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ON

VARIATIONAL METHODS IN ANALYSIS AND MATHEMATICAL PHYSICS

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A SYSTEMATIC APPROACH TO THE VARIATIONAL FORMULATION IN PHYSICS AND ENGINEERING

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1 - Introduction

It is known that the equations of many physical theories have a "spontaneous" variational formulation in the sense that there is a variational principle from which the equations can be deduced.

It is also known that there are other physical theories in which this does not happen: either a "spontaneous" variational formulation does not exist (and one must resort to more or less acceptable transformations of the equations) or a variational formulation does not exist at all.

We are going to show the mathematical structure of the equations of many physical theories in order to show the reason for the existence of a "spontaneous" variational formulation in some cases and for its lack in other cases. In the latter case we shall show how the equation may be worked to admit a variational formulation.

Moreover we shall present a result recently obtained by the present author according to which every nonlinear problem (i.e. equation plus additional conditions) admits an "integrating operator" that makes the problem of such a kind that a variational formulation exists.

We shall take an inductive approach: we shall examine at first four equations of different physical theories and analyse them to show that they have a common mathematical structure. We shall find it very useful to construct a general diagram that may be used to display the variables and the equations of every physical theory. Later we shall show that such a diagram has its roots in a simple algebraic-topological structure common to many physical theories.

<u>Example 1: deflection of a beam.</u>

A beam fixed at one end and simply supported to the other is under the action of a vertical load. The load is distributed and $q(x)$ will denote the vertical load for unit length. The corresponding deflection will be denoted by $\eta(x)$, Fig.1.

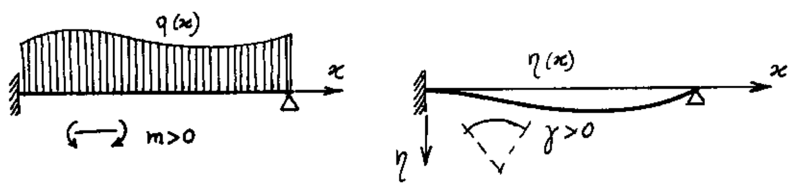


Fig. 1: Deflection of a beam under a vertical load.

The beam is the physical system, the load $q(x)$ is the input, i.e. the source of the deformation, the deflection is the output, i.e. the configuration of the physical system. The fundamental problem is the determination of the configuration variable $\eta(x)$ when the source variable $q(x)$ is assigned.

To link the source with the configuration variables it is useful to introduce two intermediate variables. The first is of geometrical nature: the curvature of the beam, we shall denote by $\gamma(x)$; the second is of statical nature: the bending moment $m(x)$.

If we limit our consideration to small displacements we can write the two equations

$$\gamma(x) = \frac{d^2}{dx^2}\eta(x) \qquad\qquad \frac{d^2}{dx^2} m(x) = q(x) \qquad (1.1)$$

where the second one expresses, in a local form, the equilibrium. The introduction of the two intermediate variables γ and m makes it possible to search for a "constitutive law" that links the geometrical variable γ , the curvature, with the statical variable m , the bending moment. Euler postulated that the two variables are proportional (the larger is the bending moment, the larger will be the curvature). We write today

$$m(x) = E\mathcal{J}(x)\,\gamma(x) \qquad (1.2)$$

where E is the Young modulus, a material parameter and $\mathcal{J}(x)$ is the moment of inertia of a normal section, a geometrical

quantity. If the beam has a variable cross section, $\mathcal{J}$ depends on x . If we consider the boundary conditions we can represent the four variables and their connecting equations by the diagram of Fig. 2.

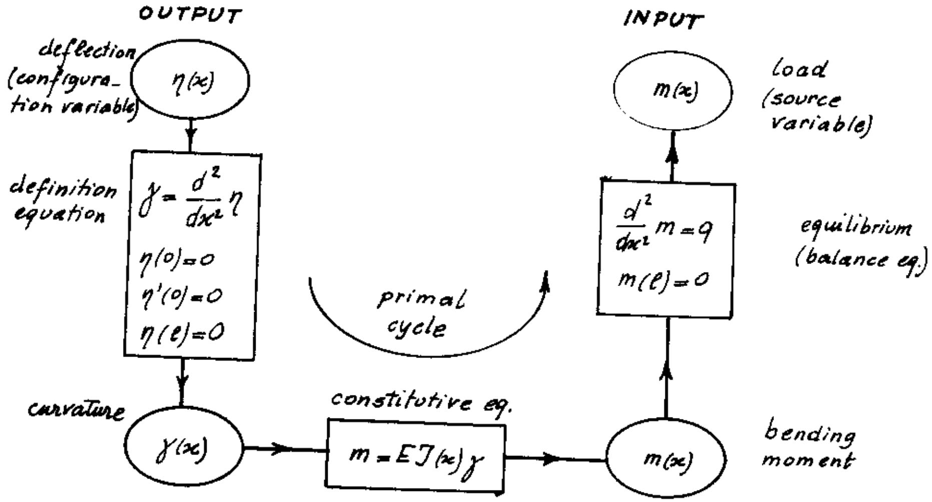


Fig. 2: The flow-chart of the bending problem.

The link between the source variable and the configuration variable is obtained crossing the flow-chart from left to right following the arrows: we obtain the problem

$$\begin{cases} \dfrac{d^2}{dx^2}\left[E\mathcal{J}(x)\,\dfrac{d^2}{dx^2}\eta(x)\right] = q(x) \qquad (1.3) \\ \eta(0)=0 \qquad \eta'(0)=0 \qquad \eta(\ell)=0 \\ m(\ell)=0. \end{cases}$$

This is an equation of the fourth order with variable coefficients with boundary conditions: it is the fundamental equation of the beam deflection.

The first thing we want to do is to give to this problem an operatorial form. The operator of the left side of the diagram (see Fig. 2) will be denoted by D. The operator of the right side will be denoted by C. The operator of the constitutive equation will be denoted by . We shall consider the four functions

$$\underbrace{\eta(x)\,,\,\gamma(x)}_{\text{geometrical variables}} \qquad \underbrace{m(x)\,,\,q(x)}_{\text{statical variables}} \tag{1.4}$$

as elements of four distinct function spaces. These function spaces will be considered of linear kind and without topology. On account of the homogeneous boundary conditions the domain of the operators D and B will be a subset of the corresponding vector spaces.

The "fundamental" problem (1.3) may be written in operatorial language:

$$B\,C\,D\,\eta = q \tag{1.5}$$

Let us introduce two bilinear forms

$$\langle q,\eta\rangle_{I} = \int_0^{\ell} q(x)\,\eta(x)\,dx \qquad \langle m,\gamma\rangle_{II} = \int_0^{\ell} m(x)\,\gamma(x)\,dx$$

With reference to Fig. 3 we may say that the two pairs of spaces H, Q and Γ, M form two dual pairs: H and Q are put in duality by the first bilinear form while Γ and M are put in duality by the second bilinear form. Here we

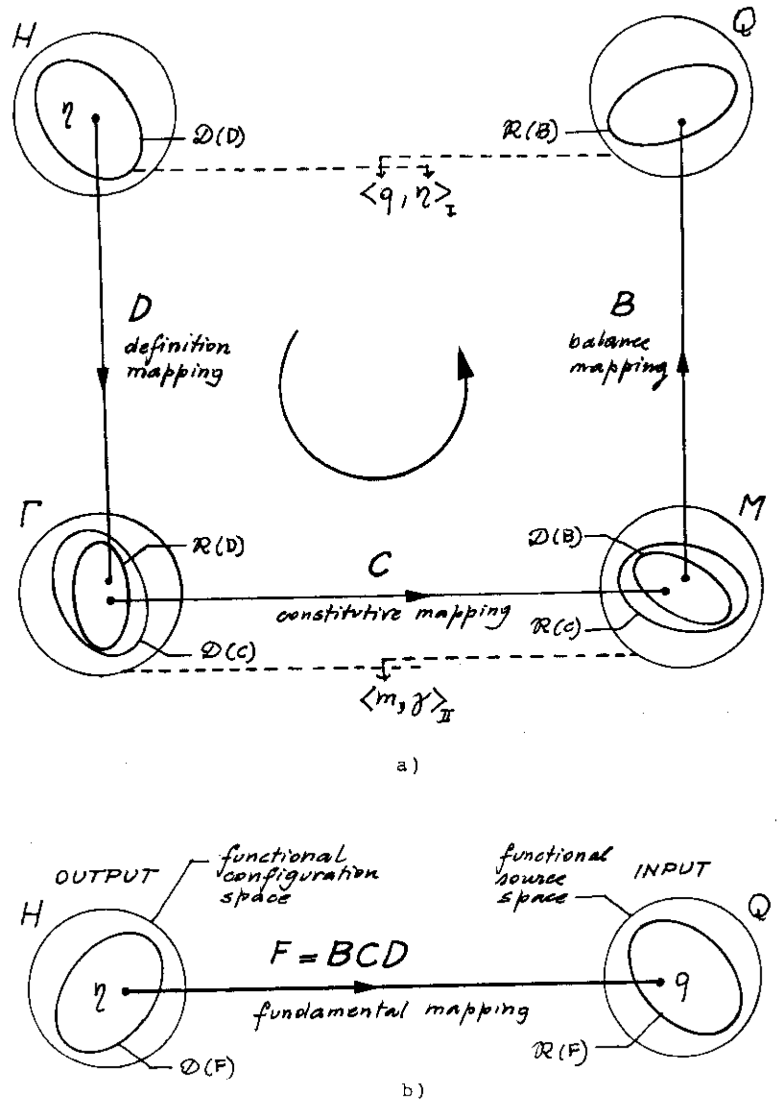


Fig. 3 - a) The four function spaces and the three mappings; b) the resultant fundamental mapping.

have two couples of spaces in duality as described in Bourbaky [1, p. .]

At this point we have three remarkable mathematical properties:

1) the "balance" operator B is the adjoint of the "definitional" operator D, i.e.

$$\langle B\,m, \eta\rangle_I = \langle m, D\eta\rangle_{II} \tag{1.7}$$

as can be seen by an integration by parts;

2) the "constitutive" operator is symmetric, i.e.

$$\langle C\gamma, \bar{\gamma}\rangle_{II} = \langle C\bar{\gamma}, \gamma\rangle_{II} \tag{1.8}$$

where γ and $\bar{\gamma}$ are two functions of the domain of C. Since the domain of C is the whole space Γ it follows that is selfadjoint.

3) the "constitutive" operator C is positive definite, i.e.

$$\langle C\gamma, \gamma\rangle_{II} > 0 \tag{1.9}$$

We then have

$$B = D^* \qquad\qquad C = C^*. \tag{1.10}$$

The fundamental problem (1.5) becomes

$$D^* C\, D\eta = q. \tag{1.10}$$

Now we can easily see that this operator is symmetric:

$$\langle D^*C\,D\eta, \bar{\eta}\rangle_I = \langle C D\eta, D\bar{\eta}\rangle_{II} = \langle C D\bar{\eta}, D\eta\rangle = \\ = \langle D^* C\bar{\eta}, D\eta\rangle_I. \tag{1.11}$$

The symmetry of an operator is the necessary condition in order to have a variational formulation. In fact

$$\langle D^*C\,D\eta - q, \delta\eta\rangle_I = \langle C D\eta, \delta\eta\rangle_{II} - \langle q, \delta\eta\rangle_I = \\ = \delta\left[\tfrac{1}{2}\langle C D\eta, D\eta\rangle_{II} - \langle q, \eta\rangle_I\right] = \delta\, V[\eta] \tag{1.12}$$

where

$$V[\eta] = \tfrac{1}{2}\langle C D\eta, D\eta\rangle_{II} - \langle q, \eta\rangle_I. \tag{1.13}$$

The functional $V[\eta]$ represent the total potential energy of the system (i.e. the internal potential energy plus the potential energy of the applied external loads). Since C is a positive definite operator, i.e.

$$\langle C\gamma, \gamma\rangle_{II} = \int_0^\ell E\, J(x)\, \gamma(x)\, \gamma(x)\, dx > 0 \tag{1.14}$$

and since the operator D is kernel free, i.e.

$$D\eta = 0 \tag{1.15}$$

implies $\eta(x) \equiv 0$, it follows that the operator D^*CD is also positive definite:

$$\langle D^*C\,D\eta, \eta\rangle_I = \langle C D\eta, D\eta\rangle_{II} = \langle C\gamma, \gamma\rangle_{II} > 0. \tag{1.16}$$

Then the functional $V[\eta]$ is bounded below and the solution of the fundamental problem (1.10) gives the minimum to the functional.

We have in this way a variational formulation of the fundamental problem.

As we have seen the variational formulation is possible because

$$B = D^* \quad \text{and} \quad C = C^* \tag{1.17}$$

The "spontaneous" variational formulation of the beam deflection is consequence of the two properties (1.17).

We shall see that this property of the operators is shared by other physical theories.

Example 2: fluid motion.

Let us consider the motion of a fluid under the conditions

a) stationary flow (no time dependence)

b) perfect fluid (no viscosity)

c) irrotational motion (no vorticity)

d) isochoric motion (constant density).

Let us denote by Ω a space region we suppose simply connected, and let us call

$$\begin{cases} \vec{v} = \text{velocity} \qquad\qquad \rho = \text{density} \\ \vec{p} = \text{momentum density} = \text{mass flow density rate} \\ \varphi = \text{velocity potential} \\ \sigma = \text{mass source density rate} \end{cases}$$

The equations of the phenomenon are

$$\begin{cases} \operatorname{div} \vec{p} = \sigma \\ \vec{p}\cdot\vec{n}\big|_{\partial_1\Omega} = \text{assigned} \end{cases} \qquad \text{balance eq. (mass balance)} \tag{1.19}$$

$$\begin{cases} \vec{v} = -\operatorname{grad} \varphi \\ \varphi\big|_{\partial_2\Omega} = \text{assigned} \end{cases} \qquad \text{definition eq.} \tag{1.20}$$

$$\vec{p} = \rho\,\vec{v} \qquad \text{constitutive eq.} \tag{1.21}$$

We may summarize these entities in the diagram of Fig. 4

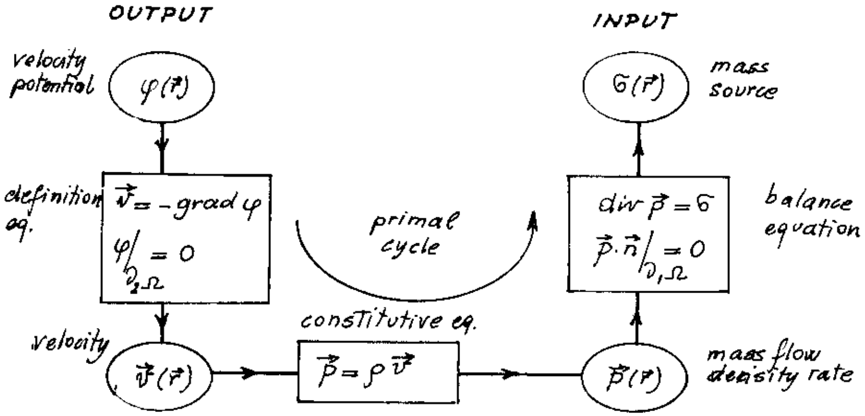


Fig. 4: Fluid flow (perfect, irrotational, isochoric, stationary).

If we introduce the operatorial notation and define the two bilinear forms

$$\langle \sigma, \varphi \rangle_I = \int_\Omega \sigma(\vec{r})\,\varphi(\vec{r})\,d\Omega \qquad \langle p, v \rangle_{II} = \int_\Omega \vec{p}(\vec{r})\cdot\vec{v}(\vec{r})\,d\Omega \tag{1.22}$$

we can easily prove that (under homogeneous boundary conditions) is

1) $B = D^*$

2) $C = C^*$ $\qquad (1.23)$

3) $\langle Cv, v\rangle_{II} > 0$

The corresponding diagram is

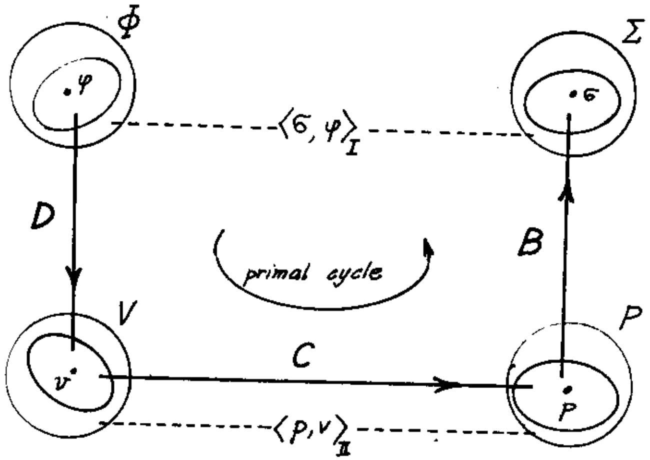


Fig. 5 - Fluid flow: the operatorial description.

In fact

$$\langle p, v\rangle_{II} = \int_\Omega \vec{p}\cdot(-\nabla\varphi)\, d\Omega = \int_\Omega (\nabla\cdot\vec{p})\,\varphi\, d\Omega + \oint_{\partial\Omega} \vec{p}\cdot\vec{n}\,\varphi\, dS.$$

On account of the boundary conditions the last term vanishes: then $B = D^*$. The other properties can be easily shown.

The fundamental problem of the fluid motion will be

$$\begin{cases} \nabla\cdot[\rho(-\nabla\varphi)] = \sigma \\ \varphi\big|_{\partial_2\Omega} = 0 \qquad \dfrac{\partial\varphi}{\partial n}\Big|_{\partial_1\Omega} = 0 \end{cases} \qquad (1.24)$$

and in operatorial notation

$$D^* C D \varphi = \sigma \qquad (1.24)$$

Proceeding as in eq. (1.12) we find the functional

$$J[\varphi] = \int_\Omega \rho\,(\nabla\varphi)^2 d\Omega - \int_\Omega \sigma\varphi\, d\Omega \qquad (1.25)$$

as it is well known [2, p.]. This gives a variational formulation to the fundamental problem (1.24).

In both examples the equations were linear: boundary conditions were made homogeneous in order to make linear the domain of the operators. In this way the operators D and B are made linear.

What does it happen when D and B become nonlinear? A differential operator may be nonlinear either because its domain is a nonlinear set or because its formal operator is nonlinear.

When the formal operators $\mathcal{D}$ and $\mathcal{B}$ are linear but the domains of D and B are nonlinear, one may easily see that the adjointness relation between B and D is replaced by (†)

$$B' = D'^* \qquad (1.26)$$

where B' and D' are the Gateaux derivatives of B and D respectively.

Let us see what happens when D and B are nonlinear.

(†) At this point the spaces will be supposed equipped with a topology compatible with the duality, see later.

Example 3: large elastic deformations.

Let us consider the large deformations of a continuum. Let us denote by

- u_h the components of the displacement vector
- e_{hk} the components of the Green strain tensor
- σ_{hk} the components of the Kirchhoff stress tensor
- f_h the components of the body forces.

The strain tensor is defined by the relation [3,p.435].

$$e_{hk} = \frac{1}{2}\left(u_{h,k} + u_{k,h} + u_{i,h}\, u_{i,k}\right) \qquad (1.27)$$

If we add the boundary condition

$$u_h \Big/_{\partial_1 \Omega} = 0 \qquad (1.28)$$

we obtain a nonlinear operator D. The equation and the boundary conditions can be written in operatorial notation

$$e = D(u) \qquad (1.29)$$

The stress tensor is linked to the body force by the balance equation [3,p.441] (°)

$$-\partial_k \left[\left(\delta_{ih} + u_{i,h} \right) \sigma_{hk} \right] = f_i \qquad (1.30)$$

to which must be added the boundary condition

$$\left[n_k \left(\delta_{ih} + u_{i,h} \right) \sigma_{hk} \right] \Big/_{\partial_2 \Omega} = 0 \qquad (1.31)$$

(°) As it is usual in mathematical physics dummy indices are summed, i.e. $\partial_k \sigma_{hk}$ means $\sum_k \partial_k \sigma_{hk}$.

Balance equations can be written

$$B_u \sigma = f. \qquad (1.32)$$

Let us introduce the bilinear forms

$$\langle f, u \rangle_I = \int_\Omega f_h\, u_h \, d\Omega \qquad (1.33)$$

$$\langle \sigma, e \rangle_{II} = \int_\Omega \sigma_{hk}\, e_{hk} \, d\Omega. \qquad (1.34)$$

We have the Gateaux derivative

$$\delta e_{hk} = \frac{1}{2}\left[\partial_k \delta u_h + \partial_h \delta u_k + u_{i,k}\, \partial_h \delta u_i + u_{i,h}\, \partial_k \delta u_i \right]. \qquad (1.35)$$

Let us perform the adjoint

$$\langle \sigma, \delta e \rangle_{II} = \langle \sigma, D'_u \delta u \rangle_{II} = \iiint_\Omega \sigma_{hk} \left[\partial_h \delta u_k + \partial_k \delta u_h + u_{i,k}\, \partial_h \delta u_i + u_{i,h}\, \partial_k \delta u_i \right]/2 \; d\Omega = \iiint_\Omega -\partial_k \left[\left(\delta_{ih} + u_{i,h} \right) \sigma_{hk} \right] \delta u_i \, d\Omega + \oiint_{\partial\Omega} n_k \left(\delta_{ih} + u_{i,h} \right) \sigma_{hk}\, \delta u_i \, ds = \langle B_u \sigma, \delta u \rangle_I \qquad (1.36)$$

Then we have

$$\langle B_u \sigma, \delta u \rangle_I = \langle \sigma, D'_u \delta u \rangle_{II}. \qquad (1.37)$$

In words: the balance operator is adjoint of the derivative of the definition operator. The constitutive mapping for an elastic material is

$$\sigma_{hk} = C_{hkrs}\, e_{rs} \qquad (1.37\ bis)$$

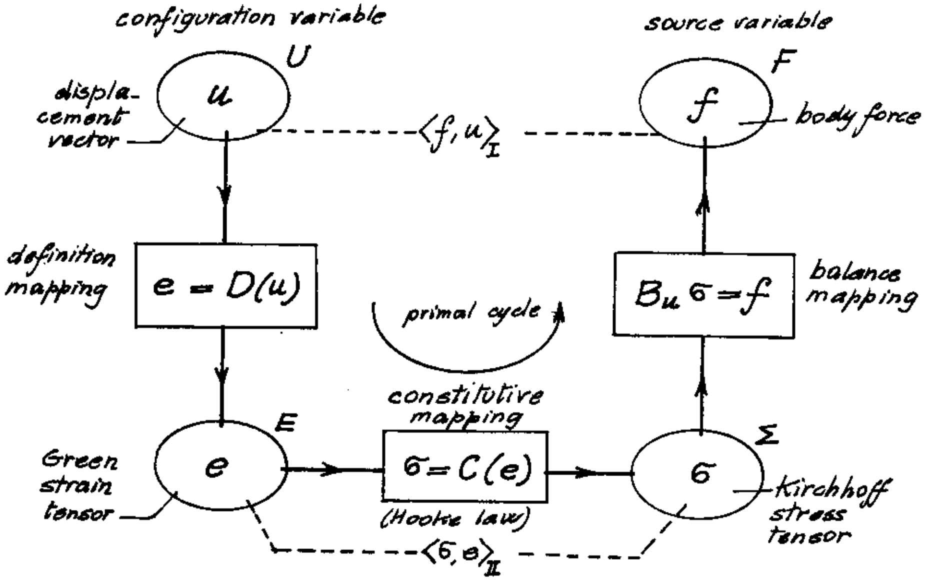


Fig. 6 - Large elastic deformations.

where $C_{hkrs}=C_{rshk}$ and $e = e_{rs}\delta_{rs}$. This constitutive mapping is linear and symmetric:

$$\langle Ce,\bar e\rangle_{II} = \int_\Omega C_{hkrs}\, e_{hk}\,\bar e_{rs}\, d\Omega = \int_\Omega C_{rshk}\,\bar e_{rs}\, e_{hk}\, d\Omega =$$

$$= \int_\Omega C_{hkrs}\,\bar e_{rs}\, e_{hk}\, d\Omega = \langle C\bar e, e\rangle_{II}. \qquad (1.38)$$

Now on account of the property (1.37bis) and of the symmetry (1.38) the fundamental problem

$$F(u) = D_u'^{*}\, C\, D(u) \qquad (1.39)$$

admits a variational formulation. In fact

$$\langle D_u'^{*} C D(u) - f, \delta u\rangle_I = \langle CD(u), D_u'\delta u\rangle_{II} - \langle f,\delta u\rangle_I =$$

$$= \langle CD(u), \delta D(u)\rangle_{II} - \delta\langle f,u\rangle_I =$$

$$= \delta\left[\tfrac{1}{2}\langle CD(u), D(u)\rangle_{II} - \langle f,u\rangle_I\right] = \delta V[u] \qquad (1.40)$$

where

$$V[u] = \frac{1}{2}\int_\Omega C_{hkrs}\, e_{hk}(u)\, e_{rs}(u)\, d\Omega - \int_\Omega f_h\, u_h\, d\Omega \qquad (1.41)$$

is the total potential energy.

In some physical theory the constitutive operator C is nonlinear yet it has a Gateaux derivative that is symmetric (i.e. it is a potential operator). So in fluid dynamics of compressible fluid flows the constitutive equation is

$$\vec p = \rho(v^2)\,\vec v. \qquad (1.42)$$

Now the first variation

$$\delta p = C_v'\,\delta v \qquad (1.43)$$

becomes

$$\delta p_h = \delta\left[\rho(v^2) v_h\right] = \frac{d\rho(v^2)}{dv^2}\,\delta(v_k v_k) v_h + \rho(v^2)\,\delta v_h =$$

$$= \frac{d\rho(v^2)}{dv^2}\, 2 v_h v_k \,\delta v_k + \rho(v^2)\,\delta v_h =$$

$$= \left[\frac{d\rho(v^2)}{dv^2}\, 2 v_h v_k + \rho(v^2)\,\delta_{hk}\right]\delta v_k . \qquad (1.44)$$

The expression in square brackets is symmetric in h,k then the whole operator C'_V is symmetric, i.e.

$$\langle C'_v \varphi, \psi\rangle_{II} = \langle C'_v \psi, \varphi\rangle_{II} . \qquad (1.45)$$

In many cases the operator C is strictly monotone, i.e.

$$\langle C(u) - C(\bar{u}),\ u - \bar{u}\rangle_{II} > 0 \qquad (1.46)$$

for $u \neq \bar{u}$. If C has a weak derivative (Gateaux derivative) this condition becomes

$$\langle C'_u \varphi, \psi\rangle_{II} > 0 \qquad (1.47)$$

i.e. the derivative is accretive. For example the nonlinear operator (1.42) is strictly monotone as long as the fluid motion remains subsonic [4,p.].

We may summarize our findings in table I.

Table I. A recurrent relation among the three operators B, C, D of physical theories.

	linear operators	nonlinear operators
1)	$B = D^*$	$B_\varphi = D'^*_\varphi$
2)	$C = C^*$	$C'_u = C'^*_u$
3)	C=accretive	C=strictly monotone

Anticipating what we shall prove later we can say that the "spontaneous" variational formulation of many fundamental equations of physics and engineering rests on the two properties

1) adjointness of balance and definition operators B and D respectively (or of their derivatives)
2) symmetry of the constitutive operator C.

Moreover if

3) C is strictly monotone
4) D is kernel free

then the functional is convex and the variational principle is a minimum principle.

This shows that the variational formulation in many physical theories rests on a peculiar property of the operators that constitute the "fundamental" operator of a theory, i.e. the operator forming the field equations or the equations of motion.

In time dependent phenomena the properties 1) and 2) on which a variational formulation is based are lost. Let us examine some examples.

Example 4 - Particle dynamics

The configuration variable is the radius vector $\vec{r}$ while the source variable is the force $\vec{f}$. The two intermediate variables are velocity $\vec{v}$ and momentum $\vec{p}$. We have the relations

$$\begin{cases} \vec{v} = \dfrac{d}{dt}\vec{r} & \text{definitional equation} \\ \dfrac{d\vec{p}}{dt} = \vec{f} & \text{balance equation (balance of momentum)} \\ \vec{p} = m\vec{v} & \text{constitutive equation} \\ \vec{f} = \vec{f}(\vec{r},\vec{v};t) + \vec{f}^{\,impr} & \text{interaction equation.} \end{cases} \qquad (1.48)$$

The diagram for three relations is shown in Fig.

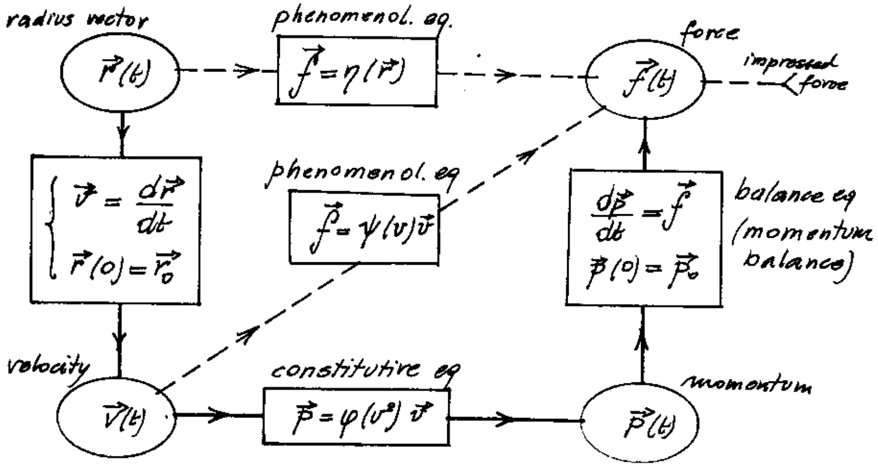


Fig. 7 - The scheme of particle dynamics.

If we introduce the two bilinear forms

$$\langle f, r\rangle_{I} = \int_0^T \vec{f}(t)\cdot\vec{r}(t)\,dt$$
$$\langle p, v\rangle_{II} = \int_0^T \vec{p}(t)\cdot\vec{v}(t)\,dt \qquad (1.49)$$

we obtain

$$\langle p, Dr\rangle_{II} = \int_0^T \vec{p}\cdot\frac{d}{dt}\vec{r}\,dt = \int_0^T \left[-\frac{d}{dt}\vec{p}\right]\cdot\vec{r}\,dt + \left[\vec{p}\cdot\vec{r}\right]_0^T. \qquad (1.50)$$

Now if we consider, at first, homogeneous initial conditions

$$\vec{r}(0) = \vec{0} \qquad \vec{p}(0) = \vec{0} \qquad (1.51)$$

we have

$$D = \begin{cases} \dfrac{d}{dt} \\ \vec{r}(0) = \vec{0} \end{cases} \qquad D^* = \begin{cases} -\dfrac{d}{dt} \\ \vec{p}\,(T) = \vec{0} \end{cases} \qquad B = \begin{cases} \dfrac{d}{dt} \\ \vec{p}(0) = \vec{0}. \end{cases} \qquad (1.52)$$

In this case, typical of evolution problems it is $D^* \neq B$. We have lost the property on which a variational formulation is possible.

At this point one may observe that the difference between B and D^* is in the minus sign in the formal operator and, what is more important, in the final condition $\vec{p}(T)=\vec{0}$ in place of the initial one $\vec{p}(0)=\vec{0}$.

One may be tempted to change artificially the physical problem substituting the final condition $\vec{p}(T)=\vec{0}$ to the initial one $\vec{p}(0)=\vec{0}$. If one does this we arrive to the following scheme:

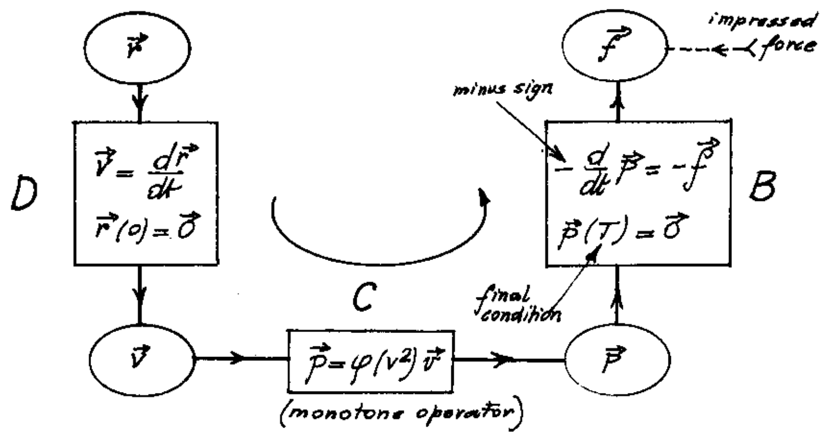


Fig. 8 - The modified scheme of particle dynamics.

With this trick we have realized the relation $B = D^*$. Since C is a symmetric operator, it follows that the fundamental problem

$$B\,C\,D\,r = f^{\text{impressed}} \qquad (1.53)$$

i.e.

$$\begin{cases} -\dfrac{d}{dt}\left[\varphi\left(\dfrac{d\vec r}{dt}\right)\dfrac{d\vec r}{dt}\right] = \vec f^{\text{impressed}} \\ \vec r(0) = \vec 0 \qquad \vec p(T) = \vec 0 \end{cases} \qquad (1.54)$$

admits a variational formulation. So if we take the newtonian constitutive law $\vec p = m\vec v$ the functional will be

$$F[\vec r] = \frac{1}{2}\langle CDr, Dr\rangle_{II} - \langle f, r\rangle_{I} = \frac{1}{2}\int_0^T m\vec v\cdot\vec v\,dt - \int_0^T \vec f\cdot\vec r\,dt \qquad (1.55)$$

that is the total energy (kinetic+potential) of the particle. We have obtained in this way the Hamilton principle.

As strange as it may appear, the Hamilton principle is based on a trick: one replaces the physical initial condition $\vec p(0) = \vec 0$ with an unphysical final one $\vec p(T) = \vec 0$. The motion of a particle is a typical initial value problem that is arbitrarily converted in a boundary value problem to the purpose of giving a variational formulation. The result is that since the momentum $\vec p$ at the final instant is unknown one can neither use numerical methods to solve initial value problems using Hamilton principle, or use it for existence proofs.

As we shall see later it is possible to replace the Hamilton principle by another variational principle in which only the initial conditions are used. In this way numerical methods can be applied.

Example 5 - Neutral meson field

The fundamental equation of the meson field, i.e. the Klein-Gordon equation is obtained by the equation (†)

(†) p_α is the fourdimensional momentum, α=0,1,2,3, $g_{\mu\nu}$ is the metric tensor of space-time, m_0, c, $\hbar$ are respectively the proper mass of the particle, the light speed and the Planck constant divided by 2π.

$$g^{\alpha\beta} p_\alpha p_\beta = (m_0 c)^2 \qquad (1.56)$$

replacing p_α by the operator $i\hbar\partial_\alpha$. One obtains

$$g^{\alpha\beta}(i\hbar\partial_\alpha)(i\hbar\partial_\beta)\psi - (m_0 c)^2 I\psi = \sigma^{interaction} \qquad (1.57)$$

that is

$$-\hbar^2 \Box\psi - (m_0 c)^2\psi = \sigma^{interaction}. \qquad (1.58)$$

If we consider besides the two functions ψ (amplitude) and σ the intermediate variables $u_\alpha = -\partial_\alpha\psi$ and v^β such that $\partial_\beta v^\beta = \sigma^{inter}$ we can construct the scheme of Fig.

Since the time is involved we have initial conditions. If one uses the usual trick of replacing one initial condition by a final condition it is obtained the adjointness relation between the operators B and D of Fig. 9.

In fact

$$\int_\Omega v^\alpha(\partial_\alpha\psi)\,d\Omega = \int_\Omega(-\partial_\alpha v^\alpha)\psi\,d\Omega + \oint_{\partial\Omega} v^\alpha n_\alpha\psi\,dV. \qquad (1.59)$$

The fundamental operator BCD is symmetric and then we have the functional

$$\mathcal{J}[\psi] = \frac{1}{2}\langle CD\psi, D\psi\rangle_{II} - \frac{1}{2}\langle M\psi,\psi\rangle_I \qquad (1.60)$$

which in explicit notation becomes

$$\mathcal{J}[\psi] = \frac{1}{2}\int_\Omega\left[\hbar^2\sqrt{-g}\,g^{\alpha\beta}\partial_\alpha\psi\,\partial_\beta\psi - (m_0 c)^2\psi^2\right]d\Omega \qquad (1.61)$$

that is the usual action functional for the Klein-Gordon field [6, p.96].

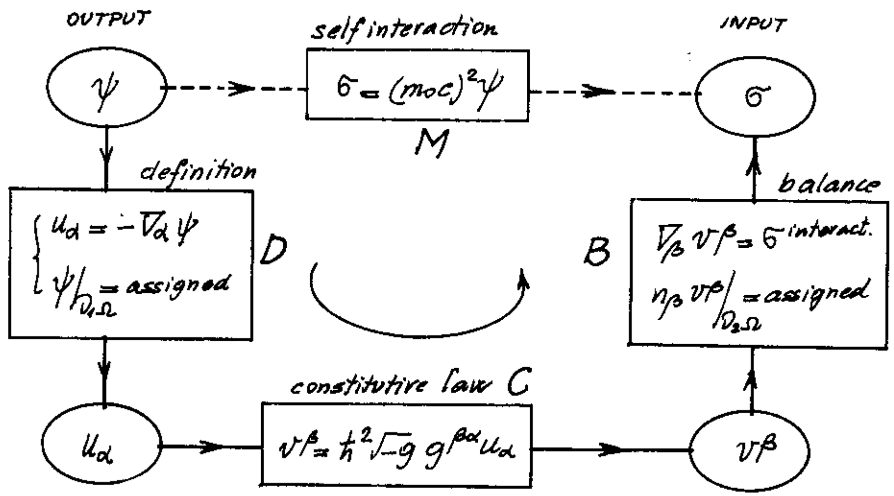


Fig. 9 - The scheme for Klein-Gordon equation (meson field).

Example 6 - Electromagnetic field

The equations of the electromagnetic field are

$$\begin{cases} \operatorname{div} \vec{B} = 0 \\ \operatorname{rot} \vec{E} + \partial_t \vec{B} = 0 \end{cases} \qquad \begin{cases} \vec{D} = \varepsilon \vec{E} \\ \vec{H} = \frac{1}{\mu} \vec{B} \\ \vec{J} = \sigma \vec{E} \end{cases} \qquad \begin{cases} \operatorname{div} \vec{D} = \rho \\ \operatorname{rot} \vec{H} - \partial_t \vec{D} = \vec{J} \end{cases} \tag{1.62}$$

first set of Maxwell equations — constitutive equations — second set of Maxwell equations

$$\begin{cases} \vec{B} = \operatorname{rot} \vec{A} \\ \vec{E} = -\partial_t \vec{A} - \operatorname{grad} V \end{cases} \tag{1.63}$$

general solution of the first set of Maxwell equation.

These may be written in space-time notation as follows: putting

$$\begin{cases} \phi_\alpha & = \text{space-time potential} = \left(\frac{V}{c}, -\vec{A}\right) \\ J^\beta & = \text{four current} = (\rho c, \vec{J}) \\ F_{\alpha\beta} & = \text{first electromagnetic tensor} \\ G^{\mu\nu} & = \text{second electromagnetic tensor} \\ \chi^{\mu\nu\alpha\beta} & = \text{constitutive tensor} = \chi^{\alpha\beta\mu\nu} \quad \text{(symmetric)} \end{cases} \tag{1.64}$$

we have

$$\begin{cases} F_{\alpha\beta} = \nabla_\alpha \phi_\beta - \nabla_\beta \phi_\alpha \\ \phi_\alpha \big|_{\partial_1 \Omega} = \text{assigned} \end{cases} \qquad \begin{cases} \nabla_\nu G^{\mu\nu} = J^\mu \\ n_\nu G^{\mu\nu} \big|_{\partial_2 \Omega} = \text{assigned} \end{cases}$$

$$G^{\mu\nu} = \frac{1}{2} \chi^{\mu\nu\alpha\beta} F_{\alpha\beta} \tag{1.65}$$

The corresponding diagram is shown in Fig. 10.

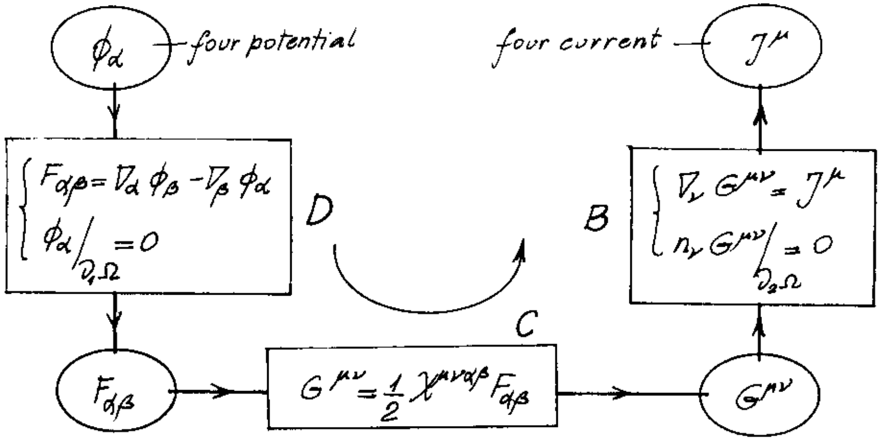


Fig. 10 - The equations of the electromagnetic field.

The fundamental problem is

$$\begin{cases} \nabla_\nu \left[\frac{1}{2} \chi^{\mu\nu\alpha\beta} \left(\nabla_\alpha \phi_\beta - \nabla_\beta \phi_\alpha \right) \right] = J^\mu \\ \phi_\alpha \big|_{\partial_1 \Omega} = 0 \qquad n_\nu G^{\mu\nu} \big|_{\partial_2 \Omega} = 0. \end{cases} \tag{1.66}$$

This is the equation of the electromagnetic waves.

The operator D has a kernel: if $\phi_\alpha = \nabla_\alpha \chi$ and $\nabla_\alpha \chi \big|_{\partial_1 \Omega} = 0$

$$\nabla_\alpha (\nabla_\beta \chi) - \nabla_\beta (\nabla_\alpha \chi) \equiv 0 \tag{1.67}$$

then the fundamental problem

$$B\,C\,D\,\phi = J \tag{1.68}$$

has many solutions. One usually eliminates the indeterminacy of ϕ adding the condition (of Lorentz)

$$\nabla_\alpha \phi^\alpha = 0 \qquad (1.69)$$

Once more the operator B is adjoint of D (if we substitute the initial condition with a final one as in particle dynamics). Then we have the functional

$$\Lambda[\phi] = \frac{1}{2}\langle CD\phi, D\phi\rangle_{II} - \langle J, \phi\rangle_{I} =$$
$$= \frac{1}{2}\int_\Omega \frac{1}{2}\chi^{\mu\nu\alpha\beta}\,\nabla_\alpha\phi_\beta\,\nabla_\mu\phi_\nu\, d\Omega - \int_\Omega J^\alpha\phi_\alpha\, d\Omega \qquad (1.70)$$

In particular for the vacuum is

$$\chi^{\mu\nu\alpha\beta} = \sqrt{\frac{\varepsilon_0}{\mu_0}}\sqrt{-g}\,\left(g^{\mu\alpha}g^{\nu\beta} - g^{\mu\beta}g^{\nu\alpha}\right) \qquad (1.71)$$

(see 5) and the functional reduces itself to

$$\Lambda[\phi] = \frac{1}{4}\int_\Omega \sqrt{\frac{\varepsilon_0}{\mu_0}}\sqrt{-g}\,(\nabla_\mu\phi_\nu)(\nabla^\mu\phi^\nu)\, d\Omega - \int_\Omega J^\alpha\phi_\alpha\, d\Omega \qquad (1.72)$$

that is the usual action for the electromagnetic field [6,p.].

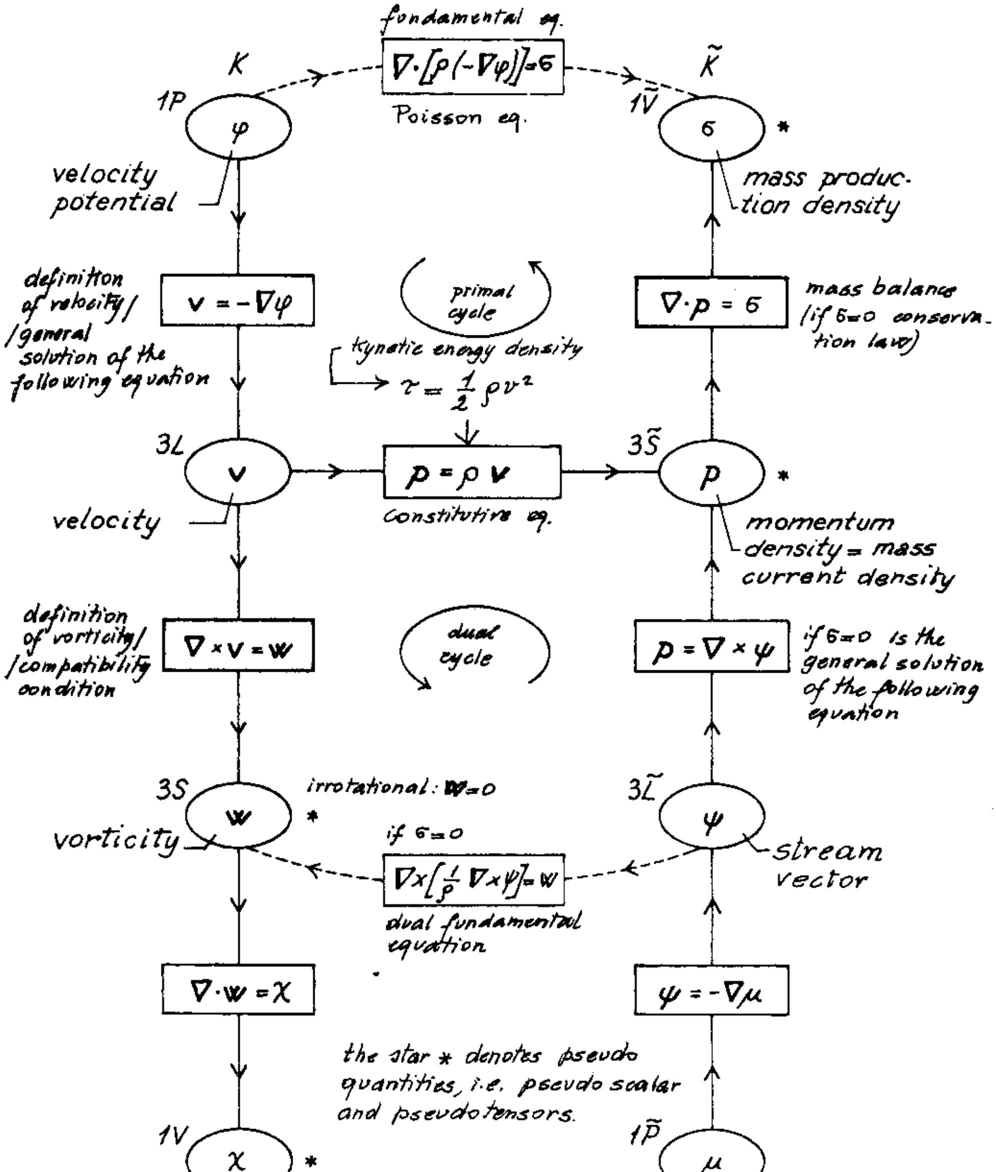
One can ask what are the roots of the existence of the relation $B=D^*$ in many physical theories. An investigation in this direction (see author's paper [7],[8],[9]) leads to the discovery that physical theories have a remarkable mathematical structure that can be described in terms of algebraic topology using the notions of cells, chains, coboundary operators or in the equivalent terms of differential geometry using the notions of differential forms and exterior differential. The adjointness of B and D can then be traced back to the adjointness of the boundary and coboundary operators in algebraic topology, i.e. to Poincaré duality theorem. It has been shown that one can construct general classification scheme for physical variables of practically every physical theory. The simple diagrams shown in the proceeding pages are pieces of this general scheme. We report in Tables II and III two of such general schemes.

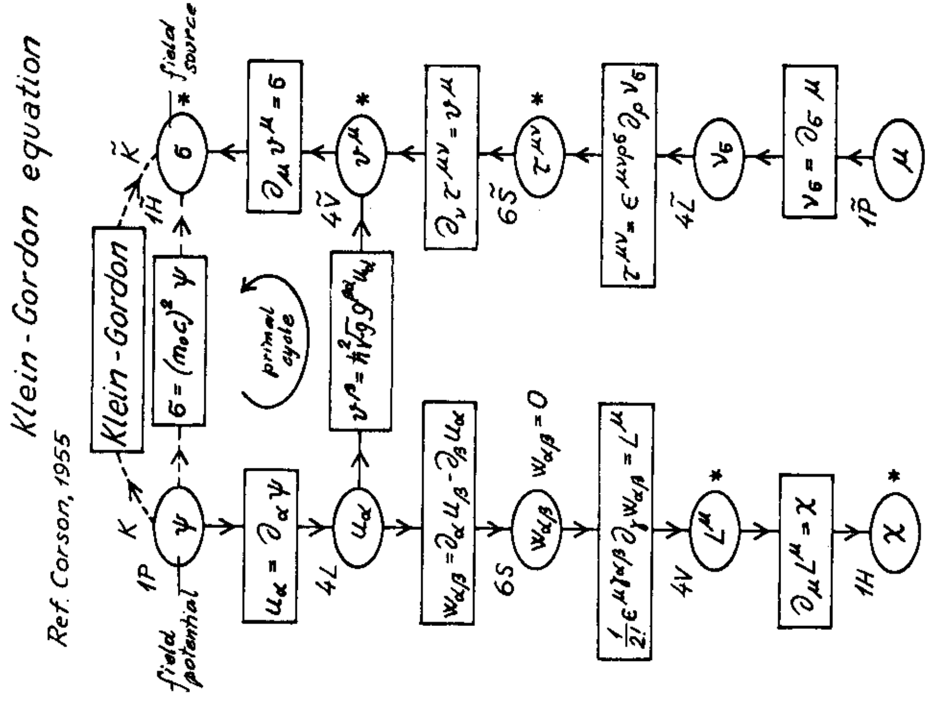
These schemes are based on the fact that in every physical theory there are physical variables that are referred to space and time simplest objects, i.e. points, lines, surfaces, volumes, time instants and intervals, denoted **P**, **L**, **S**, **V**, **I**, **T** respectively. This explains the four ellyptic boxes for variables depending only on space coordinates and the five boxes for those depending also on time.

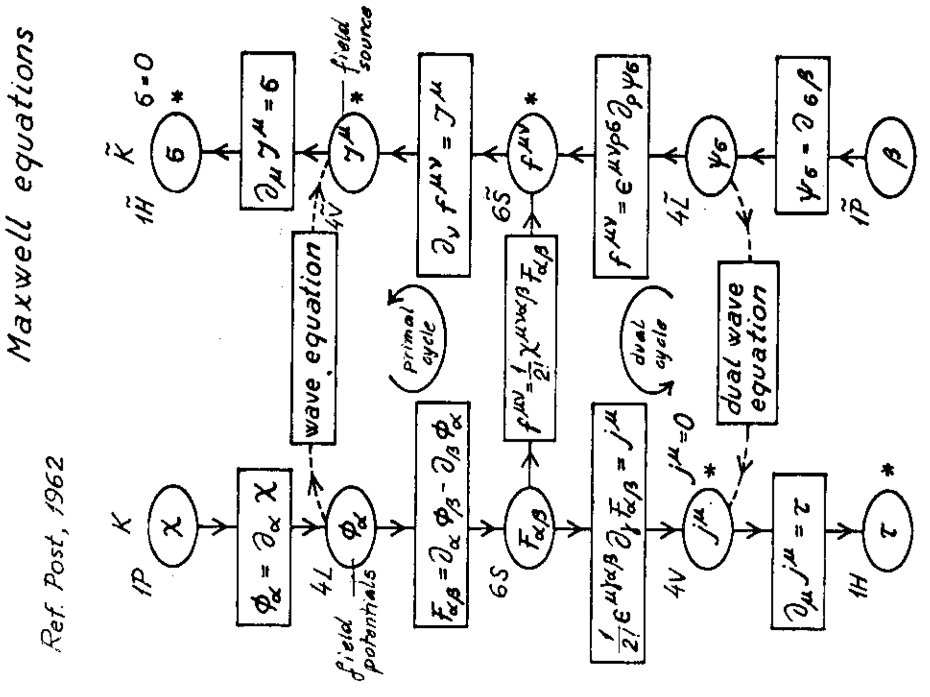
Table II **fluid motion**

perfect, irrotational, stationary, isochoric

$\mu, \nu = 0$ $\quad w = 0$ $\quad \partial_t = 0$ $\quad \rho = const.$







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