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ON

VARIATIONAL METHODS IN ANALYSIS AND MATHEMATICAL PHYSICS

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OPTIMAL CONTROL OF DISTRIBUTED SYSTEMS

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OPTIMAL CONTROL OF DISTRIBUTED SYSTEMS

This set of basic lectures correspond essentially to the book of J.L. LIONS [3].
The aim of these short notes is to give the main result. For detailed proofs and analysis we refer to the book quoted above.

OPTIMAL CONTROL OF ELLIPTIC SYSTEMS

1 Formulation of the problem

An example

The pressure of water in a phreatic is governed by the diffusion equation

$$(1.1)\qquad \sum_{i=1}^{2} \frac{\partial}{\partial x_i}\left(k(x)\frac{\partial y}{\partial x_i}\right) = v \qquad x \in \Omega \subset \mathbb{R}^2$$

$$(1.2)\qquad y|_\Gamma = 0 \qquad \Gamma = \partial\Omega$$

y is the water pressure
k is the porosity
v is the input / output flow rate (distributed)

(1.3) $y(x;v) \equiv y(v)$ is the solution of the problem

An optimal control problem would be to optimize the flow rate in order to get a satisfactory behaviour of the phreatic

$$(1.4)\qquad J(v) = \int_\Omega |y(x) - z_d(x)|^2 dx + \nu \int_\Omega |v(x)|^2 dx$$

Remark The example (1.1) (1.2) is quite academic! In reality we must add more realistic boundary conditions.

Abstract formulation of the problem

(1.5)
- $a(\varphi,\psi)$ is a bilinear form on $V\times V$ which is
  - continuous

    $$|a(\varphi,\psi)| \le M\|\varphi\|\,\|\psi\|$$
  - coercive

    $$a(\varphi,\varphi) \ge \alpha\|\varphi\|_V^2$$
- $v\in E$ Hilbert space
- $B\in\mathcal{L}(E,V')$

The abstract state equation is then:

(1.6) $$a(y,\varphi)=(Bv,\varphi)\qquad\forall\,\varphi\in V$$

or equivalently

(1.6)' $$Ay=Bv \quad \text{in } V'$$

<u>Theorem 1</u> The problem (1.6) <u>under assumptions</u> (1.5) <u>has a unique solution</u> $y(v)$ <u>in</u> V.

Back to the example:

$$V=\{\varphi\in H^1(\Omega)\mid \varphi|_\Gamma=0\}=H_0^1(\Omega)$$

this is a Hilbert space equipped with the norm

(1.7) $$\|\varphi\|_V=|\nabla\varphi|_{L^2(\Omega)}=\int_\Omega|\nabla\varphi(x)|^2\,dx$$

<u>Note</u>. The fact that the norm (1.7) is equivalent to the norm of $H_0^1(\Omega)$ is due to the Poincaré's inequality.

If $\varphi\in H^1(\Omega)$ and $\varphi=0$ on $\Gamma_0\subset\Gamma$ (meas $\Gamma_0\neq0$) then there exists a constant C, depending on Ω, such that

(1.8) $$|\varphi|_{L^2(\Omega)}\le C\,|\nabla\varphi|^2_{L^2(\Omega)}.$$

If $k(\cdot)\in L^\infty(\Omega)$ and satisfies

(1.9) $$k(x)\ge\alpha>0 \quad \text{a.e. in } \Omega,$$

we get the coercivity of the following bilinear form

(1.10) $$a(\varphi,\psi)=\int_\Omega k(x)\nabla\varphi(x)\nabla\psi(x)\,dx.$$

It follows from (1.9) that

$$a(\varphi,\varphi)=\int_\Omega k(x)|\nabla\varphi(x)|^2\,dx\ge\alpha\,|\nabla\varphi|^2_{L^2(\Omega)}.$$

Then the operator B is the canonic injection: $L^2(\Omega)\to V'$ defined by

$$(Bv,\varphi)_{V',V}=\int_\Omega v\varphi\,dx \ !$$

Now we introduce the cost functional.

(1.11) Let $C\in\mathcal{L}(V;F)$ and $N\in\mathcal{L}(E;E)$ be given, we assume that N satisfies

$$(Nv,v)_E\ge\nu\|v\|_E^2.$$

The cost functional is given by

(1.12) $$J(v)=\|Cy(v)-z_d\|_F^2+(Nv,v)_E$$

where z_d "desired state" is given in F.

<u>Problem of control</u>

Let K a closed domain in E

To find $u \in K$ such that

(1.13) $J(u) \leq J(v) \quad \forall v \in K$

Thm 2 Under assumptions of thm 1 and (1.11) there exists a unique solution u to (1.13)

Proof Function (1.12) is strictly convex, it is coercive and continuous. Then there exists a unique solution to the inf problem.

Back to the example (continued)

$F = L^2(\Omega)$, $C \in \mathcal{L}(V; L^2(\Omega))$ $\quad N = \nu I \quad$ (I = Identity).

2. Examples

2.1 Boundary control – (Neumann) – Distributed observation

(2.1)
$$\begin{cases} -\Delta y + y = f & f \in L^2(\Omega) \\ \dfrac{\partial y}{\partial n} = v & v \in L^2(\Gamma) \end{cases}$$

(2.2)
$$J(v) = \int_\Omega |y(x;v) - z_d(x)|^2 + \nu \int_\Gamma |v(\gamma)|^2 d\gamma$$

The standard way to get the variational formulation is obtained via Green's formula

$$\int_\Omega -\Delta y \cdot \varphi \, dx = -\int_\Gamma \frac{\partial y}{\partial n} \varphi \, dx + \int_\Omega \nabla y \cdot \nabla \varphi \, dx$$

$$= -\int_\Gamma v \varphi \, dx + \int_\Omega \nabla y \nabla \varphi \, dx$$

(by using (2.1))

Then the variational formulation is

(2.3)
$$\int_\Omega \nabla y \nabla \varphi \, dx + \int_\Omega y \varphi \, dx = \int_\Omega f \varphi \, dx + \int_\Gamma v(\gamma) \varphi(\gamma) \, d\gamma .$$

It must be noticed that B is defined by

(2.4)
$$(Bv, \varphi)_{V',V} = \int_\Gamma v(\gamma) \varphi(\gamma) \, d\gamma$$

The continuity of B comes from the continuity of the mapping

(2.5)
$$\begin{cases} \varphi \xrightarrow{\gamma} \varphi|_\Gamma \\ H^1(\Omega) \to L^2(\Gamma) \end{cases}$$

Remark The general trace theorem says that

$\varphi \xrightarrow{\gamma} \varphi|_\Gamma$

is continuous from $H^m(\Omega) \to H^{m-1/2}(\Gamma)$. ■

2.3. Boundary control – (Fourier)

(2.6)
$$\begin{cases} -\Delta y + y = f \\ \dfrac{\partial y}{\partial n} + \beta y = v & \beta(.) \text{ given in } L^\infty(\Gamma). \end{cases}$$

Standard Green's formula leads to the following formulation

(2.7)
$$\int_\Omega \nabla y . \nabla \varphi \, dx + \int_\Omega y \varphi \, dx + \beta \int_\Gamma y \varphi \, d\gamma = \int_\Omega f \varphi \, dx + \int_\Gamma v \varphi \, d\gamma$$

For the same reasons as previously the bilinear form

$$a(y,\varphi) = ((y,\varphi))_{H^1(\Omega)} + \beta \int_\Gamma y \varphi \, d\gamma$$

is continuous

Coercivity is obtained after the additional assumption

(2.8) $\quad \beta(x) \geq 0 \quad$ a.e. on Γ

2.4 Boundary control (Dirichlet).

Following the previous approach it would be natural to introduce the following problem

(2.9) $\quad \begin{cases} -\Delta y + y = 0 \\ y|_\Gamma = v \end{cases}$ (2.10)

The main difficulty is in what space (2.10) takes sense?

Answer 1 If v is given in $L^2(\Gamma)$ eq. (2.10) may not have a sense

Answer 2 If v is given in $H^{1/2}(\Gamma)$ there is no problem but the space of control is quite "complicated".

A possible way to avoid difficulties is to take a weak solution of (2.9) (2.10).

Multiplying (2.9) by φ we get

(2.11) $$-\int_\Omega y\,\Delta\varphi\,dx + \int_\Omega y\varphi\,dx = \int_\Gamma \frac{\partial y}{\partial n}\varphi\,d\gamma - \int_\Gamma y\frac{\partial\varphi}{\partial n}\,d\gamma$$

Let be φ solution of

$$\begin{cases} -\Delta\varphi + \varphi = \psi \\ \varphi|_\Gamma = 0 \end{cases}$$

The (2.11) becomes

(2.12) $$\int_\Omega y\psi\,dx = -\int_\Gamma y\frac{\partial\varphi}{\partial n}\,d\gamma = -\int_\Gamma v\frac{\partial\varphi}{\partial n}\,d\gamma$$

So the weak solution of (2.9) (2.10) is the unique $y \in L^2(\Omega)$ such that

(2.13) $$\int_\Omega y\psi\,dx = -\int_\Gamma v\frac{\partial\varphi}{\partial n}\,d\gamma \quad \forall \psi \in L^2(\Omega)$$

Nota Difficulty is removed because we have the sequence of mapping

$$\psi \longrightarrow \varphi \longrightarrow \frac{\partial\varphi}{\partial n}$$

which are continuous on

$$L^2(\Omega) \longrightarrow H^2(\Omega) \longrightarrow L^2(\Gamma)$$

the Riesz's theorem claims that y exists and is unique because the mapping

$$\psi \longrightarrow \int_\Gamma v\frac{\partial\varphi}{\partial n}\,d\gamma$$

is a continuous linear form on $L^2(\Omega)$. ■

2.5 Boundary observation (Distributed control)

A typical example is

(2.14) $$\begin{cases} -\Delta y + y = v \\ \frac{\partial y}{\partial n} = 0 \end{cases}$$

(2.15) $$J(v) = \int_\Gamma |y(\gamma;v) - z_d(\gamma)|^2\,d\gamma + \nu\int_\Omega |v|^2\,dx$$

There is Existence and uniqueness of control because the mappings

$$v \longrightarrow y(v) \longrightarrow y(v)|_\Gamma$$
$$L^2(\Omega) \longrightarrow H^1(\Omega) \longrightarrow L^2(\Gamma)$$

are linear and continuous.

3. Necessary conditions for optimality

By using an abstract formulation of the control problem we have the following situation

(3.1) $$Ay = f + Bv$$

(3.2) $$J(v) = \mathcal{J}(y(v),v) = \|Cy(v) - z_d\|_F^2 + (Nv,v)_E$$

(3.3) $$\min_{v \in K} J(v)$$

(where f is given in V', this includes non homogeneous boundary conditions).

Let us introduce the following Lagrange's function

(3.4) $$\mathcal{L}(y,v,p) = \|Cy - z_d\|_F^2 + (Nv,v)_E + 2\langle p, f + Bv - Ay\rangle_{V,V'}$$

Now we want to choose y such that $y = y(v)$, this will be satisfied if

(3.5) $$\frac{\partial \mathcal{L}}{\partial p}(y,v,p) = 2(f + Bv - Ay) = 0$$

and we want to choose $p = p(v)$ such that

(3.6) $$\frac{\partial \mathcal{L}}{\partial y}(y(v),v,p(v)) = 0 .$$

It follows from (3.4)(3.5)(3.6) that

(3.7) $$\frac{d}{dv}\mathcal{L}(y(v),v,p(v)) = \frac{\partial \mathcal{L}}{\partial y}\frac{dy(v)}{dv} + \frac{\partial \mathcal{L}}{\partial v} + \frac{\partial \mathcal{L}}{\partial p}\frac{dp(v)}{dv} = \frac{\partial \mathcal{L}}{\partial v}(y(v),v,p(v))$$

Let us calculate

$$\frac{\partial \mathcal{L}}{\partial y}\cdot\tilde{y} = 2(Cy - z_d, C\tilde{y})_F + 2\langle p, -A\tilde{y}\rangle_{V,V'}$$

then if we introduce the adjoint state $p(v)$ solution of

(3.8) $$A^* p = C^* \Lambda_F (Cy(v) - z_d)$$

where Λ_X stands for the canonical isomorphism $X \to X'$

then (3.6) is satisfied. Now by simple calculation we have

(3.9) $$\frac{dJ(v)}{dv}\cdot\tilde{v} = \frac{\partial \mathcal{L}}{\partial v}(y(v),v,p(v))\cdot\tilde{v} = 2(Nv,\tilde{v})_E + 2\langle p, B\tilde{v}\rangle_{V,V'} = 2\left(Nv + \Lambda_E^{-1}B^* p(v)\right)$$

Then we have the following

Theorem 3.1 The optimal control $u \in K$ is given as the solution of the following problem

(3.10) $$Ay(u) = f + Bu$$

(3.11) $$A^* p(u) = C^* \Lambda_F (Cy(u) - z_d)$$

(3.13) $$\left(Nu + \Lambda_E^{-1} B^* p(u), v - u\right)_E \geq 0 \quad \forall v \in K$$

We will see in the parabolic case what is the exact interpretation of (3.13).

Chapter II

CONTROL OF PARABOLIC SYSTEMS

1 Parabolic equations

We suppose we are in the classical situation were

$$V \hookrightarrow H \hookrightarrow V'.$$

And we will use the space X being an Hilbert space

(1.1) $$L^2(0,T;X) = \{ t \to \varphi(t) \mid \int_0^T \|\varphi(t)\|_X^2 \, dt < +\infty \}$$

equipped with the natural inner product

(1.2) $$(\varphi,\psi)_{L^2(0,T;X)} = \int_0^T (\varphi(t),\psi(t))_X \, dt.$$

Then being given a family $t \to A(t)$ of operators

$$A(t) \in \mathcal{L}(V,V')$$

which is

- measurable
- continuous:

(1.3) $$\exists M : \quad \|A(t)\|_{\mathcal{L}(V,V')} \le M \quad \forall t \in [0,T]$$

- coercive in the following sense

(1.4) $$\begin{cases} \exists \alpha>0, \exists \lambda \text{ such that} \\ \forall \varphi \in V \quad \langle A(t)\varphi,\varphi\rangle_{V',V} + \lambda|\varphi|_H^2 \geq \alpha\|v\|_V^2 . \end{cases}$$

and

(1.5) $$\begin{cases} f \text{ given in } L^2(0,T;V') \\ y_0 \text{ given in } H \end{cases}$$

then we may consider the following equation

(1.6) $$y'(t) + A(t)y(t) = f(t) \quad \text{on }]0,T[$$

(1.7) $$y(0) = y_0$$

<u>Theorem 1</u> <u>Under the previous assumption there exists a unique y which satisfies</u>

(1.8) $$y \in L^2(0,T;V), \quad y' \in L^2(0,T;V')$$

<u>and which is solution of (1.5) (equation i) with initial condition (1.6)</u>

<u>Remark 1.1</u>. It follows from (1.9) that $y(t)$ is (a modification on a set with measure 0) is continuous

$$[0,T] \to H.$$

<u>Corollary</u> <u>The mapping</u>

$$\{y_0, f\} \longrightarrow y \text{ sol. of } (1.6)\,(1.7)$$

$$H \times L^2(0,T;V') \longrightarrow W(0,T)$$

<u>where</u>

$$W(0,T) = \{ \varphi \in L^2(0,T;V) \mid \varphi' \in L^2(0,T;V') \}$$

<u>is affine continuous.</u>

2. Optimal control problem.

We consider the following situation:

The control $v(.)$ belongs to the space

(2.1) $$\mathcal{U} = L^2(0,T;E)$$

and

(2.2) $\mathcal{U}_{ad}$ is a closed convex subset of $\mathcal{U}$,

A family of "control operators" $B(t)$ is given

(2.3) $$\begin{cases} \forall t \in [0,T] & B(t) \in \mathcal{L}(E;V') \\ \exists \beta > 0 \text{ such that } & \|B(t)\|_{\mathcal{L}(E;V')} \leq \beta \quad \forall t \in [0,T] \end{cases}$$

Then it is possible to define the solution of the controlled system

(2.4) $$\begin{cases} y'(t) + A(t)\,y(t) = f(t) + B(t)\,v(t) \\ y(0) = y_0 \end{cases}$$ (2.5)

which will be denoted by

$y(t;v)$ or simply $y(v)$.

Theorem 2.1

The mapping $v \longrightarrow y(v)$ is affine continuous from

$$\mathcal{U} = L^2(0,T;E) \longrightarrow W(0,T)$$

Now we have to introduce an observation by means of the family of operators $C(.)$:

(2.6) $$\begin{cases} \forall t \in [0,T] \quad , \quad C(t) \in \mathcal{L}(V';F) \\ \exists \gamma \quad : \quad \|C(t)\|_{\mathcal{L}(V';F)} \leq \gamma \end{cases}$$

where F is a given Hilbert space ("space of observation").

Furthermore let $N(t) \in \mathcal{L}(E;E)$ a family of operators such that

(2.7) $$\begin{cases} N \text{ is symmetric and} \\ \exists \nu > 0 \text{ such that } \quad (N(t)v,v)_E \geq \nu\|v\|_E^2 \quad \forall t \in [0,T] \end{cases}$$

Then we define the following "criterion" or "cost" or "economic function" $J(v)$ by

(2.8) $$J(v) = \int_0^T \|C(t)y(t;v) - z_d(t)\|_F^2\,dt + \int_0^T (N(t)v(t),v(t))_E\,dt$$

(z_d given in $L^2(0,T;F)$).

Optimal control problem

(2.9) $$\begin{cases} \text{To find } u \in \mathcal{U}_{ad} \text{ such that} \\ J(u) \leq J(v) \quad \forall v \in \mathcal{U}_{ad} \end{cases}$$

Theorem 2.2 There exists a unique optimal control u solution of (2.9).

Proof. Under assumption (2.7) and theorem 2.1 the functional $J(v)$ is quadratic, coercive, continuous on $\mathcal{U}$; the existence and optimality come from classical results in minimization theory. ∎

Theorem 2.3

Let J' be the (Frechet or Gateaux) derivative of J, then u is solution of (2.9) iff

$$(2.10) \qquad (J'(u), v-u)_{\mathcal{U}} \geq 0 \qquad \forall v \in \mathcal{U}_{ad}.$$ ∎

Let introduce now the following Lagrange's function:

$$(2.11) \qquad \begin{cases} \mathcal{L}(y,v,p) = \int_0^T \|Cy - z_d\|^2 dt + \int_0^T (Nv,v)_E \, dt \\ \qquad + 2\int_0^T \langle f + Bv - y' - Ay, p \rangle_{V',V} \, dt \end{cases}$$

where $p \in W(0,T)$.

It is clear that

$$(2.12) \qquad \frac{\partial \mathcal{L}}{\partial p}(y,v,p) = 0 \Longrightarrow y = y(v) \text{ the state corresponding to } v.$$

Now we want to determine $p(u)$ such that

$$(2.13) \qquad \frac{\partial \mathcal{L}}{\partial y}(y,v,p(u)) = 0.$$

Before to do that we have to notice that

$$(2.14) \qquad \begin{cases} \int_0^T \langle y' + Ay, p \rangle_{V',V} \, dt = (y(T), p(T))_H - (y_0, p(0)) \\ \qquad + \int_0^T \langle -p' + A^* p, y \rangle_{V',V} \, dt \end{cases}$$

By combining this result and Theorem 2.3 we have the following

Theorem 2.4 The optimal control u is uniquely determined by the set of equations and inequality

$$(2.18) \qquad \begin{cases} y'(u) + Ay(u) = f + Bv \\ y(0;u) = y_0 \end{cases}$$

$$(2.19) \qquad \begin{cases} -p'(u) + A^* p(u) = C^* \Lambda_F (Cy(u) - z_d) \\ p(T;u) = 0 \end{cases}$$

$$(2.20) \qquad (\Lambda_{\mathcal{U}}^{-1} B^* p(u) + Nu, v - u)_E \geq 0 \qquad \forall v \in \mathcal{U}_{ad}$$

Remark 2. If $\mathcal{U}_{ad} = \mathcal{U}$ then (2.20) is replaced by

$$(2.21) \qquad u = -N^{-1} \Lambda_{\mathcal{U}}^{-1} B^* p(u)$$ ∎

3. Examples

Example 1. Distributed control and observation.

$V = H^1(\Omega) \quad H = L^2(\Omega) \quad E = F = L^2(\Omega).$

$$(3.1) \qquad \begin{cases} \frac{\partial y}{\partial t} - \Delta y = f + v(x,t) & \text{in } Q = \Omega \times]0,T[\\ \frac{\partial y}{\partial n}\Big|_\Sigma = 0 & \text{on } \Sigma = \Gamma \times]0,T[\\ y(x,0) = y_0(x) \end{cases}$$

$$(3.2) \qquad J(v) = \int_Q |y(x,t;v) - z_d(x,t)|^2 dx\,dt + \nu \int_Q |v(x,t)|^2 dx\,dt$$

In this case the adjoint state is given by

$$(3.3)\quad \begin{cases} -\dfrac{\partial p}{\partial t} - \Delta p = y - z_d & \text{in } Q \\ \dfrac{\partial p}{\partial n}\Big|_\Sigma = 0 \\ p(T) = 0 \end{cases}$$

and

$$(3.4)\quad \frac{dJ(v)}{dv} = p + \nu v \quad (\in L^2(Q))$$

The optimality condition is then

$$(3.5)\quad \left(p(u) + \nu u,\ v - u\right)_{L^2(Q)} \geq 0 \quad \forall v \in \mathcal{U}_{ad} \subset L^2(Q)$$

Remark 3.1 If $\mathcal{U}_{ad} = \mathcal{U} = L^2(Q)$ then (3.5) gives

$$(3.6)\quad u(x,t) = -\frac{1}{\nu}\, p(x,t)$$

Then, as in this case, the regularity of p is

$$p \in H^{2,1}(Q) = \{ \varphi \in L^2(0,T;H^2(\Omega)) \mid \varphi' \in L^2(0,T;L^2(\Omega)) \}$$

then

$$u \in H^{2,1}(Q)$$

∎

Remark 3.2

If we consider the case

$$\mathcal{U}_{ad} = \{ v \in L^2(Q) \mid v(x,t) \geq 0 \ \text{ a.e in } Q \}$$

Then we have the following interpretation of (3.5):

$$Q = Q_0 \cup Q_+$$

where

$$Q_0 = \{ (x,t) \in Q \mid u(x,t) = 0 \}$$

and

$$Q_+ = \{ (x,t) \in Q \mid u(x,t) > 0 \}$$

Then for

$(x,t) \in Q_+$ we have $u(x,t) = -\frac{1}{\nu}\, p(x,t)$ $(u(x,t) > 0!)$

and for

$(x,t) \in Q_0$ we have $\begin{cases} (u(x,t) = 0 \ !) \ \text{and} \\ p(x,t) + \nu u(x,t) > 0 \end{cases}$

∎

Example 2 (Boundary observation and control)

$$(3.7)\quad \frac{\partial y}{\partial t} - \Delta y = f \quad \text{in } Q$$

$$(3.8)\quad \frac{\partial y}{\partial n} + k y = v \quad \text{on } \Sigma$$

$$(3.9)\quad y(0) = y_0$$

Here we have $V = H^1(\Omega)$ $\mathcal{U} = L^2(\Sigma) = L^2(0,T;L^2(\Gamma))$

Remark 3.3 The "variational" formulation of (3.7) (3.8) is

$$\frac{d}{dt}\left(y(t),\varphi\right)_{L^2(\Omega)} + a(y(t),\varphi) = (f(t),\varphi)_{L^2(\Omega)} + \int_\Gamma v(\gamma,t)\varphi(\gamma)\,d\gamma$$

$$\forall \varphi \in V$$

where

$$a(y,\varphi) = \int_\Omega \nabla y \cdot \nabla \varphi \, dx + k \int_\Gamma y(\gamma,t)\,\varphi(\gamma)\,d\gamma$$

which is coercive under the hypothesis

$$k \geq 0 \ .$$

By taking

$$F = L^2(\Gamma)$$

we consider the following cost

$$(3.10) \qquad J(v) = \int_\Sigma |y(\gamma,t;v) - z_d(\gamma,t)|^2 \, d\gamma \, dt + \nu \int_\Sigma |v(\gamma,t)|^2 \, d\gamma \, dt.$$

Then the adjoint state is defined by

$$(3.11) \qquad \begin{cases} -\dfrac{\partial p}{\partial t} - \Delta p = 0 & \text{in } Q \\ \dfrac{\partial p}{\partial n} + kp = y - z_d & \text{on } \Sigma \\ p(T) = 0 \end{cases}$$

and the gradient of J is given by

$$(3.12) \qquad \frac{dJ(v)}{dv} = 2\left(p|_\Sigma + \nu v \right) \quad (\in L^2(\Sigma))$$

The optimality condition is then

$$\int_\Sigma \big(p(\gamma,t;u) + \nu u(\gamma,t)\big)\big(v(\gamma,t) - u(\gamma,t)\big) \, d\gamma \, dt \geq 0$$

$$\forall v \in U_{ad} \subset L^2(\Sigma)$$

Chap III Synthesis of the linear-quadratic control problems (Parabolic case)

1 The classical Riccati equation

We will consider first the standard parabolic problem in finite interval of time.

The optimal control problem is the following:

$$(1.1) \qquad y'(t) + A(t)y(t) = f(t) + B(t)v(t) \qquad y(\cdot) \in W(0,T)$$

$$(1.2) \qquad y(0) = y_0$$

where

$v \in L^2(0,T;E)$, $B(t) \in \mathcal{L}(E;V')$ satisfies $\|B(t)\|_{\mathcal{L}(E,V')} \leq B$

the cost function is

$$(1.3) \qquad J(v) = \int_0^T \|C(t)y(t;v) - z_d(t)\|_F^2 + \int_0^T (N(t)v(t), v(t))$$

the optimal control u is characterized by

$$(1.4) \qquad \begin{cases} -p'(u) + A^*(t)p(u) = C^*(t)\Lambda_F(C(t)y(u) - z_d) \\ p(T) = 0 \end{cases}$$

$$(1.5) \qquad \Lambda_u^{-1} B^*(t) p(t;u) + N u(t) = 0$$

which corresponds to the unconstrained case.

The synthesis problem:

We are looking for a closed-loop solution of the control problem i.e.:

$$(1.6) \qquad u(t) = \Phi(y(t))$$

We will show that the function Φ is affine and more precisely that

(1.7) $$u(t) = -N^{-1}\Lambda_{\mathcal{U}}^{-1} B^* p(t)$$

with

(1.8) $$p(t) = P(t)\, y(t) + r(t)$$

where $P(t) \in \mathcal{L}(H;H)$ is solution of a Riccati-type equation and $r(t)$ is solution of a parabolic equation.

Let us consider now that we have the same control problem as previously but on the time interval $]s,T[$ $(s > 0)$. The state equation is

(1.9) $$+y' + Ay = f + Bv \quad \text{on }]s,T[$$

(1.10) $$y(s) = h \qquad h \text{ given in } H,$$

The cost $J(v) = J(v;s,h)$ is

(1.11) $$J(v;s,h) = \int_s^T \|Cy(t) - z_d\|_F^2\, dt + \int_s^T (Nv,v)\, dt.$$

It is clear that there exists a unique control $u(s,h) \in L^2(s,T;E)$ such that

(1.12) $$J(u(s,h);s,h) \leq J(v;s,h) \quad \forall v \in L^2(s,T;E),$$

which is characterized by

(1.13) $$-p' + A^* p = C^*\Lambda_F(C^* y(u) - z_d) \quad \text{on }]s,T[$$

(1.14) $$p(T) = 0$$

(1.15) $$\Lambda_{\mathcal{U}}^{-1} B^* p(u) + Nu = 0 \quad \underline{\text{with}} \quad u = u(s,h).$$

By introducing the following notation

$$D_1 = BN^{-1}\Lambda_{\mathcal{U}}^{-1}B^*$$

$$D_2 = C^*\Lambda_F C$$

we have the following system of optimality

(1.16) $$y' + Ay - D_1 p = f \quad \text{on }]s,T[$$

(1.17) $$-p' + A^* p - D_2 y = g \qquad g = -C^*\Lambda_F z_d$$

(1.18) $$y(s) = h, \quad p(T) = 0$$

which has a unique solution $(y(\cdot;s,h), p(\cdot,s,h))$.

<u>Theorem 1.1</u> <u>The mapping</u>

$$h \to p(s;s,h)$$

<u>is affine continuous from</u> $H \to H$.

<u>Then there exists</u> $P(s) \in \mathcal{L}(H;H)$ <u>and</u> $r(s) \in H$ <u>such that</u>

(1.19) $$p(s;s,h) = P(s)\, h + r(s)$$

<u>Proof.</u> (cf [3])

The proof can be done in two steps.

(i) By taking $f = 0$ and $z_d = 0$ we are reduced to the linear case.

(ii) It is easy to show that

$$h \to (y(\cdot;s,h), p(\cdot;s,h))$$

$$H \to W(s,T) \times W(s,T)$$

is linear and continuous, by showing that

$$h \to u(\cdot;s,h)$$

$$H \to L^2(s,T;E)$$

is linear and continuous.

Now the problem consists in getting an equation for P (and r).

Formal calculations

We simply substitute p in eq. (1.16) (1.17) by

$$p(t) = P(t)\,y(t) + r(t)$$

this gives

(1.20) $$y' + Ay - D_1 P y - D_1 r = f$$

(1.21) $$-(Py + r)' + A^*(Py + r) - D_2 y = g.$$

Developping (1.21) we get

$$-P'y - Py' + A^* P y - D_2 y - r' + A^* r = g$$

and then by using (1.20)

(1.22) $$-P'y - PAy - PD_1 P y + A^* P y = D_2 y - Pf - r' + A^* r = g.$$

This equation is valid for any y solution of an optimal control problem corresponding to a given data h. But P, r, g do not depend on h. Thus we have

(1.23) $$(-P' + PA + A^* P + PD_1 P)\, y = D_2 y,$$

(1.24) $$-r' + A^* r = g + Pf.$$

Thus, at least formally, we are led to consider the following Riccati equation:

(1.25) $$-P' + PA + A^* P + PD_1 P = D_2$$

(1.26) $$P(T) = 0$$

The last condition (1.26) comes from the fact that

$$\lim_{s \to T} p(s; s, h) = 0 = \lim_{s \to T} P(s)\,h + r(s)$$

which implies $P(T) = 0$ and $r(T) = 0$

Remark 1.1

It is possible to get a direct justification of (1.25) (1.26) under assumptions

(i) $B \in \mathcal{L}(E; H)$ (distributed control)

and the mapping

(ii) $I : V \to H$ is compact.

This is the case considered in LIONS [3].

For more general hypotheses see M. SORINE [8] [9] [

2. Reduction of dimension

This section is devoted to the obtaining of so called Chandrasekhar type equations which have been first obtained in the finite dimensional case by (LINDQUIST, T. KAILATH) and the results have been somehow extended by CASTI and L. LJUNG [7] to infinite dimensional case. The following presentation will be formal. Precise justifications require very long developments (see essentially M. SORINE [8] [9] [10]).

2.1 Derivation of K-L equations

We start from the Riccati's equation

(2.1) $$\begin{cases} -P' + PA + A^*P + PD_1P = D_2 \\ P(T) = 0 \end{cases}$$

(2.2) $$D_1 = B\Lambda_u^{-1} N^{-1} B^* \qquad D_2 = C^*\Lambda_F C$$

and we assume that

(2.3) $$\begin{cases} \text{all operators } A, B, C, N \text{ are independants} \\ \text{of time.} \end{cases}$$

Then it is possible to take the derivative of (2.1) which gives

(2.4) $$-P'' + P'A + A^*P' + P'D_1P + PD_1P' = 0$$

the final condition being in this case given by writing that

$$-P'(T) + P(T)A + A^*P(T) + P(T)D_1P(T) = -P'(T) = D_2$$

because of final condition on P.

By letting

$$M = P'$$

we get the following equation for M

(2.5) $$\begin{cases} -M' + MA + A^*M + MD_1P + PD_1M = 0 \\ M(T) = -D_2 = -C^*\Lambda_F C \end{cases}$$

Then as $M(T)$ can be factorized, is it possible to get a factorization of $M(t)$ for any t? To do that let us set (formally):

(2.6) $$M(t) = -L^*(t)\Lambda_F L(t),$$

then by plugging this relation in (2.5) we get

(2.7) $$\begin{cases} -\frac{dL^*}{dt}\Lambda_F L - L^*\Lambda_F \frac{dL}{dt} + L^*\Lambda_F LA + A^*L^*\Lambda_F L + \\ \quad + L^*\Lambda_F L D_1 P + PD_1L^*\Lambda_F L = 0, \\ L(T) = C. \end{cases}$$

Let us suppose now that there exists L solution of

(2.8) $$\begin{cases} -\frac{dL}{dt} + L(A + D_1P) = 0 \\ L(T) = C \end{cases}$$

then it is easy to see that L is also a solution of (2.7). Now we have to eliminate P. This will be done by introducing the gain of the controller. We know that the optimal control is given by

$$u(t) = -\Lambda_u^{-1} N^{-1} B^* P\, y(t)$$

or

$$Bu(t) = -D_1 P\, y(t)$$

Let us define

(2.9) $$K = \Lambda_u^{-1} B^* P$$

Then we can derive a differential equation for K, which is

(2.10)
$$\begin{cases} \dfrac{dK}{dt} - \Lambda_u^{-1} B^* \dfrac{dP}{dt} = \dfrac{dK}{dt} + \Lambda_u^{-1} B^* \left(L^* \Lambda_F L \right) = 0 \\ K(T) = \Lambda_u^{-1} B^* P(T) = 0 \; . \end{cases}$$

Then finally we have obtained the following set of two equations in K and L

(2.11)
$$\begin{cases} -\dfrac{dK}{dt} - \Lambda_u^{-1} B^* (L^* \Lambda_F L) = 0 \\ K(T) = 0 \; , \end{cases}$$

(2.12)
$$\begin{cases} -\dfrac{dL}{dt} + L(A+BK) = 0 \; , \\ L(T) = C \; . \end{cases}$$

<u>Remark 2.1</u> The interest of (2.11) and (2.12) lies in the fact that the dimension of this problem is reduced, comparing to the size of P, in many circumstances.

Let us consider the finite dimensional case:

$$Y = \mathbb{R}^n, \quad E = \mathbb{R}^m, \quad F = \mathbb{R}^l$$

then

$P \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^n)$, is symmetric then the Riccati equation corresponds to a set of

$$\frac{n(n+1)}{2}$$ differential equations.

By looking at eq (2.12) and (2.9)

$$L(t) \in \mathcal{L}(V; F) \; ,$$
$$K(t) \in \mathcal{L}(V; E) \; ,$$

Then the system (2.11) (2.12) is a set of

$$n(m+l)$$

differential equations.

Then if $(m+l) \ll \frac{n}{2}$ we have an important reduction of the differential system obtained. In the case of distributed systems this is practically always the case because, usually, the control and observation are pointwise.

<u>Remark 2.2</u> As we have mentioned before all this calculus are formal. For instance we see from (2.12) that

$$L(t) \in \mathcal{L}(V'; F)$$

but $$L(T) \in \mathcal{L}(V; F) \not\subset \mathcal{L}(V'; F) \; !$$

2.2. <u>Interpretation of the K–L system.</u>

The equation of evolution of the optimal state is

(2.13) $$y' + (A+BK)\, y = 0$$

(2.14) $$y(s) = h$$

Let be $G(t;s)$ the semi-group associated to eq (2.13), i.e.:

(2.15) $$y(t) = G(t;s)\, h \; ,$$

by definition we have

(2.16) $$\frac{\partial G}{\partial t}(t;s) + (A+BK)\, G(t,s) = 0 \; .$$

Now if we compute the derivative

$$-\frac{\partial G}{\partial s}(t;s)$$

we get the following equation

(2.17)
$$\begin{cases} -\dfrac{\partial G}{\partial s}(t;s) + G(t;s)(A+BK) = 0 \\ G(t;t) = I \end{cases}$$

In particular we have

$$(2.18)\quad \begin{cases} -\dfrac{\partial G}{\partial s}(T;s) + G(t;s)\,(A+BK) = 0 \\ G(T,T) = I . \end{cases}$$

By multiplying this eq by C we get

$$(2.19)\quad \begin{cases} -\dfrac{\partial}{\partial s}\, CG(T;s) + CG(T;s)\,(A+BK) = 0 \\ CG(T,T) = I . \end{cases}$$

Then if we define

$$(2.20)\qquad L(s) = CG(T;s)$$

we have

$$L(s).h = CG(T;s)\,h = Cy(T;s),$$

where $y(t;s)$ is the solution of

$$(2.21)\quad \begin{cases} \dfrac{\partial y}{\partial t}(t;s) + (A+BK)\,y(t;s) = 0 \\ y(s;s) = h \end{cases}$$

that is to say that $L(s).h$ is the observation of the optimal state at time T, eq. (2.19) with notation (2.20) is then

$$\begin{cases} -\dfrac{\partial L}{\partial t} + L\,(A+BK) = 0 \\ L(T) = C \end{cases}$$

which is precisely eq. (2.12).

§3. Algebraic Riccati equation

Now we want to see what happens when $T \to \infty$?

More precisely we want to study the following control problem.

$$(3.1)\qquad y' + Ay = Bv$$

$$(3.2)\qquad y(0) = h$$

$$(3.3)\qquad J(v) = \int_0^\infty \|Cy(t;v), y(t;v)\|_F^2 + \int_0^\infty (Nv(t), v(t))^2\,dt$$

where A, B, C, N are time independent.

First of all to be sure that the problem (3.1) (3.3) has a solution it is necessary to assume that, h being given,

$$(3.4)\quad \begin{cases} \text{There exists } v \in L^2(0,\infty;E) \text{ such that} \\ \qquad Cy(v) \in L^2(0,\infty;F) . \end{cases}$$

Actually we may pose the following definition

Definition 3.1 The pair (A,B) is said to be C-stabilizable if

$$(3.5)\quad \begin{cases} \forall h \in H, \text{ there exists a control } v(t) \in L^2(0,\infty,E) \\ \text{such that} \\ \qquad Cy(v,h) \in L^2(0,\infty;F) \end{cases}$$

Definition 3.2 The pair (A,B) is C-stabilizable by feedback if

$\forall h \in H$

there exists a feedback

$$v(t) = K(y(t)) \qquad K : H \to E$$

such that

the equation

$$(3.6)\quad \begin{cases} y' + Ay + BK(y) = 0 \\ y(0) = h \end{cases}$$

has a solution such that

$$Cy \in L^2(0,\infty;F)$$

Theorem 3.1 Under assumption (3.1)...(3.5) there exists a unique $u \in L^2(0,\infty;E)$ solution of

$$J(u) \le J(v) \quad \forall v \in L^2(0,\infty;E) = \mathcal{U}$$

Proof By hypothesis (3.5) there exists $v_h \in \mathcal{U}$ such that

$$J(v_h) < +\infty,$$

then there exists a minimizing sequence u_n such that

$$(3.7)\quad J(u_n) \xrightarrow[n\to\infty]{} \inf_{w\in\mathcal{U}} J(w),$$

then for large n we have

$$J(u_n) \le J(v_h)$$

then u_n is bounded in $L^2(0,\infty;E)$, and we may extract a subsequence u_μ such that

$$(3.8)\quad \begin{cases} u_\mu \to u \quad \text{in } L^2(0,\infty:E) \text{ weakly} \\ y_\mu = y(u_\mu) \to y(u) \quad \text{in } L^2(0,\infty;V) \text{ weakly.} \end{cases}$$

Then if we define

$$J_T(v) = \int_0^T \{ \|Cy(t;v)\|_F^2 + (Nv,v)_E \}\, dt$$

$$J_T(u) \le \liminf_\mu J_T(u_\mu) \le \liminf_\mu J(u_\mu) = \inf_{v\in\mathcal{U}}$$

then

$$J_T(u) \le \inf_{v\in\mathcal{U}} J(v)$$

then

$$\lim_{T\to\infty} J_T(u) = J(u) = \inf J(v)$$

Uniqueness results from strict convexity of $v \to \int_0^\infty (Nv,v)$ in $\mathcal{U}$.

Theorem 3.2 Under the same assumptions that previously the optimal control is characterized by

$$(3.9)\quad \begin{cases} y'(u) + Ay(u) + D_1 p(u) = 0 & y \in L^2(0,\infty;V) \\ -p'(u) + A^* p(u) - D_2 y(u) = 0 & p \in L^2(0,\infty;V \\ y(0) = y_0 & \\ p(u;t) = P\, y(t;u) & \forall t \ge 0 \end{cases}$$

where

$$(3.10)\quad P \in \mathcal{L}(H;H), \quad P = P^*, \quad P \ge 0.$$

Then the optimal control is given by

$$(3.11)\quad u(t) = -N^{-1}\Lambda_{\mathcal{U}}^{-1} B^* P\, y(t;u) \quad \text{a.e. in }]0,\infty[$$

Proof

By considering the optimal control on $]0,T[$ associated to previous one (on $]0,\infty[$) we may consider

$$\begin{cases} u_T \text{ solution of} \\ \inf_{v\in L^2(0,T;E)} J_T(v) \end{cases}$$

and the corresponding state y_T and adjoint state p_T

which are defined by

$$(3.12)\quad \begin{cases} y_T' + A y_T + \dots p_T = 0 \\ -p_T' + A^* p_T - D_2 y_T = 0 \\ y_T(0) = h, \quad p_T(T) = 0 \end{cases}$$

and we know (cf. § 1) that

$$(3.13)\quad \begin{cases} \text{there exists } P_T(t) \text{ such that} \\ P_T(t) \in \mathcal{L}(H,H), \quad P_T(t) = P_T^*(t), \quad P_T(t) \geq 0 \ \text{a.e. on }]0,T[\end{cases}$$

and

$$(3.14)\quad \begin{cases} p_T(t) = P_T(t)\, y_T(t) \\ u_T(t) = -N^{-1}\Lambda_E^{-1} B^* p_T(t) = -N^{-1}\Lambda_E^{-1} B^* P_T(t)\, y_T(t). \end{cases}$$

As the operators are time independent we may translate all the equations in time, then

$$(3.15)\quad P_T(t) = P_{T-t}(0) \qquad 0 \leq t < T$$

(this is a direct consequence of the construction of $P_T(t)$).

Furthermore we have

$$J_{T_1}(v) \leq J_{T_2}(v) \qquad 0 \leq T_1 \leq T_2, \ \forall v \in L^2(0,\infty;E)$$

then as we know, (from § 1), that

$$J_T(u_T) = (P_T(0)\, h, h)$$

then we have

$$(P_{T_1}(0)\, h, h) \leq (P_{T_2}(0)\, h, h) \qquad \forall h \in H, \ T_1 \leq T_2$$

So with relation of order associated to the positive definiteness in H we have

$$(3.16)\quad P_{T_1} \leq P_{T_2} \quad \text{if} \quad T_1 \leq T_2.$$

At first let us recall the following

Lemma. Let be $G_n \in \mathcal{L}(H;H)$ a sequence of self adjoint operators in X such that

$$\|G_n\| \leq c \quad \text{constant indep. of } n,$$

then if the sequence is monotonic, then

$$G_n \xrightarrow[n\to\infty]{} G \quad \text{in } \mathcal{L}(H;H).$$

To apply this Lemma it is sufficient to prove that

$$\|P_T(0)\|_{\mathcal{L}(H;H)} \leq c$$

But

$$(P_T(0)\, y_0, y_0) = J_T(u_T) \leq J_T(u) \leq J(u)$$

where u is the optimal control in the infinite time interval.
Let be $h, \bar{h} \in H$ then if

$$\begin{cases} u_h \text{ is the optimal control in }]0,\infty[\\ \text{associated to initial condition } y(0) = h \end{cases}$$

we have

$$(P_T(0)\, h, \bar{h}) \leq \frac{1}{4}\left\{ \left(P_T(0)(h+\bar{h}), h+\bar{h}\right) + \left(P_T(0)(h-\bar{h}), h-\bar{h}\right) \right\}$$

$$= \frac{1}{4}\left\{ J_T(u(h+\bar{h})) + J_T(u(h-\bar{h})) \right\}$$

$$\leq \frac{1}{4}\left\{ J(u(h+\bar{h})) + J(u(h+\bar{h})) \right\}$$

By applying the Banach Steinhaus theorem we get

$$\|P_T(0)\, h\|_H \leq c_0 \qquad \forall h \in H$$

and then, by the same argument,

$$\|P_T(0)\|_{\mathcal{L}(H;H)} \leq c_1.$$

Then

$$(3.17)\quad \lim_{T\to\infty} P_T(t) = P \quad \text{in } \mathcal{L}(H;H).$$

Let us denote by $\tilde{y}_T$ the function defined on $[0,+\infty[$ by

$$\tilde{y}_T(t) = \begin{cases} y_T(t) & \text{if} \quad t \in [0,T[\\ 0 & \text{if} \quad t \geq T \end{cases}$$

Now, as we have

$$J_T(\tilde{u}_T) = \int_0^\infty \left\{ \| C\tilde{y}_T(t) \|_F^2 + (N\tilde{u}_T, \tilde{u}_T)_E \right\} dt \leq J_T(u) \leq J(u)$$

then

(3.18) $$\|\tilde{u}_T\|^2_{L^2(0,\infty;E)} \leq \frac{J(u)}{\nu} = c_0$$

then, by using the state equation we have

(3.19) $$\int_0^\infty \|\tilde{y}_T(t)\|_V^2 \, dt \leq c_1 ,$$

and by energy estimate on the adjoint state we have also

(3.20) $$\int_0^\infty \|\tilde{p}_T(t)\|^2 \, dt \leq c_2 .$$

let T_n be a sequence of final time such that

$$T_n \to \infty ,$$

then by extracting eventually a subsequence we have

(3.21) $$\begin{cases} u_{T_n} \longrightarrow w & \text{in } L^2(0,\infty;E) \text{ weakly} , \\ y_{T_n} \longrightarrow y(w) & \text{in } L^2(0,\infty;V) \quad " , \\ p_{T_n} \longrightarrow p(w) & \text{in } L^2(0,\infty;V) \quad " . \end{cases}$$

let be T fixed then

(3.22) $$J_T(w) \leq \liminf_{T_n \to \infty} J_T(\tilde{u}_{T_n}) \leq \liminf_{T_n \to \infty} J_{T_n}(\tilde{u}_{T_n}) \leq \liminf_{T_n \to \infty} J_{T_n}(u) = J(u)$$

Then

$$J_{T_n}(w) \longrightarrow J(w)$$ (this is an increasing sequence bounded from above)

Then

$$J(w) \leq J(u)$$

have

(3.23) $$w = u .$$

Thus, as

(3.24) $$J(\tilde{u}_{T_n}) \to J(u)$$

we have

$$\int_0^\infty \|C\tilde{y}_{T_n}\|_F^2 \, dt + \int_0^\infty (N\tilde{u}_{T_n}, \tilde{u}_{T_n})_E \, dt \longrightarrow \int_0^\infty \|Cy(u)\|_F^2 \, dt + \int_0^\infty (Nu,u)_E$$

and, by (3.21),

(3.23) $$\liminf \int_0^\infty \|C\tilde{y}_{T_n}\|_F^2 \, dt \geq \int_0^\infty \|Cy(u)\|_F^2 \, dt .$$

Suppose that

$$\liminf \int_0^\infty (N\tilde{u}_{T_n}, \tilde{u}_{T_n})_E > \int_0^\infty (Nu,u)_E \, dt$$

then we will have, from (3.24), we will have

$$\liminf_{n \to \infty} \int_0^\infty \|C\tilde{y}_{T_n}\|_F^2 dt < \int_0^\infty \|Cy(u)\|^2_F dt$$

which is contradiction with (3.23), then

$$\liminf_{n \to \infty} \int_0^\infty (N\tilde{u}_{T_n}, \tilde{u}_{T_n}) \, dt = \int_0^\infty (Nu,u) \, dt$$

but by the argument applied to

$$\limsup_{n \to \infty} \{ -J(\tilde{u}_{T_n}) \}$$

we have finally

$$\lim_{n \to \infty} \int_0^\infty (N\tilde{u}_{T_n}, \tilde{u}_{T_n}) \, dt = \int_0^\infty (Nu,u) \, dt$$

then

(3.24) $$\tilde{u}_{T_n} \to u \quad \text{strongly in } L^2(0,\infty;E)$$

and, as a consequence

(3.25) $\qquad \tilde{y}_{T_n} \longrightarrow y(u)$ strongly in $L^2(0,\infty;V)$

Furthermore for any $T>0$

(3.26) $\qquad \tilde{y}_{T_n} \longrightarrow y(u)$ in $W(0,T)$ (strongly)

then it is easy to show this is true also for p_T:

(3.27) $\qquad \tilde{p}_{T_n} \longrightarrow p(u)$ in $W(0,T)$. (strongly)

Let be t fixed, then because of (3.26) (3.27) we have

(3.28) $$\begin{cases} \tilde{y}_{T_n}(t) \longrightarrow y(t;u) & \text{strongly in } H \\ \tilde{p}_{T_n}(t) \longrightarrow p(t;u) & \text{strongly in } H \end{cases}$$

and then by using (3.17) we can take the limit in eq:

$$\tilde{p}_{T_n}(t) = P_{T_n}(t)\,\tilde{y}_{T_n}(t) = P_{T_n-t}(0)\,\tilde{y}_{T_n}(t)$$

that is to say

(3.29) $\qquad p(t;u) = P\, y(t;u)$

Thus, at least formally it is possible to take the limit in the Riccati equation in order to get the following

Stationary Riccati eq:

(3.30) $$\boxed{PA + A^*P + PD_1P = D_2}$$

Remark 3.1.

Uniqueness of solution of (3.30) is not at all obvious, at the contrary there is in general not uniqueness, even if we impose the conditions

P symmetric $\quad P \geq 0$.

Remark 3.2

The set of equations (3.9) corresponds to the set of optimality conditions of the following control problem

(3.31) $$\begin{cases} y' + Ay = f + Bv &]0,T[\\ y(0) = y_0 \end{cases}$$

(3.32) $$J(v) = \int_0^T \|Cy(v)\|_F^2\,dt + \int_0^T (Nv,v)_E\,dt + \big(Py(T;v), y(T;v)\big)_H$$

If u_T is the optimal control of (3.31) (3.32) then

$$u_T(t) = u(t) \quad \text{for } t \in]0,T[$$

where u is the optimal control on $]0,\infty[$

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