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Notes on Bruhat-Tits-Buildings

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Introduction

These notes are the basis of a course of five lectures at the School on Algebraic Groups & Arithmetic Groups in Trieste 1996. They contain

- a detailed introduction to the theory for SL_2 over a local field in §1 to §3
- a description of affine Weyl groups in §4
- the construction of lattices for SL_n and for split groups in §5 and the connection of the arising open and compact groups with a Coxeter complex
- the general definition of the building $\mathcal{T}$ associated to a semi simple group G over a local field in §6
- the description of further properties of the building in §7

The notes are preliminary in many respects. In particular they are too short, since many results come without a complete proof. On the other hand, the notes are too long, since it will be unlikely that all the material mentioned in the notes can be explained in five hours. I hope that in spite of these shortcomings the general outline of the theory becomes visible.

Notation: We denote by K a locally compact commutative field with discrete valuation v and finite residue field, i.e.: $v : K^* \longrightarrow \mathbb{Z}$ is a surjective homomorphism such that $v(x+y) \geq \inf(v(x), v(y))$ for all $x, y \in K$, where $v(0) := \infty$. Let $\mathcal{O} := \{x \in K/v(x) \geq 0\}$ be the ring of integers in K and $\mathcal{P} := \{x \in K/v(x) \geq 1\}$ be the maximal ideal of $\mathcal{O}$. Then there is a $\pi \in \mathcal{P}$ such that $\mathcal{P} = \pi\mathcal{O}$ and $\mathcal{O}/\mathcal{P} =: k$ is a finite field. The ring $\mathcal{O}$ is a principal ideal domain and $x\mathcal{O} = \pi^{v(x)}\mathcal{O}$. The units of $\mathcal{O}$ are $\mathcal{O}^* := \{x \in \mathcal{O}/v(x) = 0\}$. By G we denote a semi simple connected algebraic group defined over K. We often write $G = G(K)$.

§1. The tree $\mathcal{T}$ associated to SL_2 over a local field

We construct for $SL_2(K)$, K a local field, a tree $\mathcal{T}$. Here we follow essentially Serre, see [S]. The properties of $\mathcal{T}$ and the methods to construct $\mathcal{T}$ will later motivate the construction of a building $\mathcal{T}$ which is associated to a semi simple group G defined over K.

1.1. Let V be a K–vectorspace of dimension 2. We denote by $GL(V)$ resp. $SL(V)$ the group of K–linear isomorphisms of V resp. the group $SL(V) = \{g \in GL(V)/\det g = 1\}$.

A lattice L in V is a finitely generated $\mathcal{O}$ submodule of V such that $K \cdot L = V$. Since $\mathcal{O}$ is a principal ideal domain there exists a basis e_1, e_2 of V such that $L = \mathcal{O}e_1 + \mathcal{O}e_2$. We call e_1, e_2 a basis of L.

If $x \in K^*$ and if L is a lattice in V then xL also is a lattice in V. Lattices L_1 and L_2 of L are called *equivalent* if there is an $x \in K^*$ such that $L_1 = xL_2$. If $V = K \times K$ we have examples $L_1 = \mathcal{O} \oplus \mathcal{O}$ and $L_2 = \mathcal{O} \oplus \mathcal{P}$ of inequivalent lattices.

1.2. Let L_1, L_2 be lattices. Then there is an $n \in \mathbb{N}$ such that $\pi^n L_2 \subset L_1$. By the elementary divisor theorem there is a basis e_1, e_2 of L_1 such that $\pi^n L_2 = \pi^a\mathcal{O}e_1 + \pi^b\mathcal{O}e_2$. We define the distance $d(L_1, L_2)$ of L_1 and L_2 by $d(L_1, L_2) = |a - b| \in \mathbb{N}$. Apparently $d(L_1, L_2)$ does not depend on the choice of π^n and by the elementary divisor theorem $d(L_1, L_2)$ does not depend on the choice of e_1, e_2. Since now $\pi^{a-n}e_1$, $\pi^{b-n}e_2$ is a basis of L_2 we have an $m \geq 0$ such that $\pi^m L_1 \subset L_2$. We see that $d(L_1, L_2) = d(L_2, L_1)$. Moreover by computation $d(xL_1, yL_2) = d(L_1, L_2)$ for all $x, y \in K^*$. Let Λ_1 resp. Λ_2 be the equivalence class of lattices determined by L_1 resp. L_2. Then we can define $d(\Lambda_1, \Lambda_2) := d(L_1, L_2)$ as the *distance of classes of lattices.*

We note that $d(\Lambda_1, \Lambda_2) = 0$ iff $\Lambda_1 = \Lambda_2$. Let $d(\Lambda_1, \Lambda_2) = l$ and represent Λ_1 by L_1. Then there is a unique representing lattice L_2 of Λ_2 such that $L_2 \subset L_1$; $L_2 \not\subset \pi L_1$ and L_1/L_2 is generated by one element. Then $L_1/L_2 \cong \mathcal{O}/\pi^l\mathcal{O}$.

1.3. We denote by $\mathcal{T}$ the topological realisation of the simplicial complex whose vertices (= 0–simplices) are the classes of lattices and where a 1–simplex is a pair of vertices

(Λ_0, Λ_1) with $d(\Lambda_0, \Lambda_1) = 1$. We call $\mathcal{T}$ the *building associated to* $SL(V)$.

1.4. We recall the definition of the topological realisation. Let $\mathcal{L}$ be the set of classes of lattices. Then $\mathcal{T}$ is the set of maps $f : \mathcal{L} \longrightarrow \mathbb{R}_{\geq 0}$ such that

i) $\operatorname{supp} f = \{\Lambda \in \mathcal{L} / f(\Lambda) \neq 0\}$ is a simplex

ii) $\sum_{\Lambda \in \mathcal{L}} f(\Lambda) = 1$.

In particular if $\operatorname{supp} f = \Lambda$ then $f(\Lambda) = 1$ and if $\operatorname{supp} f = (\Lambda_0, \Lambda_1)$ then $f(\Lambda_1) = t$, $f(\Lambda_2) = 1 - t$, $0 < t < 1$. We view f as a point t on the line connecting Λ_0 and Λ_1. We define a topology on $\mathcal{T}$ which after identifying $\Lambda_0 \cup (\Lambda_0, \Lambda_1) \cup \Lambda_1$ with $[0,1]$ induces the ordinary topology on $[0,1]$.

1.5. Let Λ_0, Λ_l be vertices. A *path* c in $\mathcal{T}$ from Λ_0 to Λ_l is a sequence $\Lambda_0, \Lambda_1, \dots, \Lambda_{l-1}, \Lambda_l$ of vertices in $\mathcal{T}$ such that $d(\Lambda_i, \Lambda_{i+1}) = 1$, $i = 0, \dots, l-1$. We denote by $c = (\Lambda_0, \Lambda_1, \dots, \Lambda_l)$ also the simplicial realisation of c. If $\Lambda_0 = \Lambda_l$ then c is called a *closed path.* If $\Lambda_i \neq \Lambda_j$ for all $i \neq j$ then c is called a *simple* path. A closed path $c = (\Lambda_0, \dots, \Lambda_l)$ is called *simple* if $(\Lambda_0, \dots, \Lambda_{l-1})$ is a simple path and $l \geq 3$. In particular we consider $(\Lambda_0, \Lambda_1, \Lambda_0)$ not as a simple closed path.

Let $c = (\Lambda_0, \dots, \Lambda_l)$ be a path with $\Lambda_0 \neq \Lambda_l$. Then there is a subset $I = \{n_0, \dots, n_t\} \subset \{0, 1, \dots, l\}$ such that $n_0 = 0$, $n_t = l$ and such that $(\Lambda_{n_0}, \Lambda_{n_1}, \dots, \Lambda_{n_t})$ is a simple path. To see this we choose $i \geq 0$ minimal such that there is a $j > i$ with $\Lambda_i = \Lambda_j$ and replace c by $(\Lambda_0, \dots, \Lambda_i, \Lambda_{j+1}, \dots, \Lambda_l)$ and so on.

The complex $\mathcal{T}$ is called *connected* if it is pathwise connected and $\mathcal{T}$ is called a *tree* if it is connected and if it contains no closed simple paths.

1.6. Proposition. — $\mathcal{T}$ *is a tree.*

Proof. We show that $\mathcal{T}$ is pathwise connected. For this let $\Lambda \neq \Lambda'$ be vertices of $\mathcal{T}$ represented by lattices $L' \subset L$ such that $L/L' \simeq \mathcal{O}/\pi^n\mathcal{O}$. Hence the preimages of $\pi^j\mathcal{O}/\pi^n\mathcal{O}$, $j \leq n$ in L define a sequence of lattices

$$L' = L_n \subset L_{n-1} \subset \dots \subset L_0 = L$$

and $L_i/L_{i+1} \simeq \mathcal{O}/\pi\mathcal{O}$. Then L_i represents a vertex Λ_i and $(\Lambda_0, \Lambda_1, \dots, \Lambda_n)$ is a path form $\Lambda_0 = \Lambda$ to Λ'. Hence $\mathcal{T}$ is connected.

Let $(\Lambda_0, \dots, \Lambda_n)$ be a simple path. If we can show that then $\Lambda_0 \neq \Lambda_n$ it follows that $\mathcal{T}$ is a tree. Now $d(\Lambda_0, \Lambda_n) = n$ implies $\Lambda_0 \neq \Lambda_n$. To prove that $d(\Lambda_0, \Lambda_n) = n$ we represent $(\Lambda_0, \dots, \Lambda_n)$ by a sequence of lattices

$$L_n \subset L_{n-1} \subset \dots \subset L_0$$

with $L_i/L_{i+1} = \mathcal{O}/\pi\mathcal{O}$. If we can show that $L_n \not\subset \pi L_0$ then by 1.2 we get $d(\Lambda_0, \Lambda_n) = n$. We assume inductively that $L_{n-1} \not\subset \pi L_0$. Then πL_{n-2} and L_n are contained in L_{n-1}. Their images in the 2-dimensional k-vectorspace $L_{n-1}/\pi L_{n-1}$ are non zero and cannot coincide for otherwise $\pi L_{n-2} + \pi L_{n-1} = L_n + \pi L_{n-1}$ and $\pi L_{n-2} = L_n$, i.e.: $\Lambda_{n-2} = \Lambda_n$ and $(\Lambda_0, \dots, \Lambda_n)$ would be not simple. Hence $L_{n-1} = L_n + \pi L_{n-2}$ and since $\pi L_{n-2} \subset \pi L_0$ an inclusion $L_n \subset \pi L_0$ would imply $L_{n-1} \subset \pi L_0$ which contradicts our assumption.

We note two consequences:

1.8 Corollary.

1) *Two different vertices* Λ, Λ' *of* $\mathcal{T}$ *are joined by a unique simple path.*

2) $-\mathcal{T}$ *is contractible.*

Often the unique simple path joining Λ and Λ' is called the connecting *geodesic.* The contraction of $\mathcal{T}$ to some vertex Λ_0 is given by shrinking of the unique geodesics from Λ_0 to points different from Λ_0.

Let Λ be a vertex of $\mathcal{T}$ represented by a lattice L. Then a vertex Λ' with $d(\Lambda, \Lambda') = 1$ can be represented by a unique lattice $L' \subset L$ such that $L' \not\subset \pi L$ and $L/L' \simeq \mathcal{O}/\pi\mathcal{O} = k$. Then $\pi L \subset L'$ and $L'/\pi L$ is a 1-dimensional k subvectorspace of

$L/\pi L \cong k \oplus k$. The number q of such subspaces is equal to the number of points of the projective line $P_1(k)$ over k. Hence $q = |k| + 1$. We have shown:

1.8. Proposition. *If* Λ *is a vertex then*

$$\#\{\Lambda' \in \mathcal{T}/\Lambda' \neq \Lambda,\ d(\Lambda', \Lambda) = 1\} = |k| + 1 .$$

For $k = \mathbb{Z}/2\mathbb{Z}$ we get the following picture of $\mathcal{T}$ containing all points of distance ≤ 4 from the middle vertex.

The chambers here have decreasing size with respect to the distance to the middle vertex.

§2. Group actions on the tree $\mathcal{T}$

2.1. If $L \subset V$ is a lattice and $g \in GL(V)$ then $g(L)$ is a lattice. Moreover for $x \in K^*$ we have $g(xL) = xg(L)$. Hence g acts on the classes of lattices and induces a permutation of the vertices of $\mathcal{T}$.

Since $GL(V)$ acts transitively on the set of basis of V it follows that $GL(V)$ acts transitively on the vertices of $\mathcal{T}$. Moreover if $G(L)$ denotes the stabilizer of a lattice L we see that $G(L)$ is isomorphic to $GL_2(\mathcal{O})$. In particular the stabilizer of a lattice is compact.

Let Λ_1, Λ_2 be vertices of $\mathcal{T}$ and $d(\Lambda_1, \Lambda_2) = n$. We can choose representing lattices L_i

with $L_1 \subset L_2$, $L_1 \not\subset \pi L_2$, $L_2/L_1 = \mathcal{O}/\pi^n\mathcal{O}$. Hence $d(g\Lambda_1, g\Lambda_2) = d(\Lambda_1, \Lambda_2)$.

Example: Let $L = \mathcal{O}e_1 \oplus \mathcal{P}e_2$. Then $Gl(L) \cong \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(K)/a,\ d \in \mathcal{O},\ c \in \mathcal{P},\ b \in \pi^{-1}\mathcal{O},\ ad - bc \in \mathcal{O}^* \right\} = \begin{pmatrix} 1 & 0 \\ 0 & \pi \end{pmatrix} GL_2(\mathcal{O}) \begin{pmatrix} 1 & 0 \\ 0 & \pi^{-1} \end{pmatrix}$. If $\Lambda_0 \in X$ is represented by $e_1\mathcal{O} + e_2\mathcal{O}$ and if L represents Λ_1 then $d(\Lambda_0, \Lambda_1) = 1$ and the element of $GL(V)$ corresponding to $\begin{pmatrix} 1 & 0 \\ 0 & \pi \end{pmatrix}$ maps to Λ_0 to Λ_1.

2.2. Proposition. *If* Λ *is a vertex of* $\mathcal{T}$ *and* $g \in GL(V)$ *then*

$$d(\Lambda, g\Lambda) \equiv v(\det g) \bmod 2 .$$

Proof. Let L be a lattice which represents Λ. Then there is a π^n such that $L_1 := \pi^n g(L) \subset L$. By the elementary divisor theorem we find a basis v_1, v_2 of L and $a, b \in \mathbb{N}$ s.t. $\pi^a v_1$, $\pi^b v_2$ is a basis of $\pi^n g(L)$. Another basis of $\pi^n gL$ is $\pi^n g(v_1)$, $\pi^n g(v_2)$. Hence here is an $A \in GL(L_1)$ st. $\pi^a v_i = A\pi^n g(v_i)$, $i = 1, 2$. With respect to the basis $\pi^a v_1$, $\pi^b v_2$ of L_1 we get an equation

$$E = Ag \begin{pmatrix} \pi^{n-a} & 0 \\ 0 & \pi^{n-b} \end{pmatrix} \quad \text{where} \quad A \in GL_2(\mathcal{O}) .$$

Hence $v(\det g) \equiv a + b \bmod 2$ and $a + b \equiv |a - b| \bmod 2$.

2.3. Corollary. *If* $g \in SL(V) = \{g \in GL(V)/\det(g) = 1\}$ *and if* $\Lambda \in \mathcal{T}$ *then* $d(\Lambda, g\Lambda) \equiv 0 \bmod 2$.

In particular if $d(\Lambda_0, \Lambda_1) = 1$, then $g(\Lambda_0) \neq \Lambda_1$ for all $g \in SL_2(V)$.

In the following we work only with $SL_2(V) := G$. We denote the stabilizer of a lattice L by G_L and let $G_\Lambda := \{g \in G/g(L) = xL,\ x \in K^*\}$ be the stabilizer of the class Λ of lattices represented by L.

2.4. Lemma. — $G_L = G_\Lambda$.

Proof. Of course $G_L \subset G_\Lambda$. If $g \in G_\Lambda$ then $gL = xL$ for some $x \in K^\bullet$. Then $x^{-1}g \in GL(L)$ i.e. $v(\det(x^{-1}g)) = 0$ and $v(x) = 0$. Therefore $xL = L$ and $g \in G_L$.

2.5. Proposition. *Let H be a closed subgroup of $G = SL(V)$. Then the following are equivalent*

i) *H is compact*

ii) *$H \subset G_\Lambda$ for some vertex Λ of $\mathcal{T}$.*

The maximal compact subgroups of G are the stabilizers G_Λ for Λ a vertex of $\mathcal{T}$.

Proof. Since a stabilizer G_Λ is compact the last claim follows from (i). Apparently i) follows from ii) since G_Λ is compact. Let L be a lattice. Then $G_L \subset GL(V)$ is a open and compact subgroup. Hence $H \cap G_L$ is open in H and since H is compact then $[H : H \cap G_L] < \infty$. Hence $\sum_{g \in H/H\cap G_L} gL = L_1$ is a lattice which is H–stable. We choose Λ to be the class represented by L_1 .

Let $\sigma = (\Lambda_0, \Lambda_1)$ be a 1-simplex of $\mathcal{T}$, i.e. Λ_0 and Λ_1 are vertices and $d(\Lambda_0, \Lambda_1) = 1$. If $g \in G = SL_2(V)$ and $g(\Lambda_0, \Lambda_1) = (\Lambda_0, \Lambda_1)$ then $g\Lambda_0 = \Lambda_0$ and $g(\Lambda_1) = \Lambda_1$, see 2.3. For the stabilizer G_σ of σ in G we get $G_0 = G_{\Lambda_0} \cap G_{\Lambda_1}$, see 2.3.

Example: $\Lambda = \mathcal{O}e_1 \oplus \mathcal{O}e_2$, $\Lambda_1 = \mathcal{O}e_1 \oplus \mathcal{P}e_2$. With respect to e_1, e_2 we identify $GL(V)$ with $GL_2(K)$. Then $P_0 := G_{\Lambda_0} = GL_2(\mathcal{O})$ and $P_1 := G_{\Lambda_1} = \{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(K)/a, d \in \mathcal{O}, c \in \mathcal{P}, b \in \mathcal{P}^{-1}\}$. Hence $G_{\Lambda_0\Lambda_1} = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(K)/a, b, d \in \mathcal{O},\ c \in \mathcal{P} \right\} = P_0 \cap P_1$.

The stabilizers of simplices of $\mathcal{T}$ are called *parahoric subgroups.* The stabilizers of 1-dimensional simplices of $\mathcal{T}$ are called *Iwahori subgroups.* Apparently $SL(V)$ acts on the set of parahoric subgroups.

2.6. Proposition. *There are two $SL(V)$-conjugacy classes of maximal parahoric subgroups of $SL(V)$. All Iwahori-subgroups of $SL(V)$ are $SL(V)$-conjugate.*

Proof. The vertices of $\mathcal{T}$ correspond bijectively to the maximal parahoric subgroups of $SL(V)$, (see 2.5), and they consist of one $GL(V)$ orbit. Fix a lattice L representing a vertex Λ . Then $GL(\Lambda) := \{g \in GL(V)/g\Lambda = \Lambda\} = GL(L) \cdot K^\bullet$ and $SL(V)\backslash GL(V)/GL(L)K^\bullet$ corresponds to the $SL(V)$-orbits of vertices. From the exact sequences

$$\begin{array}{ccccccccc} 1 & \longrightarrow & SL(V)/\{\pm I\} & \longrightarrow & GL(V)/K^\bullet & \xrightarrow{\det} & K^\bullet/K^{\bullet 2} & \longrightarrow & 1 \\ 1 & \longrightarrow & SL(L) & \longrightarrow & GL(L) & \longrightarrow & \mathcal{O}^\bullet & \longrightarrow & 1 \end{array}$$

we see that there are two such orbits and the first claim holds. We may choose P_0 and P_1 from the example as represtatives of the orbits.

The Iwahori subgroups correspond bijectively to the 1–simplices of $\mathcal{T}$. If $(\Lambda_0, \Lambda_1), (\Gamma_0, \Gamma_1)$ are two 1–simplices there is a $g \in GL(V)$ with $g(\Lambda_0) = \Gamma_0$ or $g(\Lambda_0) = \Gamma_1$. Say $g(\Lambda_0) = \Gamma_0$. Then $d(\Gamma_0, g\Lambda_1) = 1$. If the lattice L_0 represents Γ_0 then there is a surjection $G_{L_0} \longrightarrow P_1(\mathcal{O}/\mathcal{P})$, see 1.8. Hence there is a $h \in SL(L)$ such that $h(\Gamma_0) = \Gamma_0$ and $h(g(\Lambda_1)) = \Gamma_1$. Therefore $hg \in SL(V)$ maps (Λ_0, Λ_1) to (Γ_0, Γ_1) and the stabilizers of these simplices are conjugate.

Let $\mathbf{P}$ be the complex whose simplices are the parahoric subgroups of $SL(V)$ and where a pair (P, P') of different parahoric subgroups is a 1–simplex if $P \cap P'$ is an Iwahori subgroup. Then we have shown.

2.7. Proposition. — *$\mathcal{T}$ is the topological realisation of $\mathbf{P}$.*

§ 3. Chambers and apartments of the tree $\mathcal{T}$

3.1. *A ray* or half apartment in $\mathcal{T}$ is a subcomplex of X such that its vertices can be labelled $\Lambda_0, \Lambda_1, \ldots, \Lambda_i$, $0 \leq i < \infty$, and $d(\Lambda_i, \Lambda_{i+1}) = 1$. We denote a ray by $\{\Lambda_i\}_{i\geq 0}$ and call Λ_0 the *origin of the ray*. Two rays $\{\Lambda_i\}_{i\geq 0}$, $\{\Gamma_j\}_{j\geq 0}$ are called *equivalent* if there are integers $i, d \in \mathbb{Z}$ such that $\Lambda_j = \Gamma_{j+d}$ for all $j \geq i$. An equivalence class of rays is called an *end of* X. We observe: If $\{\Lambda_i\}_{i\geq 0}$ is a ray and if $g \in SL(V)$, then $\{g\Lambda_i\}_{i\geq 0}$ is a ray.

3.2. Let e_1, e_2 be a basis of V. For $i \in \mathbb{N}$ put $L_i = \mathcal{O}e_1 \oplus \pi^i\mathcal{O}e_2$. The lattices L_i represent vertices Λ_i and by definition of d we have $d(\Lambda_i, \Lambda_{i+1}) = 1$, $i \geq 0$. Hence $\{\Lambda_i\}_{i\geq 0}$ is a ray. We show that all rays occur in this way. So let $\{\Lambda_i\}_{i\in\mathbb{N}}$ be a ray. We can choose lattices L_i representing Λ_i with the following properties:

$$L_{i+1} \subset L_i; \quad L_{i+1} \not\subset \pi L_i \quad L_i/L_{i+1} \xrightarrow{\sim} \mathcal{O}/\pi\mathcal{O}$$

Hence there is a system of natural maps of exact sequences of compact groups

$$\begin{array}{ccccccccc}
0 & \longrightarrow & L_i & \longrightarrow & L_0 & \longrightarrow & \mathcal{O}/\pi^i\mathcal{O} & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & L_{i+1} & \longrightarrow & L_0 & \longrightarrow & \mathcal{O}/\pi^{i+1}\mathcal{O} & \longrightarrow & 0
\end{array}$$

Since $\mathcal{O} = \varprojlim_i \mathcal{O}/\pi^i\mathcal{O}$ and $\varprojlim_i L_i = \bigcap_i L_i V$ and since $\varprojlim -$ is exact on the category of compact groups we find an exact sequence

$$0 \longrightarrow \bigcap L_i \longrightarrow L_0 \xrightarrow{f} \mathcal{O} \longrightarrow 0 .$$

Since L_0 is free of rank 2 we find an $e_1 \in \bigcap_i L_i$ and an e_2 such that $f(e_2) = \pi$ with $\bigcap_i L_i = \mathcal{O}e_1$. Then $L_0 = \mathcal{O}e_1 \oplus \mathcal{O}e_2$. We observe that e_1 is unique up to $\mathcal{O}^*$ and that $L_j \cap Ke_1 = \mathcal{O}e_1$ for j sufficiently large. Since $L_0/L_j \xrightarrow{\sim} \mathcal{O}/\pi^j\mathcal{O}$ we get that e_1, $\pi^j e_2$ is a basis of L_j for all j.

We note that Ke_1 depends only on almost all j and that Ke_1 does not depend on the lattices choosen to represent the ray $\{\Lambda_i\}_{i\geq 0}$. Hence we have a natural map which associates to the end represented by $\{\Lambda_i\}_{i\geq 0}$ the Borel subgroup $B \subset SL(V)$ such that

$$B = \{A \in SL(V) | g(Ke_1) = Ke_1\} .$$

3.3. Propositon.

i) *Let Λ_0 be a vertex of $\mathcal{T}$. Then G_{Λ_0} acts transitively on the set of rays with origin Λ_0.*

ii) *There is a natural bijection between the ends of $\mathcal{T}$ with the set of Borel subgroups of $SL(V)$.*

Proof. Let $\{\Lambda_i\}_{i\geq 0}$ be a ray and represent $\Lambda = \Lambda_0$ by a lattice $L = L_0$. By 3.4 we find a basis e_1, e_2 of L_0 such that the ray $\{\Lambda_i\}_{i\geq 0}$ is represented by a sequence of lattices as in 3.2. Since $G_L = G_\Lambda$ is transitive on the set of basis of L the first claim follows.

By 3.2 we have a well defind map $\{\text{ends of } \mathcal{T}\} \longrightarrow \{\text{Borel subgroups of } SL(V)\}$. If conversely a Borel subgroup $B \subset SL(V)$ is given we have a corresponding line $Ke_1 \subset V$ which is stabilized by B. Choose some $e_2 \in V$ such that e_1, e_2 is a basis of V and put $L_0 := \mathcal{O}e_1 \oplus \mathcal{O}e_2$. Define $L_n := \pi^n L_0 + L_0 \cap D = \mathcal{O}e_1 \oplus \pi^n\mathcal{O}e_2$. Then $\bigcap_i L_i = L_0 \cap Ke_1$ determines B. Let $\Lambda_i \in X$ be represented by L_i. Then $\{\Lambda_i\}_{i\geq 0}$ is a ray which determines B. Instead of e_1 we can choose $e_1' = \alpha e_1$, $\alpha \in K^*$ and instead of e_2 we can take $e_2' = \beta e_1 + \gamma e_2$ with $\gamma \in K^*$, $\beta \in K$. Then there is a $b \in GL(V)$ with $be_1 = e_1'$, $be_2 = e_2'$ and we construct a corresponding sequence of lattices $L_n' = bL_n$. By computation then $b\Lambda_i = \Lambda_{i+d}$, $d = v(\gamma) - v(\alpha)$, for all sufficiently large i. Hence B determines a well defined end of $\mathcal{T}$ and our claim holds.

3.4. An *apartment* A *of* $\mathcal{T}$ is a subcomplex of $\mathcal{T}$ such that the set of vertices of A can be labelled by Λ_i, $i \in \mathbb{Z}$ with $d(\Lambda_i, \Lambda_{i+1}) = 1$ for all $i \in \mathbb{Z}$. A 1-simplex $(\Lambda_i, \Lambda_{i+1})$ is called a *chamber* of A.

The labelling of the vertices of A is uniquely determined by the choice of two vertices of A which have distance 1 and by the choice of their names Λ_0 and Λ_1. We write an apartment A as $A = \{\Lambda_i\}_{i\in\mathbb{Z}}$. Let $g \in SL(V)$ (or $g \in GL(V)$) and let A be an apartment. Then gA is an apartment.

Let $A = \{\Lambda_i\}_{i\in\mathbb{Z}}$ be an apartment. Then A defines two different rays $\{\Lambda_i\}_{i\leq 0}$ and $\{\Lambda_i\}_{i\geq 0}$ and hence two different ends of $\mathcal{T}$. Of course, the set of the two ends does not depend on the labelling of A. Let $\{B_1, B_2\}$; $B_i \in SL(V)$, be the ends determined by A. Then B_i is the stabilizer of a 1 dim subspace V_i of V and $V_1 \oplus V_2 = V$. If $V_i = Ke_i, i = 1,2$ then in matrix notation with respect to the basis e_1, e_2 the group B_1 is $\left\{\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \in SL_2(K)\right\}$ and B_2 is $\left\{\begin{pmatrix} a & 0 \\ c & d \end{pmatrix} \in SL_2(K)\right\}$ and $B_1 \cap B_2 \cong T = \left\{\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \middle| t \in K^*\right\}$. Hence A determines a well defined split torus T of $SL(V)$.

Conversely let $T \subset SL_2(V)$ be a split torus, i.e. there is a basis of V such that T has diagonal form with respect to this basis. Then there are exactly two Borel subgroups $\{B_1, B_2\}$ of $SL(V)$ such that $T = B_1 \cap B_2$. By 3.4 the B_i defines two different ends. Let $\{\Lambda_i\}_{i\geq 0}$ represent B_1 and $\{\Gamma_i\}_{i\geq 0}$ represent B_2. Then for sufficiently large n we have $\Lambda_i \neq \Gamma_i$ for all $i \geq n$. The vertices Λ_i, Γ_j, $i,j \geq n$ together with the vertices of the unique geodesic from Λ_n to Γ_n define an apartment. By the uniqueness of geodesics this in independent on the choice of n. Hence we have shown:

3.5. Proposition. *There is a natural bijection between the set of K-split tori of $SL(V)$ with the set of apartments of $\mathcal{T}$.*

The typical example of an appartment A is given by a sequence (L_i), $i \in \mathbb{Z}$, of lattices as follows: Let $\{e_1, e_2\}$ be a basis of V. Put $L_i = \mathcal{O}e_1 \oplus \pi^i\mathcal{O}e_2$, $i \in \mathbb{Z}$ and let L_i represent the vertex Λ_i. Then $A = \{\Lambda_i\}_{i\in\mathbb{Z}}$ is an apartment. Since split tori or sets of basis $\{e_1 e_2\}$ are $SL(V)$–conjugate, Prop. 3.6 implies that all apartments are of the form gA, $g \in SL(V)$.

3.6. We denote the apartment A associated to a split torus $T \subset SL(V)$ by $A(T)$. Then the normalizer $N(T)$ of $T = T(K)$ in $SL(V)$ acts on $A(T)$ by affine maps. Let $\mu : N(T)(K) \longrightarrow \mathrm{Aff}(A(T))$ be the corresponding map. To make the action explicit we represent $A = \{\Lambda_i\}_{i\in\mathbb{Z}}$ by a sequence of lattices $L_i = e_1\mathcal{O} \oplus \pi^i\mathcal{O}e_2$, $i \in \mathbb{Z}$, where e_1, e_2 is an $\mathcal{O}$-basis of L_0 which is unique up to units and order of the basis. Then

$t = \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \in SL_2(K)$ maps Λ_i to $\Lambda_i + 2v(t)$ and $w = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ maps Λ_i to Λ_{-i}. Hence the image W of N in $Aut(A)$ is generated by a reflection s_0 at Λ_0 and a translation $t : \Lambda_i \longrightarrow \Lambda_{i+2}$. Since $(\Lambda_{-1}, \Lambda_0, \Lambda_1)$ is a fundamental domain for the occurring translations and s_0 acts, we see that (Λ_0, Λ_1) is a fundamental domain for W. Moreover W is generated by the reflections s_0 at Λ_0 and s_1 and Λ_1. Thus the structure of $A(T)$ as a simplical complex can be described with the affine action of W. on $\mathbb{R} \xrightarrow{\sim} A(T)$. On $\mathbb{R}$ we view W as group generated by the reflections at all $i \in \mathbb{Z} \subset \mathbb{R}$.

3.7. Let $A_0 \subset \mathcal{T}$ be an apartment. We have a natural surjection $f : G \times A_0 \longrightarrow \mathcal{T}$ sending $(g, b) \in G \times A_0$ to gb. We give $G \times A_0$ the simplicial structure and topology of a disjoint sum. Then f is continuous, surjective, simplicial and G–equivariant and induces an equivalence

$$G \times A_0/(\sim) \xrightarrow{\sim} \mathcal{T}.$$

We describe the equivalence relation. To $b \in A$ be associate $P_b \subset G(K)$, such that P_b is the subgroup fixing b. Then $(g, x) \sim (h, y)$ if there is an $n \in N$ such that $y = \nu(n)x$ and $g^{-1}hn \in P_x$. Then $g^{-1}hP_yh^{-1}g = P_x$. The action of $N(T)(K)$ via μ here is necessary to get rid of the unnatural labelling which reflects the various ways to identify A_0 with $(\mathbb{R}, Z)$. We get with this definition of the quivalence using 3.6 and 3.3:

3.8. Proposition. *We have a natural identification*

$$G \times A_0/(\sim) \xrightarrow{\sim} \mathcal{T}.$$

This description of $\mathcal{T}$ will be generalized in the following chapters. For this we first generalize A to a simplical tessellation of an euclidean vectorspace which results from the action of a Coxeter group W and then we introduce generalisations of the groups P_b which will define the equivalence relation. The building $\mathcal{T}$ will be defined by the left side of 3.8.

§4. Coxeter Groups and Coxeter Complexes

We recall some essential definitions and results on Coxeter groups. For proofs we refer to [B, Chap. IV, V].

4.1. Let $(E, \langle\ ,\ \rangle)$ be a finite dimensional euclidean l-dimensional $\mathbb{R}$-vectorspace with positive definite scalarproduct $\langle\ ,\ \rangle$. Let $Iso(E)$ be the group of isometrical maps $f : E \longrightarrow E$. We view E as a subgroup of $\mathrm{Iso}(E)$ where $v \in E$ corresponds to the translation $w \longmapsto w + v$. Let $O(E)$ be the group of linear maps $f : E \longrightarrow E$ which preserve $\langle\ ,\ \rangle$. Then we have a natural action of $O(E)$ on E and on the translations in $Iso(E)$. We get a semidirect product decomposition

$$E \rtimes O(E) \xrightarrow{\sim} Iso(E)$$

A *(affine) hyperplane* $H \subset E$ is an affine subspace of dimension $l-1$, i.e. $H = h + W$ where $W \subset E$ is a linear subspace of dimension $l-1$ and $h \in H$. We choose $0 \neq \alpha \in W^{\perp}$. Then $H = \{v \in E / \langle v, \alpha\rangle - \langle v, h\rangle = 0\}$. We define the *reflection* s_H *with respect to* H by $s_H(v) = v - (\langle v, \alpha\rangle - \langle v, h\rangle)\frac{2\alpha}{\langle \alpha,\alpha\rangle}$. Then $s_H^2 = Id$ and s_H fixes exactly H. Moreover s_H does not depend on the choice of $\alpha \in W^{\perp}$ and the choice of h.

We have an affine map $a : E \longrightarrow \mathbb{R}$ with $a(v) = \langle \alpha, v\rangle - \langle h, v\rangle$. If conversely $a : E \longrightarrow \mathbb{R}$ is a non constant affine map we can write $a(v) = L(v) - r$ where $L : V \longrightarrow \mathbb{R}$ is linear $L \neq 0$ and $a(0) = -r \in \mathbb{R}$. Now $L(v) = \langle v, \alpha\rangle$ for some $0 \neq \alpha \in E$ and we introduce $s_a(v) := v - a(v)\frac{2\alpha}{\langle \alpha,\alpha\rangle}$, $v \in V$, a reflection at the hyperplane $H_a = \{v \in E / a(v) = 0\}$. We write $a = (\alpha, r)$ and note that s_a is independent of the choice of α.

4.2. Let $\mathcal{H} := \{H_i\}_{i\in I}$ be a family of (affine) hyperplanes of E. The family is said to be *locally finite* if every $v \in E$ has a neighbourhood $\mathcal{U}$ in E such that $|\{i \in I | H_i \cap \mathcal{U} \neq \emptyset\}|$ is finite. A subgroup $W \subset Iso(E)$ is called a *Coxeter group* if it is generated by the reflections s_H contained in W such that the set of hyperplanes $\mathcal{H} = \{H / s_H \in W\}$ is a locally finite family.

We have a natural homomorphism $p : Iso(E) \longrightarrow O(E)$ sending an affine isometry to its linear part. Let $p(W) = {}^vW$ be the image (= vectorial part of W in $O(E)$). Then reflections at parallel hyperplanes have identical image in ${}^vW$ and ${}^vW$ is generated by these images, which are reflections at hyperplanes through $0 \in E$.

4.3. A Coxeter group W is called *essential* if $E^{{}^vW} = \{v \in E / u(v) = v \text{ for all } u \in {}^vW\} = \{0\}$ and it is called *irreducible* of the representation of ${}^vW$ on E is irreducible. The investigation of Coxeter groups can be reduced to the study of essential and irreducible Coxeter groups, see [B: Chap. V, § 3, no 8].

4.4. Let W be an essential and irreducible Coxeter group. Denote by $\mathcal{H}$ the set of hyperplanes such that $s_H \in W$. The connected components of $E \setminus \bigcup_{H \in \mathcal{H}} H$ are called *chambers* of W. Let C be one chamber with topological closure $\overline{C}$. Then the following hold:

- if $|W| = \infty$ then $\overline{C}$ is a ℓ-dimensional simplex bounded by $\ell + 1$ hyperplanes
- if $|W| < \infty$ then C is a cone bounded by ℓ hyperplanes
- W acts simply transitive on the set of chambers
- W is generated by the set S of reflections at the bounding hyperplanes (= *walls*) of C
- ${}^vW \subset O(E)$ is a finite reflection group.
- If $x \in \overline{C}$ then $W_x = \{w \in W / w(x) = x\}$ is generated by the $s_H \in W$ with $s_H(x) = x$.
- there exist points $x \in E$ such that W_x maps isomorphically to ${}^vW$. These points are called *special points.*

4.5. We continue to assume that W is essential and irreducible. Replacing W by a conjugate subgroup of $Iso(E)$ we can assume that $W_x = {}^vW$. Hence we can view ${}^vW$ as a subgroup of W. We recall that a lattice L in E is a subgroup (of E) generated by ℓ linear independent vectors. Then we have

— If $|W| = \infty$ then there is a lattice $L \subset E$ such that $W = L \rtimes {}^vW$

— If $|W| < \infty$ then $W = W_0 \xrightarrow{\sim} {}^vW$.

The cones corresponding to W_0 intersect with a sphere $S^{\ell-1}$ of radius 1 with center 0 and give a tessellation of this sphere. Hence a finite Coxeter group $W = W_0$ is called *spherical*. If $|W| = \infty$ then W is called *euclidean*. The closed chambers of W give a tessellation of E consisting of congruent closed ℓ-dimensional simplices.

The sphere $S^{\ell-1}$ resp. E together with the simplicial structure coming from the simplicial structure of the closed chambers is called the *Coxeter complex* of W.

4.6. The Coxeter groups W can be classified, see [B. Ch. VI, §4, Thm. 1, Ch. VI, §2 Prop. 8]. We describe the result for $|W| = \infty$.

Let R be an irreducible and reduced root system in the vectorspace $V = E^* = \mathrm{Hom}_{\mathbb{R}}(E, \mathbb{R})$. Let $R^\vee$ be the dual root system in E and denote by s_α the reflection in E given by $s_\alpha(x) = x - \alpha(x)\alpha^\vee, \alpha \in R$. The group generated by these reflections is denoted by ${}^vW$. If identify V and E with the help of a ${}^vW$-invariant euclidean scalarproduct $\langle\, , \rangle$ on E, then $\alpha^\vee = \frac{2\alpha}{\langle \alpha, \alpha \rangle}$ for $\alpha \in R$.

The reflection at the (affine) hyperplane $H_{\alpha,k} = \{x \in E/\alpha(x) = k\}$ is denoted by $s_{\alpha,k}$. We have $s_{\alpha,k}(x) = x - (\alpha(x) - k)\alpha^\vee = t(k\alpha^\vee) \circ s_\alpha$, where $t(k\alpha^\vee)$ denotes the translation by $k\alpha^\vee$. In particular $s_{\alpha,0} = s_\alpha$. We write $a = (\alpha, -k)$, $s_a = s_{\alpha,k}$, and view a as affine map $a(x) = \alpha(x) - k$. Then $R_{\mathrm{aff}} := \{a : E \longrightarrow \mathbb{Z} \text{ affine } / a = (\alpha, k), \alpha \in R; k \in \mathbb{Z}\}$ is called the set of *affine roots* of R. The group generated by the s_a, $a \in R_{\mathrm{aff}}$ is denoted by W and is called the *affine Weyl group* of R. Of course ${}^vW \subset W$ and ${}^vW$ acts on the lattice Λ in E generated by the $\alpha^\vee \in R^\vee$. Hence $\Lambda \rtimes {}^vW$ is defined and one has $W = \Lambda \rtimes {}^vW$, see [B: Chap. VI §2]. The classification mentioned above says that every essential irreducible euclidean Coxeter group C is isomorphic to exactly one such W given by an irreducible and reduced root system R. Moreover the isomorphism comes from a isomorphism of the underlying affine spaces.

4.7. We can describe a chamber of the tessellation of E given by W as follows: Let $\alpha_1, ..., \alpha_\ell$ be a basis of R and denote by $\tilde{\alpha} = \sum_{i=1}^{\ell} n_i \alpha_i$ the largest (positive) root,

$n_i \geq 1$, $n_i \in \mathbb{Z}$. Then $C = \{x \in E/\alpha^i(x) > 0,\ \tilde{\alpha}(x) < 1\}$ is a chamber for W. Let $w_1, ..., w_\ell \in E$ be such that $\langle \alpha_j, w_i \rangle = \delta_{ij}$. The w_i then are a basis of the weights $P^\vee$ for the root system $R^\vee$ in E. Then the hyperplanes spanned by the w_i are the walls of a Weyl chamber in E for ${}^vW$ and $\overline{C}$ has another wall given by the affine hyperplane $\{x \in E/\tilde{\alpha}(x) = 1\}$. Hence $\overline{C}$ has vertices $\{0, \frac{w_1}{n_1}, ..., \frac{w_l}{n_l}\}$ and W is generated by the $l+1$ reflections at the hyperplanes bounding $\overline{C}$. Since $x \in V$ is special if for every α_i there is a k such that $x \in H_{\alpha_i,k}$ we see that $\frac{w_j}{n_j}$ is special iff $n_j = 1$.

4.8. The affine Weyl groups we need later are constructed from a non reduced root system R. Bruhat and Tits have defined a finer notion of equivalence in this situation and give a classification of the arising Weyl groups (as abstract groups these are the old ones) together with the lattice of translations given by the root system R. In general the arising lattice of translations now is not proportional to the lattice of translations given by the reduced root system of R. The classification is given in terms of extended Dynkin diagramms where the direction of double or threefold arrows can be reversed and where some vertices may get in addition a $\star$ indicating the simple roots whose double is also a root. For all this see [Br–T 3: 1.4.5].

§5. Construction of compact and open subgroups

Let G/K be a semi simple group defined over K. Then by definition there is a $n \in \mathbb{N}$ and a finite set of polynomials with coefficients in K in variables $X_{ij}, i, j \in \{1, ...n\}$, such that $G(K) \subset GL_n(K)$ is the set of common zeros of the polynomials. Since $GL_n(\mathcal{O})$ is an open and compact subgroup of $GL_n(K)$ we get an open and compact subgroup $\Gamma := G(K) \cap GL_n(\mathcal{O})$ of $G(K)$. If Γ_1 and Γ_2 are open and compact subgroups then $\Gamma_1 \cap \Gamma_2$ is open and compact and of finite index in Γ_1 and Γ_2. This implies that every open and compact subgroup of $G(K)$ is conjugate in $GL_n(K)$ to a subgroup of $GL_n(\mathcal{O})$.

There are many different possibilities to view G as a subgroup of some GL_n. For the adjoint group G_{ad} of G with K–Lie algebra $\mathfrak{g}$ we have the natural embedding

$$\mathrm{Ad} : G_{ad}(K) \hookrightarrow GL(\mathfrak{g}).$$

Composing with the natural map $G \longrightarrow G_{ad}$ we get a homomorphism with finite kernel

$$G(K) \longrightarrow GL(\mathfrak{g})$$

and all open compact subgroups Γ of $G(K)$ occur as stabilizers of lattices $L \subset \mathfrak{g}$. Here L is a finitely generated $\mathcal{O}$ submodule of $\mathfrak{g}$ such that $L \cdot K = \mathfrak{g}$.

The "good" compact open subgroups should posess an $\mathcal{O}$–Liealgebra L and should occur as stabilizers under the adjoint action of L. Hence we are motivated to construct $\mathcal{O}$-Lie algebras L such that $L \otimes_{\mathcal{O}} K = \mathfrak{g}$. We explain this for K–split groups and start with the example $SL_n(K)$.

5.1. In $SL_n(K)$ we consider the maximal split torus $T = \{t = \begin{pmatrix} t_1 & & 0 \\ & \ddots & \\ 0 & & t_n \end{pmatrix} / t_i \in K^{\bullet}, \prod_{i=1}^n t_i = 1\}$. We define $\varepsilon_i(t) = t_i =: t^{\varepsilon_i}$. We identify the $\mathbb{R}$–vector space spanned by the ε_i with $\mathbb{R}^n$. Let $V \subset \mathbb{R}^n$ be the subspace $V := \{(x_1, ..., x_n) \in \mathbb{R}^n | \sum x_i = 0\}$. It has a basis $\varepsilon_i - \varepsilon_{i+1}, i = 1, ..., n-1$, and V is identified with $X^*(T) \otimes_{\mathbb{Z}} \mathbb{R}$ where $X^*(T)$ denotes the group of characters of T. The roots $\varepsilon_i - \varepsilon_j =: \alpha_{ij}$, $i \neq j$; $i, j \in \{1_i, ..., n\}$ are in V and $t \in T$ acts on $u_{ij} = 1 + e_{ij} x$ by $t u_{ij} t^{-1} = 1 + \alpha_{ij}(t) x e_{ij}$. Here $x \in K$ and e_{ij} is the $n \times n$ matrix with all coefficients zero except the coefficient in i–th column and j–th row and this coefficient is 1. The elements of the normalizer $N(T)$ of T in $SL_n(K)$ have exactly one non zero entry in each row and column. The vectorial Weyl group ${}^vW = N(T)/T$ acts as group of permutations of the $(\varepsilon_1, ..., \varepsilon_n)$.

Let L be a lattice in K^n, i.e. L is a finitely generated $\mathcal{O}$ –submodule with $K \cdot L = K^n$. By the elementary divisor theorem there is a basis $v_1, ..., v_n$ of K^n such that $L = \mathcal{O}v_1 + ... + \mathcal{O}v_n$ and the stabilizer G_L of L in $SL_n(K)$ can be identified with $SL_n(\mathcal{O})$. In particular $GL_n(K)$ acts transitively on the set of lattices. As in 2.5 (proof) we see that the stabilizer of L in $SL_n(K)$ is a maximal compact subgroup of $SL_n(K)$. Lattices L_1, L_2 are called *equivalent* if $xL_1 = L_2$ for some $x \in K^{\bullet}$. Apparently equivalent lattices have the same stabilizers.

We consider the lattices $L = \pi^{r_1}\mathcal{O} \oplus ... \oplus \pi^{r_n}\mathcal{O}, r_i \in \mathbb{Z}$ in K^n. The corresponding lattice classes are uniquely determined by $L_0 = \mathcal{O} \oplus ... \oplus \mathcal{O}$ and $(r_1 + t, ..., r_n + t) mod\, t, t \in \mathbb{Z}$.

The stabilizer of the lattice L in $SL_n(K)$ is $t\,SL_n(\mathcal{O})t^{-1}$ where $t = \begin{pmatrix} \pi^{r_1} & & 0 \\ & \ddots & \\ 0 & & \pi^{r_n} \end{pmatrix} \in GL_n(K)$. Apparently $tSL_n(\mathcal{O})t^{-1}$ contains $T(\mathcal{O}) = \left\{ \begin{pmatrix} t_1 & & 0 \\ & \ddots & \\ 0 & & t_n \end{pmatrix} \in SL_n(K) / t_i \in \mathcal{O}^{\bullet} \right\}$ and the groups $U_{i,j,k} = \left\{ 1 + e_{ij} x / x \in \pi^k \mathcal{O} \right\}$, where $\nu(x) \geq k = r_i - r_j$ (recall: $v : K \longrightarrow \mathbb{Z}$ is the valuation of K). These subgroups generate $tSL_n(\mathcal{O})t^{-1}$ which of course only depends on the class of the lattice.

We put $E = V^* = \mathrm{Hom}_{\mathbb{R}}(V, \mathbb{R})$ and view V^* as quotient of $(\mathbb{R}^n)^*$. Here the standard basis of $(\mathbb{R}^n)^*$ is denoted by $e_1, ..., e_n$ and $e_1, ..., e_n$ is the dual basis of the standard basis $\varepsilon_1, ..., \varepsilon_n$ of $\mathbb{R}^n$. We have an action of the affine Weyl group W attached to the set of affine roots $a_{ij,k} = \alpha_{ij} + k$, $k \in \mathbb{Z}$ on E, see §4. Let $A(T)$ be the arising Coxeter complex. Recall, that the chamber C given by the set of positive roots, $\alpha_1, ..., \alpha_{n-1}$ has vertices $(0, w_1, ..., w_{n-1})$. Here we use $R = R^{\vee}$ and the fact that $\alpha_1 + ... + \alpha_{n-1} = \tilde{\alpha}$ is the largest root. The w_j have been computed in [B: Chap. V Planche I]. One has $w_j = e_1 + ... + e_j - \frac{j}{n}\sum_{i=1}^n e_i$ i.e. in coordinates $w_j = \left(\underbrace{1 - \frac{j}{n}, ..., 1 - \frac{j}{n}}_{j}, -\frac{j}{n}, ..., -\frac{j}{n} \right)$ and $\langle w_j, \alpha_i \rangle = \delta_{ij}$. The reflection s_{α_i} acts on $x = (x_1, ..., x_n) \in E$ as $s_{\alpha_i}(x) = (x_1, ..., x_{i-1}, x_{i+1}, x_i, x_{i+2}, ... x_n), i = 1, n-1$. The reflection at the wall $H_{\tilde{\alpha},1} = \{e \in E / \langle \tilde{\alpha}, e \rangle = 1\}$ is given by $s(x_1, ..., x_n) = (x_n + 1, x_2, ..., x_{n-1}, x_1 - 1)$.

To the class of lattices represented by $\pi^{r_1}\mathcal{O} \oplus ... \oplus \pi^{r_n}\mathcal{O} = L$ we attach the well defined vertex $v(L) = -(r_1 - \frac{r}{n} ..., r_n - \frac{r}{n}) \in E$ where $r = \sum_{i=1}^n r_i$. Note that $v(L)$ determines the class of L. If $t \in T(K)$, $t = \begin{pmatrix} t_1 & & 0 \\ & \ddots & \\ 0 & & t_n \end{pmatrix}$, $\prod_{i=1}^n t_i = 1$ then tL is the lattice $\pi^{r_1 + v(t_1)}\mathcal{O} \oplus ... \oplus \prod^{r_n + v(t_n)} \mathcal{O}$ and $\sum_{i=1}^n v(t_i) = 0$. Hence to $t(L)$ corresponds the point $-(r_1 + v(t_1) - \frac{r}{n}, ..., r_n + v(t_n) - \frac{r}{n})$. The coroots $\alpha^{\vee} \in R^{\vee}(= R)$ span the lattice $\{(u_1, ..., u_n) \in \mathbb{Z}^n / \sum_{i=1}^n u_i = 0\}$ in E. Hence $v(tL) = v(L) + q$ where $q \in \mathbb{Z}(R^{\vee})$ is in the lattice spanned by the coroots. We observe that $v(\underbrace{\pi^{-1}\mathcal{O}\oplus\ldots\oplus\pi^{-1}\mathcal{O}}_{j}\oplus\mathcal{O}\oplus\ldots\oplus\mathcal{O}) = w_j$. In obvious matrix block notation the stabilizer of the lattice corresponding to w_j is

$$P_j = \left\{ \begin{pmatrix} \mathcal{O} & \mathcal{P}^{-1} \\ \mathcal{P} & \mathcal{O} \end{pmatrix} \in SL_n(K) \right\},$$

where in the upper left corner $\mathcal{O}$ stands for a $j\times j$–matrix with coefficients in $\mathcal{O}$ and the vertex 0 corresponds to $SL_n(\mathcal{O})$. From this and with the action of $T(K)$ we see that v establishes a bijection between the classes of lattices $\pi^{r_1}\mathcal{O}\oplus\ldots\oplus\pi^{r_n}\mathcal{O}$ with the set of vertices of the Coxeter complex given by T and by the action of the affine Weyl group W of R.

We note that the chamber C viewed as a simplex can be represented by the flag of lattices $L_0 = \pi\mathcal{O}\oplus\ldots\oplus\pi\mathcal{O} \subset L_1 = \mathcal{O}\oplus\pi\mathcal{O}\oplus\pi\mathcal{O}\ldots\oplus\pi\mathcal{O} \subset \ldots \subset L_j = \underbrace{\mathcal{O}\oplus\ldots\oplus\mathcal{O}}_{j}\oplus\pi\mathcal{O}\oplus\ldots\oplus\pi\mathcal{O} \subset L_{n-1} = \mathcal{O}\underbrace{\oplus\ldots\oplus}_{n-1}\pi\mathcal{O}$. Here the L_i correspond to the vertices of C.

5.2. Let $\mathfrak{g}$ be a simple split K–Lie algebra. Then there is a Cartan subalgebra $\mathfrak{h}$, a reduced irreducible system of roots $R\subset\mathfrak{h}^* = \mathrm{Hom}_{\mathbb{R}}(\mathfrak{h},\mathbb{R})$ and root spaces $\mathfrak{g}_\alpha = \{x\in\mathfrak{g}/[HX] = \alpha(H)X\}$ of dimension 1 such that $\mathfrak{g} = \mathfrak{h}\oplus\prod_{\alpha\in R}\mathfrak{g}_\alpha$. Put $l = \dim\mathfrak{h}$. Choose R^+ a system of positive roots with basis $\alpha_1,\ldots,\alpha_l$. A *Chevalley basis* of $(\mathfrak{g},\mathfrak{h})$ is a basis $H_{\alpha_1},\ldots,H_{\alpha_l}$ of $\mathfrak{h}$ and vectors $0\neq X_\alpha\in\mathfrak{g}_\alpha$ such that the following hold.

1) $[H_{\alpha_i}X_\alpha] = \langle\alpha_i,\alpha^\vee\rangle X_\alpha$
2) $[X_{\alpha_i},X_{-\alpha_i}] = -H_{\alpha_i}$ and $[X_\alpha,X_{-\alpha}]$ is a $\mathbb{Z}$–linear combination of the H_{α_i}.
3) If $\alpha,\beta\in R$ and $\alpha+\beta\in R$, then $[X_\alpha,X_\beta] = \pm(r+1)X_{\alpha+\beta}$ when $\beta-r\alpha,\ldots,\beta,\ldots,\beta+q\alpha$, $r,q\in\mathbb{N}$ is the α–string through β.

We note that $\langle\alpha,\beta^\vee\rangle = n(\alpha,\beta)\in\{0,-1,-2,-3\}$. Since the H_{α_i} and X_α span a $\mathbb{Z}$–Lie subalgebra $\mathfrak{g}_{\mathbb{Z}}$ we can construct a lattice $\mathfrak{g}_0 = \mathfrak{g}_{\mathbb{Z}}\otimes_{\mathbb{Z}}\mathcal{O}$ of $\mathfrak{g}$.

5.3. Let G_{ad} be the adjoint connected algebraic group acting as automorphisms of the Lie–algebra $\mathfrak{g}$. Over an algebraic closure $\overline{K}$ of K the group $G_{ad}(\overline{K})$ is generated by the

elements $exp\,ad\,X_\alpha t$, $\alpha\in R$, $t\in\overline{K}$. Since the X_α come from a Chevalley basis of $\mathfrak{g}$ the *adjoint Chevealley* group G_{ad} is defined over $\mathbb{Z}$.

Let $Aut_0(\mathfrak{g},\mathfrak{h})$ be the subgroup of G_{ad} leaving $\mathfrak{h}$ invariant. Let T_{ad} denote the split torus in G_{ad} corresponding to $\mathfrak{h}$. Then there is an exact sequence

$$1\longrightarrow T_{ad}\longrightarrow Aut_0(\mathfrak{g},\mathfrak{h})\longrightarrow {}^vW\longrightarrow 1,$$

where $T_{ad} = T_{ad}(K) = \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}(R),K^*)$, see [B : Ch. V III, §5 n° 2]. Here ${}^vW$ is the vectorial Weyl group of the root system R and $\mathrm{Aut}_0(\mathfrak{g},\mathfrak{h}) = N(T)(K)$ is the normalizer of T_{ad} in G_{ad}. It is known (see loc. cit. n°. 3) that $G_{ad}(K)$ is simply transitive on the set of Chevalley basises for $\mathfrak{g}$. Therefore T_{ad} is simply transitive on Chevalley basises for $(\mathfrak{g},\mathfrak{h})$ with some choice of a basis $\alpha_1,\ldots,\alpha_l$ of R.

5.4. Let Λ be a lattice in $V = \mathbb{R}(R)$ such that

$$\mathbb{Z}(R)\subset\Lambda\subset\mathbb{Z}(P)$$

where $\mathbb{Z}(P)$ is the lattice spanned by the weights P and put $T(\overline{K}) = \mathrm{Hom}_{\mathbb{Z}}(\Lambda,\overline{K}^*)$. Then T is a torus, defined over K with group of characters $X^*(T) = \Lambda$. For T there exists a group G/K with maximal split torus T such that G covers G_{ad}, i.e. over $\overline{K}$ there is an exact sequence

$$1\longrightarrow\mu\longrightarrow G\longrightarrow G_{ad}\longrightarrow 1.$$

with $\mu(\overline{K}) = \mathrm{Hom}_{\mathbb{Z}}(\Lambda/\mathbb{Z}(R),\overline{K}^*)$. If $\Lambda = \mathbb{Z}(P)$ the group G is called the simply connected covering of G_{ad} or the *simply connected Chevalley group of type* R. The name "simply connected" is motivated by the fact that $G(\mathbb{C})$ viewed as topological space with the topology given by the usual one of $\mathbb{C}$ is simply connected. For all this, see [D–G].

We have $T(K) = \mathrm{Hom}_{\mathbb{Z}}(\Lambda,K^*) = \mathrm{Hom}_{\mathbb{Z}}(\Lambda,\mathbb{Z})\otimes_{\mathbb{Z}}K^*$ with $X_*(T) = \mathrm{Hom}_{\mathbb{Z}}(\Lambda,\mathbb{Z})$, the group of cocharacters of G. If $\Lambda = \mathbb{Z}(P)$ we get $\mathrm{Hom}_{\mathbb{Z}}(\Lambda,\mathbb{Z}) = \mathbb{Z}(R^\vee)$ by the definition of the weights P.

5.5. We assume now that G is a simply connected split Chevalley group with split maximal torus T, $X_*(T) = \mathbb{Z}(R^\vee), X^*(T) = \mathbb{Z}(P)$ and vectorial Weyl group ${}^vW$. We put $E = X_*(T) \otimes \mathbb{R}$, $V = X^*(T) \otimes \mathbb{R}$ chose a ${}^vW$-invariant metric on V and E^* and consider the action of the affine Weyl group $\mathbb{Z}(R^\vee) \rtimes {}^vW =: W$ on E. According to §4 we then have the notion of a Coxeter complex $A(T)$ with a chamber C determined by a choice of a system of positive roots $\alpha_1, ..., \alpha_l$. The chamber has vertices $\left(0, \frac{w_1}{n_1}, ..., \frac{w_\ell}{n_\ell}\right)$, see §4.

As explained the end of §3.7 we want to attach open and compact subgroups P_i to the $\ell + 1$ vertices of C. We describe these as stabilzers in $G(K)$ of $\mathcal{O}$-lattices in $\mathfrak{g}$. We fix a Chevalley basis $\{H_{\alpha_i}, X_\alpha\}_{i=1,...,\ell}, \alpha \in R$, and denote the $\mathcal{O}$-Liealgebra spanned by the basis by $\mathfrak{g}_0$. Let P_0 be the stabilizer in $G(K)$ of $\mathfrak{g}_0$ with respect to the adjoint action.

Let $\tilde{\alpha} = \sum_{j=1}^{\ell} n_j \alpha_j$ be the largest positive root. The $\alpha \in R$ is written uniquely $\alpha = \sum_{j=1}^{\ell} m_j \alpha_j$, $m_j \in \mathbb{Z}\ |m_j| \le n_j$. We define for $j \in \{1, ..., \ell\}$ and $\alpha \in R$ an integer

$$n(j, \alpha) = \begin{cases} -1 & m_j = n_j \\ 0 & 0 \le m_j < n_j \\ 1 & m_j < 0 \end{cases}$$

and consider the $\mathcal{O}$-span of the basis $\left\{H_{\alpha_i}, \pi^{n(j,\alpha)} X_\alpha\right\} i = 1, ..., \ell, \alpha \in R$, and see that this is an $\mathcal{O}$ Lie algebra. Let $P_i \subset G(K)$ be the stabilizer of this $\mathcal{O}$-lattice. We note that for $G = SL_{\ell+1}$ we have $\tilde{\alpha} = \alpha_1 + ... + \alpha_\ell$. Hence since $n_j = 1$ for all j and we get exactly the groups P_i of §5.1.

Let U_α be the root group for $\alpha \in R$ of G. Then U_α is isomorphic to the root group for $\alpha \in R$ in G_{ad} since μ does not meet U_α. The choice of $X_\alpha \in \mathfrak{g}_\alpha$ determines an isomorphism $U_\alpha = U_\alpha(U) \xrightarrow{\xi_\alpha} K$. Hence there is a filtration $U_{\alpha,k}$ of U_α with $U_{\alpha,k} = \{x \in U_\alpha(K)/v(\xi_\alpha(x)) \ge k\}$. We view (α, k) as affine map $a : A(T) \longrightarrow \mathbb{R}$ with $a(b) = \alpha(b) + k$. If $\alpha = \sum_{j=1}^{\ell} m_j \alpha_j$ then $\alpha\left(\frac{w_j}{n_j}\right) = \frac{m_j}{n_j}$ and $a\left(\frac{w_j}{n_j}\right) \ge 0$ iff $k \ge n(j, \alpha)$. Hence P_j is the subgroup of $G(K)$ generated by $T(\mathcal{O})$ and all the U_α with $a\left(\frac{w_j}{n_j}\right) \ge 0$. More generally we attach to $b \in A(T)$ the group P_b generated by $T(\mathcal{O})$ and all U_a with $a(b) \ge 0$.

We note that $T(K)$ acts by conjugation on the P_b. If P_b comes from a lattice with basis $\{\mathcal{O}H_{\alpha_i}, \mathcal{O}\pi^{r_\alpha} X_\alpha\}, i = 1, ..., l, \alpha \in R$ then $t\,P_b t^{-1}$ comes from the lattice $(\mathcal{O}H_{\alpha_i}, \mathcal{O}\pi^{r_\alpha + v(\alpha(t))} X_\alpha)$. Let b determine the root groups $U_{\alpha,k}$, $a = \alpha + k$. Then $t\,P_b t^{-1}$ determines root groups $U_{\alpha,k+v\alpha(t)} = \{x \in U_\alpha/\alpha(x) - v(\alpha(t)) \ge k\}$. We therefore define a map $\mu : T(K) \longrightarrow \{\text{translations of } A(T)\}$ as follows. First we map $t \in T(K) = \mathrm{Hom}(\mathbb{Z}(R), K^*)$ to $v(t) \in \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}(R), \mathbb{Z})$ by composing with $-v : K^* \longrightarrow \mathbb{Z}$. We note that $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}(P), \mathbb{Z}) = \mathbb{Z}[R^\vee]$ canonically. Hence $v(t) \in \mathbb{Z}(R^\vee)$ and $\mu(t)$ is defined as translation on $A(T) = E = \mathbb{Z}(R^\vee) \otimes_{\mathbb{Z}} \mathbb{R}$ given by $x \longmapsto x + v(t)$. With this notation $t\,P_b t^{-1} = P_{b+v(t)}$.

The map $\mu : T(k) \longrightarrow \mathrm{Aff}(E)$ can be extended to a homomorphism $\nu : N(T)(K) \longrightarrow A(E)$ such that the linear part of $\mu(n)$, $n \in N(T)(K)$ acts as prescribed by the action of $\overline{n} \in {}^vW$ on E. We get $nP_b n^{-1} = P_{\nu(n)b}$. We note that if b is an interior point of a simplex of $A(T)$ with vertices $b_1, ..., b_r$ then $P_b = \bigcap_{i=1}^{r} P_{b_i}$.

Summarizing we have attached to every simplex of $A(T)$ a compact subgroup of $G(K)$ and the attachment is compatible with the action of $N(T)(K)$ on both sides. The description of the groups depends on a fixed choice of a Chevalley lattice $\mathfrak{g}_0$ in $\mathfrak{g}$ with stabilizer P_0 corresponding to $0 \in A(T)$ given by the point zero in E. The same situation already occurred for SL_2: the apartments where described with respect to a choice of 0.

§6. Construction of Buildings

Let G/K be an semi simple connected algebraic group over a local field K with maximal K-split torus S. We construct at first a Coxeter complex $A(S)$ and then describe how the complexes are glued together to obtain the Bruhat-Tits building of $G(K)$. For simplicity we assume that G is almost simple.

6.1. We let $X^*(S)$ be the group of K-rational characters of S and put $E := \mathrm{Hom}_{\mathbb{Z}}(X^*(S), \mathbb{R})$. For $s \in S = \mathrm{Hom}_{\mathbb{Z}}(X^*(S), K^*) = X_*(S) \otimes_{\mathbb{Z}} K^*$ we define $\mu(s) \in E$ by $\chi(\mu(s)) = -v(\chi(s))$ where $v : K \longrightarrow \mathbb{Z}$ is the discrete valuation of K and where

$\chi \in X^*(S)$. The group $X^*(Z)$ of K-rational characters of the centralizer Z of S in G is of finite index in $X^*(S)$. Hence μ extends to a homomorphism denoted by $\mu : Z(K) \longrightarrow E$. Let $N(S)$ be the normalizer of S in G. We write $N(K) = N(S)(K)$. Then $N(K)/Z(K) =: {}^vW$ is the (vectorial) Weyl group generated by reflections at the K-roots of S. These K-roots from a root system R (not necessarily reduced) and the roots generate $X^*(S) \otimes \mathbb{R}$. Since ${}^vW$ acts on $X^*(S)$ we get a linear action of $w \in {}^vW$ on E written $v \longmapsto w \cdot v$. There is a natural extension of μ to a homomorphism $\nu : N(S)(K) \longrightarrow \mathrm{Aff}(E)$ into the group of affine bijections such that

a) if $s \in Z(K)$ then $\nu(s)v = v + \mu(s)$

b) if $n \in N(K)$ represents $w \in {}^vW$ then the linear part of $\nu(n)$ is the map $v \longmapsto w \cdot v$.

To see this we introduce $Z_c = \ker \mu$ and observe that $N(K)/Z_c$ is a group extension of ${}^vW$ by $\Lambda := \mu(Z(K))$ which is a lattice with $X_*(S) \subset \mu(Z(K)) \subset \mathrm{Hom}_{\mathbb{Z}}(X^*(Z), \mathbb{Z})$. Thus $N(K)/Z_c$ represents a class in $H^2({}^vW, \Lambda)$ which gets trivial in $H^2({}^vW, \Lambda \otimes_{\mathbb{Z}} \mathbb{R})$ and we have a map $N(S)(K)/Z_c \longrightarrow \Lambda \otimes_{\mathbb{Z}} \mathbb{R} \rtimes {}^vW$. The semi direct product acts by affine maps on $\Lambda \otimes_{\mathbb{Z}} \mathbb{R} = E$. Hence we get a map $\nu : N(S)(K) \longrightarrow \mathrm{Aff}(E)$ with properties a) and b). Since ${}^vW$ acts irreducibly on E the map ν is unique.

6.2. Let $R \subset X^*(S)$ be the system of K-roots of (G, S). Any $\alpha \in R$ defines a reflection s_α on E and a root group $U_\alpha \subset G$ which is connected and unipotent (of dimension 1 if G is split, but not in general) We abbreviate $U_\alpha = U_\alpha(K)$.

If $x \in U_\alpha \backslash (1)$ there is a unique element $m_\alpha(x) \in U_{-\alpha} x U_{-\alpha} \cap N(K)$ whose image in ${}^vW$ is s_α see [B - T]. Then $\nu(m_\alpha(x))$ is the affine reflection s_x on E with

$$s_x(v) = v - (\langle \alpha, v \rangle - \varphi_\alpha(x))\alpha^\vee$$

where $\varphi_\alpha(x) \in \mathbb{R}$ and $\alpha^\vee \in E$ is the coroot of α. By the uniqueness of $m_\alpha(x)$ one gets $m_\alpha(x^{-1}) = m_\alpha(x)^{-1}$ and $m_{w\alpha}(nxn^{-1}) = n\, m_\alpha(x) n^{-1}$ if $n \in N(K)$ represents $w \in {}^vW$. Consequently $t := \varphi_{w\alpha}(nxn^{-1}) - \varphi_\alpha(x)$ is independent of x. If we choose $n = m_\alpha(u), u \in U_\alpha$, then we get $\varphi_{-\alpha}(m(u)xm(u)^{-1}) - \varphi_\alpha(x) = 2\varphi_\alpha(u)$. We put $\varphi_\alpha(1) = \infty$.

Example: If $G = SL_n(K)$ and $\alpha = \varphi_i - \varphi_j$ then $U_\alpha = \{1 + te_{ij}/t \in K\}$. We get $\varphi_\alpha(t) = v(t)$.

We introduce $U_{\alpha,r} = \{x \in U_\alpha(K)/\varphi_\alpha(x) \geq r\}$, $r \in \mathbb{R}$.

6.3. Proposition. *We use the above notation. Then*

(i) $U_{\alpha,r}$ *is a subgroup of* $U_\alpha(K)$.

(ii) *If* $\alpha, \beta \in R$ *and* $\alpha \notin -\mathbb{R}^\bullet \beta$ *then the commutator* $[U_{\alpha,r}, U_{\beta,s}]$ *is contained in the subgroup generated by* $U_{p\alpha+q\beta, pr+qs}$ *where* p, q *are integers* ≥ 1.

The proof in the general case is difficult, see [Br-T 4]. For simply connected split groups the proof follows from what we have observed in §5 and standard properties of Chevalley groups, see [Br-T 3: 6.1.3 b)].

6.4. For $\alpha \in R$ and $2\alpha \notin R$ put $U_{2\alpha} = \{1\}$ and for $2\alpha \in R$ write $U_{2\alpha} = U_{2\alpha}(K)$. Then we have a filtration $U_{\alpha,r}U_{2\alpha}$ of U_α, $\alpha \in R$, $r \in \mathbb{R}$. Let J_α be the discrete set of $r \in \mathbb{R}$ such that the filtration jumps i.e.

$$U_{\alpha,r} \not\subset \bigcup_{s>r} U_{\alpha,s}U_{2\alpha}.$$

Then (α, r) is called an *affine root.* We view (α, r) as affine map $a : E \longrightarrow \mathbb{R}$ given by $a(v) = \langle \alpha, v \rangle - r$. The corresponding reflection s_a then is given by

$$s_a(v) = v - a(v)\alpha^\vee.$$

We write $U_a = U_{\alpha,r}$. The group generated by the s_a for affine roots $a = (\alpha, r)$ in the space of affine bijections of E is denoted by W and is called *the affine Weyl group.* This is the affine Weyl group of some root system (see §4) which can differ from R. We note that for $n \in N(S)(K)$ we get $nU_a n^{-1} = U_{\nu(n)a}$. For simply connected split groups we get as set of affine root R_{aff} as described in §5 and W is the affine Weyl group attached to R. In general

W is different from $\nu(N(S)(K)) =: \tilde{W}$. If G is simply connected then $W = \nu(N(S)(K))$.

6.5. Let $A = A(S)$. The affine Weyl group acts as Coxeter group on A and A together with the arising Coxeter complex is called an *apartment* of G. For $b \in A$ let P_b be the group generated by $H := Z(K)_c$ and all U_a where $a(b) \geq 0$. Then P_b is an open and compact subgroup of $G(K)$. Let $N := \nu^{-1}(W) \subset N(S)(K)$. Then $N/H \xrightarrow{\sim} W$.

We consider the equivalence relation on $G(K) \times A$ given by:
$(g,x) \sim (h,y)$ if there is an $n \in N$ such that $y = \nu(n)x$ and $g^{-1}hn \in P_x$. For a proof that this is an equivalence relation, see [B-T 3: 7.4.1.].

6.6. Definition: – We denote by $\mathcal{T}$ the quotient of $G \times A$ by the above equivalence relation. $\mathcal{T} = B\mathcal{T}(G,K)$ is called the *Bruhat-Tits building* of G over the local field K.

6.7. We observe the following, see [Br-T 3: 7.4]

- The action of $G(K)$ is compatible with the equivalence relation and defines an action of $G(K)$ on $\mathcal{T}$.
- The natural map $1 \times A \longrightarrow \mathcal{T}$ is injective. We identify A with its image in $\mathcal{T}$. Then $\mathcal{T} = \bigcup_{g \in G(K)} gA$. The sets gA are called apartments of $\mathcal{T}$.
- The natural simplicial structure of A given by W induces a simplical structure on $\mathcal{T}$.
- The group N is the stabilizer of A in G and H is the subgroup of G fixing A.
- If A' and A'' are apartments there is an element $g \in G(K)$ such that $g(A') = A''$ and such that g fixes $A' \cap A''$ pointwise. Moreover $A' \cap A''$ is the union of closed facets of A' (and of A'').

6.8. On an apartment A we can choose a ${}^{v}W$-invariant positive scalar product which can be normalized by the Killing form or by the prescription of the length of short roots. The corresponding distance d extends to a map $d : \mathcal{T} \times \mathcal{T} \longrightarrow \mathbb{R}$. Here one uses the last

observation of 6.7, see [Br-T 3: 7.4.8]. One has the following result [Br-T 3: 7.4.20]:

6.9. Proposition.

i) *$d : \mathcal{T} \times \mathcal{T} \longrightarrow \mathbb{R}$ is $G(K)$ invariant and gives $\mathcal{T}$ the structure of a locally compact metric space.*

ii) *If $x, y \in \mathcal{T}$ then there is an apartment A containing x and y. The "geodesic" $[x,y] = \{tx + (1-t)y/t \in [0,1]\}$ is contained in every apartment containing x and y. In particular if $g \in G(K)$ fixes x and y then g fixes all of* [x,y].

iii) *The map $[0,1] \times \mathcal{T} \times \mathcal{T} \longrightarrow \mathcal{T}$ with $(t,x,y) \longrightarrow tx + (1-t)y$ is continuous. In particular $\mathcal{T}$ is contractible.*

6.10. Remark. If G is semi simple then $\mathcal{T}$ is constructed by the same procedure where an apartment A is replaced by a product of apartments A_i which are given by the simple factors of the root system for G. Then $\mathcal{T}$ is a poly-simplicial complex. If G is reductive the apartment A has an additional factor $X_*(S_1) \otimes \mathbb{R}$ where S_1 is the maximal central split torus of G.

§7. Some properties of buildings

There is a combinatorial approach to buildings starting with a Tits system (G,B,N,S). This will be explained now.

The root groups U_α together with the affine map φ_α of §6 give rise to a Tits system and the building associated to this system is the building constructed in §6.

7.1. A *Tits system* is a quadruple (G,B,N,S) where G is a group, B and N are subgroups and S is a subset of $N/B \cap N$ such that the following hold:

i) $B \cup N$ generates G and $B \cap N$ is a normal subgroup of N

ii) S generates $W := N/B \cap N$ and S consists of elements of order 2

iii) for all $s \in S$ and $w \in W$ one has

$$sBw \subset BwB \cup BswB$$

iv) for all $s \in S$ one has $sBs \neq B$.

A Tits system (G, B, N, S) is called *affine* if W is a euclidean Coxeter group.

Example: $G = SL_2(\mathbb{Q}_p)$, $B = \left\{ \begin{pmatrix} \mathcal{O} & \mathcal{O} \\ \mathcal{P} & \mathcal{O} \end{pmatrix} \in SL_2(\mathcal{O}) \right\}$, $N = \left\{ \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} \in G \right\} \cup \left\{ \begin{pmatrix} 0 & b \\ c & 0 \end{pmatrix} \in G \right\}$, $H = \left\{ \begin{pmatrix} \mathcal{O} & 0 \\ 0 & \mathcal{O} \end{pmatrix} \in G \right\}$ $N = s_1 H \cup s_2 H$ with $s_1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ $s_2 = \begin{pmatrix} 0 & p \\ -p^{-1} & 0 \end{pmatrix}$. Here W is the infinite dihedral group.

7.2. Let (G, B, N, S) be a Tits system. If X is a subset of W let W_X be the subgroup of W generated by X. Put $B_X := \bigcup_{w \in W_X} BwB$. Then B_X is called a *parahoric* subgroup of G of type X. The map $X \longmapsto B_X$ gives a bijection between the subsets of S with the set of subgroups of G containing B. The groups B_X are self normalizing in G. A subgroup of G which is is conjugate to some B_X is called a *parahoric subgroup of type* X.

7.3. Let (G, B, N, S) be an irreducible Tits system which means that W is an irreducible Coxeter group where $\ell + 1 = |S| < \infty$. We denote by $\Delta = \Delta(G, B, N, S)$ the simplical complex whose simplices correspond bijectiveley to the set of parabolic subgroups $\sigma \longleftrightarrow P_\sigma$ where $\dim \sigma = |type(P_\sigma)| - 1$ and $\sigma \supset \tau \Longleftrightarrow P_\sigma \subset P_\tau$. We identify Δ with its topological realisation. Let C the simplex corresponding to the minimal parabolic subgroup B with closure $\overline{C}$. Then $\bigcup_{w \in W} w\overline{C} = A$ is called an apartment of Δ. The group G acts on Δ. Since N acts on A and $H := B \cap N$ fixes A pointwise we get an action of $W = N/N \cap B$ on A. If the apartment A together with the W–action is an euclidean Coxeter complex then Δ is called an *affine building.*

7.4. Let $G = G(K)$ be an almost simple connected group over a local field K and denote by R_{aff} the affine root system with Weyl group $W = N/H$, $N \subset N(S)(K)$, associated to G as is §6. Let $B = P_C$ be the subgroup of $G(K)$ generated by Z_c and the U_a with $a(C) \geq 0, a \in R_{aff}$. Let $S \in W$ be the set of affine reflections at the walls of C.

7.5. Proposition. *Let G be simply connected. Then $(G(K), B, N(K), S)$ is a Tits system. The associated building coincides with the building constructed in § 6.*

For $SL_n = G$ there is an elementary proof in [Br]. A direct proof of the first claim in 7.5 without the theory of buildings was given by Hijikata [Hi]. For split groups see also [I–M]. The general case is handled in [Br–T 3: Thm. 6.5], [R 1: Thm. 10.18]. There the results are extended to non simply connected groups.

If G is simply connected the subgroups of the form $BW_X B$ and their conjugates are called *parahoric subgroups.* The group B and its conjugates are called *Iwahori subgroups.*

For $SL_n(K)$ we have open and compact subgroups $P_{v_i} = P_i$ corresponding to the vertices $v_0, ..., v_{n-1}$ of a chamber C, and the P_i are stabilizers of lattices L_i in K^n. Since $GL_n(K)$ acts transitively on lattices in K^n and since a maximal compact subgroup of $SL_n(K)$ is a stabilizer of some lattice we see that the P_i are maximal compact subgroups.

Let [L] be a lattice class represented by the lattice L. If $g \in GL_n(K)$ we can attach to [gL] the well defined number $v(\det g) \bmod n$. Hence to L_i as above we attach the number $i \bmod n$. We see that the P_i represent all $SL_n(K)$-conjugacy classes of maximal compact subgroups of $SL_n(K)$. This generalizes as follows:

7.6. Proposition. *Let G/K be simply connected and allmost simple with K-rank l. Then there are $l+1$ $G(K)$- conjugacy classes of maximal compact subgroups of $G(K)$. If C is a chamber in $\mathcal{T}$ with vertices $v_0, ..., v_l$ then the P_{v_i} are representatives of all the conjugacy classes.*

If G is not simply connected then $G(K)$ acts on the simply connected covering G_{sc} and by transport of structure also on the building $\mathcal{T}$ for G_{sc}. The maximal compact subgroups of $G(K)$ then occur as normalizers of certain parahoric subgroups for $G_{sc}(K)$. Note that in general the action of G_{ad} on $\mathcal{T} = \mathcal{T}(G_{sc})$ is not simplicial.

7.8. The group $G(K)$ admits several decompositions some of which can be viewed as analogs of well known decompositions in the theory of real Lie groups see [He]. To formulate the decompositions we introduce some notation. For this we use the notions which have been introduced in §6.

We choose a special vertex $v = 0$ in $E = \mathrm{Hom}(X^*(S), \mathbb{R})$ and a chamber C with $v \in \overline{C}$. We then have ${}^vW = W_0 \subset W$ and ${}^vW$ is the vectorial Weyl group of a vectorial root system ${}^vR$ with set of positive ${}^vR^+$ roots (given by the choice of C). Let $U = U(K)$ be the subgroup of $G(K)$ generated by the U_α, $\alpha \in {}^vR^+$. Let $\tilde{C} = \mathbb{R}_+^* C$ be the cone in E generated by C. Put $Y = \nu(Z(K)) \subset E$ and $Y_+ = Y \cap \tilde{C}$ and $Z(K)_+ = \nu^{-1}(Y_+)$. Let $B \subset G(K)$ be the group fixing C pointwise and let $G(K)^0$ be the subgroup of $G(K)$ fixing $0 = v$. Note that in the simply connected case B is an Iwahori subgroup and $G(K)^0$ is a special maximal subgroup. Then the following hold:

7.9. Proposition. Let $G, B, G(K)^0, Y, Y_+$ be as above. Then one has

(i) *Bruhat decomposition: — $G(K) = BN(K)B$ and the map $bnB \longmapsto \nu(n)$ establishes a bijection $B\backslash G(K)/B \xrightarrow{\sim} \hat{W}$*

(ii) *Iwasawa decomposition: — $G(K) = G(K)^0 \cdot Z(K)U(K)$ and the map $G(K)^0 Z U(K) \longmapsto \nu(Z)$ establishes a bijection $G(K)^0\backslash G(K)/U(K) \xrightarrow{\sim} Y$*

(iii) *Cartan decomposition: — $G(K) = G(K)^0 Z(K)_+ G(K)^0$ and the map $KzK \longmapsto \nu(z)$ establishes a bijection*

$$G(K)^0\backslash G(K)/G(K)^0 \xrightarrow{\sim} Y_+ .$$

For proofs and generalisations of there results see [Br–T 3: §4]. Now that (i) follows for simply connected group from 7.5.

7.10. If we compare what has been explained for the building $\mathcal{T}$ of G — say G simply connected — with the theory for SL_2 we see that the following still is missing:

— the description of the simplicial structure of a neighbourhood (= link) of a facett F of a chamber C in $\mathcal{T}$

— a generalization of the notion "*ends*"

For the first point, see [Br] and [R 1]. The description of the links requires (as in the SL_2-case) the notion of certain group-schemes over $\mathcal{O}$ and their reduction mod $\mathcal{P}$, see [Br–T 4]. The second point is dealt with in [T 1] and for the connection with affine buildings, see [R 1: §9] and [B–S].

7.11. We have not touched any application of the theory of buildings. For this the reader should consult the survey articles [T 2], [R 2], [R S] and further literature mentioned there. In particular the theory of buildings, leads to a new proof of the classification of semi simple groups over p-adic fields, which was achieved first by Kneser using a case by case analysis, see [K], [Br-T 2], [T 3].

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