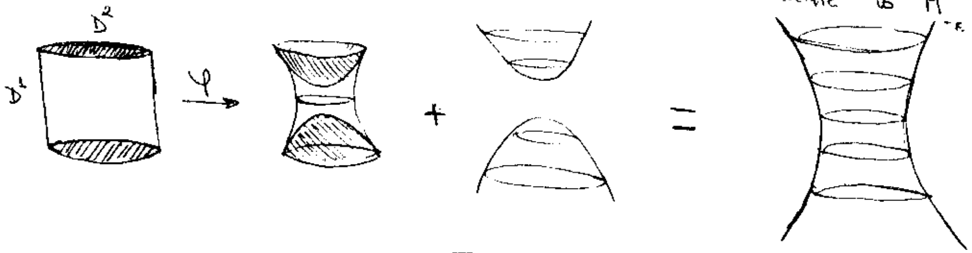
$f: \mathbb{R}^3 \to \mathbb{R}$, $f(x,y,z) = x^2 + y^2 - z^2$

f has a single critical point $(0,0,0)$ with 0 critical level. Besides, at $(0,0,0)$ we have two direction in which f increases, one direction in which it increases.

We represent again the sets M_a.



To obtain M_{ε} from $M_{-\varepsilon}$, attach a $D^1 \times D^2$ handle to $M_{-\varepsilon}$



by gluing $\varphi(\partial D^1 \times D^2) =$  to it.

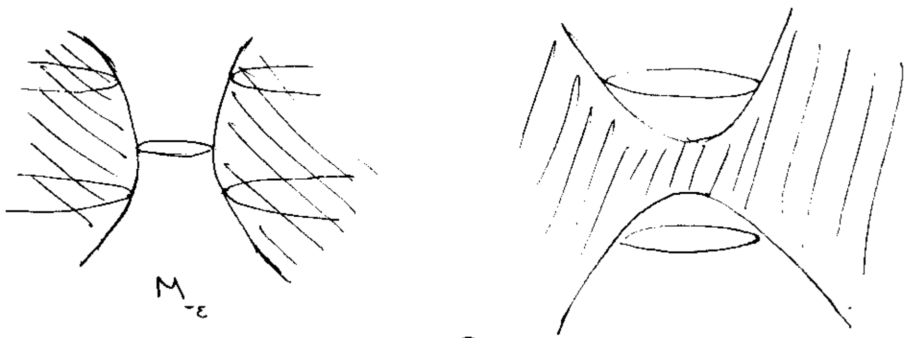
Example 4.

$g: \mathbb{R}^3 \to \mathbb{R}$, $g(x,y,z) = z^2 - x^2 - y^2$.

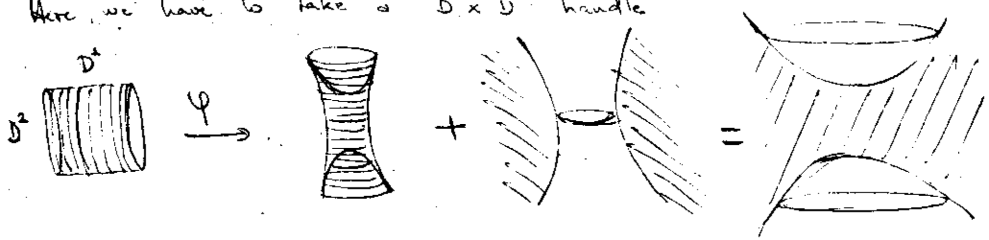
$g = -f$, where f is the function defined in example 3. Then, 0 is the only critical level of f with two directions of decreasing, one of increasing. Furthermore,

$$M_{-\varepsilon} = \{x \in \mathbb{R}^3;\ g(x) \le -\varepsilon\} = \{x \in \mathbb{R}^3,\ f(x) \ge \varepsilon\}$$

$$M_{\varepsilon} = \{x \in \mathbb{R}^3;\ g(x) \le \varepsilon\} = \{x \in \mathbb{R}^3;\ f(x) \ge -\varepsilon\}$$



Here, we have to take a $D^2 \times D^1$ handle



The next exercise is a warning: we must avoid some degenerate situations.

Exercise 5 - Analyze, as before, the topological changes of M_a for the following functions

(i) $f(x) = x^3$
(ii) $f(x,y) = x^3 - y^3$
(iii) $f(x,y) = x^2 - y^3$.

Let us summarize. We take $f: \mathbb{R}^n \to \mathbb{R}$. Then:

(i) As a varies without crossing a critical level of f, nothing happens to the topology of M_a.

(ii) As a crosses a critical level of f, a modification of that topology occurs (except for degenerate situations as )

(iii) When crossing the critical level c, we can obtain $M_{c+\varepsilon}$ from $M_{c-\varepsilon}$ by attaching a handle $D^k \times D^l$ to $M_{c-\varepsilon}$, where

k is the number of decreasing directions of f (= number of negative eigenvalues of $f''(c)$)

l is the number of increasing directions of f (= number of positive eigenvalues of $f''(c)$)

and $k + l = n$

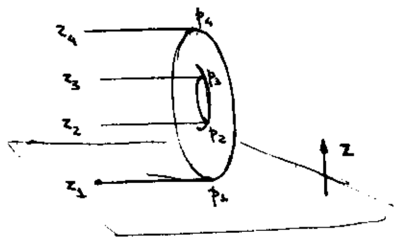
(if $k + l < n$ we are in a degenerate situation!)

The same situation happens when f is defined on a manifold of $\mathbb{R}^n$

Let us see a more sophisticated example.

Example 5 -

Let M be a torus



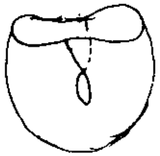
We consider the function $z: M \to \mathbb{R}$, $z(p)$ is the height of the point $p \in M$. We have the following

M_a

ϕ if $a < z_1$

 if $z_1 < a < z_2$

 if $z_2 < a < z_3$

 if $z_3 < a < z_4$

 if $a > z_4$

(i) a crosses z_1

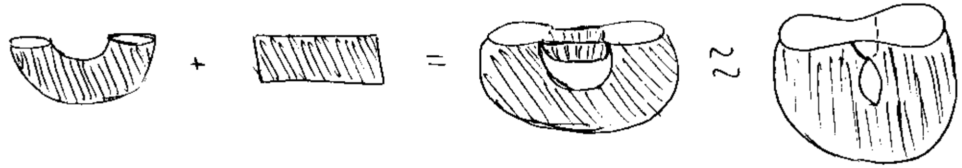
The critical point p_1 has 2 directions of increasing. We attach, then, a $D^0 \times D^2$ handle

(ii) a crosses z_2.

p_2 has 1 direction of increase, 1 direction of decrease $\Rightarrow$ $D^1 \times D^1$ handle 

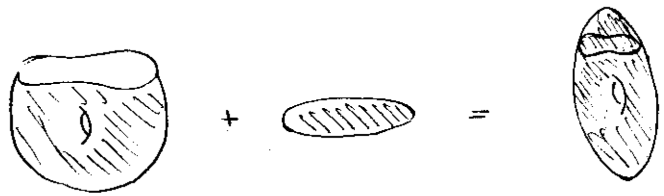
(iii) a crosses z_3

Again, p_3 has 1 direction of increase, 1 direction of decrease $\Rightarrow$ $D^1 \times D^1$ handle 



(iv) a crosses z^4

p_4 has 2 directions of decrease. We attach a $D^2 \times D^0$ handle



That is what Morse theory is all about. In the next chapter we will show that we can put this geometrical observations in a precise and rigorous form. We need to:

(1) Define manifolds, submanifolds

(2) Define properly "attaching a handle".

(3) Show, through a local study (Morse lemma) that any nondegenerate situation can be treated by the model case $f(x_1, x_2 \ldots x_n) = f(0,0 \ldots, 0) + x_1^2 + \ldots + x_k^2 - x_{k+1}^2 \ldots$

(4) Prove the theorem in this model case

(5) Prove it in general.

Chapter II - Deformation Lemmas

We start by showing that no topological changes occurs in M_a while it doesn't cross a critical value. First, we need some preliminairies.

2.1 - Preliminaires

2.1.1 - Review of some basic facts about O.D.E's

Definition - $V : \mathbb{R}^N \to \mathbb{R}^N$ is called a locally Lipschitz function if

$\forall x_0 \in \mathbb{R}^N$, $\exists\ \mathcal{U}(x_0)$ open neighborhood of x_0 and $K > 0$ such that

(2.1) $\quad \| V(x) - V(y) \| \leq K \|x-y\| \quad \forall x, y \in \mathcal{U}(x_0)$

V is Lipschitz if the inequality (2.1) is satisfied for all $x, y \in \mathbb{R}^n$

We have the following fundamental theorem of the ODE's.

Theorem 2.1 - Let V be a locally Lipschitz continuous function.

(i) For any $x_0 \in \mathbb{R}^n$, there is a unique solution of the initial value problem

$$(2.2) \quad \begin{cases} \frac{dx}{dt}(t) = V(x(t)) \\ x(0) = x_0 \end{cases}$$

denoted by $X(t, x_0)$ (X is called a flow)

(ii) this solution is defined on a maximal interval of time $(t$ such that $-\infty \leq t^-(x_0) < 0 < t^+(x_0) \leq +\infty$

Either $t^+(x_0) = +\infty$ ($t^-(x_0) = -\infty$) or

$$\lim_{\substack{t \to t^+(x_0) \\ (\text{resp. } t \to t^-(x_0))}} \| X(t, x_0) \| = +\infty$$

(iii) The application $\eta_t : \mathbb{R}^N \to \mathbb{R}^N$, $\eta_t(x_0) = X(t, x_0)$ is continuous

(iv) $X(-t, X(t, x_0)) = x_0$.

The proof of this result can be found in, eg.,

[1] Theory of Ordinary Differential Equations by E. Coddington, N. Levinson

We also have the

Theorem 2.2 - Suppose $V : \mathbb{R}^N \to \mathbb{R}^N$ localy Lipschitz and that

$$\| V(x) \| \leq C \qquad \forall x \in \mathbb{R}^N$$

Then,

(i) $t^-(x_0) = -\infty$ and $t^+(x_0) = +\infty \quad \forall x_0 \in \mathbb{R}^N$

(ii) $\eta_t : \mathbb{R}^N \to \mathbb{R}^N$ is a homeomorphism for all $t \in \mathbb{R}$.

(iii) $\eta_t(x_0) : [0,1] \times \mathbb{R}^N \to \mathbb{R}^N$ is a continuous deformation of $\mathbb{R}$

$$(t, x_0) \mapsto \eta_t(x_0)$$

from $\eta_0 = I$.

Proof -

(i) Let $x_0 \in \mathbb{R}^N$ and $x(t)$ the solution of (2.2). We have that

$$\|x'(t)\| \leq m \qquad \forall\, t \in (t^-(x_0), t^+(x_0))$$

$$x(t) = x_0 + \int_0^t x'(s)\, ds$$

$$\|x(t)\| \leq \|x_0\| + |t|\, m$$

Therefore, $\|x(t)\|$ is bounded if $t \in (t^-(x_0), t^+(x_0))$ and, from Theorem 2.1, part (ii), $t^+(x_0) = \infty$, $t^-(x_0) = -\infty$.

(ii) $\eta_t : \mathbb{R}^N \to \mathbb{R}^N$ is a continuous function for all $t \in \mathbb{R}$. Besides, Theorem 2.1, part (iv) says that $\eta_t \circ \eta_{-t} = \eta_{-t} \circ \eta_t = Id$ and so, η_t is invertible with inverse η_{-t} continuous.

(iii) It is clear that $\eta_0 = Id$. Also, $\eta(t,x)$ is continuous in $x_0 \in \mathbb{R}^N$ and in $t \in \mathbb{R}$.

<u>Theorem 2.3</u> - Theorems 2.1 and 2.2 hold if $\mathbb{R}^N$ is replaced by E infinite dimensional Banach space.

See also [1] for the proof (which is the same as for the previous case)

2.1.2 - <u>The Palais - Smale condition</u>

Let $f \in C^1(\mathbb{R}^N; \mathbb{R})$. We say that f satisfies the Palais-Smale condition (P.S.) if

$$(P.S.) \begin{cases} \forall\, c_1, c_2 \in \mathbb{R},\ \forall \{x_n\} \subset \mathbb{R}^N \\ -\infty < c_2 \leq f(x_n) \leq c_2 < +\infty \\ \qquad f'(x_n) \to 0 \end{cases} \Longrightarrow \exists \{x_{n_k}\} \subset \{x_n\},\ x_{n_k} \to x$$

If $-\infty$ (resp. $+\infty$) is replaced by 0 then we say that f satisfies $(P.S.^+)$ (resp. $(P.S.^-)$).

<u>Lemma 2.4</u> - Let $f \in C^1(\mathbb{R}^N, \mathbb{R})$ verifies (P.S.) and we suppose $a \in \mathbb{R}$ is a regular value of f. Then, there exists $\varepsilon > 0$, $\delta > 0$ such that

$$(2.3) \qquad a - \varepsilon \leq f(x) \leq a + \varepsilon \Longrightarrow \|f'(x)\| \geq \delta$$

Proof - We argue by contradiction and we suppose that (2.3) is not true. Then, we can take two sequences $\varepsilon_n \to 0$, $\delta_n \to 0$ and $x_n \in \mathbb{R}^N$ such that

$$(2.4) \qquad a - \varepsilon_n \leq f(x_n) \leq a + \varepsilon_n$$

$$(2.5) \qquad \|f'(x_n)\| < \delta_n$$

By (P.S.), we have $x_{n_k} \to x$ and, passing to the limit in (2.4), (2.5), we obtain

$$f(x) = a$$

$$f'(x) = 0$$

which contradicts the fact that a is a regular value of f.

2.2 - Deformation near a regular value.

We introduce the notation $[f \leq a] = \{x \in \mathbb{R}^N,\ f(x) \leq a\}$ and similar ones for others inequalities or equalities. The next theorem shows that the sets $[f \leq b]$ are homeomorphic if b is near a, regular value of $f \in C^2(\mathbb{R}^N; \mathbb{R})$.

Theorem 2.5. ("Local deformation theorem") - Suppose $f \in C^2(\mathbb{R}^N, \mathbb{R})$ satisfies (P.S.) and let $a \in \mathbb{R}$ be a regular value of f. Then there exists a $\bar{\varepsilon} > 0$ such that, for all $\varepsilon \in (0, \bar{\varepsilon})$ we can construct a continuous mapping $\eta : [0,1] \times \mathbb{R}^N \to \mathbb{R}^N$ with the following properties:

(i) $\eta_0(x) = x \quad \forall x \in \mathbb{R}^N$

(ii) $\eta_t(x) = x \quad$ if $f(x) \notin (a - \bar{\varepsilon}, a + \bar{\varepsilon})$

(iii) $f(\eta_t(x)) \geq f(\eta_{t'}(x)) \geq f(x) \quad$ if $0 < t' \leq t \leq 1$.

(iv) $\eta_1([f \leq a - \varepsilon]) = [f \leq a + \varepsilon]$

$\eta_1([f = a - \varepsilon]) = [f = a + \varepsilon]$

Proof - We take $\bar{\varepsilon}, \delta$ as in lemma 2.4 and we consider, for $0 < \varepsilon < \bar{\varepsilon}$,

$$A = [f \leq a - \bar{\varepsilon}] \cup [f \geq a + \bar{\varepsilon}]$$

$$B = [a - \varepsilon \leq f \leq a + \varepsilon]$$

A and B are closed disjoint sets. We define

$$g(x) = \frac{d(x, A)}{d(x, A) + d(x, B)}, \text{ where } d(x, A) = \inf_{y \in A} d(x, y)$$

We leave as an exercise to the reader to check that

- g is well defined and is a Lipschitz continuous function
- $0 \leq g \leq 1$.
- $g(x) = 0 \iff x \in A$.
- $g(x) = 1 \iff x \in B$

Let

$$V(x) = \begin{cases} g(x) \dfrac{f'(x)}{\|f'(x)\|^2} & \text{if } g(x) > 0 \\ 0 & \text{if } g(x) = 0 \end{cases}$$

Then

- $V : \mathbb{R}^N \to \mathbb{R}^N$ is well defined and Lipschitz continuous (prove it)
- $\| V(x) \| = \dfrac{g(x)}{\|f'(x)\|} = \begin{cases} 0 & \text{if } g(x) = 0 \\ \leq \frac{1}{\delta} & \text{if } g(x) > 0 \end{cases}$

Consider the flow

$$(2.6) \quad \begin{cases} \dfrac{dx}{dt} = 2\varepsilon V(x) \\ \\ x(0) = x_0 \end{cases}$$

Theorem 2.2 applies and (2.6) has a unique solution $\eta_t(x_0)$ for all $x_0 \in \mathbb{R}^N$, $t \in \mathbb{R}$.

<u>Claim</u> : $\eta_t : [0,1] \times \mathbb{R}^N \to \mathbb{R}^N$ satisfies the requirements of theorem 2.5

Indeed,

(i) $\eta_0 = Id$.

(ii) If $f(x) \notin (a-\bar{\varepsilon}, a+\bar{\varepsilon})$ then $g(x) = 0$ and $\eta_t(x) = x \quad \forall t \in \mathbb{R}$.

(iii) $\dfrac{d}{dt} f(\eta_t(x)) = f'(\eta_t(x)) \dfrac{d}{dt} \eta_t(x) = 2\varepsilon \, g(\eta_t(x)) \geq 0$

Thus, $f(\eta_t(x))$ is a nondecreasing function of t.

(iv) We have

$$(2.7) \quad f(\eta_t(x)) = f(x) + \int_0^t \frac{d}{ds} f(\eta_s(x))\, ds = f(x) + 2\varepsilon \int_0^t g(\eta_s(x))\, ds$$
$$\leq f(x) + 2\varepsilon t$$

Thus, $\eta_1([f \leq a-\varepsilon]) \subset [f \leq a+\varepsilon]$. Besides, if $f(x) = a-\varepsilon$, then $\eta_t(x) \in B$ for $t \in [0,1]$ and we have equality in (2.7). This proves that $\eta_1([f = a-\varepsilon]) \subset [f = a+\varepsilon]$.

To obtain the inclusions on the other way, we consider the homeomorphisms η_{-t} for $t \in [0,1]$. and the same reasoning shows us that

$$\eta_{-1}([f \leq a+\varepsilon]) \subset [f \leq a-\varepsilon]$$
$$\eta_{-1}([f = a+\varepsilon]) \subset [f = a-\varepsilon]$$

But $\eta_1 \circ \eta_{-1} = Id$ and this ends the proof.

The arguments used in the proof of theorem 2.5 lead us to the following result.

<u>Theorem 2.6</u> - ("Global deformation theorem") - Let $f \in C^2(\mathbb{R}^N; \mathbb{R})$ satisfies (P.S.) and let $a, b \in \mathbb{R}$, $a < b$ such that $[a,b]$ does not contain any critical value of f. Then, there exists $\varepsilon > 0$, $\eta : [0,1] \times \mathbb{R}^N \to \mathbb{R}^N$ continuous satisfying:

(i) $\eta_0 = id$.

(ii) η_t is a homeomorphism for all $t \in [0,1]$

(iii) $\eta_t(x) = x$ if $f(x) \notin [a-\epsilon, b+\epsilon]$, $t \in [0,1]$

(iv) $\eta_t[f \le a] = [f \le b + t(b-a)]$ $\forall t \in [0,1]$

(v) $\eta_1[f \le a] = [f \le b]$

(vi) $\eta_1[f = a] = [f = b]$.

As a corollary of this result we have the

<u>Theorem 2.7</u> - Let $f \in C^2(\mathbb{R}^N; \mathbb{R})$ satisfies (P.S). Assume $a < b$, $[a,b]$ not containing any critical values. Then,

$$[a \le f \le b] \text{ is homeomorphic to } [f = a] \times [0,1]$$

Proof - We leave to the reader the verification that

$$h : [f=a] \times [0,1] \longrightarrow [a \le f \le b]$$
$$(x,t) \longmapsto \eta_t(x)$$

is a homeomorphism.

<u>Remarks</u>

1 - Theorems 2.5, 2.6 and 2.7 apply as well to infinite dimensional Hilbert spaces, with the same proofs as given here. We also have the same results in infinite dimensional Banach spaces. For the

proofs, we have to introduce the notion of a pseudo-gradient vect field, which replaces the gradient vector field considered here. (See, eg,

[2] P. H. Rabinowitz - Variational Methods for Nonlinear Eigenvalue Problems, C.I.M.E. (1974)

Using the pseudo-gradient vector field we can also allow $f \in C^1(E; \mathbb{R})$

2 - A word of aware. In infinite dimensional spaces, $f \in C^1(E; \mathbb{R})$ should be understood as:

- $f'(x) \in E^*$ (the dual space of E) exists for all $x \in E$
- $f'(x)$ is a continuous function with respect to the norm

$$\| f'(x) \|_* = \sup_{\xi \in E} \frac{\langle f'(x), \xi \rangle_{E^*, E}}{\| \xi \|_E}$$

So, for example, in the Palais-Smale condition the convergence $f'(x) \to 0$ means convergence in E^*.

With the aid of the remarks we have made, the next result are still valid on infinite dimensional Banach spaces.

<u>Definition</u> - Let X be a topological space and $B \subset X$. B is called a deformation retract of X if there exists $r : X \to B$ continuous such that $r(X) = B$ and $r(x) = x$ for all $x \in B$. r is called a retract from X to B.

Theorem 2.8 - Suppose $f \in C^2(\mathbb{R}^N, \mathbb{R})$ satisfies (P.S.) and that there exists $a \in \mathbb{R}$ such that f has no critical values in $[a, +\infty)$. Then, $[f \leq a]$ is a deformation retract of $\mathbb{R}^N$.

Proof - We consider g a Lipschitz continuous function of $\mathbb{R}^N$ into $\mathbb{R}$ such that

$$g \equiv 1 \text{ on } [f \geq a]$$
$$g \equiv 0 \text{ on } [f \leq a-\varepsilon]$$

where $\varepsilon > 0$ has been choosed so that $f'(x) \neq 0$ on $[a-\varepsilon \leq f \leq a]$

We argue as before and we consider $\tilde{\eta}_t(x_0)$ the solution of

$$\begin{cases} \dfrac{dx}{dt} = -g(x)\dfrac{f'(x)}{\|f'(x)\|^2} \\ x(0) = x_0 \end{cases}$$

We have that $\left|\frac{d}{dt} f(\tilde{\eta}_t(x_0))\right| \leq 1$. Thus, $f(\tilde{\eta}_t(x))$ is bounded in a finite time t and $\tilde{\eta}_t(x_0)$ is defined for all $t \in \mathbb{R}$.

Moreover, if $b > a$ and $T = b-a$ then,

$$\tilde{\eta}_T[f = b] = [f = a]$$

Let $t^+ = \begin{cases} t & \text{if } t \geq 0 \\ 0 & \text{if } t < 0 \end{cases}$ and $h(x) = \tilde{\eta}((f(x)-a)^+, x)$

It is easy to see that h is continuous and it is a retract from $\mathbb{R}^N$ into $[f \leq a]$.

Existence of critical points of "minimax type"

To end this chapter, we give two important results in the critical point theory which are direct consequences of the deformation theorems we have proved. Taking into account remarks 1 and 2, we state these results as follows.

We consider E a Banach space, $S_\rho = \{x \in E,\ \|x\| = \rho\}$ and $f \in C^1(E; \mathbb{R})$ such that

$$(f1) \quad \begin{cases} f(0) = 0 \\ \exists \rho > 0,\ \alpha > 0 \text{ with } f|_{S_\rho} \geq \alpha \\ \exists e \in E,\ \|e\| > \rho \text{ and } f(e) < \alpha \end{cases}$$

We have the

Theorem 2.9 - (Mountain Pass Lemma) - Let E be a Banach space and $f \in C^1(E; \mathbb{R})$ satisfying (P.S.$^+$) and (f1). If

$$\Gamma = \{g \in C([0,1]; E) \ ;\ g(0) = 0 \text{ and } g(1) = e\}$$

then

$$c = \inf_{g \in \Gamma} \sup_{t \in [0,1]} f(g(t))$$

is a critical value of f.

Proof 1 -

Let $\varepsilon_1 = m\cdot(c, c - f(e)) > 0$ and suppose c is not a critical value of f.

Then we can find $0 < \varepsilon < \bar\varepsilon < \varepsilon_1$ and $\tilde\eta_1$ homeomorphism (just take the solution of (2.6) with -2ε instead of 2ε) such that $\tilde\eta\,[f \le c+\varepsilon] = \tilde\eta\,[f \le c-\varepsilon]$

$$\tilde\eta(x) = x \quad \text{if} \quad f(x) \notin (c-\bar\varepsilon, c+\bar\varepsilon)$$

This implies that, if $g \in \Gamma$ then $g(0), g(1) \notin c-\bar\varepsilon, c+\bar\varepsilon$ and $\tilde\eta\circ g \in \Gamma$.

Let $g_\varepsilon \in \Gamma$ be such that $\sup_{t\in[0,1]} f(g_\varepsilon(t)) \le c+\varepsilon$

Then $\tilde\eta\circ g_\varepsilon \in \Gamma$ and $\sup_{t\in[0,1]} f(\tilde\eta(g_\varepsilon(t)) \le c-\varepsilon$ which contradicts the definition of c.

Proof 2 - We now give a more direct proof of the fact that f has a critical value $c \ge \alpha$, but we don't get the characterization obtained in proof 1.

Suppose again that f has no critical values in $[\alpha, +\infty)$. Then, by (P.S.$^+$), we can find $0<\alpha'<\alpha$ such that f has

no critical values in $[\alpha', +\infty)$. (see Lemma 2.4)

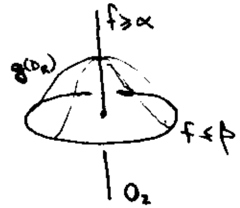
By Theorem 2.8 there exists a retract of E into $A = [f \le \alpha']$.

But $\{0, e\} \subset A$ and $S_e \notin A$, so A is not arcwise connected, which is a contradiction.

We consider now a geometrical situation of "linking type".

Let $f \in C^1(\mathbb{R}^3, \mathbb{R})$, $\alpha, \beta \in \mathbb{R}$, $R > 0$ be such that

$$f(0,0,z) \ge \alpha > \beta \ge f(x,y,0) \qquad \text{if} \quad \begin{array}{l} z \in \mathbb{R} \\ x^2+y^2 = R^2 \end{array}$$



and Let $D_R = \{(x,y,0) \in \mathbb{R}^3;\ x^2+y^2 \le R^2\}$

$$\Gamma = \{ g \in C(D_R, \mathbb{R}^3),\ g|_{S_R} = Id \}$$

Theorem 2.10 - Under these assumptions and if f verifies (P.S.) then f has a critical value $c \ge \alpha$.

Proof - We will assume the following result:

(2.8) $\forall g \in \Gamma,\ g(D_R) \cap O_z \ne \phi$

(whose proof is a simple application of degree theory.)

We define $c = \inf_{g \in \Gamma} \max_{u \in g(D_R)} f(u)$

Then, (2.8) implies that $c \geq \alpha$. Suppose c is not a critical value of f. By Theorem 2.5 we can find a homeomorphism $\tilde{\eta} : \mathbb{R}^3 \to \mathbb{R}^3$ such that $\tilde{\eta}\,[f \leq c+\varepsilon] \subset [f \leq c-\varepsilon]$

$$\tilde{\eta}\big|_{[f \leq \beta]} = Id.$$

Take $g_\varepsilon \in \Gamma$ such that $g_\varepsilon(D_R) \subset [f \leq c+\varepsilon]$. We have that $\tilde{\eta}(g_\varepsilon(D_R)) \subset [f \leq c-\varepsilon]$ with $\tilde{\eta} \circ g_\varepsilon \in \Gamma$, which contradicts the definition of c.

Chapiter III - Local Morse Theory.

Let $\mathcal{U} \subset \mathbb{R}^n$ be an open set and $f \in C^2(\mathcal{U}; \mathbb{R})$. Then, the Hessian of f at x, $H_f(x) = f''(x)$ is the symmetric matrix

$$H_f(x) = \begin{pmatrix} \frac{\partial^2 f}{\partial x_1^2}(x) & \frac{\partial^2 f}{\partial x_1 \partial x_2}(x) & \cdots & \frac{\partial^2 f}{\partial x_1 \partial x_n}(x) \\ \frac{\partial^2 f}{\partial x_1 \partial x_2}(x) & & & \vdots \\ \vdots & & & \vdots \\ \frac{\partial^2 f}{\partial x_1 \partial x_n}(x) & & & \frac{\partial^2 f}{\partial x_n^2}(x) \end{pmatrix}$$

which is a continuous function of x.

We will suppose that $x_0 \in \mathcal{U}$ is a non-degenerate critical point of f, that is

(3.1) $f'(x_0) = 0$

(3.2) $H_f(x_0)$ is invertible

Condition (3.2) is equivalent to

$J_f(x_0)$ (the determinant of $H_f(x_0)$) $\neq 0$.

Before stating Morse Lemma, we give some definitions and prove a preliminaire result.

Definition

1- Local change of variable - A C^1-mapping $y=h(x)$ defined on a neighborhood of $p\in\mathbb{R}^n$ into $\mathbb{R}^n$ is called a local change of coordinates if $J_h(p)\neq 0$.

Then, using the inverse function theorem, we see that a local change of coordinates is a diffeomorphism from a neighborhood of p into a neighborhood of $h(p)$.

2- We consider $\mathcal{M}_n$ the space of $\mathbb{R}^n\times\mathbb{R}^n$ matrices $A=(a_{ij})_{i,j=1,n}$ with norm $\|A\|=\left(\sum_{i,j=1}^{n}|a_{ij}|^2\right)^{1/2}$ and

$\mathcal{M}_n^s\subset\mathcal{M}_n$ the subspace of symmetric $n\times n$ matrices.

$GL(n;\mathbb{R})\subset\mathcal{M}_n$ the subspace of invertible $n\times n$ matrices.

$\mathcal{D}_n\subset\mathcal{M}_n$ the subspace of diagonal $n\times n$ matrices.

We define also $B_\varepsilon(A)=\{B\in\mathcal{M}_n,\ \|B-A\|<\varepsilon\}$.

Lemma 3.1 - Let $A\in\mathcal{D}_n$, $A=\mathrm{diag}(a_1,a_2,\dots,a_n)$ where $a_i=\pm1$. Then, there exists $\varepsilon>0$ and a C^1-mapping $m: B_\varepsilon(A)\cap\mathcal{M}_n^s\to GL(n;\mathbb{R})$ such that $m(A)=I$ and

$${}^t m(B)\, B\, m(B) = A.$$

Proof - We prove the lemma by induction on n.

$n=1$.

We take $m(b)=\sqrt{\frac{a}{b}}$ with b and a having the same sign.

Suppose the result true for $n-1$.

Take $\varepsilon>0$ such that b_{11} has the same sign as a_1 for all $B\in B_\varepsilon(A)\cap\mathcal{M}_n^s$. Let

$$T=\begin{pmatrix} 1 & -\frac{b_{12}}{b_{11}} & \dots & -\frac{b_{1n}}{b_{11}}\\ 0 & 1 & & 0\\ \vdots & & \ddots & \\ 0 & 0 & & 1\end{pmatrix}\frac{1}{\sqrt{|b_{11}|}}$$

Then, ${}^tTBT=\begin{pmatrix} a_1 & 0\dots 0\\ 0 & \\ \vdots & B_1\\ 0 & \end{pmatrix}$, where T, B_1 are C^∞ functions of B.

Hence, ${}^tTBT$ is continuous on B and we can find $\varepsilon, \varepsilon_1>0$ such that

$$\|B-A\|<\varepsilon \Rightarrow \|{}^tT(B)BT(B)-{}^tT(A)AT(A)\|<\varepsilon_1$$

which implies that $\|B_1-\mathrm{diag}(a_2\dots a_n)\|<\varepsilon_1$. By the induction hypothesis, let ${}^tT_1B_1T_1=\mathrm{diag}(a_2\dots a_n)$. We take

$\hat{T}=\begin{pmatrix} 1 & 0\dots 0\\ 0 & \\ \vdots & T_1\\ 0 & \end{pmatrix}$ and $m(B)=T\hat{T}$

We state now the

<u>Morse Lemma</u> - Let $\mathcal{U} \subset \mathbb{R}^n$ be an open set, $f \in C^3(\mathcal{U}, \mathbb{R})$ and $x_0 \in \mathcal{U}$ be a non-degenerate critical point of f. Then, there exists a local change of coordinates $y = h(x)$ such that $h(x_0) = 0$ and

$$f(x) = f(h^{-1}(y)) = f(x_0) + y_1^2 + \cdots + y_k^2 - y_{k+1}^2 - \cdots - y_n^2$$

where k = number of positive eigenvalues of $H_f(x_0)$
$n-k$ = number of negative eigenvalues of $H_f(x_0)$

Proof - We denote by $(.)$ the scalar product of $\mathbb{R}^n$. Taking $g(x) = f(x_0 + x) - f(x)$ we can assume $x_0 = 0$ and $f(x_0) = 0$.

<u>Step 1</u> - We show first that there exists a C^1-function $Q(x) = (q_{ij}(x))_{i,j=1,n}$ from $\mathcal{U}$ to $\mathcal{M}_n^s$ such that

(3.3) $\quad f(x) = (Q(x)\, x, x)$

(3.4) $\quad Q(0) = H_f(0)$

In fact,

$$f(x) = \int_0^1 \frac{d}{dt}(f(tx_1, \ldots, tx_n))\, dt = \int_0^1 \sum_{i=1}^n \frac{\partial f}{\partial x_i}(tx)\, x_i\, dt$$

$$\frac{\partial f}{\partial x_i}(tx) = \int_0^1 \frac{d}{ds}\left(\frac{\partial f}{\partial x_i}(stx)\right) ds = \int_0^1 \sum_{j=1}^n \frac{\partial^2 f}{\partial x_i \partial x_j}(stx)\, t x_j\, ds$$

So, $$f(x) = \sum_{i,j=1}^n \left(\int_0^1 \int_0^1 \frac{\partial^2 f}{\partial x_i \partial x_j}(stx)\, ds\, dt\right) x_i x_j$$

and if $q_{ij}(x) = q_{ji}(x) = \int_0^1 \int_0^1 \frac{\partial^2 f}{\partial x_i \partial x_j}(stx)\, ds\, dt$, then $Q(x)$ is a C^1-mapping satisfying (3.3) and (3.4)

<u>Step 2</u> - $H_f(0)$ is an invertible symmetric matrix. Thus, using spectral theory, we can find a unitary matrix T (${}^tT = T^{-1}$) such that ${}^tT\, H_f(0)\, T = \mathrm{diag}(q_1, \ldots q_n)$. We assume, without loss of generality, that $\mathrm{sign}(q_1) = \mathrm{sign}(q_2) = \cdots = \mathrm{sign}(q_k) = 1$ and $\mathrm{sign}(q_{k+1}) = \cdots = \mathrm{sign}(q_n) = -1$.

Let $\tilde{q}_i = \frac{1}{\sqrt{|q_i|}}$ $i = 1 \ldots n$ and $\tilde{D} = \mathrm{diag}(\tilde{d}_1, \tilde{d}_2 \cdots \tilde{d}_n)$. Then if $\tilde{T} = T\tilde{D}$, we have that ${}^t\tilde{T} H_f(0) \tilde{T} = \mathrm{diag}(a_1, a_2, \ldots, a_n)$ where $a_1 = a_2 = \cdots = a_k = 1$ and $a_{k+1} = \cdots = a_n = -1$.

Therefore, taking $x = \tilde{T}\tilde{x}$ and $\tilde{Q}(\tilde{x}) = {}^t\tilde{T} Q(\tilde{T}\tilde{x})\tilde{T}$, we write

$$f(x) = f(\tilde{T}\tilde{x}) = (Q(\tilde{T}\tilde{x})\tilde{T}\tilde{x}, \tilde{T}\tilde{x}) = (\tilde{Q}(\tilde{x})\tilde{x}, \tilde{x})$$

where $\tilde{Q}(0) = \mathrm{diag}(a_1, a_2 \ldots a_n)$

<u>Step 3</u> - Using step 2, we can restrict ourselves to the case

$f(x) = (Q(x)x, x)$ where $Q(0) = \mathrm{diag}(a_1, \ldots a_n)$

We take a neighborhood $\mathcal{U}$ of 0 such that $\|Q(x) - Q(0)\| < \varepsilon$

sufficiently small and we apply lemma 3.1. Thus, there exists a C^1-function $P(x) = m(Q(x))$ with $P(0) = I$, $P(x)$ inversible and

$$ {}^{t}P(x)\, Q(x)\, P(x) = Q(0) \qquad \forall x \in \mathcal{U} $$

Set $y = h(x) = [P(x)^{-1}]\, x \qquad (x = P(x)\, y)$

It follows that $h(0) = 0$ and

$$ \lim_{t \to 0} \frac{h(tx) - h(0)}{t} = \lim_{t \to 0} P(tx)^{-1} x = x $$

So, $h'(0) = I$ and h is a local change of coordinates. Besides,

$$ f(x) = (Q(x)\, P(x)\, y,\ P(x)\, y) = (Q(0)\, y, y) = y_1^2 + \dots + y_k^2 - y_{k+1}^2 - \dots - y_n^2 . $$

and that ends the proof.