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AUTUMN COURSE

ON

VARIATIONAL METHODS IN ANALYSIS AND MATHEMATICAL PHYSICS

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TOPICS IN CRITICAL POINTS THEORY

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§0. Preliminaries

We list below some background material, we will use later on.

a) Sobolev spaces.

Throughout in the following we will use these notations:

$\mathbb{R}^n$ is the Euclidean n-space; for $x \in \mathbb{R}^n$, $|x| = (\sum x_i^2)^{1/2}$; Ω is an open subset of $\mathbb{R}^n$.

For a multi-index $\beta = (\beta_1, \dots, \beta_n)$, $\beta_i \in \mathbb{N}$, $|\beta| = \beta_1 + \dots + \beta_n$.

$D_i = \partial/\partial x_i$, $D_{ij} = \partial^2/\partial x_i \partial x_j$ and if β is a multi-index

$$D^\beta = \partial^{|\beta|}/\partial x_1^{\beta_1} \partial x_2^{\beta_2} \dots \partial x_n^{\beta_n}$$

The norm in the Banach space $L^p(\Omega)$, $p \geq 1$, will be

denoted by $\|\cdot\|_p$, i.e.

$$\|u\|_p = \left(\int_\Omega |u|^p \right)^{1/p}.$$

We list below some well known facts on L^p spaces. Let $p > 1$ and set $p' = p/p-1$ $\left(\frac{1}{p} + \frac{1}{p'} = 1 \right)$. $L^p(\Omega)$ is reflexive, the dual of $L^p(\Omega)$ being isomorphic to $L^{p'}(\Omega)$. $L^2(\Omega)$ is a Hilbert space, with scalar product

$$(u, v)_2 = \int_\Omega uv$$

When there is no ambiguity, we will write (u,v) for $(u,v)_2$.

If $u \in L^p(\Omega)$ and $v \in L^{p'}(\Omega)$, then $uv \in L^1(\Omega)$ and

$$\int_\Omega uv \leq \|u\|_p \|v\|_{p'} \qquad \text{(Hölder's inequality)}$$

If Ω is bounded and $1 \leq p \leq q$ then $L^q(\Omega) \subset L^p(\Omega)$ and

$$\|u\|_p \leq c_\Omega \|u\|_q$$

If $1 \leq p$ and u_n is a Cauchy sequence in $L^p(\Omega)$ then u_n has a subsequence converging almost everywhere (a.e.) in Ω.

We also indicate with $L^\infty(\Omega)$ the Banach space of bounded functions on Ω with norm

$$\|u\|_\infty = \sup_\Omega |u|$$

Obviously by sup we mean the essential supremum.

The Sobolev spaces $W^{k,p}(\Omega)$ are defined for $p \geqq 1$ and non-negative integer k in the following way: $W^{k,p}(\Omega)$ is the space of all functions u having weak derivatives $D^\beta u$ for all $|\beta| \leqq k$, and $D^\beta u \in L^p(\Omega)$ for all $|\beta| \leqq k$.

$W^{k,p}(\Omega)$ is a Banach space under the norm

$$(0.1) \qquad \|u\|_{k,p} = \sum_{|\beta| \leq k} \|D^\beta u\|_p$$

The space $W_0^{k,p}(\Omega)$ is defined as the closure of $C_0^\infty(\Omega)$ under the norm (0.1)

For $p=2$, the spaces $W^{k,2}(\Omega)$ and $W_0^{k,2}(\Omega)$ are Hilbert spaces. We will write $H^k(\Omega)$, $H_0^k(\Omega)$, for $W^{k,2}(\Omega)$ $W_0^{k,2}(\Omega)$ resp.)

In the following we will interested only with the case in which Ω is a <u>bounded</u> domain of $\mathbb{R}^n$. We will also require that Ω has a Lipschitz continuous boundary $\partial\Omega$. From now on we will always assume these two conditions above, even if sometimes they would be not strictly necessary.

We state now some imbedding theorems concerning the connection between Sobolev spaces $W^{k,p}(\Omega)$ and $L^p(\Omega)$ spaces.

0.1. <u>Theorem</u>. <u>Let</u> $p \geqq 1$, <u>k be a non-negative integer and $\Omega \subset \mathbb{R}^n$. Then:</u>

(i) <u>if</u> $kp < n$, <u>then</u> $W^{k,p}(\Omega) \subset L^q(\Omega)$ <u>for any</u> $1 \leq q \leq np/(n-kp)$; <u>and it is</u>

$$\|u\|_q \leqq c \, \|u\|_{k,p}$$

(ii) <u>if</u> $kp = n$, <u>then</u> $W^{k,p}(\Omega) \subset L^q(\Omega)$ <u>for any</u> $1 \leqq q < \infty$

(iii) <u>if</u> $kp > n$ <u>and</u> $n + p(h+\alpha) < kp$, <u>then</u> $W^{k,p}(\Omega) \subset C^{h,\alpha}(\bar\Omega)$ <u>and</u>

$$\|u\|_{C^{h,\alpha}} \leqq c \, \|u\|_{k,p} \quad (+)$$

0.2. <u>Remark</u>. The above imbeddings are also valid if we replace $W^{k,p}(\Omega)$ with $W_0^{k,p}(\Omega)$. In this case the result holds without any smoothness hypothesis on $\partial\Omega$.

0.3. <u>Theorem</u>. <u>Let</u> $kp < n$ <u>and</u> $1 \leqq q < np/(n-kp)$. <u>Then the imbedding of $W^{k,p}(\Omega)$ in $L^q(\Omega)$ is compact</u>.

<u>Notation</u>: (if $p<n$) sometime we set $p^* = np/(n-p)$ so that, in reference with 0.3-(i), one has $W^{1,p}(\Omega) \subset L^{p^*}(\Omega)$.

(+) As usual, $C^{h,\alpha}(\Omega)$ denotes the space of the functions $u \in C^h(\bar\Omega)$ whose derivatives of order h are Hölder continuous. $C^{h,\alpha}(\bar\Omega)$ is a Banach space under the norm $\longrightarrow$ 1/.

The following is often referred (for $p=2$) as the Poincaré-inequality:

0.4. <u>Theorem</u>. <u>If</u> $u \in W_0^{1,p}(\Omega)$ <u>then</u>

$$\|u\|_p \leq c \, \|\mathrm{grad}\, u\|_p \, .$$

In the preceding theorems, as well as later, we will denote by c (or $c_1, c_2, \ldots$) a constant which possibly depends on n, p, Ω, etc., but is <u>independent on u</u>.

Let us remark that from Theorem 0.4 it follows that in $W_0^1(\Omega)$ the scalar product

$$((u,v)) = \int_\Omega \nabla u \cdot \nabla v$$

induces a norm equivalent to $\|\cdot\|_{1,2}$. In the following we will always understand that

$$\|u\|_{1,2} = \left(\int_\Omega |\nabla u|^2\right)^{1/2}$$

For more details on Sobolev spaces we refer, for ex., to ~~Ad~~ [Ad] or to [Sob].

b) <u>Weak solutions of elliptic equations.</u>

Let Ω be a bounded domain of $\mathbb{R}^n$ with smooth boundary $\partial\Omega$, and $h \in L^2(\Omega)$. By a weak solution of the Dirichlet b.v.p.

$$(0.2) \qquad \begin{cases} -\Delta u = h(x) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

we mean a function $u \in W_0^1(\Omega)$ such that

$$\int_\Omega D_i u \, D_i \varphi = \int_\Omega h \varphi \qquad \text{for all } \varphi \in C_0^\infty(\Omega).$$

More in general, we can define a weak solution of the Dirichlet b.v.p

$$(0.3) \qquad \begin{cases} Lu = -\sum D_i (a_{ij}(x) D_j u) = h & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

as a function $u \in W_0^1(\Omega)$ such that

$$\int_\Omega a_{ij} D_j u \, D_i \varphi = \int h \varphi \qquad \text{for all } \varphi \in C_0^\infty(\Omega)$$

Above, we could take $a_{ij} \in L^\infty(\Omega)$.
It is clear that if (the a_{ij} are smooth and) u is a strong solution of (0.3), i.e. $u \in C^2(\overline{\Omega})$ and satisfies pointwise (0.3), then u is a weak solution of (0.3).

The operator L is elliptic if $\exists\, \nu > 0$ such that

$$a_{ij}(x)\,\xi_i\,\xi_j \geq \nu|\xi|^2$$

for all $\xi \in \mathbb{R}^n$ and a.e. $x \in \Omega$.
We will also assume $a_{ij} = a_{ji}$, in such a way that the bilinear form

$$\int_\Omega a_{ij}\, D_j u\, D_i v \equiv a(u,v) \qquad u, v \in W_0^1(\Omega)$$

is symmetric.
It is easy to see, as consequence of the Riesz representation theorem, that if $a_{ij} \in L^\infty(\Omega)$ and $h \in L^2(\Omega)$ then (0.3) has a unique weak solution $u \in W_0^1(\Omega)$ and one has

$$\|u\|_{1,2} \leq c\, \|h\|_2$$

Now, let us assume, for simplicity, that both a_{ij} and $\partial\Omega$ are smooth (say C^∞). Then one has:

0.5. Theorem. Let $u \in W_0^1(\Omega)$ be a weak solution of (0.5)
(i) if $h \in L^p(\Omega)$, $1 < p < \infty$, then $u \in W^{2,p}(\Omega)$ and

$$\|u\|_{2,p} \leq c\, \|h\|_p$$

(ii) if $h \in C^{0,\alpha}(\overline{\Omega})$ and $u \in C^{0,\alpha}(\Omega)$, then $u \in C^{2,\alpha}(\overline{\Omega})$ and

$$\|u\|_{C^{2,\alpha}} \leq c\, \|h\|_{C^{0,\alpha}}$$

For more details on point b) we refer, for ex. to [G-T] ~~or~~, to [Giu] or to [Lad-Ur].

c) Nemitski operators.

Let Ω be a bounded domain in $\mathbb{R}^n$ and $g : \Omega \times \mathbb{R} \to \mathbb{R}$ be a function verifying the following (Caratheodory) condition

(0.4) $g(x,s)$ is continuous in s for a.e. $x \in \Omega$, and ~~and~~ measurable in x for all $s \in \mathbb{R}$.

To g one can associate an operator (usually called the Nemitskii operator of g) $g_\#$ defined on the class of real valued functions on Ω, by setting

$$g_\#(u)(x) = g(x, u(x)).$$

We will be meanly interested in the case in which $g_\#$ acts on L^p spaces.

0.6 Theorem. Suppose g satisfies (0.4) and

(0.5) $|g(x,s)| \leq c_1 + c_2\,|s|^{p/q}$ $\qquad p, q \geq 1$

Then $g_\#$ maps $L^p(\Omega)$ in $L^q(\Omega)$ and is continuous.

The proof of theorem 0.6 as well as ~~the~~ a ~~more~~ complete discussion of the Nemitski operators can be found, for ex, in ~~[Kra]~~[Va].

We now ~~consider~~ discuss a situation we will widely ~~use~~ consider in the following sections.

~~Let g satisfy (0.4) and the growth conditions~~

~~(0.6) $|g(x,s)| \le c_1 + c_2 |s|^p$~~

Let $n > 2$ and ~~We~~ denote by $\mathcal{G}$ the class of all functions $g : \Omega \times \mathbb{R} \to \mathbb{R}$ satisfying (0.4) and such that

(0.6) $$|g(x,s)| \le c_1 + c_2 |s|^p$$

where $1 \le p < \frac{n+2}{n-2}$.

Let, for some $\varepsilon > 0$,

$$p = \frac{n+2}{n-2} \cdot \frac{1}{1+\varepsilon}$$

$$\alpha = \frac{2n}{n-2} \cdot \frac{1}{1+\varepsilon}$$

$$\beta = \frac{\alpha}{p} = \frac{2n}{n+2}$$

Recall that, since $\alpha < 2^*$, then the imbedding $W_0^1(\Omega) \xrightarrow{i} L^\alpha(\Omega)$ is compact. Moreover, if $g \in \mathcal{G}$, then

$$g_\# : L^\alpha(\Omega) \longrightarrow L^\beta(\Omega)$$

is continuous. Define the operator $B : W_0^1(\Omega) \longrightarrow W_0^1(\Omega)$ by setting

(0.7) $$((Bu, v)) = \int g(x,u)\, v \qquad \text{for } v \in W_0^1(\Omega)$$

~~Call the functional $b : W_0^1(\Omega) \to \mathbb{R}$ by~~

We remark that the right-hand side in (0.7) makes sense because for $u, v \in W_0^1(\Omega)$ one has

$$g(x,u) = g_\#(u) \in L^\beta(\Omega)$$

$$v \in L^{2^*}(\Omega)$$

so that: $g(x,u)v \in L^1(\Omega)$.

Let

$$G(x,u) := \int_0^u g(x,s)\, ds$$

From

$$|G(x,u)| \le c_3 |u| + c_4 |u|^{p+1}$$

it follows that $G(x,u) \in L^1(\Omega)$, for $u \in W_0^1(\Omega)$.

Let

$$b(u) := \int_\Omega G(x,u)$$

0.7. Lemma. If $g \in \mathcal{G}$ then $b : W_0^1(\Omega) \to \mathbb{R}$ is C^1 and $b'(u) = B(u)$. Moreover B is compact.

Proof. For $u, v \in W_0^1(\Omega)$ it results

$$b(u+v) - b(u) - ((Bu, v)) =$$

$$= \int_\Omega dx \int_u^{u+v} g(x,s)\, ds - \int_\Omega g(x,u)\, v =$$

$$= \int_\Omega dx \int_0^1 (g(x, u+\xi v) - g(x,u))\, v \, d\xi =$$

$$= \int_0^1 d\xi \int_\Omega (g(x,u+\xi v) - g(x,u))\, v$$

The last integral can be estimated as follows:

$$(0.8) \qquad \int |g(x,u+\xi v) - g(x,u)|\,|v| \leq$$

$$\leq \| g_\#(u+\xi v) - g_\#(u) \|_\beta \cdot \|v\|_{2^*}$$

Since $W_0^1(\Omega) \subset L^{2^*}(\Omega) \subset L^\alpha(\Omega)$ and $g_\#$ is continuous from L^α to L^β, from (0.8) it follows that

$$b(u+v) - b(u) - ((B(u), v)) = o(\|v\|_{1,2})$$

and thus $b'(u) = B(u)$.
The compactness of B follows directly from the compactness of the imbedding of $W_0^1(\Omega)$ into $L^\alpha(\Omega)$. ∎

In the case $n=2$ we can consider again the class $\mathcal{G}$, understanding that in (0.6) every $p \geq 1$ is allowed. Then the same result as Lemma 0.7 holds: the proof is similar, making use of the stronger form of the Sobolev imbedding theorem.

§ 1. Extrema.

In this section we will study some easy results concerning the existence of extrema of functionals in Banach spaces, or, more in general, in a subset of a Banach space ("constrained extrema"). Simple applications to a class of elliptic Dirichlet problems will be discussed.

Let E be a Banach space with norm $\|\cdot\|$; the strong (resp. weak) convergence will be denoted by $\to$ (resp. $\rightharpoonup$). Let $f: E \to \mathbb{R}$ be given; a point $u^* \in E$ is a "local" extremum for f if there exists a neighborhood U of u^* such that for all $u \in U$ either $f(u) \leq f(u^*)$ or $f(u) \geq f(u^*)$. In the former case u^* is said a local maximum, in the latter a local minimum. By a minimum (resp. maximum) of f on E we mean a point $u^* \in E$ such that $f(u) \geq f(u^*)$ (resp. $f(u) \leq f(u^*)$) for all $u \in E$.

We recall that a point $v \in E$ on which f is Fréchét-differentiable is called a stationary point of f if $f'(v) = 0$. It is easy to see that if u^* is a local extremum and f is Fréchét-differentiable at u^*, then u^* is a stationary point of f on E.

A simple criterion of existence of extrema is given in the following:

~~1.1 Theorem. Let $f \in C(E, \mathbb{R})$ satisfy to:~~
~~(1.1) $f(u) \to \infty$ as $\|u\| \to \infty$,~~
~~(1.2) f is~~

1.1. Theorem. Let E be reflexive and let $f: E \to \mathbb{R}$ be weakly lower semi-continuous (w.l.s.c.) (namely if $u_n \rightharpoonup u$ then $f(u) \leq \liminf f(u_n)$). Moreover, suppose that

$$(1.1) \qquad f(u) \to +\infty \quad \text{as} \quad \|u\| \to \infty .$$

Then f is bounded from below on E and attains its minimum in a point $u^* \in E$. If f is Fréchét-differentiable at u^*, it results $f'(u^*) = 0$.

Proof. By contradiction, let $u_n \in E$ be such that $f(u_n) \to -\infty$. From (1.1) it follows that $\|u_n\|$ is bounded, and hence there exists $u \in E$ such that $u_n \rightharpoonup u$ (without relabeling). Since f is w.l.s.c., it follows that

$$f(u) \leq \liminf f(u_n) = -\infty$$

which is impossible. Let $m = \inf\{ f(u) : u \in E \}$, and let $u_n \in E$ be a sequence such that $f(u_n) \to m$. Then $\|u_n\|$ is bounded and there exists $u^* \in E$ with $u_n \rightharpoonup u^*$. Again by the w.l.s.c. of f we deduce that $f(u^*) \leq \liminf f(u_n) = m$, and hence $f(u^*) = m$, as required ■

As simple application of theorem 1.1, we consider the nonlinear Dirichlet problem

$$(1.2) \qquad -\Delta u = g(x,u) \quad \text{in } \Omega \qquad u = 0 \quad \text{on } \partial\Omega$$

where Ω is a bounded domain in $\mathbb{R}^n$, with boundary $\partial\Omega$, which – for semplicity – will be assumed C^∞.

On g we assume

$$(g1.1) \qquad g: \Omega \times \mathbb{R} \to \mathbb{R} \text{ is Hölder continuous and bounded}$$

We will work in the Hilbert space $E := W_0^1(\Omega)$. Let $G(x,u) = \int_0^u g(x,s)\,ds$ and $f: E \to \mathbb{R}$ be

$$f(u) = \frac{1}{2}\|u\|_{1,2}^2 - \int_\Omega G(x,u)$$

From (g1.1) it follows (cf. Lemma 0.7) that f is C^1 and $f'(u) = u - B(u)$ (cf. notations of §0-a). Hence the stationary points of f are the $u \in E$ such that

$$((u,v)) - \int g(x,u)\,v = 0 \qquad \forall v \in E$$

namely the weak solutions of (1.2).

It is easy to verify that (g1.1) implies hypothesis (1.1) on f. Moreover the compact imbedding of E into $L^2(\Omega)$ (cf. Theorem 0.3) permits to show that f is w.l.s.c. As application of Theorem 1.1 we then deduce:

1.2. Proposition. If g satisfies (g1.1), then (1.2) has at least one weak solution.

Actually we know more on the solution.

1.3. Proposition. Under the hypothesis above, the weak solutions of (1.2) are in fact classical solutions.

Proof. Let $u \in W_0^1(\Omega)$ be any solution of (1.2). Since $g(\cdot, u(\cdot)) \in L^\infty(\Omega)$, theorem 0.7 implies: $u \in W^{2,p}(\Omega)$ $\forall p > 1$. Taking $p > \frac{n}{2}$, we infer, by theorem 0.3, that $u \in C^{0,\alpha}(\bar\Omega)$ $0<\alpha<1$. Now, g is Hölder continuous and hence $g(\cdot, u(\cdot))$ is so too, and from Theorem 0.5-ii, the conclusion follows. ∎

1.4. Remark. It is an easy exercise to verify that the same conclusions hold for the Dirichlet b.v.p.

$$\begin{aligned} Lu &= g(x,u) && \text{in } \Omega \\ u &= 0 && \text{on } \partial\Omega \end{aligned}$$

where $L := -\sum D_i(a_{ij} D_j)$, $a_{ij} = a_{ji}$ smooth, is an uniformly elliptic operator.

Proposition 1.2 can be used jointly with truncation arguments in several cases.
For example, suppose g satisfies

(g 1.2) g is Hölder-continuous and $\exists\ s' < 0 < s''$ such that $g(x, s') > 0 > g(x, s'')$ $\forall x \in \Omega$.

Letting

$$\hat g(x,s) = \begin{cases} g(x, s') & \text{for } s < s' \\ g(x, s) & \text{for } s' \le s \le s'' \\ g(x, s'') & \text{for } s > s'' \end{cases}$$

it is clear that $\hat g$ satisfies (g1.1). Thus the problem

$$\begin{cases} -\Delta u = \hat g(x,u) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

has a solution $u \in C^{2,\alpha}(\bar\Omega)$. We claim that

$$s' < u(x) < s'' \qquad \forall x \in \Omega$$

so that u is in fact a solution of (1.2), in view of the definition of $\hat g$. Let $x^* \in \Omega$ be the point where u attains the maximum. If $u(x^*) \ge s''$ then

$$-\Delta u(x^*) = \hat g(x^*, u(x^*)) = \hat g(x^*, s'') < 0$$

which is impossible. Similar argument shows that $\min_\Omega u(x) > s'$.

As a next example, we suppose g has the form

(1.3) $\quad g(x,u) = \lambda u - g_1(x,u)$

where g_1 is Hölder continuous an satisfies

(g 1.3) $\quad g_1(x,u) = o(|u|)$ at $u=0$ uniformly in $x \in \Omega$.

(g 1.4) $\quad \dfrac{g_1(x,u)}{u} \to +\infty$ as $u \to \mp\infty$

The problem (1.2) has now the "trivial solution" $u \equiv 0$ and we we are looking for non-trivial solutions.

Let λ_1 be the first eigenvalue of

$$-\Delta u = \lambda u \quad \text{in } \Omega$$
$$u = 0 \quad \text{on } \partial\Omega$$

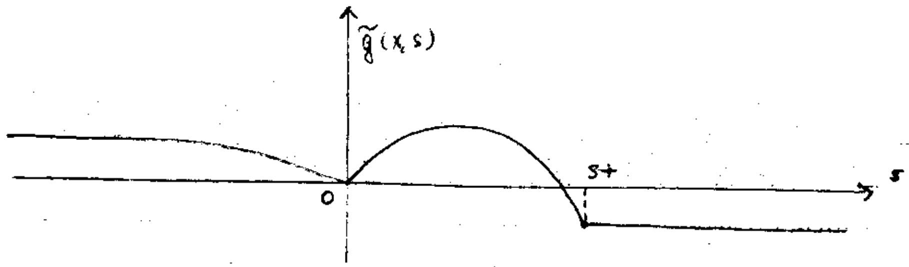
and denote by φ the corresponding eigenfunction, taken such that $\varphi > 0$ in Ω and normalized by $\|\varphi\|_2 = 1$.

1.5 Theorem. Suppose g has the form (1.3), with g_1 Hölder continuous, satisfying (g 1.3) and (g1.4+) (resp. (g 1.4−)). Moreover let $\lambda > \lambda_1$. Then (1.2) has at least a solution $\bar{u} > 0$ in Ω (resp. $\underline{u} < 0$ in Ω).

The proof is carried out in two steps. First g is substituted by $\tilde{g}$, defined as a Hölder-continuous function ~~to~~ satisfying

$\tilde{g}(x,s) > 0$ and bounded for $s < 0$
$\tilde{g}(x,s) = g(x,s)$ for $0 \leq s \leq s^+$
$\tilde{g}(x,s) = g(x,s^+)$ for $s > s^+$

Above we have assumed (g1.4+) holds and denoted by $s^+ > 0$ a value such that $g(x,s^+) < 0 \quad \forall x \in \Omega$ - such s^+ exists in view of (g1.4+).



Remark $\tilde{g}$ is bounded. The same arguments used in Prop. 1.2 show (1.2) has a solution $\bar{u}(x)$ with $0 \leq \bar{u}(x) < s^+ \quad \forall x \in \Omega$. Such $\bar{u}$ is found as minimum of the corresponding functional.

Next, we show that ~~on~~ such a minimum the functional assumes negative values, so that $\bar{u} \not\equiv 0$. In fact, using the properties of g_1, we evaluate the functional in points of the type $\alpha\varphi$, $\alpha \in \mathbb{R}$, positive and small enough. One has

$$f(\alpha\varphi) = \frac{1}{2}\alpha^2 \|\varphi\|_{1,2}^2 - \frac{1}{2}\lambda\alpha^2 \|\varphi\|_2^2 + o(\alpha^2)$$

$$= \frac{1}{2}(\lambda_1 - \lambda)\alpha^2 + o(\alpha^2)$$

and hence the conclusion follows, provided $\lambda > \lambda_1$ ∎

For results similar to those of Theorem 1.5 we refer, for ex., to [Amb 1]

1.6 Remark. If $\lambda < \lambda_1$, then (1.2) might have only the trivial solution. This is the case, for example, when $g(x,u) = \lambda u - u^3$. In fact, any solution of (1.2) has to satisfy

$$\|u\|^2_{1,2} = \lambda \|u\|^2_2 - \|u\|^4_4$$

Since it is known that $\lambda_1 \|u\|^2_2 \le \|u\|^2_{1,2}$ $\forall u \in E$, (variational carachterization of λ_1), it follows that $u \equiv 0$.

§2. A minimax theorem.

The purpose of this section is to discuss an abstract result concerning the existence of stationary points using minimax arguments.
The example we have in mind is the case of a functional f having the origin as a strict local minimum. Assuming that f is negative somewhere and satisfies a compactness condition, we will show f possesses at least another stationary point. The corresponding stationary level is found as inf-sup on a suitable class of sets.

Let $f \in C^1(E;\mathbb{R})$. The following notations will be used:

$f^{-1}([a,b]) = \{u \in E : a \le f(u) \le b\}$ (similar meaning for $f^{-1}(]a,b[)$, etc.)

$K = \{u \in E : f'(u) = 0\}$

$K_c = \{u \in E : f'(u) = 0,\ f(u) = c\}$

A value $c \in \mathbb{R}$ such that $K_c \neq \emptyset$ is said a critical level.
A fundamental rôle in the following arguments is plaied by certain deformations, which will be defined as solutions of suitable differential equations.
First some preliminaries are needed. Let A be an open subset of the Hilbert space E, and $F \in C^{0,1}(A;E)$.
For every $x \in A$, we consider the Cauchy problem

$$(2.1) \qquad \begin{cases} \alpha'(t) = F(\alpha(t)) \\ \alpha(0) = x \end{cases}$$

Since F is $C^{0,1}$ ______, the usual contraction mapping theorem shows that (2.1) has a unique solution $\alpha(t,x)$, defined for t in a suitable neighborhood of $t=0$, which depends continuously in x. We will denote by $(t^-(x), t^+(x))$ the maximal interval of t for which $\alpha(t,x)$ is defined; namely, $t^\mp(x)$ are such that no other curves $\tilde{\alpha}(t,x)$ are defined in an interval I strictly containing $(t^-(x),t^+(x))$, and satisfying (2.1).

2.1. Lemma. If $t^+(x)<\infty$ (resp. $t^-(x)>-\infty$), then $\alpha(t,x)$ has no limit points as $t\uparrow t^+(x)$ (resp. $t\downarrow t^-(x)$).

Proof. If not, we could continue $\alpha(t,x)$ for $t>t^+(x)$ by the solution of the Cauchy problem

$$\begin{cases}\tilde{\alpha}'(t)=F(\tilde{\alpha}(t))\\ \tilde{\alpha}(t^+(x))=\lim\limits_{t\uparrow t^+(x)}\alpha(t,x)\end{cases}$$

contradicting the maximality of $t^+(x)$. ■

2.2. Lemma. Let $F\in C^1(A,E)$ and $\mathcal{C}$ be a closed subset of A. Suppose there exists $m>0$ such that $\|F(u)\|\le m$ for all $u\in\mathcal{C}$, and let $x\in\mathcal{C}$ be given such that $\alpha(t,x)\in\mathcal{C}$ for all $t\in(t^-(x),t^+(x))$. Then $t^\mp(x)=\mp\infty$.

Proof. Suppose that $t^+(x)<+\infty$ (the same argument applies to $t^-(x)$) and let $t_n\uparrow t^+(x)$. It results

$$(2.2)\quad \alpha(t_n,x)-\alpha(t_m,x)=\int_{t_m}^{t_n}\alpha'(s,x)\,ds=\int_{t_m}^{t_m}F(\alpha(s,x))\,ds$$

Since F is bounded on $\mathcal{C}$ and $\alpha(s,x)\in\mathcal{C}$ for all $s\in(t^-(x),t^+(x))$, it follows from (2.2)

$$\|\alpha(t_n,x)-\alpha(t_m,x)\|\le m\,|t_n-t_m|$$

Therefore $\alpha(t_n,x)$ is a Cauchy sequence and it converges for $t_n\uparrow t^+(x)$, in contradiction with Lemma 2.1. ■

We introduce now a compactness condition, due to Palais and Smale [P-S]: we say that f satisfies the (P-S) condition (on E) provided

$$(\text{P-S})\quad \begin{cases}\text{every sequence } u_n\in E \text{ such that } f(u_n) \text{ is}\\ \text{bounded and } f'(u_n)\to 0 \text{ has a converging}\\ \text{subsequence}\end{cases}$$

We state explicitly the following consequence of (P-S):

2.3. Lemma. Let f satisfy (P-S) and let $c\in\mathbb{R}$ be such that $K_c=\emptyset$. Then there exists $\delta>0$ such that $\|f'(u)\|^2\ge 2\delta$ for all $u\in f^{-1}([c-\delta,c+\delta])$.

Proof. Otherwise, we can find a sequence $u_n\in E$ such that $f(u_n)\to c$ and $f'(u_n)\to 0$. By (P-S) it follows that there exists a subsequence with $u_n\to\bar{u}$ (without relabeling). Evidently, one has $f(\bar{u})=c$ and $f'(\bar{u})=0$, in contradiction with the assumption $K_c=\emptyset$. ■

We are now in position to state our deformation lemma.

2.4. Lemma. Let $f \in C^1(E;\mathbb{R})$ satisfy (P-S) and let $c \in \mathbb{R}$ be such that $K_c = \emptyset$. Then, given $0 < \varepsilon \le 1$, there exists $\alpha \in C([0,1]\times E; E)$ and a constant $\delta \in]0,\varepsilon[$ such that for all d, $\delta < d < \varepsilon$, it results:

(i) $\alpha(0,u) = u$ for all $u \in E$;

(ii) $\alpha(t,u) = u$ for all $t \in [0,1]$ and all $u \notin f^{-1}([c-d,c+d])$

(iii) $f(\alpha(t,u)) \le f(u)$ for all $t \in [0,1]$ and all $u \in E$

(iv) $f(\alpha(1,u)) < c-\delta$ for all $u \in f^{-1}((-\infty, c+\delta])$.

Proof. First of all we use Lemma 2.3 and take $\delta > 0$ according to it. Obviously we can assume $\delta < \varepsilon \le 1$. Let $d \in]\delta,\varepsilon[$ be fixed and consider $g \in C^{0,1}(E;\mathbb{R})$ such that: $g(u) = 0$ for all $u \notin f^{-1}([c-d,c+d])$, $g(u) = 1$ for all $u \in f^{-1}([c-\delta,c+\delta])$ and $0 \le g(u) \le 1$ otherwise. Moreover, let $\varphi \in C^1(\mathbb{R}^+,\mathbb{R}^+)$ be a function such that: $s^2\varphi(s)$ is nondecreasing, $\varphi(s) = 1$ for all $s \in [0,1]$ and $s^2\varphi(s) = 2$ for all $s \ge 2$. Let

$$F(u) = -g(u)\,\varphi(\|f'(u)\|)\,f'(u)$$

Let us assume, for the moment, that $f \in C^2(E;\mathbb{R})$, so that $F \in C^{0,1}(E;\mathbb{R})$. We want to show that the solution $\alpha(t,x)$ of

$$\begin{cases} \alpha' = F(\alpha) \\ \alpha(0) = x \end{cases}$$

gets the deformation of the lemma.
First of all, $\alpha(t,x)$ is defined for all $t \in \mathbb{R}$, because F is bounded (cf. Lemma 2.2). It is readily verified that (i) and (ii) hold in view of the definition of α (resp. of g).

Letting $h(t) = f(\alpha(t,u))$, it results

$$h'(t) = (f'(\alpha(t,u)), \alpha'(t,u)) = (f'(\alpha(t,u)), F(\alpha(t,u)))$$

$$= -g(\alpha(t,u))\,\varphi(\|f'(\alpha(t,u))\|)\,\|f'(\alpha(t,u))\|^2 \le 0.$$

Then (iii) follows. Lastly, let $u \in f^{-1}([c-\delta,c+\delta])$. If $f(\alpha(1,u)) > c-\delta$ then, $f(\alpha(t,u))$ being non-increasing, $f(\alpha(t,u)) \in f^{-1}([c-\delta,c+\delta])$ for all $t \in [0,1]$. Now it is:

$$(2.3)\quad f(\alpha(0,u)) - f(\alpha(1,u)) = -\int_0^1 h'(s)\,ds =$$

$$= \int_0^1 g(\alpha(s,u))\,\varphi(\|f'(\alpha(s,u))\|)\,\|f'(\alpha(s,u))\|^2\,ds$$

For $v \in f^{-1}([c-\delta,c+\delta])$ one has: $\|f'(v)\|^2 \ge 2\delta$ (Lemma 2.2) and hence $\varphi(\|f'(v)\|)\|f'(v)\|^2 \ge 2\delta$, because $s^2\varphi(s)$ is non decreasing. Then, from (2.3) we deduce:

$$f(u) - f(\alpha(1,u)) \ge 2\delta$$

Thus:

$$f(\alpha(1,u)) \le f(u) - 2\delta \le c+\delta-2\delta = c-\delta.$$

This proves (iv) under the assumption $f \in C^2(E;\mathbb{R})$. To handle the more general case $f \in C^1(E;\mathbb{R})$, we have to substitute $\nabla f(u)$ with a "pseudo-gradient vector field" (cf. [Pa]). More precisely, it is possible to show that, given $f \in C^1(E;\mathbb{R})$, for any $u \in E-K$, there exists $V(u)$

such that V is Lipschitz-continuous and satisfies to

(2.4) $\quad \|V(u)\| \leq 2\|f'(u)\| \quad$ and $\quad \|f'(u)\|^2 \leq (V(u), f'(u))$.

Such V is called "pseudo gradient vector field". It is possible to show that, in view of (2.4), substituting in the definition of F, f' with V, the statements of the Lemma are again true. We do not carry out the details, referring to [Pa.] as well as to [ ] or to [Cl] ■.

We are now ready to state the main result of this section. We consider a functional $f \in C^1(E;\mathbb{R})$ verifying

(f1) $\quad f(0)=0$ and there exist $r, \rho > 0$ such that $f(u) > 0$ for all $0 < \|u\| \leq r$ and $f(u) \geq \rho > 0$ for all $\|u\| = r$

(f2) $\quad$ there exists $u_0 \in E$, $u_0 \neq 0$ such that $f(u_0) = 0$.

By (f1) $u=0$ is a stationary point of f. Our goal is to show that under (f1-2) and (P-S) f has at least another stationary point. We explicitally remark that an important feature of the result we are going to expose, will be that f is not assumed to be bounded from below or from above, as well as no simmetry is required on f.

In corrispondence of u_0 in (f2) we set

$$\Gamma = \{ \gamma \in C([0,1];E) \; : \; \gamma(0)=0, \; \gamma(1)=u_0 \}$$

and

(2.5) $$c = \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} f(\gamma(t))$$

Remark that $c < +\infty$.

2.5. Theorem. Let $f \in C^1(E;\mathbb{R})$ satisfy (f1-2) and (P-S) and let c be given by (2.5). Then $c \geq \rho > 0$ and is a critical level for f. Hence f has at least one non trivial stationary point.

Proof. Set $B_\rho = \{u \in E : \|u\| \leq \rho\}$ and $S_\rho = \{u \in E : \|u\| = \rho\}$. By (f1-2) it follows that $u_0 \notin B_\rho$ and therefore

(2.6) $\quad \gamma([0,1]) \cap S_\rho \neq \emptyset \quad$ for all $\gamma \in \Gamma$

By (2.6) it follows:

$$c \geq \inf_{u \in S_\rho} f(u) \geq \rho > 0$$

If $K_c = \emptyset$, we apply the deformation Lemma 2.4 to find $0 < d < \delta < \min\{\rho, 1\}$ and a deformation α verifying (i-iv). Denote by α_1 the mapping $\alpha(1,\cdot)$; we claim that for all $\gamma \in \Gamma$ it results $\alpha_1 \circ \gamma \in \Gamma$. In fact, by (ii) we have:

$$\alpha_1 \circ \gamma(0) = \alpha(1,0) = 0 \quad \text{and} \quad \alpha_1 \circ \gamma(1) = \alpha(1,u_0) = u_0$$

because $f(0) = f(u_0) = 0 < c - d$. Now, by the definition of c, we can choose $\gamma \in \Gamma$ such that

$$\max_{t \in [0,1]} f(\gamma(t)) \leq c + \delta$$

Using (iv), we deduce

$$\max_{t \in [0,1]} f(\alpha_1 \circ \gamma(t)) \leq c - \delta \tag{2.7}$$

Since $\alpha_1 \circ \gamma \in \Gamma$, then (2.7) is in contradiction with the definition (2.5) of c. ■

Theorem 2.5 is contained in the paper [Amb-Ra]. Further improvements can be found, among others, in [Ni] and [Be-Ra].

§3. Superlinear problems.

In spite of its simplicity, theorem 2.5 can be usefully employed to solve several interesting problems. We will discuss here one of those, concerning a superlinear elliptic boundary value problem. Other applications will be given later.

Let Ω be a bounded domain of $\mathbb{R}^n$ with smooth boundary $\partial\Omega$. Consider the Dirichlet boundary value problem:

$$(*) \qquad \begin{cases} -\Delta u = g(x,u) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

On $g : \Omega \times \mathbb{R} \to \mathbb{R}$ we assume:

(g3.1) g is Hölder-continuous

(g3.2) $|g(x,s)| \leq a_1 + a_2 |s|^p$ where p satisfies, in the case $n > 2$, the inequality $1 \leq p < \frac{n+2}{n-2}$. (If $n = 2$ any $p > 1$ is allowed).

~~(g3) $g(x,s) = o(s)$ at $s \to 0$ uniformly in $x \in \Omega$.~~

~~(g4) there exist $a > 0$ and $\theta \in [0, \frac{1}{2}[$ such that~~

Working again in $E := W_0^1(\Omega)$, we set $G(x,u) = \int_0^u g(x,s)\,ds$. Remark, that, if (g3.1 – 3.2) hold, then $g \in \mathcal{G}^0$ (cf. notations of part (c) of §0). According to Lemma 0.7, the functional defined on E by

$$f(u) = \frac{1}{2}\|u\|^2 - b(u) \quad , \quad b(u) = \int_\Omega G(x,u)$$

is C^1 (C^2 provided $g \in C^1(\mathbb{R}\times\mathbb{R})$), $f'(u) = u - B(u)$, and therefore $f'(u) = 0$ if and only if u is a weak solution of (*).

On the beahaviour of g at 0 and ∞ we will assume:

(g3.3) $g(x,0) = 0$ and $g(x,s) = o(s)$ at $s = 0$ uniformly in $x \in \Omega$.

(g3.4) $\exists\, a > 0$ and $0 < \theta < \frac{1}{2}$ such that

$$G(x,s) \leq \theta\, s\, g(x,s) \quad \text{for } s \geq a \text{ and } \forall x \in \Omega.$$

3.1. Remark. From (g3.4) it follows easily that g is superlinear as $s \to +\infty$, in the sense that

(3.1) $$G(x,s) \geq \text{const.}\ s^{\frac{1}{\theta}} \quad \text{for } s \geq a$$

In fact, one has: $\partial G(x,s)/\partial s = g(x,s) \geq \frac{1}{s}\cdot\frac{1}{\theta}\cdot G(x,s)$ for $s \geq a$.

We are now in position to state our main result of this section, concerning the existence of non-trivial solutions of (*).

3.2. Theorem. If (g3.1 – 3.4) hold, then (*) has at least a solution $\bar{u} > 0$ in Ω.

Before to prove theorem 3.2, we first show, in completion of Proposition 1.3, the following regularity result.

3.3. Proposition. If (g3.1 – 3.2) hold, then the weak-solutions of (*) are classical solutions.

Proof of Proposition 3.3. We will employ the so called "bootstrap procedure": if $u \in W_0^1(\Omega)$, we increase the regularity of u showing – by successive steps – $u \in W^{2,q}(\Omega)$ with $q > n/2$. Let $n > 2$. By theorem 0.1 we first get $u \in L^{2n/n-2}$. In the growth condition (g3.2), let $p = \frac{n+2}{n-2}\cdot\frac{1}{1+\varepsilon}$. We use theorem 0.6 to obtain $g_\#(u) \in L^{q_1}$, $q_1 = \frac{2n}{n+2}(1+\varepsilon)$. By theorem 0.5 we infer that $u \in W^{2,q_1}$ and, by theorem 0.1, that $u \in L^r$, $r = nq_1/n-2q_1$ (if $n > 2q_1$, otherwise we have done). Again, using (g3.2) and Theorem 0.6, we deduce

$$g_\#(u) \in L^{q_2}, \text{ with } q_2 = \frac{n q_1}{n - 2q_1}\cdot\frac{1}{p}$$

and, by theorem 0.5, $u \in W^{2,q_2}$. Since $q_2 > q_1(1+\varepsilon)$, after a finite number of steps, one has that $u \in W^{2,\tilde{q}}(\Omega)$ with $2\tilde{q} > n$. At this point, Theorem 0.1-(iii) implies $u \in C^{0,\alpha}(\overline{\Omega})$ and the conclusion follows as in Proposition 1.3. The case $n=2$ is similar, using the stronger form of the Sobolev imbedding. ∎

<u>Proof of theorem 3.2.</u> We will assume that g satisfies the additional assumption:

$$(3.2) \qquad g(x,s) = 0 \qquad \forall\, s<0$$

The general case will turn out to be a consequence of the fact that, by the maximum principle, a possible solution of (*) is non-negative on Ω provided (3.2) holds.

Using the notation of §0-(c), we consider

$$f(u) = \frac{1}{2}\|u\|_{1,2}^2 - b(u)$$

We first show that (f1) holds. In fact, by (g 3.2) and (g 3.3) it follows that for all $\varepsilon>0$ $\exists\, \delta>0$ such that

$$|G(x,s)| \leq c_1 |s|^{p+1} \qquad \text{for } s>\delta$$

$$|G(x,s)| \leq \varepsilon s^2 \qquad \text{for } s \leq \delta$$

Hence $b(u) = o(\|u\|_{1,2}^2)$ at $u=0$ and (f1) follows.

Let $\bar{u} \in E$, $\bar{u}>0$ in Ω be fixed. Using (3.1) we have

$$f(\rho\bar{u}) = \frac{1}{2}\rho^2\|\bar{u}\|_{1,2}^2 - \int G(x,\rho\bar{u}) \longrightarrow -\infty \quad \text{as } \rho\to\infty$$

Thus (f2) holds. Lastly we show that (P-S) is satisfied. Let $u_n \in E$ be such that $f(u_n) \leq c$ and $f'(u_n) \to 0$. Set $\Omega_a = \{x \in \Omega : u_n(x) \geq a\}$. Using (g 3.4) and recalling we assumed (3.2), one has

$$c \geq \frac{1}{2}\|u_n\|_{1,2}^2 - \int_{\Omega_a} G(x,u_n) - \int_{\Omega\setminus\Omega_a} G(x,u_n) \geq$$

$$\geq \frac{1}{2}\|u_n\|_{1,2}^2 - \theta\int_{\Omega_a} u_n g(x,u_n) - c_2 \geq$$

$$\geq \frac{1}{2}\|u_n\|_{1,2}^2 - \theta\int_{\Omega} u_n g(x,u_n) - c_3$$

Hence

$$(3.3) \qquad \|u_n\|_{1,2}^2 \leq 2\theta\int_\Omega u_n g(x,u_n) + c_4$$

Set $w_n := f'(u_n) = u_n - B(u_n)$. We have

$$((w_n, u_n)) = \|u_n\|_{1,2}^2 - ((B(u_n), u_n))$$
$$= \|u_n\|_{1,2}^2 - \int_\Omega u_n g(x,u_n)$$

Since we are assuming $w_n \to 0$, it follows:

(3.4) $\int_\Omega u_n g(x, u_n) \leq \|u_n\|^2_{1,2} + \varepsilon \|u_n\|_{1,2}$

Combining (3.3) and (3.4) we get

$$\|u_n\|^2_{1,2} \leq 2\theta \|u_n\|^2_{1,2} + c_5 \|u_n\|_{1,2} + c_6,$$

and since $2\theta < 1$, we conclude $\|u_n\|_{1,2} \leq$ const. Then, passing eventually to a subsequence, we have that $u_n \rightharpoonup u^* \in E$. Since B is compact (cf. Lemma 0.7), $B(u_n) \to B(u^*)$ and hence from $u_n = B(u_n) + w_n$, we conclude that u_n is converging.
All the assumption of Theorem 2.5 are satisfied. Therefore f has a nontrivial stationary point $\bar{u} \not\equiv 0$ which is a solution of (*). ■

It is rather natural to ask whether or not the growth condition (g3.2) is necessary. The answer is given by the following identity, due to Pohozaev [Po]

3.4. Lemma. Let $u \in C^2_0(\Omega)$ be such that

(3.5) $$\begin{cases} -\Delta u = g(u) & \text{in } \Omega \subset \mathbb{R}^n \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

Then the following identity holds

(3.6) $$n \int_\Omega G(u) + \frac{2-n}{2} \int_\Omega u g(u) = \frac{1}{2} \int_{\partial\Omega} u_\nu^2 (x \cdot \vec{\nu})\, d\sigma$$

where $x = (x_1, \dots, x_n)$, $\vec{\nu}$ is the outward pointing unit normal at $\partial\Omega$ and u_ν is the derivative along $\vec{\nu}$.

Postponing the proof of this lemma, we first show how from (3.6) it follows:

3.5 Theorem. If $x \cdot \vec{\nu} \geq 0$ on $\partial\Omega$, then the problem

(3.7) $$\begin{cases} -\Delta u = |u|^p & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

has no nontrivial solutions for $p \geq \frac{n+2}{n-2}$.

Proof of Theorem 3.5. By the maximum principle, it follows that any solution u of (3.10) is non-negative. Using (3.9), with $g(u) = u^p$, we obtain

$$\frac{n}{p+1} \int u^{p+1} + \frac{2-n}{2} \int u^{p+1} = \frac{1}{2} \int u_\nu^2 (x \cdot \vec{\nu})\, d\sigma \geq 0$$

Hence, if $u \not\equiv 0$: $2n + (2-n)(p+1) \geq 0$, namely $p \leq \frac{n+2}{n-2}$. If $p = \frac{n+2}{n-2}$, then (3.9) implies $u_\nu \equiv 0$ and thus $u \equiv 0$. ■

Proof of Lemma 3.4. Let $V(x) = (x \cdot \nabla u) \nabla u$. Then, by straightforward calculations, we find:

$$\operatorname{div} V(x) = (x \cdot \nabla u) \Delta u + |\nabla u|^2 + \frac{1}{2} \sum x_i \frac{\partial}{\partial x_i} (|\nabla u|^2)$$

Using the Gauss-Ostrogradski formula, we get

(3.8) $$\int_\Omega (x \cdot \nabla u) \Delta u = \int_{\partial\Omega} u_\nu (x \cdot \nabla u)\, d\sigma - \int_\Omega |\nabla u|^2 - \frac{1}{2} \int_\Omega \sum x_i \frac{\partial}{\partial x_i} (|\nabla u|^2)$$

Since $u=0$ on $\partial\Omega$ then $\nabla u = u_\nu \vec{\nu}$ and hence

$$(3.9)\qquad \int_{\partial\Omega} u_\nu\,(x\cdot\nabla u)\,d\sigma = \int_{\partial\Omega} u_\nu^2\,(x\cdot\vec{\nu})\,d\sigma$$

Moreover, taking $W(x) = \frac{1}{2}\,x\,|\nabla u|^2$, we have

$$\operatorname{div} W(x) = \frac{n}{2}|\nabla u|^2 + \frac{1}{2}\sum x_i \frac{\partial}{\partial x_i}\left(|\nabla u|^2\right)$$

and hence

$$(3.10)\qquad \frac{1}{2}\int_\Omega \sum x_i \frac{\partial}{\partial x_i}\left(|\nabla u|^2\right) = \frac{1}{2}\int_{\partial\Omega}|\nabla u|^2\,(x\cdot\vec{\nu})\,d\sigma - \frac{n}{2}\int_\Omega |\nabla u|^2$$

Substituting (3.9) and (3.10) into (3.8) (recalling that $|\nabla u|^2 = u_\nu^2$, we obtain

$$(3.11)\qquad \int_\Omega (x\cdot\nabla u)\,\Delta u = \frac{1}{2}\int_{\partial\Omega} u_\nu^2\,(x\cdot\vec{\nu})\,d\sigma - \frac{2-n}{2}\int_\Omega |\nabla u|^2$$

Since $u=0$ on $\partial\Omega$, it results $\int|\nabla u|^2 = -\int u\,\Delta u$ and thus, from (3.11):

$$\int_\Omega (x\cdot\nabla u)\,\Delta u = \frac{1}{2}\int_{\partial\Omega} u_\nu^2\,(x\cdot\vec{\nu})\,d\sigma - \frac{n-2}{2}\int_\Omega u\,\Delta u$$

Now, we use the fact that $-\Delta u = g(u)$ to find

$$(3.12)\qquad -\int_\Omega (x\cdot\nabla u)\,g(u) = \frac{1}{2}\int_{\partial\Omega} u_\nu^2\,(x\cdot\vec{\nu})\,d\sigma + \frac{n-2}{2}\int_\Omega u\,g(u)$$

Lastly, by integrating by parts one gets (remark that $G(u(x))=0$ on $\partial\Omega$):

$$\int_\Omega g(u)\,(x\cdot\nabla u) = \int_\Omega \sum x_i \frac{\partial}{\partial x_i} G(u) = -n\int_\Omega G(u)$$

Putting this last equality in (3.12), the lemma follows. ∎

Theorem 3.2 is taken from [Amb-Ra]. Further remarks on the Pohozaev-identity can be found in [Ra 1].

We end this section by sketching the result concerning a case in which we can use both Theorems 1.1 and 2.5.
Suppose g satisfies $(g\,3.1)$ – $(g\,3.3)$ and

$(g\,3.5)$ $\exists\, s^+>0$ such that $g(x,s^+)\le 0$,

$(g\,3.6)$ $g(x,s)$ is positive in a deleted neighborhood of zero.

Consider the problem

$$(3.13)\qquad \begin{cases} -\Delta u = \lambda\, g(x,u) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

where $\lambda\in\mathbb{R}$ is a parameter.

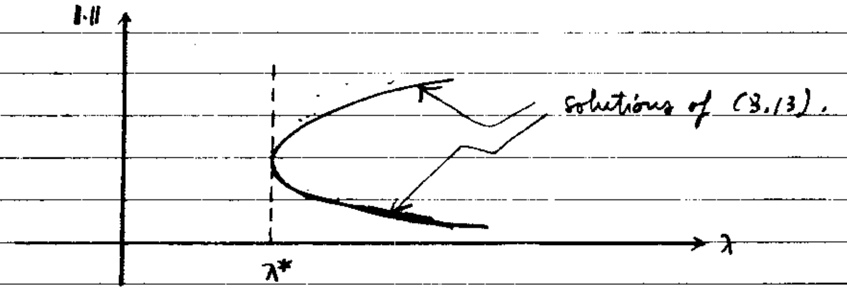
3.6 Theorem. *Suppose $(g\,3.1)$ – $(g\,3.3)$ and $(g\,3.5)$–3.6) hold. Then $\exists\,\lambda^*>0$ such that $\forall\lambda>\lambda^*$ the problem (3.13) has at least two non-trivial solutions $\underline{u}$, $\overline{u}$, both positive in Ω.*

Proof (Sketch). By possibly modifying g for $s<0$ and $s>s^+$ as in Theorems 1.5 and 3.2, we can assume g is bounded and $g(x,s) =$

$= o(s^2)$ at $s = 0$ uniformly in $x \in \Omega$. The corresponding functional

$$f_\lambda(u) = \frac{1}{2}\|u\|_{1,2}^2 - \lambda \int G(x,u)$$

has a minimum $\underline{u}$ and taking λ large enough, $f_\lambda(\underline{u}) < 0$. ~~we then~~ After, we use (§ 3.3) to find, via Theorem 2.5, another stationary point $\bar{u}$ for f, with $f_\lambda(\bar{u}) > 0$. ■



For more details on (3.13) we refer to [Amb-Ra].