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AUTUMN COURSE

ON

VARIATIONAL METHODS IN ANALYSIS AND MATHEMATICAL PHYSICS

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POSITIVE SOLUTIONS OF SEMILINEAR ELLIPTIC PROBLEMS

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hese are much harder problems due to the difficulty in obtaining priori bounds. Some progress has been done by the work of mbrosetti-Rabinowitz [7], Brezis-Turner [8], Crandall-Rabinowitz 9], Nussbaum [10], Puel [11], Turner [12]. There is still a lot o be done on this subject and many questions require a more omplete study, and we hope that these lectures will motivate some tudents to take them up.

CHAPTER 1

LINEAR EIGENVALUE PROBLEMS WITH AN INDEFINITE WEIGHT

1.1 SOME DIFFERENTIAL CALCULUS. Let U be an open subset of a Hilbert space H, and $f: U \to R$ and $g: U \to R$ be C^1 real-valued functions defined in U. Consider the hypersurface

$$S = \{x \in U: g(x) = 0, g'(x) \neq 0\}$$

which of course is supposed non-empty. Problem is

$$\text{Ext} \{f(x): x \in S\}$$

which consists in looking for extrema of f with the subsidiary condition g = 0. This is a classical question in the Calculus of Variations, which is solved using Differential Calculus as follows.

We say that $x_o \in S$ is a critical point of f restricted to S if for all C^1 paths $\alpha: (-\varepsilon,\varepsilon) \to S$, with $\alpha(0) = x_o$, one has

(1) $$\frac{d}{dt} f(\alpha(t))\Big|_{t=0} = 0.$$

Observe that points $x_o \in S$ where $f|S$ has a local maximum or a local minimum are critical points.

Proposition 1.1. $x_o \in S$ is a critical point of $f|S$ iff

(2) $$f'(x_o) = \lambda g'(x_o)$$

for some $\lambda \in R$.

Introduction

The past twenty years have witnessed a tremendous development of the theory of boundary value problems for semilin elliptic equations - positive solutions. We do not intend to sur it in this set of lecture notes which were rather prepared to be an introductory course on the subset for presentation at the Lat American School of Differential Equations held at the University of São Paulo, July 1981. For example we have nothing on bifurcati problems or existence of branches of positive solutions; these questions will be eventually discussed by other lecturers in thi conference. Extensive references may be seen in Krasnosel'skii's book [1] and in the paper by Amann [2] for the literature up to 1975. In the present work we concentrate in a single aspect of t theory: the existence of positive solutions of the Dirichlet problem. Our aim here is threefold. First we develop in detail t theory of eigenvalue problems with weight, which has shown to be an important tool in treating these problems; see Ambrosetti [3] Ambrosetti-Mancini [4], Berestycki [5], Kato-Hess [6], among othe The possibility of dealing with indefinite weights has given us more general results, see Section 2. Second we show that subline problems can be studied efficiently via the method of monotone iteration; this line of research was intensively pursued by Herbe Keller, Donald Cohen and others. In the present form is essentiall due to the work of Herbert Amann and David Sattinger; see referen [2]. Third we study some special cases of superlinear problems.

Remark. λ is called a Lagrange multiplier and in case of existence it is given by

$$\lambda = f'(x_0)\cdot g'(x_0)\,\|g'(x_0)\|^{-2}.$$

In this section we use the dot notation for the inner-product in H.

Lemma 1.2. *Let $\omega \in H$, with $\omega \perp g'(x_0)$. Then there is a C^1 path $\alpha: (-\varepsilon,\varepsilon) \to S$, with $\alpha(0) = x_0$, such that $\alpha'(0) = \omega$.*

Proof. Let H_2 be the one-dimensional subspace of H generated by $g'(x_0)$, and $H_1 = H_2^{\perp}$. Then $H = H_1 \oplus H_2$, which means that each $x \in H$ has a unique decomposition $x = u+v$, $u \in H_1$, $v \in H_2$. So $x_0 = u_0+v_0$. First we proceed to obtain a Cartesian representation of S. For that purpose it is more convenient to work with the representation of H as the Cartesian product $H_1 \times H_2$: each $x = u+v \in H$ is represented by $(u,v) \in H_1 \times H_2$. Define $\hat{g}: U_1 \times U_2 \to R$ by $\hat{g}(u,v) = g(u+v)$, where U_1 and U_2 are neighborhoods of u_0 in H_1 and v_0 in H_2, respectively. These neighborhoods are chosen in such a way that $u+v \in U$ for all $u \in U_1$ and all $v \in U_2$. It is easy to see that g is a C^1 mapping, and its partial derivatives are given by

$$\hat{g}_1'(u_1,v_1)\cdot u = g'(x_1)\cdot u \qquad \hat{g}_2'(u_1,v_1)\cdot v = g'(x_1)\cdot v.$$

Consequently $\hat{g}_2'(u_0,v_0)$ is an isomorphism between H_2 and R. Thus by the Implicit Function Theorem there are neighborhoods $V_1 \subset U_1$ of u_0, $V_2 \subset U_2$ of v_0 and a C^1 function $h: V_1 \to V_2$ such that $v_0 = h(u_0)$ and

(3) $$\hat{g}(u, h(u)) = 0, \qquad \forall\, u \in V_1.$$

Now define $p: V_1 \to H_1 \times H_2$ by $p(u) = (u, h(u))$. This is the Cartesian representation of S. Observe that

$$p'(u_1)u = (u, h'(u_1)u), \quad \forall\, u \in H_1.$$

It follows from (3) that

$$\hat{g}_1'(u_0,v_0)\cdot u + \hat{g}_2'(u_0,v_0)\cdot[h'(u_0)u] = 0$$

and since $\hat{g}_1'(u_0,v_0)\cdot u = 0$ and $\hat{g}_2'(u_0,v_0) \neq 0$ we obtain $h'(u_0)u = 0$ for all $u \in H_1$. So

$$p'(u_0)u = (u,0).$$

Now given $\omega \perp g'(x_0)$ define

$$\alpha(t) = p(u_0 + \omega t) \quad \text{for} \quad |t| < \varepsilon,$$

where $\varepsilon > 0$ is chosen in such a way that $u_0+\omega t \in V_1$, for all $|t| < \varepsilon$. Clearly

$$\alpha(0) = p(u_0) = (u_0, h(u_0)) = (u_0,v_0)$$

$$\alpha'(t)\big|_{t=0} = p'(u_0)\cdot\omega = (\omega,0).$$ Q.E.D.

Proof of Proposition 1.1. x_0 being a critical point of $f|S$ gives

$$f'(x_0)\cdot\alpha'(0) = 0 \quad \text{(from (1))}$$

for all C^1 paths $\alpha: (-\varepsilon;\varepsilon) \to S$, with $\alpha(0) = x_0$. Then by Lemma 1.2

$$f'(x_0)\cdot\omega = 0 \quad \text{for all } \omega \perp g'(x_0)$$

which gives (2). Next observe that

(4) $$g'(x_0)\cdot\alpha'(0) = 0 \quad (\text{from } g(\alpha(t)) = 0)$$

for all C^1 paths $\alpha: (-\epsilon,\epsilon) \to S$ with $\alpha(0) = x_0$. So if (2) is assumed we obtain from (4) that $f'(x_0)\cdot\alpha'(0) = 0$ which shows that x_0 is a critical point of $f|S$. Q.E.D.

1.2 SPECTRAL ANALYSIS OF COMPACT SYMMETRIC OPERATORS

Let $T: H \to H$ be a compact symmetric linear operator in a Hilbert space H. [Inner product: (,). Norm $\|\cdot\|$]. An operator is said to be compact if it is continuous and takes bounded sets into relatively compact sets. [Of course such a definition makes sense for not necessarily linear operators between Banach spaces]. In the present case the notion of a compact operator is equivalent to that of complete continuity. An operator is said to be completely continuous if it takes weakly convergent sequences into strongly convergent ones.

Lemma 1.3. If

(5) $$\lambda_1 \equiv \sup_{\|x\|=1} (Tx,x) > 0$$

then there exists $\phi_1 \in H$, with $\|\phi_1\| = 1$, such that

(6) $$(T\phi_1,\phi_1) = \lambda_1, \quad T\phi_1 = \lambda_1\phi_1$$

Proof. Since $(Tx,x) \leq \|T\|$ for $\|x\| = 1$, we have $\lambda_1 < \infty$. Let us take $\|x_n\| = 1$ such that $(Tx_n,x_n) \to \lambda_1$. Passing to subsequences we may assume that $x_n \rightharpoonup \phi_1$ [$\rightharpoonup$ designates weak convergence, while $\longrightarrow$ means strong convergence] and $Tx_n \to T\phi_1$, in view of the compactness of T. So we obtain the first part of (6). For the second part we use Proposition 1.1 with $f(x) = (Tx,x)$ and $g(x) = \frac{1}{2}[\|x\|^2-1]$. Observe that $\|\phi_1\| \leq \liminf \|x_n\| = 1$ and in fact $\|\phi_1\| = 1$, for otherwise $x = \phi_1/\|\phi_1\|$ would lead to $(Tx,x) > \lambda_1$, which is impossible. So ϕ_1 is a critical point of $f|S$. Then there is $\lambda \in R$ such $T\phi_1 = \lambda\phi_1$. Taking inner product with ϕ_1 and using the first equality in (6) we obtain $\lambda = \lambda_1$. Q.E.D.

By completely similar arguments we prove

Lemma 1.3': If

(7) $$\lambda_{-1} = \inf_{\|x\|=1} (Tx,x) < 0$$

then there exists $\phi_{-1} \in H$, $\|\phi_{-1}\| = 1$, such that

(8) $$(T\phi_{-1},\phi_{-1}) = \lambda_{-1}, \quad T\phi_{-1} = \lambda_{-1}\phi_{-1}$$

Remark. Observe that λ_1 defined by (5) is the largest eigenvalue of T. Similarly λ_{-1} is the smallest one. This procedure can be repeated to obtain other eigenvalues of T, considering its restriction to the orthogonal complement of $R\phi_1$. This possibility rests on the following easily proved fact: "Let $V \subset H$ be a subspace of H such that $T(V) \subset V$. Then $T: V^{\perp} \to V^{\perp}$ is compact and symmetric". So we obtain

Proposition 1.4. If

(9) $$\lambda_n \equiv \sup \left\{(Tx,x) : \|x\| = 1,\ x \perp \phi_1,\dots,\phi_{n-1}\right\} > 0$$

then there exists $\phi_n \in H$, with $\|\phi_n\| = 1$, such that

$$(T\phi_n,\phi_n) = \lambda_n \qquad T\phi_n = \lambda_n\phi_n.$$

A similar statement holds when

(10) $$\lambda_{-n} \equiv \inf \left\{(Tx,x) : \|x\| = 1,\ x \perp \phi_{-1},\dots,\phi_{-(n-1)}\right\} < 0.$$

Before proceeding we collect in the following lemma some facts about compact linear operators. Proofs are immediate.

Lemma 1.5. (a) Eigenvectors corresponding to different eigenvalues are orthogonal.

(b) If λ is an eigenvalue of T, then the eigenspace $N(T-\lambda I)$ has finite dimension.

(c) The eigenvalues of T cannot accumulate at a point $\lambda \neq 0$.

Next we claim that all the eigenvalues $\neq 0$ of T are obtained by the process described in Proposition 1.4. To prove that we need the following result.

Lemma 1.6. For each $x \in H$ one has

(11) $$Tx = \sum_{i=-\infty}^{\infty}{}' \lambda_i (x,\phi_i)\phi_i .$$

[Σ' means that $i = 0$ does not appear in the summation].

Indeed suppose that there is an eigenvalue $0 \neq \lambda \neq \lambda_i$ for all i. By Lemma 1.5 (a) a corresponding eigenfunction ϕ ($T\phi = \lambda\phi$) is orthogonal to all ϕ_i. By (11) $T\phi = 0$, which is impossible. In the proof of Lemma 1.6 one needs

Lemma 1.7. Let $T: H \to H$ be a symmetric operator in H. Then

(12) $$\sup_{\|x\|=1} |(Tx,x)| = \sup_{\|x\|=1} \|Tx\|$$

[The expression in the right side of (12) is the norm $\|T\|$ of T, and let us denote by η the left side of (12)].

Proof. The only issue is the proof of $\|T\| \leq \eta$. For any $x,y \in H$ $\eta\|x+y\|^2 \geq (T(x+y),x+y) = (Tx,x) + (Ty,y) + 2(Tx,y)$, and $-\eta\|x-y\|^2 \leq (T(x-y),x-y) = (Tx,x) + (Ty,y) - 2(Tx,y)$. Subtracting the second inequality from the first we have

(13) $$4(Tx,y) \leq 2\eta(\|x\|^2 + \|y\|^2).$$

Now for $\|x\| = 1$, either $Tx = 0$ or $Tx \neq 0$. In the latter case take $y = Tx/\|Tx\|$ in (13) to obtain $\|Tx\| \leq \eta$, which holds for all $\|x\|=1$, in either case. Q.E.D.

Proof of Lemma 1.6. Let

$$x_n = x - \sum_{i=-n}^{n}{}' (x,\phi_i)\phi_i .$$

It suffices to prove that $Tx_n \to 0$. Since $x_n \perp \phi_i$ for $i = -n,\dots,-1,1,\dots,n$ we have by Proposition 1.4 and Lemma 1.7 that

$$\|Tx_n\| \le \max\{|\lambda_{-n}|, \lambda_n\}\|x_n\|.$$

So we have the conclusion because $\|x_n\|^2 = \|x\|^2 - \sum_{i=-n}^{n} (x,\phi_i)^2 \le \|x\|^2$.

Q.E.D.

Formula (9) for the n-th positive eigenvalue presents the inconvenience of requiring the knowledge of all previous positive ones. Similar statement for λ_{-n} in (10). The next result improves this situation.

__Proposition 1.8__. __For each positive__ n

(14) $$\lambda_n = \inf_{F_{n-1}} \sup_{\substack{\|x\|=1 \\ x \perp F_{n-1}}} (Tx,x)$$

__where the infimum is taken over all subspaces__ F_{n-1} __of__ H __with dimension__ n-1. __A similar formula holds for__ λ_{-n}

(15) $$\lambda_{-n} = \sup_{F_{n-1}} \inf_{\substack{\|x\|=1 \\ x \perp F_{n-1}}} (Tx,x)$$

__Proof__. Let us denote by Λ_n the right side of (14). Taking F_{n-1} = subspace generated by $\phi_1,\dots,\phi_{n-1}$, we see by Proposition 1.4 that $\Lambda_n \le \lambda_n$. On the other hand let $h_1,\dots,h_{n-1}$ be any system of mutually orthogonal vectors in H, and F_{n-1} the subspace generated by them. Choose a vector $\phi = \sum_{i=1}^{n} \alpha_i \phi_i$ in such a way that $(\phi,h_j) = 0$ for $j=1,\dots,n-1$, and normalize it, i.e. $\sum_{i=1}^{n} \alpha_i^2 = 1$. Now $\|\phi\| = 1$ and

$$(T\phi,\phi) = \sum_{i=1}^{n} \alpha_i^2 \lambda_i \ge \lambda_n.$$

which implies

$$\sup_{\substack{\|x\|=1 \\ x \perp F_{n-1}}} (Tx,x) \ge \lambda_n, \quad \forall F_{n-1},$$

and consequently $\Lambda_n \ge \lambda_n$. Q.E.D.

Formula (14) has still a set back since the sup (Tx,x) has to be taken over x in a subspace of infinite dimension. This situation is improved by the following result.

__Proposition 1.9__. __For each__ n __positive__

(16) $$\lambda_n = \sup_{F_n} \inf_{\substack{\|x\|=1 \\ x \in F_n}} (Tx,x)$$

__where the supremum is taken over all subspaces__ F_n __of__ H __with dimension__ n. __A similar statement holds for__ λ_{-n}

(17) $$\lambda_{-n} = \inf_{F_n} \sup_{\substack{\|x\|=1 \\ x \in F_n}} (Tx,x)$$

__Proof__. Let us denote by Γ_n the right side of (16). Taking F_n the subspace generated by $\phi_1,\dots,\phi_n$ one has for all $x = \sum_{i=1}^{n} \alpha_i \phi_i \in F_n$, $\|x\|^2 = \Sigma \alpha_i^2 = 1$, that

$$(Tx,x) = \sum_{1}^{n} \alpha_i^2 \lambda_i \ge \lambda_n$$

So $\Gamma_n \ge \lambda_n$. Conversely given any subspace F_n of dimension n, choose a $x \in F_n$ such that $x \perp \phi_1,\dots,\phi_{n-1}$. By Proposition 1.4, $(Tx,x) \le \lambda_n$. Consequently

$$\inf_{x\in F_n} (Tx,x) \le \lambda_n, \quad \forall F_n,$$

which implies $\Gamma_n \le \lambda_n$. Q.E.D.

Remark. Observe that all the inf's a sup's in the above characterizations of λ_n and λ_{-n} are actually assumed. So they could be replaced by min's and max's.

1.3 THE EIGENVALUE PROBLEM WITH AN INDEFINITE WEIGHT: THE VARIATIONAL FORMULATION

Let us consider the eigenvalue problem

(18) $$Lu = \mu m u \quad \text{in} \quad \Omega, \quad u = 0 \quad \text{on} \quad \partial\Omega,$$

where

$$Lu = -\sum_{i,j=1}^{N} \frac{\partial}{\partial x_j}\left(a_{ij}(x)\frac{\partial u}{\partial x_i}\right) + a_0(x)u$$

is a strongly elliptic operator in a bounded domain Ω of R^N, i.e., there is a constant $c_0 > 0$ such that

(19) $$\sum_{i,j=1}^{N} a_{ij}(x)\xi_i\xi_j \ge c_0|\xi|^2, \quad \forall\, x \in \bar\Omega, \quad \forall\, \xi \in R^N.$$

We assume $a_{ij} = a_{ji}$ and $a_0(x) \ge 0$ in Ω. The weight function $m:\bar\Omega \to R$ is assumed to be in $L^r(\Omega)$ with $r > N/2$. [In particular, this is the case if m is continuous in $\bar\Omega$. However we should allow discontinuous functions m viewing future applications]. We emphasize that m could change sign in Ω. One assumes $a_{ij} \in L^\infty(\Omega)$ and $a_0 \in L^{N/2}(\Omega)$.

Let us recall Poincaré's inequality

$$\|u\|_{L^2} \le c\|u\|_{H^1}, \quad \forall\, u \in H_0^1(\Omega),$$

which implies that the inner products

(20) $$\int \nabla u\cdot\nabla v \qquad \int [\nabla u\cdot\nabla v + uv]$$

are equivalent in H_0^1.

Problem (18) is studied here in its variational formulation:

(21) $$\begin{cases} a[u,v] = \mu\int muv, & \forall\, v \in H_0^1, \\ u \in H_0^1 \end{cases}$$

where $a: H_0^1 \times H_0^1 \to R$ is the bilinear form defined by

$$a[u,v] = \int\left[\sum a_{ij}(x)\frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_j} + a_0(x)uv\right]dx.$$

[Since $H_0^1 \subset L^{2^*}$, where $\frac{1}{2^*} = \frac{1}{2} - \frac{1}{N}$, we see, using the hypotheses on a_0 and m, that expressions like $\int muv$ and $\int a_0 uv$, for $u,v \in H_0^1$ are well defined]. This form defines in H_0^1 an inner-product equivalent to the original one [in (20)] since

$$a[u,v] = a[v,u], \quad |a(u,v)| \le c\|u\|_{H^1}\|v\|_{H_1}, \quad a[u,u] \ge c\|u\|^2_{H_1}.$$

Let us denote this inner-product by $(\cdot,\cdot)_a$ and corresponding norm by $\|\cdot\|_a$.

For fixed $u \in H_0^1$, the map $v \longmapsto \int muv$ is a bounded linear functional in H_0^1. So by the Riesz-Fréchet representation theorem, there is an element in H_0^1, denote it by Tu, such that

(22) $$(Tu,v)_a = \int muv.$$

Clearly $T: H_o^1 \to H_o^1$ is linear symmetric and bounded. Moreover T is compact; indeed let (u_n) be a bounded sequence in H_o^1. Passing to a subsequence we may assume that $u_n \rightharpoonup u$ in H_o^1, $u_n \to u$ in L^s, for $s < 2^*$, using the Rellich-Kondrachov theorem on compact imbeddings of Sobolev spaces. Use (22) with u replaced by u_n-u and v by Tu_n-Tu:

$$(Tu_n-Tu, Tu_n-Tu)_a = \int m(u_n-u)(Tu_n-Tu).$$

Using Hölder's inequality, with $\frac{1}{s} = 1 - \frac{1}{r} - \frac{1}{2^*}$, we get

$$\|Tu_n-Tu\|_a^2 \leq \|m\|_{L^r}\|u_n-u\|_{L^s}\|Tu_n-Tu\|_{L^{2^*}}$$

which gives

$$\|Tu_n-Tu\|_a \leq c\|u_n-u\|_{L^s} \Longrightarrow Tu_n \to Tu \text{ in } H_o^1.$$

Problem (21) may be rewritten as

$$(u,v)_a = \mu(Tu,v)_a \quad \forall\, v \in H_o^1$$

or

$$\mu Tu = u \quad \text{or} \quad Tu = \frac{1}{\mu}u.$$

So we can apply the theory developed in the previous section to describe the eigenvalues and eigenvectors (in H_o^1) of problem (18).

<u>Proposition 1.10</u>. <u>The eigenvalue problem</u> (21) <u>has a double sequence of eigenvalues</u>

$$\cdots \leq \mu_{-2} \leq \mu_{-1} < 0 < \mu_1 \leq \mu_2 \leq \cdots$$

<u>whose variational characterizations are</u>

(23) $$\frac{1}{\mu_n} = \sup_{F_n} \inf_{\substack{\|u\|_a=1 \\ u\in F_n}} \int mu^2, \quad \frac{1}{\mu_{-n}} = \inf_{F_n} \sup_{\substack{\|u\|_a=1 \\ u\in F_n}} \int mu^2,$$

<u>where F_n varies over all n-dimensional subspaces of H_o^1. The corresponding eigenfunctions ϕ_n are such that</u>

(24) $$a[\phi_n, v] = \mu_n \int m\phi_n v, \quad \forall\, v \in H_o^1$$

and

(25) $$a[\phi_n,\phi_n] = 1 \qquad \frac{1}{\mu_n} = \int m\phi_n^2.$$

By the results of the previous section the eigenvalues above do not accumulate, except at $+\infty$ or $-\infty$, and the situation is described by

<u>Proposition 1.11</u>. <u>Let</u> $\Omega_\pm = \{x \in \Omega: m(x) \gtrless 0\}$. <u>Then</u>

(a) $|\Omega_+| = 0 \Longrightarrow$ <u>there is no positive</u> μ_n.

(b) $|\Omega_-| = 0 \Longrightarrow$ <u>there is no negative</u> μ_{-n}.

(c) $|\Omega_+| > 0 \Longrightarrow$ <u>there is a sequence of positive</u> $\mu_n \to +\infty$

(d) $|\Omega_-| > 0 \Longrightarrow$ <u>there is a sequence of negative</u> $\mu_{-n} \to -\infty$.

[$|\cdot|$ denote the Lebesgue measure of the set].

<u>Proof</u>. (a) and (b) follow readily from Proposition 1.4. To prove (c), let $B_1,\ldots,B_n$ be a set pairwise disjoint balls in Ω such that the sets $B_j \cap \Omega_+$ have positive measure. Let $u_1,\ldots,u_n$ be C_o^∞ functions with supports in the corresponding balls and such that $\int mu_j^2 > 0$. [u_j can be obtained as an $L^{2r'}$-approximation to the

characteristic function of the set $B_j \cap \Omega_+$, where $1/r' + 1/r = 1$]. Let us normalize them so as to have $\int mu_j^2 = 1$ for $j = 1,\ldots,n$. We claim that $\mu_n > 0$. Let F_n be the subspace generated by the mutually orthogonal functions u_j. So for $u = \Sigma \alpha_j u_j \in F_n$ we have

$$\int mu^2 = \sum_1^n \alpha_j^2 \int mu_j^2 = \sum_1^n \alpha_j^2$$

and

$$a[u,u] = \sum_1^n \alpha_j^2 a[u_j,u_j] \le c\Sigma\alpha_j^2$$

where $c = \max_{j=1,\ldots,n} a[u_j,u_j]$. Hence

$$\int mu^2 \ge \frac{1}{c} a[u,u], \quad \forall\, u \in F_n.$$

The result follows from this and (23). A similar proof for (d).

Q.E.D.

The next two results explain how $\mu_n = \mu_n(m)$ varies as a function of m. Both of them follow readily from Proposition 1.10.

Proposition 1.12 A. Let $m, \hat{m}: \bar{\Omega} \to R$ be L^r functions, with $r > N/2$, such that $m(x) \le \hat{m}(x)$ for $x \in \Omega$. Let us suppose that for a given n, $n = \pm 1, \pm 2, \ldots$, the eigenvalues $\mu_n(m)$ and $\mu_n(\hat{m})$ exist. Then

$$\mu_n(m) \ge \mu_n(\hat{m}). \tag{26}$$

Moreover, if $m(x) < \hat{m}(x)$ on a subset of positive measure in Ω, then one has strict inequality in (26).

Proposition 1.12 B. $\mu_n(m)$ is a continuous function of m, in the norm of $L^{N/2}(\Omega)$. In other words, if $m_j \in L^r(\Omega)$ converges in the $L^{N/2}$ - norm to $m \in L^r(\Omega)$, then $\mu_n(m_j)$ converges to $\mu_n(m)$.

Remark. If $m(x) \equiv 1$ in Ω we use the following notation $\mu_j(1) = \lambda_j$, $j = 1,2,\ldots$. Observe that there are no λ_j with negative j. It is easy to see that

$$\mu_j(\alpha) = \lambda_j \alpha^{-1} \quad \text{for } \alpha > 0.$$

This relation will be used extensively later on for values of α being equal to eigenvalues λ_k.

1.4. A THEOREM OF THE KREIN-RUTMAN TYPE

Theorem 1.13. Let $m: \Omega \to R$ be an L^r-function, with $r > N/2$, (not necessarily positive). Suppose that $m > 0$ on a subset of Ω with positive measure. Then the first positive eigenvalue μ_1 of (21) is simple, and ϕ_1 can be taken > 0 in Ω. A similar statement holds if $m < 0$ on a subset of Ω with positive measure.

Remark. The above result is due to Manes-Micheletti [13]. A similar result for non self-adjoint problems with m continuous has been proved recently by Hess-Kato [6].

Proof. First step. Let $\omega \in H_o^1$ be an eigenfunction corresponding to the first positive eigenvalue μ_1. At this stage we do not know whether or not some other μ_j's are equal to μ_1, and consequently ω

may be a linear combination of ϕ_j's. In any case ω is a solution of

(27) $$a[\omega,v] = \mu_1\int m\omega v, \quad \forall\ v \in H_0^1.$$

We first claim that ω does not change sign in Ω. Indeed, suppose it does, and let $\omega^+ = \max(\omega,0)$ and $\omega^- = \min(\omega,0)$. We know that ω^+ and ω^- are also in $H_0^1(\Omega)$, see Stampacchia [14]. So

$$\int m\omega^2 = \int m(\omega^+)^2 + \int m(\omega^-)^2 \equiv \alpha_1+\alpha_2$$

$$a[\omega,\omega] = a[\omega^+,\omega^+] + a[\omega^-,\omega^-] \equiv \beta_1 + \beta_2, \quad \beta_1,\beta_2 > 0.$$

A simple arithmetic shows that either

$$\text{(i)}\ \frac{\alpha_1+\alpha_2}{\beta_1+\beta_2} = \frac{\alpha_1}{\beta_1} = \frac{\alpha_2}{\beta_2}, \quad \text{or} \quad \text{(ii)}\ \frac{\alpha_1+\alpha_2}{\beta_1+\beta_2} < \max\left(\frac{\alpha_1}{\beta_1}, \frac{\alpha_2}{\beta_2}\right).$$

Applying this to

$$\frac{1}{\mu_1} = \frac{\int m\omega^2}{a[\omega,\omega]} = \frac{\int m(\omega^+)^2 + \int m(\omega^-)^2}{a[\omega^+,\omega^+] + a[\omega^-,\omega^-]}$$

we conclude that ω^+ and ω^- are also eigenfunctions corresponding to μ_1. So they are solution of (27) and accordingly to a result of Stampacchia [15, Corollary 8.1 p. 238] $\omega^+ > 0$ in Ω(a.e.) and $\omega^- < 0$ in Ω(a.e.), which is impossible. The fact that ϕ_1 can be taken > 0 in Ω it follows from the afore-mentionned result by Stampacchia.

<u>Second step</u>. We now prove that the geometric multiplicity of μ_1 is 1. Indeed, let ω_1 and ω_2 be eigenfunctions corresponding to μ_1. By the previous step we know that for each $\alpha \in R$ the eigenfunction $\omega_1 + \alpha\omega_2$ has a definite sign. So the sets

$$A = \{\alpha \in R: \omega_1 + \alpha\omega_2 \geq 0\} \quad \text{and} \quad B = \{\alpha \in R: \omega_1 + \alpha\omega_2 \leq 0\}$$

are non-empty, closed and $A \cup B = R$. Consequently there is $\bar\alpha \in A \cap B \longrightarrow \omega_1 + \bar\alpha\omega_2 = 0$, which says that ω_1 and ω_2 are linearly dependent.

<u>Third step</u>. The algebraic multiplicity of μ_1 is 1, i.e., μ_1 is a simple eigenvalue. Indeed, referring to Section 3, our claim is to prove that $N(I-\mu_1T)^2 = N(I-\mu_1T)$. Let $u \in N(I-\mu_1T)^2$. Then $u-\mu_1Tu=t\phi_1$ for some $t \in \mathbb{R}$. Taking inner product with ϕ_1: $(u-\mu_1Tu,\phi_1)_a = (u,\phi_1-\mu_1T\phi_1)_a = 0 = t(\phi_1,\phi_1)_a$, which implies $t = 0$. So $u\in N(I-\mu_1T)$ and the proof is complete. Q.E.D.

1.5 <u>A MAXIMUM PRINCIPLE</u>.

Let us start with some considerations on the Dirichlet problem

(28) $$Lu = h \ \text{ in } \ \Omega, \quad u = 0 \text{ on } \partial\Omega$$

where the operator L has been introduced in Section 3. We shall discuss the variational formulation of (28) namely

(29) $$\begin{cases} a[u,v] = \int hv, \quad \forall\ v \in H_0^1 \\ u \in H_0^1(\Omega) \end{cases}$$

For the needs of the present discussion we assume that $h \in L^\sigma$ where $\sigma = 2N/(N+2)$. Under this assumption it follows from the Sobolev imbedding theorem that the map $v \longmapsto \int hv$ is a bounded linear functional in H_0^1. So by the Riesz representation theorem there exists $\bar h \in H_0^1$ such that $\int hv = (\bar h,v)_a$, $\forall\ v \in H_0^1$. Such an $\bar h$

is the unique solution of (29) [recall that the bilinear form a is coercive in H_0^1]. In this way we have defined a bounded linear mapping $S: L^\sigma \to H_0^1$ by $S(h) = \bar{h}$. Applying Theorem 8.1 [16, p. 168] we see that $\bar{h} \geq 0$ whenever $h \geq 0$. Observe that this statement is a maximum principle and it has a crucial assumption: $a_0(x) \geq 0$ in Ω. The next result shows how this hypothesis can be relaxed somehow.

Theorem 1.14. Let $u \in H_0^1(\Omega)$ be a solution of

(30) $$Lu - \lambda mu = h \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega$$

where $h \in L^\sigma(\Omega)$ and $h \geq 0$ in Ω. Suppose that $m \in L^\infty(\Omega)$ is positive on a subset of Ω with positive measure. Assume also that $0 \leq \lambda < \mu_1(m)$. Then $u \geq 0$ in Ω. Moreover, if $h > 0$ in a set of positive measure then $u > 0$ in Ω.

Proof. a) (30) is to be understood in the weak form

(31) $$a[u,v] - \lambda \int muv = \int hv, \quad \forall\, v \in H_0^1.$$

Without loss of generality we may assume that $|m(x)| < 1$ for x and that fact will be used later. (31) may be rewritten as

(32) $$a_\lambda[u,v] = \lambda \int (m+1)uv + \int hv, \quad \forall\, v \in H_0^1$$

where $a_\lambda[u,v] = a[u,v] + \lambda \int uv$.

b) Given $\lambda \geq 0$ let us denote by $\mu_{1,\lambda}$ the first positive eigenvalue of

$$\begin{cases} a_\lambda[u,v] = \mu \int (m+1)uv, & \forall\, v \in H_0^1 \\ u \in H_0^1 \end{cases}$$

We claim that $0 \leq \lambda < \mu_1(m) \Longrightarrow \lambda < \mu_{1,\lambda}$. [To simplify the notation: $\alpha = \mu_{1,\lambda}$] Indeed, suppose by contradiction that $\alpha \leq \lambda$. Then

$$\frac{1}{\alpha} = \sup \frac{\int (m+1)v^2}{a[v,v] + \lambda \int v^2} \leq \sup \frac{\int (m+1)v^2}{a[v,v] + \alpha \int v^2}$$

Hence, for any given $\varepsilon > 0$ there is a $v \in H_0^1$ such that

$$\frac{1}{\alpha} - \varepsilon \leq \frac{\int (m+1)v^2}{a[v,v] + \alpha \int v^2} \Longrightarrow \left(\frac{1}{\alpha} - \varepsilon\right) - \alpha\varepsilon \frac{\int v^2}{a[v,v]} \leq \frac{\int mv^2}{a[v,v]}$$

By the variational characterization of the eigenvalues we get

$$\left(\frac{1}{\alpha} - \varepsilon\right) - \alpha\varepsilon \frac{1}{\mu_1(1)} \leq \frac{1}{\mu_1(m)}, \quad \forall\, \varepsilon > 0,$$

which implies $\mu_1(m) \leq \alpha \leq \lambda < \mu_1(m)$: impossible!

c) In analogy with the mapping S defined in the beginning of the present section let us denote by S_λ the operator associated with the bilinear form a_λ. Then (32) may be written as

$$u = \lambda S_\lambda((m+1)u) + S_\lambda h.$$

Let $W: H_0^1 \to H_0^1$ be the operator defined by $W(u) = \lambda S_\lambda((m+1)u)$. This operator is compact in view of the compact imbedding $H_0^1 \subset L^\sigma$. By part b) we see that the spectral radius of W is less than 1. So

(33) $$u = (I-W)^{-1} S_\lambda h = \sum_{j=0}^{\infty} W^j(S_\lambda h)$$

and the positivity of u follows from the positivity of S_λ and W. The last statement follows readily from the strong maximum principle. Q.E.D.

The result of the prceding theorem is sharp in the sense that problem (30) cannot have a positive solution if $\lambda > \mu_1(m)$. In fact, we have

Proposition 1.15. Let $u \in C^1(\bar{\Omega})$ be a solution of (30), where $h \in L^\sigma(\Omega)$ and $h \geq 0$ in Ω. Suppose that $m \in L^\infty(\Omega)$ is positive on a set of positive measure, and $\lambda \geq \mu_1(m)$. Then $u > 0$ [i.e., $u(x) \geq 0$ in Ω and $u(x') > 0$ at some point $x' \in \Omega$] implies $\lambda = \mu_1(m)$, $h = 0$ and $u = t\phi_1$, where ϕ_1 is the eigenfunction of (21) corresponding to $\mu_1(m)$.

Remark. If the solution $u \in H_0^1$ and the coefficients of L have the appropriate regularity for the use of the L^p-theory for the Dirichlet problem we can conclude that in fact $u \in C^1(\bar{\Omega})$.

In the proof of Proposition 1.15 we use the following form of the strong maximum principle.

Proposition 1.16. Let $u \in C^1(\bar{\Omega})$ and assume that $a[u,\psi] \leq 0$ for all $\psi \in H_0^1(\Omega)$, $\psi \geq 0$. Suppose that $\partial\Omega$ satisfies the interior sphere condition at $x_0 \in \partial\Omega$. Then the outward normal derivative $\frac{\partial u}{\partial \nu}(x_0)$ is > 0, provided u is not constant in Ω, and $u(x_0) \geq 0$. The latter restriction is not necessary when $a_0(x) \equiv 0$.

Proof. Let $B_R(y) \subset \Omega$ be an open ball of radius R centered at y and

such that $\{x_0\} = \partial B_R \cap \partial\Omega$. Let $0 < \rho < R$ be fixed and consider the auxiliary function

$$v(x) = e^{-\alpha r^2} - e^{-\alpha R^2}, \quad r = |x-y|,$$

defined in the ring $\Gamma = B_R(y) \setminus \overline{B_\rho(y)}$. α will be choosen later. Let $\omega(x) = u(x) - u(x_0) + \varepsilon v(x)$, for some small $\varepsilon > 0$, in Γ. We assume that u is not constant. By the strong maximum principle [16, Theorem 8.19, p. 188] it follows that $u(x_0) > u(x)$ for all $x \in \Gamma \setminus \{x_0\}$. So on $\partial B_R(y)$ we have $\omega(x) \leq 0$ and on $\partial B_\rho(y)$ we can choose ε small enough so that $\omega(x) < 0$. On the other hand, an easy computation shows that for $\alpha > 0$ large $a[\omega,\psi] \leq 0$ for $\psi \in H_0^1$, $\psi \geq 0$. (Proceed as in [16, p. 33] to show that $a[v,\psi] \leq 0$; here we have to integrate by parts throwing all the derivatives over v). Using again Theorem 8.19 of [16] one concludes that $\omega(x) < 0$ on Γ. Then normal derivative

$$\frac{\partial \omega}{\partial \nu}(x_0) \geq 0 \longrightarrow \frac{\partial u}{\partial \nu}(x_0) \geq -\varepsilon \frac{\partial v}{\partial \nu}(x_0) > 0. \qquad \text{Q.E.D.}$$

Proof of Proposition 1.15. Equation (30) may be written as (32), from which follows $a_\lambda[u,v] \geq 0$ for all $0 \leq v \in H_0^1$. Proposition 1.16 then implies that $u > t\phi_1$ for some $t > 0$. Let $t_0 = \sup\{t \geq 0 : u \geq t\phi_1\}$. One has

$$a[t_0\phi_1, v] + \lambda \int t_0\phi_1 v = \mu_1(m) \int m t_0 \phi_1 v + \lambda \int t_0 \phi_1 v \leq \lambda \int (m+1) t_0 \phi_1 v, \quad \forall\ 0 \leq v \in H_0^1.$$

Subracting this expression from (32) and denoting by $\omega = u - t_0\phi_1$ one has

$$a[\omega, v] + \lambda \int \omega v \geq \lambda \int (m+1) \omega v + \int h v, \qquad \forall\ 0 \leq v \in H_0^1.$$

Applying again Proposition 1.16 we obtain $\frac{\partial\omega}{\partial\nu} < 0$. But this implies that for $t' > 0$ sufficiently small $\omega > t'\phi_1 \longrightarrow u > (t_o + t')\phi_1$, contradicting the maximality of t_o. Q.E.D.

The conclusion of Theorem 1.14 can be strengthened if we assume regularity on the coefficients of L to apply the L^p theory of the Dirichlet problem.

Theorem 1.17. Same assumptions of Theorem 1.14 plus the following: (i) regularity of the coefficients of L, (ii) $h \in L^p(\Omega)$ with $p > N$. Then the solution $u \in C^{1,\alpha}(\bar{\Omega})$ and $\frac{\partial u}{\partial \nu} < 0$ on $\partial\Omega$ if $h > 0$ in a set of positive measure in Ω.

Proof. a) By a standard bootstrap argument we conclude that $u \in C^{1,\alpha}(\bar{\Omega})$. b) If $h \in L^p(\Omega)$ with $p > N$ then Sh (see beginning of this Section 5) belongs to $C^{1,\alpha}(\bar{\Omega})$ and Proposition 1.16 can be applied to conclude that $\frac{\partial}{\partial\nu}(Sh) < 0$ on $\partial\Omega$, if $h \geq 0$. c) Using part b) we conclude that the operator W (see proof of Theorem 1.14) has a similar property. Finally the result follows using (33).

Q.E.D.

CHAPTER 2

SUBLINEAR ELLIPTIC PROBLEMS

2.1 EXISTENCE OF POSITIVE SOLUTIONS. Let us start posing the following problem. Let $g: \bar{\Omega}\times R^+ \to R$ be a C^1 function. Suppose

(1) $$\liminf_{s\downarrow 0} \frac{g(x,s)}{s} > \lambda_1$$

and

(2) $$\limsup_{s\to+\infty} \frac{g(x,s)}{s} < \lambda_1$$

Prove that the Dirichlet problem

(3) $$-\Delta u = g(x,u) \text{ in } \Omega,\ u = 0 \text{ on } \partial\Omega$$

has a positive solution $u \in C^{2,\alpha}(\Omega)$.

We prove a result which includes the problem above as a special case. Besides the theory of linear eigenvalue problems with an indefinite weight function developed in the previous chapter, the other main tool is the method of sub and supersolutions, which we expound next.

Let L be an elliptic operator as in Section 3 of the previous chapter. We make all the assumptions on Ω and on the coefficients of L in order to be able to apply Schauder's theory and the L^p theory of the Dirichlet problem. Let $g: \bar{\Omega}\times R^+ \to R$ be a C^α function (i.e. a Hölder continuous function with exponent $0<\alpha<1$). We consider the Dirichlet problem

(4) $$Lu = g(x,u) \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega.$$

A function $\bar{u} \in C^2(\bar{\Omega})$ is said to be a supersolution of (4) if

$$L\bar{u} \geq g(x,\bar{u}) \text{ in } \Omega, \quad \bar{u} \geq 0 \text{ on } \partial\Omega.$$

A function $\underline{u} \in C^2(\bar{\Omega})$ is said to be a subsolution of (4) if

$$L\underline{u} \leq g(x,\underline{u}) \text{ in } \Omega, \quad \underline{u} \leq 0 \text{ on } \partial\Omega.$$

The result below is due to Amann [17] and [2], see also Sattinger [18] and [19].

Theorem 2.1. Suppose that problem (4) has a subsolution $\underline{u}$ and a supersolution $\bar{u}$, with $\underline{u} \leq \bar{u}$. Assume that $g: \bar{\Omega}\times R^+ \to R$ is a C^α function and that there is a constant $k \geq 0$ such that

(5) $$g(x,s_1) - g(x,s_2) \geq -k(s_1-s_2)$$

for all $x \in \bar{\Omega}$ and $s_1 > s_2$ with $|s_1|,|s_2| \leq \max(\|\underline{u}\|_{C^0}, \|\bar{u}\|_{C^0})$. Then problem (4) has solutions $U,V \in C^{2,\alpha}(\bar{\Omega})$ such that $\underline{u} \leq U \leq V \leq \bar{u}$. These solutions are obtained by an iteration scheme (see the Proof). Moreover any solution u of (4) with $\underline{u} \leq u \leq \bar{u}$ is such that $U \leq u \leq V$. [It is not stated that $U \neq V$].

Proof. By the Schauder theory, for each given $u \in C^{2,\alpha}(\bar{\Omega})$ there is a unique $\omega \in C^{2,\alpha}(\bar{\Omega})$ such that

(6) $$L\omega + k\omega = g(x,u) + ku \quad \text{in} \quad \Omega, \quad \omega = 0 \text{ on } \partial\Omega.$$

This defines a mapping $T: C^{2,\alpha}(\bar{\Omega}) \to C^{2,\alpha}(\bar{\Omega})$ which is monotone in the interval $[\underline{u},\bar{u}]$, that is, if $\underline{u} \leq u_1,u_2 \leq \bar{u}$, then

(7) $$u_1 \leq u_2 \longrightarrow Tu_1 \leq Tu_2$$

Indeed, writing (6) with $u = u_1$ and $u = u_2$ and subtracting one from the other we obtain

$$L(\omega_1-\omega_2) + k(\omega_1-\omega_2) = g(x,u_1) - g(x,u_2) + k(u_1-u_2).$$

Hence using (5) and the maximum principle we obtain (7). For the construction of solution U and V proceed by iteration as follows

$$u_n = Tu_{n-1}, \quad u_0 = \underline{u} \quad ; \quad v_n = Tv_{n-1}, \quad v_0 = \bar{u}.$$

We claim that

$$\underline{u} = u_0 \leq u_1 \leq u_2 \leq \dots \leq v_2 \leq v_1 \leq v_0 = \bar{u}, \quad \text{in } \Omega.$$

The proof of the monotonicity of the sequences (u_n) and (v_n) is done by induction. First observe that

$$L(\underline{u}-u_1)+k(\underline{u}-u_1) = L\underline{u}+k\underline{u} - [g(x,\underline{u})+k\underline{u}] \leq 0, \text{ in } \Omega$$

and

$$\underline{u} - u_1 = \underline{u} \leq 0 \quad \text{on } \partial\Omega.$$

This implies by the maximum principle that $\underline{u}-u_1 \leq 0$ in $\bar{\Omega}$. A similar argument gives $v_1 \leq \bar{u}$. The monotonicity of T gives the rest. So there are functions U and V such that

(8) $$u_n \to U \quad \text{and} \quad v_n \to V \qquad \text{pointwisely.}$$

By the Lebesgue convergence theorem, it follows that U and V are in $L^p(\Omega)$ and the convergence in (8) is in the L^p sense for any $p \geq 1$. Using the a priori estimates for solutions of linear elliptic equations in the Sobolev spaces we get

$$\|u_n-u_m\|_{W^{2,p}} \leq c\left\{\|g(x,u_{n-1}) - g(x,u_{m-1})\|_{L^p} + \|u_{n-1}-u_{m-1}\|_{L^p}\right\}$$

where C is a constant independent of n and m. A similar expression for the sequence (v_n). So the convergence in (8) is in the norm of $W^{2,p}(\Omega)$. Next we use the Sobolev imbedding theorem (take $p > N$) to conclude that U and V are in $C^{1,\alpha}(\bar{\Omega})$ and the convergence in (8) is in the norm of $C^{1,\alpha}(\bar{\Omega})$. By the Schauder estimates it follows that U and V are in $C^{2,\alpha}(\bar{\Omega})$ and the convergence in (8) is in the norm of $C^{2,\alpha}(\bar{\Omega})$. So U and V are solutions of (4). The remaining assertions of the theorem follow easily from (7). Q.E.D.

Now let us consider the Dirichlet problem (4) with a C^{α} function $g: \bar{\Omega}\times R^+ \to R$ which satisfies the following conditions:

(c_0) There is a continuous function $g_0: \bar{\Omega} \to R$, with $g_0(x) \geq 0$, and $s_0 > 0$ such that

$$g(x,s) \geq g_0(x)s, \quad \forall\, x \in \bar{\Omega}, \quad \forall\, 0 < s < s_0.$$

(c_∞) There are continuous functions g_∞, $c: \bar{\Omega} \to R$, with $c(x) \geq 0$ such that

$$g(x,s) \leq g_\infty(x)s + c(x), \quad \forall\, x \in \bar{\Omega}, \quad \forall\, s \geq 0.$$

Theorem 2.2. Suppose that g satisfies the hypotheses just stated. Assume also that

(9) $$\mu_1(g_0) < 1$$
and
(10) $$\mu_1(g_\infty) > 1$$

[The latter assumption is made only if $g_\infty(x_0) > 0$ for some $x_0 \in \Omega$. Otherwise it should be dropped, and the remaining assumptions on g will suffice]. Then the Dirichlet problem (4) has a positive solution.

Remarks. 1) Conditions (c_0) and (9) are satisfied, for instance, if $\lim_{s\downarrow 0} g(x,s)s^{-1} = +\infty$. In particular if $g(x,0) > 0$.

2) If $g(x,0) = 0$, then $u \equiv 0$ is a trivial solution.

3) The function g could assume negative values. If there is an $s_0 > 0$ such that $g(x,s_0) \leq 0$ for all $x \in \Omega$, then (c_∞) and (10) are superfluous. Indeed, in this case $\bar{u} \equiv s_0$ is a supersolution of (4).

4) Hypotheses (c_0) and (9) in the previous theorem implies that $g_0(x) > 0$ on a subset of positive measure in Ω.

5) Theorem 2.2 is essentially due to Amann [2].

Examples of functions g satisfying the assumptions of Theorem 2.2: i) $g(x,s) = \lambda e^{-s}$ for any $\lambda > 0$. ii) $g(x,s) = \lambda s - f(x,s)$, where $\lambda > \mu_1(1)$ and $f: \bar{\Omega}\times R^+ \to R$ is a C^{α} function such that $\lim_{s\downarrow 0} f(x,s)s^{-1} = 0$ and $\lim_{s\to\infty} f(x,s)s^{-1} = +\infty$. iii) the function g of the Problem in the beginning of the present section.

Proof of Theorem 2.2. a) Suppose that $g_\infty(x_0) > 0$ for some $x_0 \in \Omega$. Choose C^1 functions $\hat{g}_\infty$, $\hat{c}: \bar{\Omega} \to R$, with $\hat{c}(x) \geq 0$, such that

(11) $$g(x,s) \leq \hat{g}_\infty(x)s + \hat{c}(x), \quad \forall\, x \in \bar{\Omega}, \quad \forall\, s \geq 0$$

and $\mu_1(\hat{g}_\infty) > 1$. This is possible in view of (10), using Proposition 1.12B. Now let $v \in C^{2,\alpha}(\bar{\Omega})$ be the solution of the Dirichlet problem

(12) $Lv = \hat{g}_\infty(x)v + \hat{c}(x)$ in Ω, $v = 0$ on $\partial\Omega$.

It follows from Theorems 1.14 and 1.17 that $v > 0$ in Ω, and $\frac{\partial v}{\partial \nu} < 0$ on $\partial\Omega$. Expressions (11) and (12) show that v is a supersolution of (4).

b) If $g_\infty(x) \leq 0$ for all $x \in \Omega$, we can proceed as in part a) above taking $\hat{g}_\infty(x) = \varepsilon$, with $0 < \varepsilon < \mu_1(1)$.

c) Choose a C^1 function $g_0: \bar{\Omega} \to R$ such that

(13) $g(x,s) \geq \hat{g}_0(x)s \quad \forall\, x \in \bar{\Omega}, \quad \forall\, 0 \leq s \leq s_0$

and $\mu_1(\hat{g}_0) < 1$. This is a consequence of Proposition 1.12B.

Now the eigenvalue problem

$$Lu = \mu\hat{g}_0(x)u \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega$$

has a positive eigenfunction $\psi \in C^{2,\alpha}(\bar{\Omega})$ corresponding to the first eigenvalue $\mu_1(\hat{g}_0)$. Let $t > 0$ be chosen in such a way that $t\psi < s_0$ and $t\psi < v$. Then using (c_0) and (13) we have

$$L(t\psi) = \mu_1(\hat{g}_0)\hat{g}_0 t\psi < g(x,t\psi),$$

which shows that $t\psi$ is a subsolution of (4). So we can apply Theorem 2.1 to finish the proof. Q.E.D.

Let us now apply Theorem 2.2 in the discussion of the nonlinear eigenvalue problem

(14) $Lu = \lambda f(x,u)$ in Ω $u = 0$ on $\partial\Omega$

where $f: \bar{\Omega}\times R^+ \to R$ is a C^α function, and let us suppose that the limits below are continuous functions:

$$f_0(x) = \liminf_{s \downarrow 0} \frac{f(x,s)}{s}, \quad f_\infty(x) = \limsup_{s \to +\infty} \frac{f(x,s)}{s}.$$

Then the following statements [warning: these are just sufficient conditions for the solvability of (14)] are immediate consequences of Theorem 2.2.

a) If $f_0(x) \equiv +\infty$ (in particular if $f(x,0) > 0$) and $f_\infty(x) \leq 0$, then problem (14) has a positive solution for all $\lambda > 0$.

b) If $f_0(x) \equiv +\infty$ and $f_\infty(x_0) > 0$ for some point $x_0 \in \Omega$, then problem (14) has a positive solution for all $0 < \lambda < \mu_1(1)/[\sup f_\infty(x)]$

c) If $0 < f_0(x) < +\infty$ in $\bar{\Omega}$ and $f_\infty(x) \leq 0$, then problem (14) has a positive solution for all $\lambda > \mu_1(1)/[\inf f_0(x)]$.

d) If $0 < f_0(x) < +\infty$ in $\bar{\Omega}$ and $f_\infty(x_0) > 0$ for some $x_0 \in \Omega$, then problem (14) has a solution for all $\mu_1(1)/[\inf f_0(x)] < \lambda < \mu_1(1)/[\sup f_\infty(x)]$. Of course if $\inf f_0 < \sup f_\infty$, we cannot say anything about the existence of positive solutions of (14), at least by the present method. Observe that the above bounds on λ for existence are not necessarily sharp. Compare with [20].

2.2 <u>UNIQUENESS OF THE POSITIVE SOLUTION</u>. In this section we establish some uniqueness results for solutions of problem (4) involving some special type of nonlinearities g. Besides other assumptions on g we always assume

(15) the function $s \longrightarrow g(x,s)/s$, for $s > 0$ is non-increasing.

It is easy to see that (15) holds for instance when $g(x,s)$ is concave in s and $g(x,0) \geq 0$.

Lemma 2.3. Under the hypotheses of Theorem 2.2 problem (4) has a maximal positive solution.

Remark. If $g(x,0) > 0$ then problem (4) has also a minimal positive solution. This follows readily from Theorem 2.1 using the fact that $\underline{u} \equiv 0$ is a subsolution which is not a solution.

Proof of Lemma 2.3. We claim that any positive solution u of (4) is such that $u \leq v$ where v is the supersolution given in (12). Once this is done we use Theorem 2.1 to conclude that $u \leq V$, and then V is the maximal solution of (4). To prove the claim we use (11) to see that

$$g(x,u) = \hat{g}_\infty(x)u + d(x), \quad d(x) \leq \hat{c}(x)$$

and consequently

$$L(v-u) = \hat{g}_\infty(x)(v-u) + \hat{c}(x) - d(x).$$

Since $\mu_1(\hat{g}_\infty(x)) > 1$ and $\hat{c}(x)-d(x) \geq 0$ we can use Theorem 1.14 to conclude that $v-u \geq 0$ as required. Q.E.D.

Theorem 2.4. Assume that the hypotheses of Theorem 2.2 and (15) above hold. Suppose also that

(16) the function $\gamma(x,s) = g(x,s)/s$ is uniformly continuous for $x \in \bar{\Omega}$, $0 < s \leq s_o$, for some $s_o > 0$. Then problem (4) has a unique positive solution.

Remark. The function γ can be extended continuously to $s = 0$ by $\gamma(x,0) = \lim g(x_n,s_n)/s_n$, where $\{(x_n,s_n)\}$ is any sequence converging to $(x,0)$. Hypothesis (16) is satisfied for instance if we assume $g \in C^1(\bar{\Omega}\times R^+)$ and $g(x,0) = 0$ for all $x \in \Omega$; in this case $\gamma(x,0) = g_s'(x,0)$, which follows from the mean value theorem.

Proof of Theorem 2.4. Let u_1 and u_2 be two positive solutions of (4). Without loss of generality we may assume $u_1 \leq u_2$, in virtue of Lemma 2.3. Then

(17) $$Lu_i = m_i(x)u_i \quad \text{in } \Omega \qquad u_i = 0 \quad \text{on } \partial\Omega$$

for $i = 1,2$, where

$$m_i(x) = \begin{cases} \dfrac{g(x,u_i(x))}{u_i(x)} & \text{in } \{x\in\Omega:\ u_i(x) \neq 0\} \\ \gamma(x,0) & \text{in } \{x\in\bar{\Omega}:\ u_i(x) = 0\} \end{cases}$$

Clearly m_i is a continuous function in $\bar{\Omega}$ and it is positive on a subset of positive measure in Ω. Moreover in view of (15) we see that $m_1 \geq m_2$ and $u_1 \neq u_2$ implies that $m_1 > m_2$ on a subset of positive measure. It follows from Proposition 1.12A that $\mu_1(m_1) < \mu_1(m_2)$. But from (17) and the fact that $u_i \geq 0$ we arrive to the contradictory statement that $\mu_1(m_1) = \mu_1(m_2) = 1$. Q.E.D.

Assumption (16) in Theorem 2.4 can be relaxed as follows.

Theorem 2.5. Assume that the hypotheses of Theorem 2.2 and (15) hold. Suppose also that

(18) any positive solution u of (4) is such that $u(x) > 0$ for all $x \in \Omega$, and

(19) there are constants $c > 0$, $s_0 > 0$ and $0 < \alpha < 2/N$ such that $g(x,s)/s \leq cs^{-\alpha}$, for $0 < s < s_0$.

Then problem (4) has a unique positive solution.

Remark. Hypothesis (18) is verified for instance if $g(x,s) \geq 0$ for all $x \in \bar{\Omega}$, $s \geq 0$. [That is a consequence of the strong maximum principle]. Observe that hypothesis (19) is weaker than (16). While (16) requires that $g(x,s)$ vanishes at $s = 0$ like const.s we see that (19) allows a behavior like const.s^{β}, with $0 < \beta < 1$. An analysis of the proof below shows that (18) can be weakened, being replaced by the requirement that $u(x) > 0$ in Ω (a.e.).

Proof of Theorem 2.5. Proceed as in the proof of Theorem 2.4. However the continuity of m_1 has to be replaced by the assertion that $m_1 \in L^r(\Omega)$ for some $r > N/2$, and then Proposition 1.12A is used again. In view of (19) we can choose $r > N/2$ and $r < 1/\alpha$.

Q.E.D.

2.3 SOME MULTIPLICITY RESULTS FOR SUBLINEAR PROBLEMS. In this section we show how to use the previous results to obtain some results on the existence of multiple solution for the Dirichlet problem

(20) $$Lu = \lambda u - f(x,u) \quad \text{in} \quad \Omega, \quad u = 0 \text{ on } \partial\Omega,$$

where L is the second order elliptic operator introduced in Section 1.3, [we also assume smoothness of $\partial\Omega$ and of the coefficients of L in order to be able to apply the Schauder theory and the L^p theory of the linear Dirichlet problem] and $f: \bar{\Omega}\times R \to R$ is a C^1 function satisfying the following assumptions:

(21) $$f(x,0) = 0 \qquad f'_s(x,0) = 0$$

(22) the function $s \longrightarrow \frac{f(x,s)}{s}$ is nonincreasing in the half-line

$(-\infty,0)$ and nondecreasing in $(0,\infty)$.

(23) $$\lim_{s\to+\infty} \frac{f(x,s)}{s} \text{ and } \lim_{s\to-\infty} \frac{f(x,s)}{s} \text{ are } > \lambda_2 - \lambda_1,$$

where λ_1 and λ_2 are eigenvalues associated with L for the Dirichlet problem, see end of Section 1.3. We emphasize that f is not supposed to be odd in the variable s. An example of a function satisfying hypotheses (21)-(23) is $f(x,s) = g(x)s^{2n+1}$, where $g: \bar{\Omega} \to R$ is a C^1 positive function and n is a positive integer.

Theorem 2.6. Let $f: \bar{\Omega}\times R \to R$ be a C^1 function satisfying assumptions (21), (22) and (23). Suppose that $\lambda_1 < \lambda < \lambda_2$. Then problem (20) has exactly three solutions.

Remark. This result extends Theorem 1.7 of Ambrosetti-Mancini [4] and Theorem 6 of Berger [21], and it is due in the present generality to Berestycki [5]. We remark however that Berger's applies to elliptic operators of higher order. The proofs in the first two papers use different techniques.

In the proof of Theorem 2.6 we use the following

Lemma 2.7. Let $g: \bar{\Omega}\times R \to R$ be a C^1 function such that $g_s'(x,s) < \lambda_2$ for $x \in \bar{\Omega}$ and $s \in R$. Then any two solutions u_1 and u_2 of the Dirichlet problem

(24) $$Lu = g(x,u) \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega,$$

are ordered: either $u_1 \le u_2$ or $u_2 \le u_1$.

Proof. From

$$L(u_1-u_2) = g(x,u_1) - g(x,u_2) \text{ in } \Omega, \quad u_1-u_2 = 0 \text{ on } \partial\Omega$$

we obtain

(25) $$L\omega = m(x)\omega \text{ in } \Omega, \quad \omega = 0 \text{ on } \partial\Omega$$

where $\omega = u_1-u_2$ and

$$m(x) = \begin{cases} \dfrac{g(x,u_1(x)) - g(x,u_2(x))}{u_1(x) - u_2(x)}, & \text{if } u_1(x) \ne u_2(x) \\ g_s'(x,u_1(x)), & \text{if } u_1(x) = u_2(x) \end{cases}$$

From (25) and the fact that $m(x) < \lambda_2$ we obtain

(26) $$1 = \mu_j(m) > \mu_j(\lambda_2) = \frac{\lambda_j}{\lambda_2}, \quad \text{for some } j \ge 1.$$

[This is the case if $m(x) > 0$ at some point. If $m(x) \le 0$ for all $x \in \Omega$, it follows from (25) that $\omega = 0$, and so in this case: $u_1 = u_2$]. It follows from (26) that $j = 1$, and consequently ω is an eigenfunction corresponding to the first eigenvalue $\mu_1(m)$. Thus the result follows readily from Theorem 1.13. Q.E.D.

Proof of Theorem 2.6. a) First we observe that (22) implies that $f_s'(x,s) \ge 0$ for all $x \in \Omega$ and all $s \in R$. Consequently $(\lambda s - f(x,s))_s' < \lambda_2$ and Lemma 2.7 tells us that any two solutions of (20) are ordered. Since $u = 0$ is a solution, then the other solutions have a definite sign in Ω; they are either ≥ 0 or ≤ 0.

b) Hypotheses (21) and (23) permit us to apply Theorem 2.2 (or its corollary in the form of the problem posed in the very beginning of Section 2.1), and conclude that there is a positive solution u_1 and a negative solution u_2. For the existence of the negative solution we look at the Dirichlet problem

$$Lv = \lambda v - \tilde{f}(x,v) \text{ in } \Omega \quad v = 0 \text{ on } \partial\Omega,$$

where $\tilde{f}(x,v) = -f(x,-v)$.

c) Finally using theorem 2.4 we have that u_1 is the only positive solution and u_2 is the only negative solution. Q.E.D.

Theorem 2.8. Let $f: \bar{\Omega}\times R \to R$ be a C^1-function satisfying assumption (21), (22) and

(27) $$\lim_{s\to\pm\infty} \frac{f(x,s)}{s} = +\infty.$$

Then, if $\lambda_2 < \lambda < \lambda_3$ and λ_2 is simple, problem (20) has at least four solutions. Moreover if f is odd then problem (20) has at least five solutions.

Remark. This result is due to Lazer-McKenna [22], with a slightly different proof. Assumption (27) may be weakened: suppose that those two limits are $> \lambda$. The proof is the same.

Proof. As in the previous theorem we already know that there are at least 3 solutions: 0, $u_1 \geq 0$ and $u_2 \leq 0$ we also know that u_1 is the only positive solution. A similar statement for u_2. The idea now is (i) to transform problem (20) in a suitable operator equation $\Phi(u) = u - T(u)$, where T is a compact operator in $C_0^{1,\alpha}(\Omega)$, (ii) to obtain an a priori bound for the solutions of $\Phi(u) = 0$, (iii) to compute the degree of Φ in some large ball in $C_0^{1,\alpha}(\bar\Omega)$), (iv) to compute the indices of the three known solutions, (v) to use the additivity of the Leray-Schauder degree and to conclude. We solve these questions in the sequel.

Lemma 2.9 (A priori bound on the solutions of (20)). Let $a > 0$ be such that

(28) $\lambda s - f(x,s) < 0$ for $s > a$, and $\lambda s - f(x,s) > 0$ for $s < -a$.

Then $\|u\|_{L^\infty} \leq a$ for all solutions of (20).

Proof. Let

$$u_a(x) = \begin{cases} u(x), & \text{if } u(x) < a \\ a, & \text{if } u(x) \geq a \end{cases}$$

The solutions u of (20) are in $C^{2,\alpha}(\bar\Omega)$ and in particular in $H_0^1(\Omega)$. By a result of Stampacchia [15, p. 196] $u_a \in H_0^1(\Omega)$, and then

$$a[u,u-u_a] = \int[\lambda u - f(x,u)](u-u_a).$$

Since $a[u_a,u-u_a] = 0$ we obtain from (28) that

$$c\|u-u_a\|^2_{H_0^1} \leq a[u-u_a,u-u_a] = \int[\lambda u - f(x,u)](u-u_a) \leq 0$$

which implies $u = u_a$ and consequently $u \leq a$ in Ω. Similarly we prove that $-a \leq u$. Q.E.D.

Modifying the nonlinearity. The above lemma allow us to replace the nonlinearity $\lambda u - f(x,u)$ by $\psi(u)[\lambda u - f(x,u)]$ where $\psi: R \to R$ is a non-negative C^∞ function with compact support and such that $\psi(s) = 1$ for $|s| \leq a$. Indeed the Dirichlet problem

(29) $Lu = \psi(u)[\lambda u - f(x,u)]$ in Ω, $u = 0$ on $\partial\Omega$

has the same solutions as problem (20).(As in Lemma 2.9 we prove that also for the solutions u of (29) one has $\|u\|_{L^\infty} \leq a$).Problem (29) has the advantage that the right side is a bounded function of u. So we can replace problem (20) by the following one:

(30) $Lu = g(x,u)$ in Ω, $u = 0$ on $\partial\Omega$

where $g: \bar\Omega\times R \to R$ is a bounded C^1 function satisfying the assumptions

(31) $g(x,0) = 0, \quad g'_s(x,0) = \lambda \qquad \lambda_2 < \lambda < \lambda_3$

(32) the function $s \longrightarrow \frac{g(x,s)}{s}$ is nondecreasing in $(-a,0)$ and nonincreasing $(0,a)$.

(33) $g(x,s) \leq 0$ for $s > a$ and $g(x,s) \geq 0$ for $s \leq -a$.

Observe that the solutions u of (30) are such that $\|u\|_{L^\infty} \leq a$.

Setting the operator equation. Let $X = C_0^{1,\alpha}(\bar\Omega)$ be the Banach space of functions $u: \bar\Omega \to R$ which are continuously differentiable and their first derivatives are Hölder continuous with exponent $0 < \alpha < 1$ satisfying the boundary condition $u = 0$ on $\partial\Omega$; the norm is the maximum of the sup of the function, of its derivatives and of the Hölder quocients. Let $K: X \to X$ be defined as follows: given $v \in X$, let u be the unique solution of the Dirichlet problem

$$Lu + ku = g(x,v) + kv \text{ on } \Omega \qquad u = 0 \text{ on } \partial\Omega,$$

for some $k > 0$ fixed.

By the Schauder theory one knows that $u \in C^{2,\alpha}$. So K is a compact mapping. The solutions of (30) are the zeros of I-K. Now we proceed to compute the degree of I-K in certain subsets of X.

Lemma 2.10. Under the above assumptions there is an $R > 0$ such that $\deg(I-K,B_R,0) = 1$, where $B_R = \{u \in X: \|u\|_X < R\}$.

Proof. We claim that there is an $R > 0$ such that $(I-tK)u \neq 0$ for all $t \in [0,1]$ and all $\|u\| = R$. Once this has been done the result follows from the homotopy invariance of the Leray-Schauder degree.

To prove that claim observe that $(I-tK)u = 0$ is equivalent to $Lu + ku = tg(x,u) + tku$ in Ω, $u = 0$ on $\partial\Omega$. By the argument used in Lemma 2.9 we see that $\|u\|_{L^\infty} \leq a$, which implies $\|u\|_X \leq$ const through a standard combination of L^p estimates, Sobolev imbedding theorems and Schauder estimates. Take R larger than this constant.

Q.E.D.

Now recall that the positive solution u_1 is obtained by the monotone iteration method (see the proof of Theorem 2.2): let us denote by $\underline{u}$ the subsolution and by $\bar u$ the supersolution obtained there. We can choose them in such a way that

(34) $L\bar u > g(x,\bar u)$ in Ω, $L\underline{u} < g(x,\underline{u})$ in Ω, $\bar u = \underline{u} = 0$ on $\partial\Omega$.

Lemma 2.11. There is an $r > 0$ such that $\deg(I-K,\mathcal{O}_1,0) = 1$ where $\mathcal{O}_1 = \left\{u \in X: \underline{u} < u < \bar u \text{ in } \Omega;\ \frac{\partial \bar u}{\partial\nu} < \frac{\partial u}{\partial\nu} < \frac{\partial \underline{u}}{\partial\nu} \text{ on } \partial\Omega;\ \|u\|_X < r\right\}$.

The proof of this lemma follows the general line of ideas of the proof of Lemma 12 in de Figueiredo [23]. It will be presented below for convenience of the reader. Let us assume Lemma 2.11 and finish the proof of Theorem 2.8.

Proof of Theorem 2.8 completed. Observe that $\mathcal{O}_1$ is an open set in X containing the solution u_1. In a similar way we obtain an open set $\mathcal{O}_2$ containing u_2. It is easy to see that $\mathcal{O}_1 \cap \mathcal{O}_2 = \emptyset$ and $0 \notin \mathcal{O}_1 \cup \mathcal{O}_2$. Now by a well know result going back to Leray-Schauder (see for instance Berestycki [24]) we have that the index $i(I-K,0,0) = (-1)^2 = 1$. Next let B_R be a ball centered at 0 in X containing both O_1 and O_2 and let B be a small ball centered at 0 and disjoint of both O_1 and O_2. Then

$$\deg(I-K,\ B_R \setminus \overline{O_1 \cup O_2 \cup B_\varepsilon}, 0, 0) = -2$$

by the additivity property of the degree. So there is at least one further solution u_3 of (30) in $B_R \setminus \overline{O_1 \cup O_2 \cup B_\varepsilon}$. If g is odd then $-u_3$ is still another solution. Q.E.D.

<u>Proof of Lemma 2.11</u>. Let $k = \max\{|g_s'(x,s)| : x \in \bar{\Omega},\ \underline{u}(x) \le s \le \bar{u}(x)$ and define a function $\tilde{g}: \bar{\Omega}\times\mathbb{R} \to \mathbb{R}$ by

$$\tilde{g}(x,s) = \begin{cases} g(x,\underline{u}(x)) + k\underline{u}(x), & \text{if } s \le \underline{u}(x) \\ g(x,s) + ks, & \text{if } \underline{u}(x) \le s \le \bar{u}(x) \\ g(x,\bar{u}(x)) + k\bar{u}(x), & \text{if } s \ge \bar{u}(x) \end{cases}$$

So $\tilde{g}$ is bounded and nondecreasing in the variable s. From the Schauder theory for the linear Dirichlet problem, we conclude that for each $v \in X$ there is a unique solution $u \in C^{2,\alpha}(\bar{\Omega})$ of

(35) $$Lu + ku = \tilde{g}(x,v) \text{ in } \Omega \qquad u = 0 \text{ on } \partial\Omega.$$

So we have defined a compact mapping $\tilde{K}: X \to X$ with the property that $\tilde{K} = K$ in $\bar{O}_1$. Now we claim that $\tilde{K}$ maps X into O_1, provided r is properly chosen. First let us show how to choose r. Since $\tilde{g}$ is a bounded function it follows from the L^p estimates that the solutions of (35) are uniformly bounded in X for all $v \in X$. Take $r > \sup\{\|\tilde{K}v\|_X : v \in X\}$. Next take $v \in X$ and let us prove that $u = \tilde{K}v \in O_1$. Clearly $\|u\|_X < r$. From (34) and (35) we have

$$(L+k)(\bar{u}-u) > g(x,\bar{u}) + k\bar{u} - \tilde{g}(x,v) \ge 0 \text{ in } \Omega, \quad \bar{u}-u = 0 \text{ on } \partial\Omega.$$

It follows from the strong maximum principle that $u < \bar{u}$ in Ω and $\frac{\partial\bar{u}}{\partial\nu} < \frac{\partial u}{\partial\nu}$ on $\partial\Omega$. A similar reasoning for the subsolution, which then completes the proof of the assertion $\tilde{K}(X) \subset O_1$. Now let $\Psi \in O_1$ and consider the compact homotopy $H_\theta(u) = \theta\tilde{K}(u) + (1-\theta)\Psi$, $0 \le \theta \le 1$. Since $u \ne H_\theta(u)$ for all $u \in \partial O_1$ and all $\theta \in [0,1]$ we conclude that $\deg(I-H_1,O_1,0) = \deg(I-H_0,O_1,0)$. But H_0 is a constant mapping and clearly $\deg(I-H_0,O_1,0) = 1$. So $\deg(I-K,O_1,0) = 1$. Q.E.D.

CHAPTER 3

SUPERLINEAR ELLIPTIC PROBLEMS

3.1 THE FIXED POINT INDEX. Let C be a closed convex subset of a Banach space X, and $W \subset C$ a relatively open subset of C, that is, $W = \mathcal{O} \cap C$ for some open subset $\mathcal{O}$ of X. Let $\Phi: \bar{W} \to C$ be a compact mapping such that $\Phi(x) \neq x$ for $\bar{W}\backslash W$. Associated with each such a mapping we define an integer $i_C(\Phi,W)$, called the fixed point index of Φ, as follows. By a theorem of Dugundji [25] the mapping Φ has a compact extension $\tilde{\Phi}: \bar{\mathcal{O}} \to C$. Then define

(1) $$i_C(\Phi,W) = \deg(I-\tilde{\Phi},\mathcal{O},0).$$

To see that this is in fact a good definition we have to settle the three following points: (i) $\tilde{\Phi}(x) \neq x$ for all $x \in \partial\Omega$, (ii) the degree in the right side of (1) is independent of the particular extension $\tilde{\Phi}$, (iii) it does not depend either of the particular open set $\mathcal{O}$. These facts are easily proved using the homotopy invariance of the degree and the excision property. The usual properties of the Leray-Schauder degree are transferred immediately to the fixed point index. So we have the following properties.

I) Normalization. Let $\Phi: \bar{W} \to W$ be a constant mapping, that is, $\Phi(x) = a \in W$ for all $x \in \bar{W}$ and some fixed $a \in W$. Then $i_C(\Phi,W) = 1$.

II) Additivity. Let W_1 and W_2 be two disjoint (relatively) open subsets of W, and $\Phi: \bar{W} \to C$ a compact mapping such that $\Phi(x) \neq x$ for all $x \in \bar{W}\backslash(W_1 \cup W_2)$. Then

$$i_C(\Phi,W) = i_C(\Phi,W_1) + i_C(\Phi,W_2).$$

III) Homotopy invariance. Let $I \subset R$ be a compact interval and $h: I\times\bar{W} \to C$ a compact mapping such that $h(t,x) \neq x$ for all $x \in \bar{W}\ W$ and all $t \in I$. Then $i_C(h(t,\cdot),W) =$ constant with $t \in I$.

IV) Excision. Let $V \subset W$ be relatively open, and $\Phi: \bar{W} \to C$ be a compact mapping such that $\Phi(x) \neq x$ for $x \in \bar{W}\backslash V$. Then $i_C(\Phi,V)=i_C(\Phi,W)$.

V) Solution property. $i_C(\Phi,W) \neq 0 \longrightarrow \exists\, x \in W$ such that $\Phi(x) = x$.

We shall apply the previous facts to the case when C is a cone. Let us recall that a cone C in a Banach space X is a closed subset of X such that (i) if $x,y \in C$ and $\alpha,\beta \geq 0$, then $\alpha x+\beta y \in C$, (ii) if $x \in C$ and $x \neq 0$, then $-x \notin C$. A cone induces a partial order in X defined as follows: $x \leq y$ if and only if $y-x \in C$. In the present section we use the following notation: for $r > 0$, $B_r = \{x \in C: \|x\| < r\}$. A cone C is said to be normal if there is a constant $k > 0$ such that $0 \leq x \leq y$ implies $\|x\| \leq k\|y\|$.

Theorem 3.1. *Let C be a cone in a Banach space X, and $\Phi: C \to C$ a compact mapping. Assume that there are real numbers $r, R > 0$ such that*

(2) $$x \neq t\Phi(x) \quad \text{for} \quad 0 \leq t \leq 1 \quad \text{and} \quad \|x\| = r,\ x \in C,$$

(3) <u>there exists a compact mapping</u> $F: \bar{B}_R \times [0,\infty) \to C$ <u>such that</u> $F(x,0) = \Phi(x)$ <u>for</u> $\|x\| = R$, $F(x,t) \neq x$ <u>for</u> $\|x\| = R$ <u>and</u> $t \geq 0$, <u>and</u> $F(x,t) = x$ <u>has no solution</u> $x \in \bar{B}_R$ <u>for</u> $t \geq t_o$.

<u>Then</u>: i) (2) $\Longrightarrow i_C(\Phi, B_r) = 1$, and ii) (3) $\Longrightarrow i_C(\Phi, B_R) = 0$.

<u>Proof</u>. i) A direct consequence of the homotopy invariance and the normalization properties of the fixed point index.
ii) Let us denote by $F_t: \bar{B}_R \to C$ the mapping $F_t(x) = F(t,x)$. The second part of (3) implies that $i_C(F_t, B_R) = \text{const}$, and the third part implies that this constant is zero. So $i_C(F_o, B_R) = 0$. Since $F_o(x) = \Phi(x)$ for $x \in \bar{B}_R \setminus B_R$, the result follows from the homotopy invariance property of the index. Q.E.D.

<u>A sufficient condition for (2)</u>:

(2') There exists a bounded linear operator $A: X \to X$ such that $A(C) \subset C$, where C is a normal cone, A has spectral radius $r_\sigma(A)$ strictly less than 1 and $\Phi(x) \leq Ax$ for $x \in C$ and $\|x\| = r$.

Indeed, suppose that $x = t\Phi(x)$ for some $\|x\| = r$ and $0 < t \leq 1$. Then $x \leq tAx$, which by iteration gives $x \leq t^n A^n x$. If C is a normal cone then $\|x\| \leq \delta \|t^n A^n x\|$ where $\delta > 0$ is some constant depending only on the cone C. We then have $1 \leq \delta t^n \|A^n\|$. Taking n^{th} root and passing to the limit we come to $1 \leq t r_\sigma(A)$, which is an absurd.

<u>Another suficient condition for (2)</u>:

(2") $\Phi(x) \not\leq x$ for all $x \in C$ and $\|x\| = r$

<u>A sufficient condition for (3)</u>:

(3') There exists $v \in C \setminus \{0\}$ such that $x \neq \Phi(x) + tv$ for $\|x\| = R$ and $t \geq 0$.

Indeed, let $F(t,x) = \Phi(x) + tv$. The first two assertions in (3) are readily seen, and the third is verified taking $t_o > (R+\mu)/\|v\|$ where $\mu = \sup\{\|\Phi(x)\|: x \in C, \|x\| \leq R\}$.

<u>Another sufficient condition for (3)</u>:

(3") $\Phi(x) \not\geq x$ for all $x \in C$ and $\|x\| = R$.

The following set of corollaries are easy consequences of Theorem 3.1.

<u>Corollary 3.2</u>. <u>Let</u> C <u>be a cone in a Banach space</u> X <u>and</u> $\Phi: C \to C$ <u>a compact mapping</u>. <u>Assume that there are numbers</u> $0 < r < R$ <u>for which</u> (2) <u>and</u> (3) <u>hold</u>. <u>Then</u> $i_C(\Phi, U) = -1$, <u>where</u> $U = \{x \in C: r < \|x\| < R\}$. <u>In particular</u>, Φ <u>has a fixed point in</u> U. [<u>Similar considerations can be made if</u> $0 < R < r$].

<u>Corollary 3.3</u>. (<u>Compression of a cone</u>). <u>Let</u> $\Phi: C \to C$ <u>be a compact mapping in the cone</u> C. <u>Assume that there are numbers</u> $0 < R < r$ <u>such that</u> (2") <u>and</u> (3") <u>hold</u>. <u>Then</u> Φ <u>has a fixed point</u> $x \in C$, <u>with</u> $R < \|x\| < r$.

<u>Corollary 3.4</u> (<u>Expansion of a cone</u>). <u>Let</u> $\Phi: C \to C$ <u>be a compact mapping in the cone</u> C. <u>Assume that there are numbers</u> $0 < r < R$ <u>such that</u> (2") <u>and</u> (3") <u>hold</u>. <u>Then</u> Φ <u>has a fixed point</u> $x \in C$, <u>with</u> $r < \|x\| < R$.

3.2 DIFFERENTIABLE MAPPINGS IN CONES. A mapping $\Phi: C \to C$ defined in a cone C of a Banach space X is said to be (right) differentiable at a point $x \in C$ if there is a bounded linear operator $T: \overline{C-C} \to X$ such that

(4) $$\lim_{\substack{h\to 0\\ h\in C}} \frac{\Phi(x+h) - \Phi(x) - Th}{\|h\|} = 0$$

[Recall that $\overline{C-C}$ is the closure of the set $C-C = \{x-y: x,y \in C\}$]. This operator T is called the (right) derivative of Φ at x and it is denoted by $\Phi'_+(x)$.

Proposition 3.5. If $\Phi: C \to C$ is a compact mapping in the cone C, which is differentiable at $x \in C$, then $\Phi'_+(x)|_C$ is a compact mapping in C.

Remark. If the cone C is generating, that is, $X = C-C$, then $\Phi_+(x): \overline{C-C} \to X$ is a compact operator, cf. Krasnoselskii [1, p. 102]

Proof of Proposition 3.5. Suppose by contradiction that there are $h_n \in C$, $\|h_n\| = 1$, such that $\|Th_n - Th_m\| \geq \varepsilon_0 > 0$ for $n \neq m$. It follows from (4) that there is a $\delta > 0$ such that

$$\|\Phi(x+h) - \Phi(x) - Th\| \leq \frac{1}{3}\varepsilon_0\|h\|, \quad \text{for } h \in C,\ \|h\| \leq \delta.$$

Using this expression with $h = h_n, h_m$ to estimate the expression $\varepsilon_0\delta \leq \|T(\delta h_n) - T(\delta h_m)\|$, we are led to

$$\|\Phi(x+\delta h_n) - \Phi(x+\delta h_m)\| \geq \frac{1}{3}\varepsilon_0\delta \quad \text{for all } n \neq m$$

which contradicts the compactness of Φ. Q.E.D.

A mapping $\Phi: C \to C$ defined in a cone C of a Banach space X is said to be asymptotically linear if there is a bounded linear operator $T: \overline{C-C} \to X$ such that

(5) $$\lim_{\substack{\|x\|\to\infty\\ x\in C}} \frac{\Phi(x) - Tx}{\|x\|} = 0.$$

In this case T is called the asymptotic derivative of Φ and it is denoted by $\Phi'(\infty)$.

Proposition 3.6. If $\Phi: C \to C$ is compact asymptotically linear, then $\Phi'(\infty)|_C$ is a compact mapping in C. [If C is a generating cone, then $\Phi'(\infty): \overline{C-C} \to X$ is a compact operator, cf. Krasnosel'skii, loc. cit.].

Proof. By contradiction: $x_n \in C$, $\|x_n\| = 1$ and $\|Tx_n - Tx_m\| \geq \varepsilon_0 > 0$. It follows from (5) that there is a $\delta > 0$ such that $\|\Phi(x) - Tx\| \leq \frac{\varepsilon_0}{3}\|x\|$ for $\|x\| \leq \delta$. As in the previous proposition we come to the contradictory statement: $\|\Phi(\delta x_n) - \Phi(\delta x_m)\| \geq \frac{1}{3}\varepsilon_0\delta$.

Q.E.D.

Proposition 3.7. Let $\Phi: C \to C$ be a compact mapping in a normal cone C, which is differentiable at 0, and $\Phi(0) = 0$. i) Suppose that the eigenvalue problem

(6) $$\Phi'_+(0)h = \lambda h, \qquad h \in C,$$

has no eigenvalue $\lambda \geq 1$; then there exists a number $r_0 > 0$ such that (2) holds for all $0 < r \leq r_0$. ii) On the other hand, suppose that $\lambda = 1$ is not an eigenvalue of (6) and that there is an eigenvalue strictly greater than 1; then there exists a number $R_0 > 0$ such that (3') holds for all $0 < R \leq R_0$.

Proof. i) By contradiction: there exist $r_n \to 0$, $x_n \in C$, $\|x_n\| = r_n$, $0 < t_n \leq 1$ such that $x_n = t_n\Phi(x_n)$. So

$$\frac{1}{t_n} x_n = \Phi(x_n) = \Phi'_+(0)x_n + o(x_n)$$

or

$$\frac{1}{t_n} y_n = \Phi'_+(0)y_n + \frac{o(x_n)}{\|x_n\|}, \quad y_n = x_n/\|x_n\|.$$

Using Proposition 3.5 we conclude that $t_n^{-1}y_n \to z \in C$ and $t_n \to t^* > 0$. So $z = t^*\Phi'_+(0)z$, which is impossible.

ii) Let $v \in C\ \{0\}$ be the eigenvector of (6) corresponding to the eigenvalue $\lambda > 1$ anounced in the hypothesis. We claim that (3') holds with this v and some positive R. Let us prove it by contradiction: there exist $R_n \to 0$, $x_n \in C$, $\|x_n\| = R_n$, $t_n \geq 0$ such that $x_n = \Phi(x_n) + t_n v$. Then $x_n = \Phi'_+(0)x_n + o(x_n) + t_n v$. Dividing by $\|x_n\|$ we get $y_n = \Phi'_+(0)y_n + \frac{o(x_n)}{|x_n|} + \frac{t_n}{\|x_n\|} v$, which shows that $\{t_n/\|x_n\|\}$ is a bounded sequence. Passing to a subsequence we may assume that it converges to α^*. Using the compactness of $\Phi'_+(0)$ it follows that $y_n \to y \in C$ and $y = \Phi'_+(0)y + \alpha^* v$. Iterating we get $\Phi'_+(0)^n\ y = y - \alpha^*(\lambda+\lambda^2 + \ldots + \lambda^n)v$, which is impossible (in the case $\alpha^* > 0$), since the left side is in C and the right side is not for large n. Indeed, if it were one would get $\|\alpha^*(\lambda+\ldots+\lambda^n)v\| \leq k\|y\|$, where k is the constant of normality of the cone C. Consequently $\alpha^* = 0$ and $y = \Phi'_+(0)y$ which is also a contradiction to the hypothesis. Q.E.D.

Proposition 3.8. Let $\Phi: C \to C$ be an asymptotically linear compact mapping in a normal cone C. (i) Suppose that the eigenvalue problem

(7) $$\Phi'(\infty)h = \lambda h, \quad h \in C,$$

has no eigenvalue $\lambda \geq 1$; then there is a number $r_0 > 0$ such that (2) holds for all $r \geq r_0$. (ii) Suppose that $\lambda = 1$ is not an eigenvalue of (7) and that there is an eigenvalue strictly greater than 1; then there is a number $R_0 > 0$ such that (3') holds for all $R \geq R_0$.

Proof. Analogous to the proof of Proposition 3.7.

Application to elliptic problems with asymptotically linear nonlinearities. We shall apply the preceding results to the Dirichlet problem

(8) $$Lu = g(x,u) \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega,$$

where L is a second order elliptic operator as in Sections 1.3 and 2.1, and $g: \bar{\Omega}\times R^+ \to R^+$ is a C^α function. We also assume that:

(i) $g(x,0) = 0$; (ii) $\lim_{s \downarrow 0} \frac{g(x,s)}{s} = a(x)$; (iii) $\lim_{s\to+\infty} \frac{g(x,s)}{s} = m(x)$,

and a(x) and m(x) are L^r functions with $r > N$. Condition (ii) is verified if we ask that the right partial derivative $g'_s(x,0)$ exists and it is continuous . Let $C_0^0(\bar{\Omega})$ be the Banach space of the continuous real valued functions in Ω which vanish on $\partial\Omega$, with the sup norm. Now let Φ: $C_0^0(\bar{\Omega}) \to C_0^0(\bar{\Omega})$ be the mapping defined as follows: given $v \in C_0^0(\bar{\Omega})$, then $u = \Phi(v)$ is the solution of

(9) $$Lu = g(x,v) \text{ in } \Omega, \quad u = 0 \quad \text{on} \quad \partial\Omega.$$

By a solution we mean a function $u \in W^{2,p}$. Since the right side of (9) is continuous, and consequently in L^p for all $p > 1$, we have by the Sobolev imbedding theorem that $u \in C^{1,\alpha}$. This shows that mapping Φ is compact. Also by the strong maximum principle $\Phi(C) \subset C$, where $C = \{u \in C_0^0(\bar{\Omega}): u \geq 0 \text{ in } \Omega\}$.

<u>Lemma 3.9</u>. <u>Mapping</u> Φ <u>is differentiable at</u> 0 <u>and its derivative</u> $\Phi_+'(0)$ <u>is defined as follows</u>: <u>given</u> $v \in C$, <u>let</u> $u = \Phi_+'(0)v$ <u>be the solution of</u>

(10) $$Lu = a(x)v \quad \text{in} \quad \Omega, \quad u = 0 \quad \text{on} \quad \partial\Omega.$$

<u>Moreover the eigenvalue problem</u> (6) <u>is equivalent to the following eigenvalue problem</u>:

(11) $$Lh = \lambda^{-1}a(x)h, \quad h \geq 0 \text{ in } \Omega, \; h = 0 \quad \text{on} \quad \partial\Omega.$$

<u>Proof</u>. It follows from (9) and hypothesis (i) that $\Phi(0) = 0$. From (9) and (10) we obtain

$$L[\Phi(v) - \Phi_+'(0)v] = g(x,v) - a(x)v \quad \text{in} \quad \Omega,$$

and the Dirichlet boundary condition for the function in brackets. From the L^p-estimates and Sobolev imbedding theorem we obtain

$$\|\Phi(v) - \Phi_+'(0)v\|_{L^\infty} \leq \|g(x,v) - a(x)v\|_{L^\infty}$$

and the result follows readily from (ii). (ii) is immediate.

Q.E.D.

<u>Lemma 3.10</u>. <u>Mapping</u> Φ <u>is asymptotically linear</u>, <u>and</u> $\Phi'(\infty)$ <u>is defined as follows</u>: <u>given</u> $v \in C$, <u>let</u> $u = \Phi'(\infty)v$ <u>be the solution of</u>

(12) $$Lu = m(x)v \quad \text{in} \quad \Omega, \quad u = 0 \quad \text{on} \quad \partial\Omega.$$

<u>Moreover the eigenvalue problem</u> (7) <u>is equivalent to the following eigenvalue problem</u>

(13) $$Lh = \lambda^{-1}m(x)h, \; h \geq 0 \text{ in } \Omega, \; h = 0 \quad \text{on} \quad \partial\Omega.$$

The proof of the above lemma is completely analogous to the proof of Lemma 3.9. Now we can use Propositions 3.7 and 3.8 and make the following assertions with respect to problem (8):

<u>Theorem 3.11</u>. <u>Suppose that</u> g: $\bar{\Omega} \times R^+ \to R^+$ <u>is a</u> C^α-<u>function satisfying hypotheses</u> (i), (ii) <u>and</u> (iii) <u>above</u>. <u>Then problem</u> (8) <u>has a positive solution</u> $u \in C^{2,\alpha}(\Omega)$ <u>if either one of the following assumptions hold</u>

(S) $\mu_1(a) > 1$ and $\mu_1(m) < 1$ (Superlinear case)

or

(s) $\mu_1(a) < 1$ and $\mu_1(m) > 1$ (sublinear case)

<u>Remark</u>. Compare this result with Theorem 2.2. Also as in Section 2.1 we may discuss the eigenvalue problem (14) of that section complementing the discussion we started there. We particularize a little bit: let f: $\bar{\Omega} \times R^+ \to R^+$ be a C^α function, and consider

the eigenvalue problem

(14) $Lu = \lambda f(x,u)$ in Ω, $u = 0$ on $\partial\Omega$.

Suppose that $f(x,0) = 0$ and the limits below exist and are L^r functions with $r > N$

$$\lim_{s \to 0} \frac{f(x,s)}{s} = f_0(x) \quad \text{and} \quad \lim_{s \to +\infty} \frac{f(x,s)}{s} = f_\infty(x).$$

Then it is easy to conclude from Proposition 3.11 that if $\inf f_0 < \sup f_\infty$, then problem (14) has a positive solution for all λ such that $\mu_1(1)/\sup f_\infty < \lambda < \mu_1(1)/\inf f_0$.

Remark. Observe that the second half of hypothesis (S) implies through Propositions 3.8 and 3.10 an a priori bound on the solutions of $Lu = g(x,u) + t\phi_1$ in Ω, $u = 0$ on $\partial\Omega$, for all $t \geq 0$, where ϕ_1 is the (positive) eigenfunction corresponding to the first eigenvalue $\mu_1(m)$ of (12). In the next section we develop a technique to obtain a priori bounds for positive solutions of superlinear problems, which are not necessarily asymptotically linear.

3.3 A PRIORI BOUNDS A LA BRÉZIS-TURNER. Let us consider the Dirichlet problem

(15) $Lu = g(x,u)$ in Ω, $u = 0$ on $\partial\Omega$.

where L is a second order elliptic operator as in Sections 1.3 and 2.1. The results of the present section are due to Brézis-Turner [8] that proved them in the more general case of second

order operators which are not self-adjoint. The function $g: \bar{\Omega} \times R^+ \to R^+$ is supposed to be continuous and to satisfy the following assumptions.

(16) $\liminf_{s \to +\infty} \frac{g(x,s)}{s} > \lambda_1$ uniformly for $x \in \bar{\Omega}$,

(17) $\lim_{s \to +\infty} \frac{g(x,s)}{s^\beta} = 0$ uniformly for $x \in \bar{\Omega}$, where $\beta = \frac{N+1}{N-1}$.

[$\lambda_1 = \mu_1(1)$, see the end of Section 1.3]. Let ϕ_1 be a positive eigenfunction corresponding to the first eigenvalue λ_1; for simplicity let us omit the index 1 and write just ϕ.

Theorem 3.12. Under the above assumptions, there is a constant $k > 0$ (independent to $t \geq 0$) such that $\|u\|_{L^\infty} \leq k$ for all solutions of

(18) $Lu = g(x,u) + t\phi$ in Ω, $u = 0$ on $\partial\Omega$.

In the proof of the above theorem one uses the Hardy-Sobolev inequality: if $u \in H_0^1(\Omega)$, then $u/\phi^\tau \in L^q$, where $q^{-1} = 2^{-1} - (1-\tau)N^{-1}$, $0 \leq \tau \leq 1$, and there is a constant $c > 0$ such that

(19) $\left\|\frac{u}{\phi^\tau}\right\|_{L^q} \leq c\|\nabla u\|_{L^2}$, $\forall u \in H_0^1(\Omega)$.

Observe that the extreme case $\tau = 0$ is the Sobolev imbedding theorem $H_0^1 \subset L^{2^*}$, where $2^* = 2N/(N-2)$. The other extreme case $\tau = 1$ is a fact already observed in Hardy-Littlewood-Polya [26], (see also Lions-Magenes [27]), that the behavior of a function $u \in H_0^1(\Omega)$ near the boundary $\partial\Omega$ is such that u/ϕ is in $L^2(\Omega)$.

Proof of Theorem 3.12. First step. It follows from (16) that there are $\lambda > \lambda_1$ and an $s_0 > 0$ such that $g(x,s) \geq \lambda s$ for $s \geq s_0$. So there is a constant $c_1 > 0$ such that

(20) $$g(x,s) \geq \lambda s - c_1 \quad \forall x \in \Omega, \ \forall s \geq 0.$$

Multiplying (18) by ϕ, integrating by parts one obtains

(21) $$\lambda_1 \int u\phi = \int g(x,u) + t\int \phi^2.$$

Using (20) it follows that

$$\lambda_1 \int u\phi \geq \lambda \int u\phi - c_1 \int \phi + t\int \phi^2,$$

which implies that there are positive constants such that

$\int u\phi \leq c$ and $\int g(x,u)\phi \leq c$, for all solutions u of (18)

$t \leq c$, if problem (18) is to have a solution $u \geq 0$.

(We use the same letter c to denote different constants).

Second step. Multiplying equation (18) by u, using the coerciveness of the bilinear form $a[u,u]$ associated with L, and estimating u by Poincaré's inequality we have

(22) $$\|\nabla u\|^2_{L^2} \leq c\int g(x,u)u + c\|\nabla u\|_{L^2}.$$

Now to estimate the first term in the right side of (22) we write

$$\int g(x,u)u = \int g(x,u)^{\alpha}\phi^{\alpha} g(x,u)^{1-\alpha}\phi^{-\alpha}u$$

where the parameter $\alpha \in (0,1)$ will be determined shortly. Using Hölder's inequality one obtains

(23) $$\int g(x,u)u \leq \left[\int g(x,u)\phi\right]^{\alpha}\left[\int \frac{g(x,u)}{\phi^{\frac{\alpha}{1-\alpha}}}\, u^{\frac{1}{1-\alpha}}\right]^{1-\alpha}$$

The first term in the right side of (23) is bounded. To estimate the second one we use (17): for each $\varepsilon > 0$ there is $c_\varepsilon > 0$ such that $g(x,s) < \varepsilon s^{\beta} + c_\varepsilon$ for all $x \in \bar{\Omega}$ and $s \geq 0$. Then

$$\int g(x,u)u \leq c\varepsilon^{1-\alpha}\left[\int \frac{u^{\beta+\frac{1}{1-\alpha}}}{\phi^{\frac{\alpha}{1-\alpha}}}\right]^{1-\alpha} + c'_\varepsilon\left[\int \frac{u^{\frac{1}{1-\alpha}}}{\phi^{\frac{\alpha}{1-\alpha}}}\right]^{1-\alpha}$$

Taking $\alpha = 2/(N+1)$ and using Hardy-Sobolev inequality we obtain

$$\int g(x,u)u \leq c\varepsilon^{\frac{N-1}{N+1}}\|\nabla u\|^2_{L^2} + c\|\nabla u\|_{L^2}.$$

Using this estimate in (22) we get $\|\nabla u\|_{L^2} \leq c$.

Third step. By the Sobolev imbedding theorem $\|u\|_{L^{2^*}} \leq c$. Using a bootstrap argument we conclude that $\|u\|_{L^\infty} \leq c$. Q.E.D.

3.4 SHARPER A PRIORI BOUNDS. In this section we describe a technique due do P.L. Lions, R.D. Nussbaum and the writer [28] for obtaining a priori bounds. It rests on two basic ideas, one is the use of Pohožaev identity [29] for that purpose and the other is the use of results by Gidas, Ni and Nirenberg [30] relative to the maximum principle. Unfortunately there are some restrictions on the class of second order operators L considered as well as on the geometry of the domain Ω. Those are due to the necessity of applying arguments based in [30]. For simplicity we restrict at the present work to the case $L = -\Delta$ and to convex domains Ω. More general cases are discussed in [28]. So let us consider the Dirichlet problem

(24) $$-\Delta u = f(u) \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega.$$

We assume the following hypotheses on the function f:

(25) $f: R^+ \to R^+$ is locally Lipschitzian and

(26) $$\liminf_{s\to+\infty} \frac{f(s)}{s} > \lambda_1,$$

where λ_1 is the first eigenvalue of the Laplacian acting in $H_0^1(\Omega)$. Assumption (26) is just one half of the superlinearity condition for problem (24); the other half, that is, the behavior of f near zero is not needed for the a priori estimates. That will be used only at the moment of proving existence of positive solutions for (24) in Section 3.5.

If Ω is convex then it follows, using arguments from [30], that the following condition holds:

(27) there are positive numbers γ, ε and η (depending only on Ω) such that for all $x \in \{y \in \bar{\Omega}: d(y,\partial\Omega) < \varepsilon\}$ there exists a measurable set I_x with (i) meas $(I_x) \geq \gamma$, (ii) $I_x \subset \{y \in \bar{\Omega}: d(y, \partial\Omega) > \eta\}$, and (iii) $u(\xi) \geq u(x)$ for all $\xi \in I_x$ and all positive solutions of (24).

Theorem 3.13. Assume that f satisfies (25) and (26) and that condition (27) holds. Suppose the following conditions in addition:

(28) $$\lim_{s\to+\infty} \frac{f(s)}{s^\sigma} = 0, \text{ where } \sigma = \frac{N+2}{N-2} \text{ if } N\geq 3 \text{ (or } \sigma < \infty \text{ if } N = 2\text{), and}$$

(29) $$\limsup_{s\to+\infty} \frac{sf(s)-\theta F(s)}{s^2 f(s)^{2/N}} \leq 0 \text{ for some } 0 \leq \theta < \frac{2N}{N-2},$$

where $F(s) = \int_0^s f$. [In the case of $N = 2$ this conditions is vacuous]. Then there is a constant c (*) such that $\|u\|_{L^\infty} \leq c$ for all positive solutions of (24).

Remarks: Condition (29) is verified in the special case when

(30) $$\limsup_{s\to+\infty} \frac{f(s)}{s^\tau} = 0, \quad \text{where} \quad \tau = \frac{N}{N-2}.$$

Indeed, taking $\theta = 0$, (29) reduces to $\limsup [f(s)s^{-\tau}]^{1/\tau} \leq 0$, which is an obvious consequence of (30). Observe that if (17) holds then (28) and (29) are verified. So the above theorem extends Theorem 3.12 at least in the special case of the Laplacian in convex domains.

The proof of Theorem 3.13 is split in a series of lemmas in the sequel.

Lemma 3.14. There is a constant $c > 0$ such that

(*) We use the same letter c to designate all constants appearing in this section.

(31) $\int u\phi \le c$ and $\int f(u)\phi \le c$, for all solutions u of (24).

This is precisely the contents of the first step in the proof of Theorem 3.12. Observe that it depends only on hypothesis (26). [The above statement implies that both u and $f(u)$ are bounded in $L^1_{loc}(\Omega)$].

Lemma 3.15. u and ∇u are bounded in the L^∞-norm in a neighborhood of $\partial\Omega$, for all solutions u of (24).

Proof. Using (27) and (31) we have that for all $x \in \{y \in \bar{\Omega}: d(y,\partial\Omega) < \varepsilon\}$

$$c \ge \int_{I_x} u(\xi)\phi(\xi) \ge u(x)\left[\min_{\xi \in I_x} \phi(\xi)\right]\gamma.$$

Also from (27) we see that there is a positive constant (independent of x) such that $\min_{I_x} \phi(\xi) \ge c$. Then

(32) $u(x) \le c$ for all $x \in \{y \in \bar{\Omega}: d(y,\partial\Omega) < \varepsilon\}$.

Next we use a theorem of de Giorgi-Nash type (see Theorem 14.1 of [31, p. 201] to deduce that there is $0 < \alpha < 1$ such that

$$\|u\|_{C^{0,\alpha}(\Omega_1)} \le c, \text{ where } \Omega_1 = \{y \in \Omega: \tfrac{\varepsilon}{4} < d(y,\partial\Omega) < \tfrac{3\varepsilon}{4}\}.$$

Now using Schauder interior estimates (see Theorem 6.2 of [16, p. 85]) we have for $\frac{\varepsilon}{4} < r_1 < \frac{\varepsilon}{2} < r_2 < \frac{3\varepsilon}{4}$ that

$$\|u\|_{C^{2,\alpha}(\Omega_2)} \le c, \text{ where } \Omega_2 = \{y \in \Omega: r_1 \le d(y,\partial\Omega) \le r_2\}.$$

Finally Schauder boundary estimates gives

$$\|u\|_{C^{2,\alpha}(\Omega_3)} \le c, \text{ where } \Omega_3 = \{y \in \Omega: d(y,\partial\Omega) \le r_2\},$$

which implies

(33) $|\nabla u(x)| \le c$ for all $x \in \{y \in \bar{\Omega}: d(y,\partial\Omega) < \frac{\varepsilon}{2}\}$.

Thus (32) and (33) give the result. Q.E.D.

Remark. (31) and (32) imply that $\|f(u)\|_{L^1} \le c$ for all positive solutions of (24).

Lemma 3.16 (Pohožaev identity [29]). Let $u \in C^2(\bar{\Omega})$ be a solution of (24). Then

(34) $$2N\int_\Omega F(u) - (N-2)\int_\Omega f(u)u = \int_{\partial\Omega} (x\cdot\nu)|\nabla u|^2,$$

where $\nu = \nu(x)$ is the outward unit normal to $\partial\Omega$ at x.

Proof. The idea is to multiply equation (24) by $x\cdot\nabla u$ and integrate, using next the divergence theorem. For that matter we first write down some terms as divergences. In fact

$$\Delta u(x\cdot\nabla u) = \frac{N-2}{2}|\nabla u|^2 + \sum_i\Big(\sum_j x_j u_{x_i} u_{x_j}\Big)_{x_i} - \frac{1}{2}\sum_j\Big(\sum_i x_j u_{x_i}^2\Big)_{x_j}$$

and

$$f(u)(x\cdot\nabla u) = \sum_j \Big(x_j F(u)\Big)_{x_j} - NF(u).$$

So we obtain

$$\frac{2-N}{2}\int_\Omega |\nabla u|^2 + \frac{1}{2}\int_{\partial\Omega}\sum_{i,j} x_j\nu_j u_{x_i}^2 - \int_{\partial\Omega}\sum_{i,j} x_j\nu_i u_{x_i}u_{x_j} = \int_{\partial\Omega}\sum_j x_j\nu_j F(u) - N\int_\Omega F(u)$$

which gives readily identity (34), by observing that

$$\int_\Omega f(u)u = -\int_\Omega \Delta u\cdot u = \int_\Omega |\nabla u|^2. \qquad \text{Q.E.D.}$$

Lemma 3.17. The following estimate on the positive solutions of (24) holds:

$$(35) \qquad \|\nabla u\|_{L^2} \le c$$

Proof. i) Case $N \ge 3$. Using (33) and (34) we deduce that

$$\left|2N\int_\Omega F(u) - (N-2)\int_\Omega uf(u)\right| \le c$$

which is used in the estimation below:

$$(36) \quad \int_\Omega |\nabla u|^2 = \int_\Omega uf(u) = \frac{1}{1-\lambda}\int_\Omega [uf(u)-\theta F(u)] - \frac{\lambda}{1-\lambda}\int_\Omega \left[uf(u)-\frac{2N}{N-2}F(u)\right]$$
$$\le c\int_\Omega |uf(u)-\theta F(u)| + c$$

where λ is chosen in such a way that $\theta = \lambda\frac{2N}{N-2}$. So $0 \le \lambda < 1$.

Now we use hypothesis (29) [n.b.: that is the only point where we use this hypothesis]: given $\varepsilon > 0$ there is $s_0 > 0$ such that $sf(s)-\theta F(s) \le \varepsilon s^2 f(s)^{2/N}$ for $s \ge s_0$. Thus we obtain from (36):

$$(37) \qquad \int_\Omega |\nabla u|^2 \le c_\varepsilon + c\varepsilon\int_\Omega u^2 f(u)^{2/N}.$$

To estimate the last term in the right side of (37) we use Hölder's inequality, the Sobolev imbedding theorem and the boundedness of $f(u)$ in L^1:

$$\int_\Omega u^2 f(u)^{2/N} \le \left|\int_\Omega f(u)\right|^{2/N} \left|\int_\Omega u^{\frac{2N}{N-2}}\right|^{\frac{N-2}{N}} \le c\|\nabla u\|^2_{L^2}.$$

This estimate together with (37) gives (35), by taking $\varepsilon > 0$ in such a way that $\varepsilon c < 1$.

ii) Case $N = 2$. Choose $0 < \gamma < 1$ and use hypothesis (28) with $\sigma = 1/(1-\gamma)$, which then implies

$$\lim_{s\to+\infty} \frac{sf(s)}{s^2 f(s)^\gamma} = 0$$

Consequently for $\varepsilon > 0$ there is an $s_0 > 0$ such that $sf(s) \le \varepsilon s^2 f(s)^\gamma$ for $s \ge s_0$. This is then used in the estimate below

$$\int_\Omega |\nabla u|^2 = \int_\Omega uf(u) \le c_\varepsilon + \varepsilon\int_\Omega u^2 f(u)^\gamma,$$

and we proceed as above. Q.E.D.

Lemma 3.18. The following estimate holds for all positive solutions of (24):

$$(38) \qquad \|u\|_{L^\infty} \le c$$

Proof. i) Case $N = 2$. It follows from (35) using the Sobolev imbedding theorem that $\|u\|_{L^p} < c$ for all $1 \le p < \infty$. Then in virtue of (28) we see that $\|f(u)\|_{L^p} \le c$ for all p. By the L^p theory of the Dirichlet problem and again Sobolev imbedding theorem we conclude that $\|u\|_{L^\infty} \le c$.

ii) Case $N \ge 3$. Multiplying equation (24) by u^p, $p > 1$, and integrating we have

$$p\int_\Omega |\nabla u|^2 u^{p-1} = \int_\Omega f(u)u^p$$

or

$$(39) \qquad \frac{4p}{(p+1)^2}\int_\Omega \left|\nabla u^{\frac{p+1}{2}}\right|^2 = \int_\Omega f(u)u^p.$$

Now using (28) we estimate the right side of (39) to get

$$(40) \qquad \int_\Omega \left|\nabla u^{\frac{p+1}{2}}\right|^2 \leq c_\varepsilon + c\varepsilon\int_\Omega u^{\sigma-1}u^{p+1}.$$

Using the Sobolev imbedding theorem to estimate the left side of (40) and Hölder's inequality to the right side we have

$$\left(\int_\Omega u^q\right)^{\frac{N-2}{N}} \leq c_\varepsilon + c\varepsilon\left(\int_\Omega u^{\frac{2N}{N-2}}\right)^{2/N}\left(\int_\Omega u^q\right)^{\frac{N-2}{N}}, \quad q = \frac{N(p+1)}{N-2}.$$

Using the Sobolev imbedding theorem once more and (35) we conclude that

$$\|u\|_{L^q} \leq c$$

by taking $\varepsilon c < 1$. We may choose $p > 1$ in such a way that $q > \frac{N}{2}\sigma$. Then $\|f(u)\|_{L^{\frac{q}{\sigma}}} \leq c$ with $\frac{q}{\sigma} > \frac{N}{2}$. By the regularity theory and Sobolev again we get

$$\|u\|_{L^\infty} \leq c. \qquad \text{Q.E.D.}$$

3.5 EXISTENCE OF POSITIVE SOLUTIONS FOR SUPERLINEAR PROBLEMS.

The following result is due to P.L. Lions, R.D. Nussbaum and the writer [28].

Theorem 3.19. *Let* $f: R^+ \to R^+$ *be a function satisfying* (25), (26), (28), *and* (29), *with* $f(0) = 0$. *Suppose that* (27) *holds and*

$$(41) \qquad \limsup_{s\to 0} \frac{f(s)}{s} < \lambda_1.$$

Then problem (24) *has at least one positive solution.*

Proof. Let $X = \{u \in C^0(\bar{\Omega}): u = 0 \text{ on } \partial\Omega\}$ and $C = \{u \in X: u \geq 0\}$. Let us denote by $\Phi: X \to X$ be mapping defined as follows: for each $v \in X$ let $u = \Phi(v)$ be the solution of $-\Delta u = f(v)$ in Ω and $u = 0$ on $\partial\Omega$. It is well known that $u \in X$ and that the mapping Φ is compact. By the strong maximum principle $\Phi(C) \subset C$. Now we apply Theorem 3.1. Using (41) let $0 < \alpha < \lambda_1$ be such that $f(s) \leq \alpha s$ for $0 \leq s \leq s_0$, some $s_0 > 0$. Define the linear operator $A: X \to X$ as follows: for each $v \in X$ let $u = Av$ be the solution of $-\Delta u = \alpha v$ in Ω and $u = 0$ on $\partial\Omega$. It follows that A is bounded and its spectral radius $r_\sigma(A) < 1$. Also by the maximum principle $A(C) \subset C$. Now observe that for $\|u\|_X \leq s_0$ one has $\Phi(u) \leq Au$. So condition (2'), which implies condition (2) of Theorem 3.1, holds. Next we verify condition (3) of that theorem. Let $F: C\times[0,\infty) \to C$ be defined as follows: for each $v \in C$ and $t \geq 0$ let $u = F(v,t)$ be the solution of $-\Delta u = f(v+t)$ in Ω and $u = 0$ on $\partial\Omega$. Lemma 3.14 implies that $u = F(u,t)$ has no solution $u \in C$ for $t \geq t_0$, some convenient $t_0 > 0$. By the a priori estimates of the previous section there is a constant $R > 0$ such that $u \neq F(u,t)$ for all $\|u\| > R$ with $u \in C$ and all $t \geq 0$. This concludes the proof of the theorem. Q.E.D.

Now we show how to obtain existence of a positive solution for (24) without assuming (29). In this case we have to proceed differently since we do not know if there are a priori bounds in this case. The idea here is to use the techniques developed in the previous section plus some variational methods.

<u>Theorem 3.20</u>. [28] <u>Let</u> $f: R^+ \to R^+$ <u>be a function satisfying</u> (25), (26) <u>and</u> (28), <u>with</u> $f(0) = 0$. <u>Assume that</u> (27) <u>and</u> (41) <u>hold</u>. <u>Then</u> (24) <u>has a positive solution</u>.

<u>Proof</u>. <u>First step</u>: Construction of a sequence of approximate problems. Let us choose, using (26), a sequence $s_n \to +\infty$ such that $s^{-1}f(s) \geq \lambda_1+\alpha$ for all $s \geq s_n$ and some fixed $\alpha > 0$. Let $1 < \gamma < (N+2)/(N-2)$ be some fixed constant, and define

$$f_n(s) = \begin{cases} 0 & \text{for } s \leq 0 \\ f(s) & \text{for } 0 \leq s \leq s_n \\ f(s_n) + \dfrac{f(s_n)}{s_n}(s-s_n)^\gamma & \text{for } s \geq s_n. \end{cases}$$

It easy to see that f_n satisfies conditions (25), (26), (28), (29) and also $\lim_{s\to+\infty} s^{-1}f_n(s) = +\infty$. Moreover there exist $0 < \theta_n < \frac{1}{2}$ and $t_n > 0$ such that

(42) $$F_n(s) \leq \theta_n s f_n(s) \quad \text{for } s \geq t_n$$

where $F_n(s) = \int_0^s f_n$. The approximate problems considered are

(43) $$-\Delta u_n = f_n(u_n) \text{ in } \Omega, \quad u_n = 0 \text{ on } \partial\Omega.$$

We shall need positive solutions of (43) given by a variational method, namely the mountain pass theorem of Ambrosetti-Rabinowitz [7, Thm 2.1].

<u>Second step</u>. Let us consider the functional $J_n: H_0^1(\Omega) \to R$ defined by

$$J_n(v) = \int_\Omega \left[\frac{1}{2} |\nabla v|^2 - F_n(v)\right].$$

In view of the conditions on f_n that is a well defined C^1 functiona[l]. Now we observe that J_n satisfies the hypotheses of Theorem 2.1 in [7], see Lemmas 3.3, 3.4 and 3.6 of that paper. In particular there exists an $R > 0$ (independent of n) such that $J_n(R\phi) \leq 0$. So

$$b_n = \inf_{g\in\Gamma} \sup_{t\in[0,1]} J_n[g(t)]$$

is a critical value of J_n, where $\Gamma = \{g \in C([0,1]; H_0^1(\Omega)): g(0)=0, g(1) = R\phi\}$. Observe that the critical point u_n corresponding to b_n which is a solution of (43) belongs in fact to $C^{2,\alpha}(\bar{\Omega})$.

<u>Third step</u>. There is a constant $c > 0$ (independent of n) such that $\|\nabla u_n\|_{L^2} \leq c$. Indeed, first observe that $0 \leq b_n \leq c$. Also by the results of Section 3.4 we see that $|\nabla u_n(x)| \leq c$ for all $x \in \partial\Omega$, uniformly in n. Using this information we obtain from Pohozaev identity

$$\frac{1}{2}\int_\Omega |\nabla u_n|^2 - \frac{N}{N-2}\int_\Omega F_n(u_n) \geq -c.$$

This together with

$$0 \leq b_n = \frac{1}{2}\int_\Omega |\nabla u_n|^2 - \int_\Omega F_n(u_n) \leq c$$

gives the result. As in Lemma 3.18 we obtain that $\|u_n\|_{L^\infty} \leq c$.

Forth step. Passage to the limit in (43). Since (41) holds uniformly in n we see that there exists an $\alpha > 0$ such that $\|u_n\|_{L^\infty} \geq \alpha$. So for large n u_n is a positive solution of (24).

Q.E.D.

Remark. A priori bounds for positive solutions of superlinear problems like (15) have been recently obtained by Gidas-Spruck [32], using different techniques.

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