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"Planetary Boundary Layer"

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Please note: These are preliminary notes intended for internal distribution only.

Introduction to Boundary Layer Meteorology

Stull

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$$p = \rho_{air} \Re T_v$$

$$\frac{\partial \rho}{\partial t} + \frac{\partial (\rho U_j)}{\partial x_j} = 0$$

$$\frac{d\rho}{dt} + \rho \frac{\partial U_j}{\partial x_j} = 0$$

$$\frac{\partial U_j}{\partial x_j} = 0$$

$$\frac{\partial U_i}{\partial t} + U_j \frac{\partial U_i}{\partial x_j} = - \delta_{i3}\, g - 2\, \varepsilon_{ijk}\, \Omega_j\, U_k - \frac{1}{\rho} \frac{\partial p}{\partial x_i} + \frac{1}{\rho} \frac{\partial \tau_{ij}}{\partial x_j}$$

$$\frac{\partial q_T}{\partial t} + U_j \frac{\partial q_T}{\partial x_j} = \nu_q \frac{\partial^2 q}{\partial x_j^2} + \frac{S_{q_T}}{\rho_{air}}$$

$$\frac{\partial q}{\partial t} + U_j \frac{\partial q}{\partial x_j} = \nu_q \frac{\partial^2 q}{\partial x_j^2} + \frac{S_q}{\rho_{air}} + \frac{E}{\rho_{air}}$$

$$\frac{\partial q_L}{\partial t} + U_j \frac{\partial q_L}{\partial x_j} = + \frac{S_{q_L}}{\rho_{air}} - \frac{E}{\rho_{air}}$$

$$\frac{\partial \theta}{\partial t} + U_j \frac{\partial \theta}{\partial x_j} = \nu_\theta \frac{\partial^2 \theta}{\partial x_j^2} - \frac{1}{\rho\, C_p} \left(\frac{\partial Q_j^*}{\partial x_j} \right) - \frac{L_p\, E}{\rho\, C_p}$$

$$\frac{\partial C}{\partial t} + U_j \frac{\partial C}{\partial x_j} = \nu_C \frac{\partial^2 C}{\partial x_j^2} + S_C$$

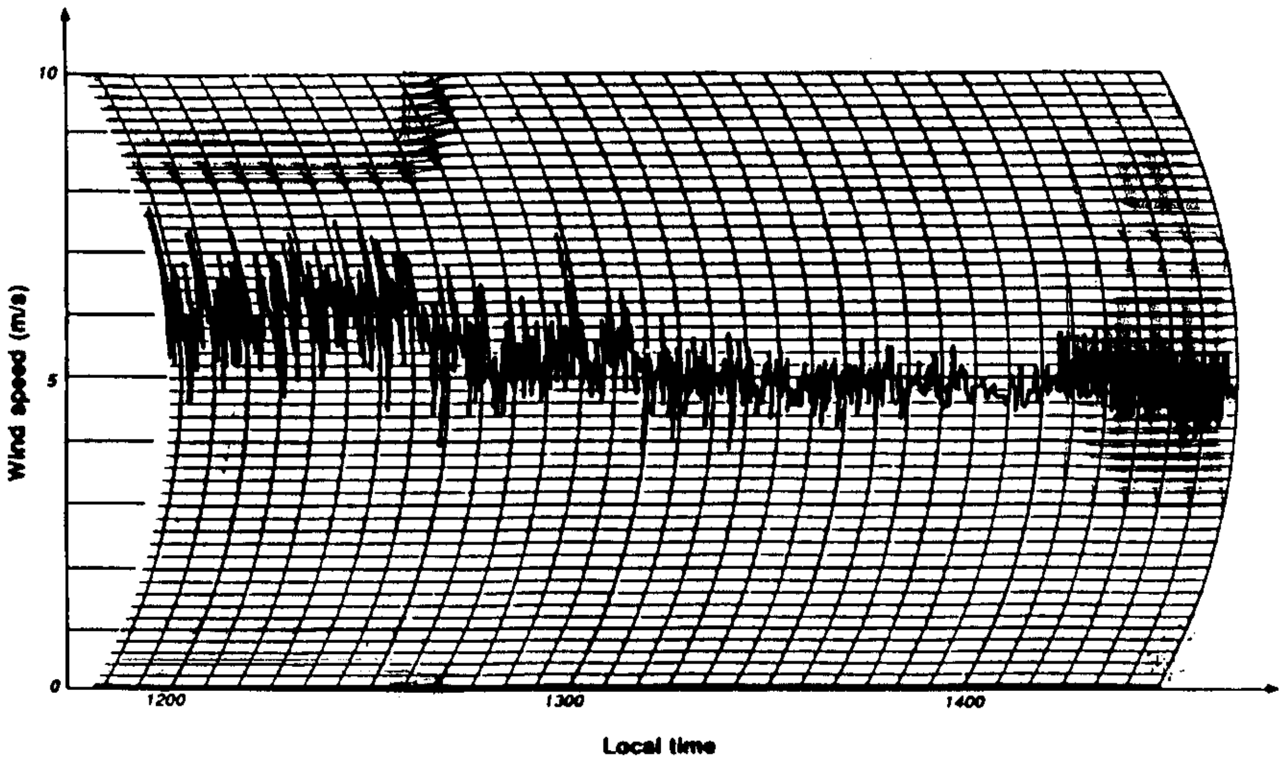


Fig. 2.1 Trace of wind speed obseved in early afternoon.

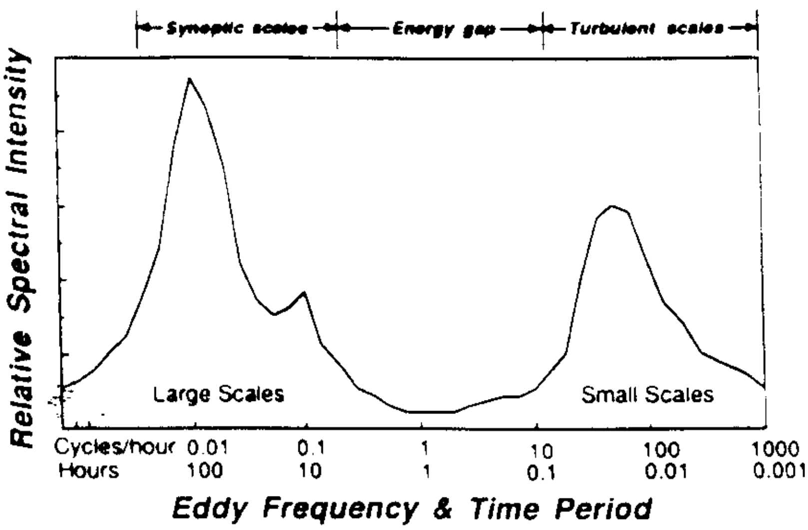


Fig. 2.2 Schematic spectrum of wind speed near the ground estimated from a study of Van der Hoven (1957).

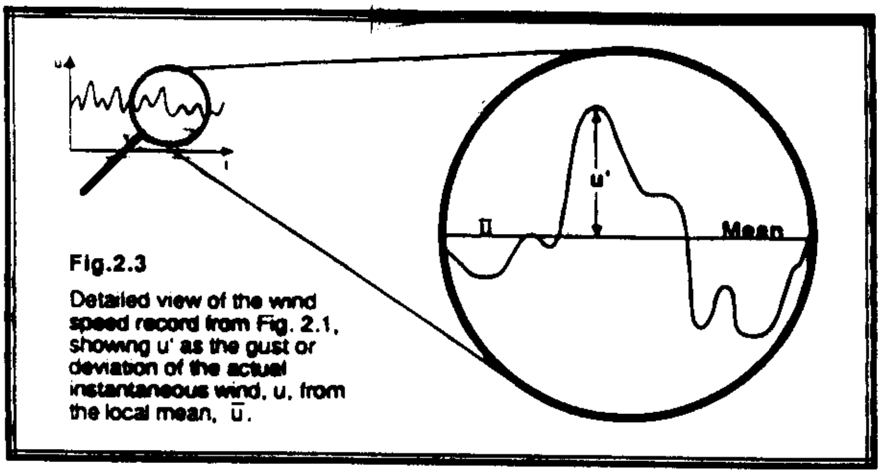


Fig.2.3

Detailed view of the wind speed record from Fig. 2.1, showing u' as the gust or deviation of the actual instantaneous wind, u, from the local mean, $\bar{u}$.

$$U = \bar{U} + u'$$

$$V = \bar{V} + v'$$

$$W = \bar{W} + w'$$

$$\theta_v = \bar{\theta}_v + \theta_v'$$

$$q = \bar{q} + q'$$

$$c = \bar{c} + c'$$

$$\frac{\bar{P}}{\Re} = \bar{\rho}\,\bar{T}_v$$

$$\frac{\partial \bar{U}_j}{\partial x_j} = 0$$

$$\frac{\partial \bar{U}}{\partial t} + \bar{U}_j \frac{\partial \bar{U}}{\partial x_j} = -f_c(\bar{V}_g - \bar{V}) - \frac{\partial(\overline{u_j'u'})}{\partial x_j}$$

$$\frac{\partial \bar{V}}{\partial t} + \bar{U}_j \frac{\partial \bar{V}}{\partial x_j} = +f_c(\bar{U}_g - \bar{U}) - \frac{\partial(\overline{u_j'v'})}{\partial x_j}$$

$$\frac{\partial \bar{q}_T}{\partial t} + \bar{U}_j \frac{\partial \bar{q}_T}{\partial x_j} = +S_{qT}/\bar{\rho}_{air} - \frac{\partial(\overline{u_j'q_T'})}{\partial x_j}$$

$$\frac{\partial \bar{\theta}}{\partial t} + \bar{U}_j \frac{\partial \bar{\theta}}{\partial x_j} = -\frac{1}{\bar{\rho}\,C_p}\left[L_v E + \frac{\partial \bar{Q}_j^*}{\partial x_j}\right] - \frac{\partial(\overline{u_j'\theta'})}{\partial x_j}$$

$$\frac{\partial \bar{C}}{\partial t} + \bar{U}_j \frac{\partial \bar{C}}{\partial x_j} = +S_c - \frac{\partial(\overline{u_j'c'})}{\partial x_j}$$

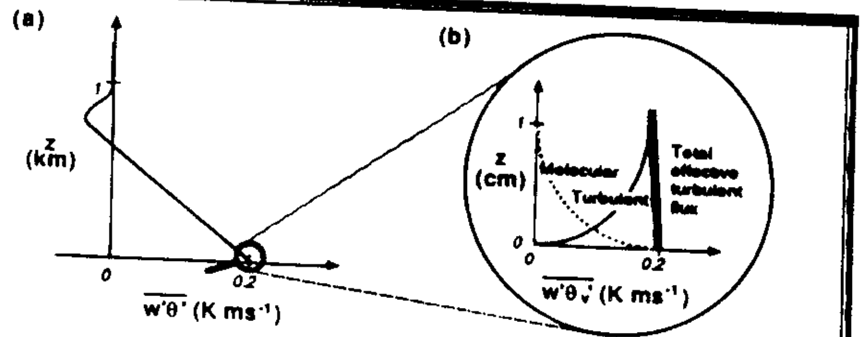


Fig. 7.1 (a) The effective turbulent heat flux using daytime convective conditions as an example, may be nonzero at the surface. (b) This effective flux, however, is the sum of the actual turbulent flux and the molecular flux.

$$
\underbrace{\frac{\partial\left(\overline{u_i'u_k'}\right)}{\partial t}}_{\text{I}} + \underbrace{\frac{\overline{U}_j\,\partial\left(\overline{u_i'u_k'}\right)}{\partial x_j}}_{\text{II}} = \underbrace{-\left(\overline{u_i'u_j'}\right)\frac{\partial \overline{U}_k}{\partial x_j}}_{\text{III}} \underbrace{-\left(\overline{u_k'u_j'}\right)\frac{\partial \overline{U}_i}{\partial x_j}}_{\text{III}} \underbrace{-\frac{\partial\left(\overline{u_i'u_j'u_k'}\right)}{\partial x_j}}_{\text{IV}}
$$

$$
+\left(\frac{g}{\overline{\theta}_v}\right)\left[\underbrace{\delta_{k3}\,\overline{u_i'\theta_v'}}_{\text{V}} + \underbrace{\delta_{i3}\,\overline{u_k'\theta_v'}}_{\text{V}}\right] + f_c\left[\underbrace{\epsilon_{kj3}\,\overline{u_i'u_j'}}_{\text{VI}} + \underbrace{\epsilon_{ij3}\,\overline{u_k'u_j'}}_{\text{VI}}\right]
$$

$$
-\frac{1}{\overline{\rho}}\left[\underbrace{\frac{\partial\left(\overline{p'u_k'}\right)}{\partial x_i}}_{\text{VII}} + \underbrace{\frac{\partial\left(\overline{p'u_i'}\right)}{\partial x_k}}_{\text{VII}} \underbrace{-\,\overline{p'\left(\frac{\partial u_i'}{\partial x_k}+\frac{\partial u_k'}{\partial x_i}\right)}}_{\text{VIII}}\right] + \underbrace{\frac{\nu\,\partial^2\left(\overline{u_i'u_k'}\right)}{\partial x_j^2}}_{\text{IX}} - \underbrace{\frac{2\nu\,\overline{\partial u_i'\,\partial u_k'}}{\partial x_j^{\,2}}}_{\text{X}}
$$

$$
\frac{\partial \overline{e}}{\partial t} + \overline{U}_j\frac{\partial \overline{e}}{\partial x_j} = +\,\delta_{i3}\frac{g}{\overline{\theta}_v}\left(\overline{u_i'\theta_v'}\right) - \overline{u_i'u_j'}\,\frac{\partial \overline{U}_i}{\partial x_j} - \frac{\partial\left(\overline{u_j'e}\right)}{\partial x_j} - \frac{1}{\overline{\rho}}\frac{\partial\left(\overline{u_i'p'}\right)}{\partial x_i} - \epsilon
$$

Fig. 5.3
The lines show modeled vertical profiles of turbulence kinetic energy, $\overline{e}$, during Day 33, Wangara, (after Therry and Lacarrere, 1983). The shaded profile applies when both shears and buoyancy are active (after Hechtel, 1988).

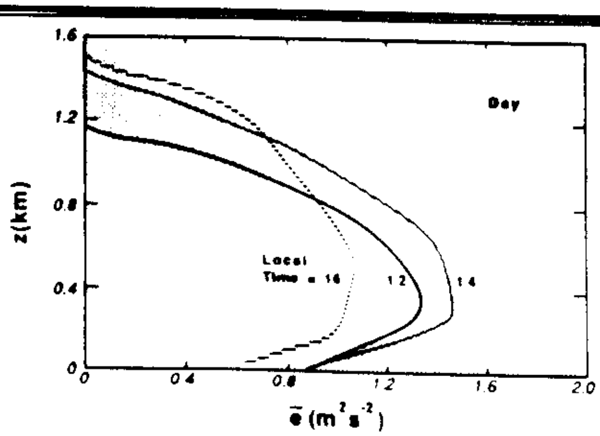
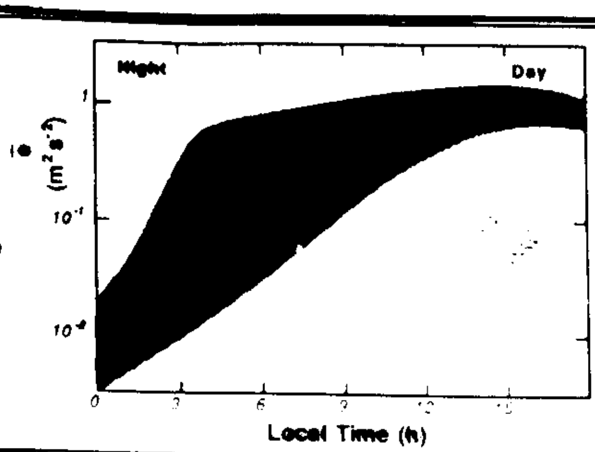


Fig. 5.2
Sample diurnal variation of observed range of turbulence kinetic energy, $\overline{e}$, in the surface layer during November. After Louis, et al. (1983).



$$\frac{\partial\left(\overline{q'u_i'}\right)}{\partial t} + \overline{U}_j \frac{\partial\left(\overline{q'u_i'}\right)}{\partial x_j} = -\overline{q'u_j'}\frac{\partial \overline{U}_i}{\partial x_j} - \overline{u_i'u_j'}\frac{\partial \overline{q}}{\partial x_j} - \frac{\partial\left(\overline{q'u_j'u_i'}\right)}{\partial x_j}$$

I II III XI IV

$$+ \delta_{i3}\left(\frac{\overline{q'\theta_v'}}{\overline{\theta}_v}\right) g + f_c \epsilon_{ij3}\left(\overline{u_j'q'}\right) - \left(\frac{1}{\overline{\rho}}\right)\left[\frac{\partial\left(\overline{p'q'}\right)}{\partial x_i} - \overline{p'\frac{\partial q'}{\partial x_i}}\right]$$

V VI VII VIII

$$+ \frac{\nu \partial^2\left(\overline{q'u_i'}\right)}{\partial x_j^2} - 2\nu\,\overline{\left(\frac{\partial u_i'}{\partial x_j}\right)\left(\frac{\partial q'}{\partial x_j}\right)}$$

IX X

$$\frac{\partial\left(\overline{\theta'u_i'}\right)}{\partial t} + \overline{U}_j \frac{\partial\left(\overline{\theta'u_i'}\right)}{\partial x_j} = -\overline{\theta'u_j'}\frac{\partial \overline{U}_i}{\partial x_j} - \overline{u_i'u_j'}\frac{\partial \overline{\theta}}{\partial x_j} - \frac{\partial\left(\overline{\theta'u_j'u_i'}\right)}{\partial x_j}$$

I II III XI IV

$$+ \delta_{i3}\left(\frac{\overline{\theta'\theta_v'}}{\overline{\theta}_v}\right) g + \left(\frac{1}{\overline{\rho}}\right)\left[\overline{\frac{p'\partial\theta'}{\partial x_i}}\right] - 2\epsilon_{u_i\theta}$$

V VIII X

$$\frac{\partial\left(\overline{c'u_i'}\right)}{\partial t} + \overline{U}_j \frac{\partial\left(\overline{c'u_i'}\right)}{\partial x_j} = -\overline{c'u_j'}\frac{\partial \overline{U}_i}{\partial x_j} - \overline{u_i'u_j'}\frac{\partial \overline{C}}{\partial x_j} - \frac{\partial\left(\overline{c'u_j'u_i'}\right)}{\partial x_j}$$

I II III XI IV

$$+\delta_{i3}\left(\frac{\overline{c'\theta_v'}}{\overline{\theta}_v}\right) g + f_c\epsilon_{ij3}\left(\overline{u_j'c'}\right) - \left(\frac{1}{\overline{\rho}}\right)\left[\frac{\partial\left(\overline{p'c'}\right)}{\partial x_i} - \overline{\frac{p'\partial c'}{\partial x_i}}\right] + \frac{\nu\partial^2\left(\overline{c'u_i'}\right)}{\partial x_j^2} - 2\nu\,\overline{\left(\frac{\partial u_i'}{\partial x_j}\right)\left(\frac{\partial c'}{\partial x_j}\right)}$$

V VI VII VII IX X

Table 6-1. Simplified example showing a tally of equations and unknowns for various statistical moments of momentum, demonstrating the closure problem for turbulent flow. The full set of equations includes even more unknowns.

Prognostic Eq. for:	Moment	Equation	Number of Eqs.	Number of Unknowns
$\overline{U_i}$	First	$\dfrac{\partial \overline{U_i}}{\partial t} = \ldots - \dfrac{\partial \overline{u_i' u_j'}}{\partial x_j}$	3	6
$\overline{u_i' u_j'}$	Second	$\dfrac{\partial \overline{u_i' u_j'}}{\partial t} = \ldots - \dfrac{\partial \overline{u_i' u_j' u_k'}}{\partial x_k}$	6	10
$\overline{u_i' u_j' u_k'}$	Third	$\dfrac{\partial \overline{u_i' u_j' u_k'}}{\partial t} = \ldots - \dfrac{\partial \overline{u_i' u_j' u_k' u_m'}}{\partial x_m}$	10	15

Table 6-2. Correlation triangles indicating the unknowns for various levels of turbulence closure, for the momentum equations only. Notice the pattern in these triangles, with the u, v, and w statistics at their respective vertices, and the cross correlations in between.

Order of Closure	Correlation Triangle of Unknowns
Zero	$\overline{U}$ $\overline{V}$ $\overline{W}$
First	$\overline{u'^2}$ $\overline{u'v'}$ $\overline{u'w'}$ $\overline{v'^2}$ $\overline{v'w'}$ $\overline{w'^2}$
Second	$\overline{u'^3}$ $\overline{u'^2 v'}$ $\overline{u'^2 w'}$ $\overline{u' v'^2}$ $\overline{u' v' w'}$ $\overline{u' w'^2}$ $\overline{v'^3}$ $\overline{v'^2 w'}$ $\overline{v' w'^2}$ $\overline{w'^3}$

Local Closure — First Order

$$\frac{\partial \overline{U}}{\partial t} = f_c(\overline{V} - \overline{V}_g) - \frac{\partial (\overline{u'w'})}{\partial z}$$

$$\frac{\partial \overline{V}}{\partial t} = -f_c(\overline{U} - \overline{U}_g) - \frac{\partial (\overline{v'w'})}{\partial z}$$

$$\frac{\partial \overline{\theta}}{\partial t} = \qquad - \frac{\partial (\overline{w'\theta'})}{\partial z}$$

$$\overline{u_j'\xi'} = -K \frac{\partial \overline{\xi}}{\partial x_j}$$

Local Closure — One-and-a-half Order

$$\frac{\partial \overline{U}}{\partial t} = f_c\left(\overline{V} - \overline{V}_g\right) - \frac{\partial (\overline{u'w'})}{\partial z}$$

$$\frac{\partial \overline{V}}{\partial t} = -f_c\left(\overline{U} - \overline{U}_g\right) - \frac{\partial (\overline{v'w'})}{\partial z}$$

$$\frac{\partial \overline{\theta}}{\partial t} = \qquad - \frac{\partial (\overline{w'\theta'})}{\partial z}$$

$$\frac{\partial \bar{e}}{\partial t} = -\overline{u'w'}\frac{\partial \overline{U}}{\partial z} - \overline{v'w'}\frac{\partial \overline{V}}{\partial z} + \left(\frac{g}{\overline{\theta}}\right)\overline{w'\theta'} - \frac{\partial\left[\overline{w'((p'/\rho) + e)}\right]}{\partial z} - \varepsilon$$

$$\frac{\partial\left(\overline{\theta'^2}\right)}{\partial t} = -2\,\overline{w'\theta'}\frac{\partial \overline{\theta}}{\partial z} - \frac{\partial (\overline{w'\theta'^2})}{\partial z} - 2\varepsilon_\theta - \varepsilon_R$$

$$\overline{u'w'} = -K_m\left(\bar{e}, \overline{\theta'^2}\right)\frac{\partial \overline{U}}{\partial z}$$

$$\overline{v'w'} = -K_m\left(\bar{e}, \overline{\theta'^2}\right)\frac{\partial \overline{V}}{\partial z}$$

$$\overline{w'\theta'} = -K_H\left(\bar{e}, \overline{\theta'^2}\right)\frac{\partial \overline{\theta}}{\partial z} - \Upsilon_c\left(\bar{e}, \overline{\theta'^2}\right)$$

$$\overline{w'\left[(p'/\bar{\rho}) + e\right]} = \left(\frac{5}{3}\right)\Lambda_4 e^{-1/2}\frac{\partial \bar{e}}{\partial z}$$

$$\overline{w'\theta'^2} = \Lambda_3 e^{-1/2}\frac{\partial \overline{\theta'^2}}{\partial z}$$

$$\varepsilon_R = 0 \qquad \varepsilon = \frac{\bar{e}^{3/2}}{\Lambda_1} \qquad \varepsilon_\theta = \frac{\bar{e}^{1/2}\,\overline{\theta'^2}}{\Lambda_2}$$

$$K = \Lambda\,\bar{e}^{1/2}$$

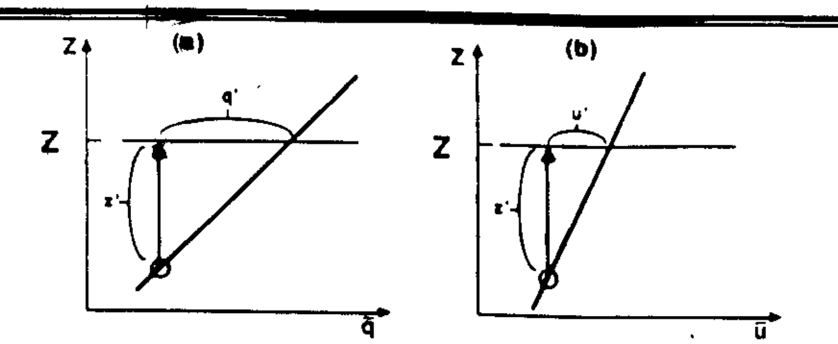


Fig. 6.1 Movement of an air parcel (shaded line) within a background having linear moisture and wind gradients (heavy line). The superposition of many such parcels, starting at different levels but all ending at level Z, forms the conceptual basis for "mixing length theory."

$$q' = -\left(\frac{\partial \bar{q}}{\partial z}\right) z'$$

$$u' = -\left(\frac{\partial \bar{U}}{\partial z}\right) z'$$

$$w' = c \left| \frac{\partial \bar{U}}{\partial z} \right| z'$$

$$R = -c\,\overline{(z')^2} \left| \frac{\partial \bar{U}}{\partial z} \right| \cdot \left(\frac{\partial \bar{q}}{\partial z} \right)$$

$$R = -l^2 \left| \frac{\partial \bar{U}}{\partial z} \right| \cdot \left(\frac{\partial \bar{q}}{\partial z} \right)$$

$$K_E = l^2 \left| \frac{\partial \bar{U}}{\partial z} \right|$$

$$K_E = k^2 z^2 \left| \frac{\partial \bar{U}}{\partial z} \right|$$

$$\frac{1}{l} = \frac{1}{k\,z} + \frac{1}{0.0004\,G\,f_c^{-1}} + \frac{\beta}{k\,L_L}$$

$$R_f = \frac{\left(\frac{g}{\overline{\theta_v}}\right)\left(\overline{w'\theta_v'}\right)}{\left(\overline{u_i'u_j'}\right)\frac{\partial \overline{U_i}}{\partial x_j}}$$

$$\overline{w'\theta'} = K\frac{\partial\overline{\theta}}{\partial z}$$

$$\overline{u_i'u_j'} = K\frac{\partial\overline{u}}{\partial z}$$

$$Ri = \frac{\frac{g}{\overline{\theta_v}}\frac{\partial\overline{\theta_v}}{\partial z}}{\left[\left(\frac{\partial\overline{U}}{\partial z}\right)^2 + \left(\frac{\partial\overline{V}}{\partial z}\right)^2\right]}$$

$$\ldots = -\frac{k\,z\,g\,(\overline{w'\theta_v'})_s}{\overline{\theta_v}\,u_*^3} + \frac{k\,z\,(\overline{u_i'u_j'})_s}{u_*^3}\frac{\partial\overline{U_i}}{\partial x_j} + \ldots - \frac{k\,z\,\epsilon|_s}{u_*^3}$$

$$\zeta = \frac{z}{L} = \frac{-k\,z\,g\,(\overline{w'\theta_v'})_s}{\overline{\theta_v}\,u_*^3}$$

$$L = \frac{-\overline{\theta_v}\,u_*^3}{k\,g\,(\overline{w'\theta_v'})_s}$$

Obukhov

$$\overline{w''u''} = -K_m \frac{\partial \bar{u}}{\partial z} = -u_*^2 \cos\mu$$

$$\overline{w''v''} = -K_m \frac{\partial \bar{v}}{\partial z} = -u_*^2 \sin\mu,$$

$$\arctan(\bar{v}/\bar{u}) = \mu \quad \text{and} \quad \bar{\rho} u_*^2 = \tau.$$

$$\bar{V} = (\bar{u}^2 + \bar{v}^2)^{1/2}$$

$$u_*^2 = \sqrt{\overline{u'w'}^2 + \overline{v'w'}^2}$$

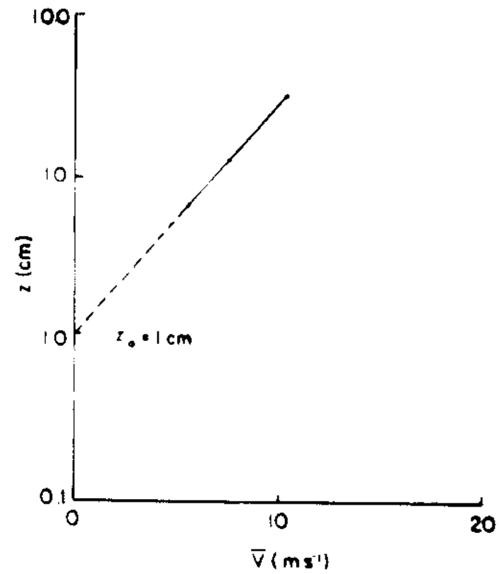
$$K_m \partial \bar{V}/\partial z = u_*^2.$$

$$K_m = kzu_*,$$

$$\partial \bar{V}/\partial z = u_*/kz.$$

$$\int_{z_0}^{z} \frac{\partial \bar{V}}{\partial z} dz = \bar{V}(z) = \int_{z_0}^{z} \frac{u_*}{kz} dz$$

$$= \frac{u_*}{k} \int_{z_0}^{z} \frac{dz}{z} = \frac{u_*}{k} \ln \frac{z}{z_0}.$$



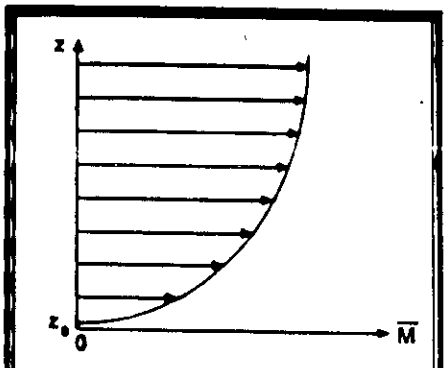


Fig. 9.4 Typical logarithmic variation of wind speed with height in the neutral surface layer.

$$z_0 = 0.032 u_*^2/g,$$

$$z_0 = (0.016 u_*^2/g) + \nu/(9.1 u_*),$$

$$z_0 = 0.5 h A^*/A'$$

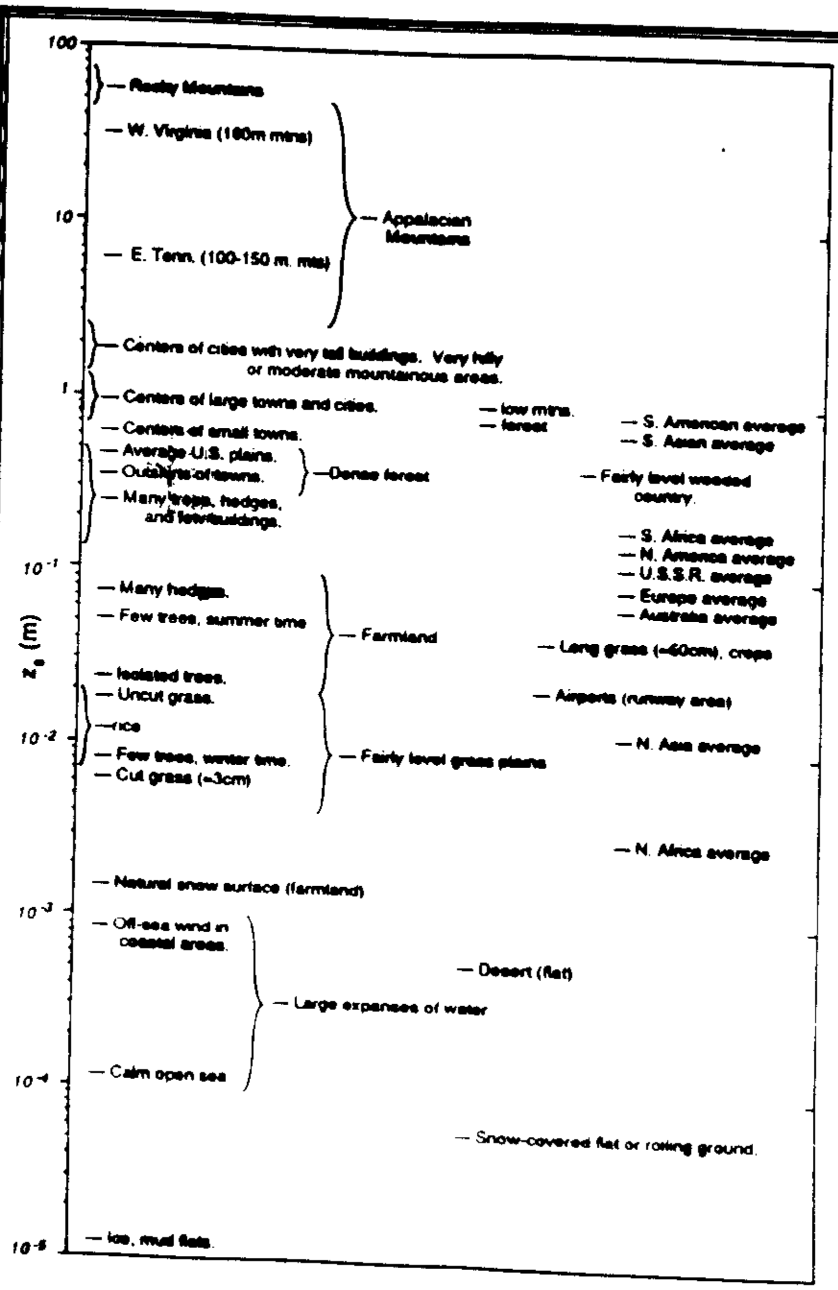
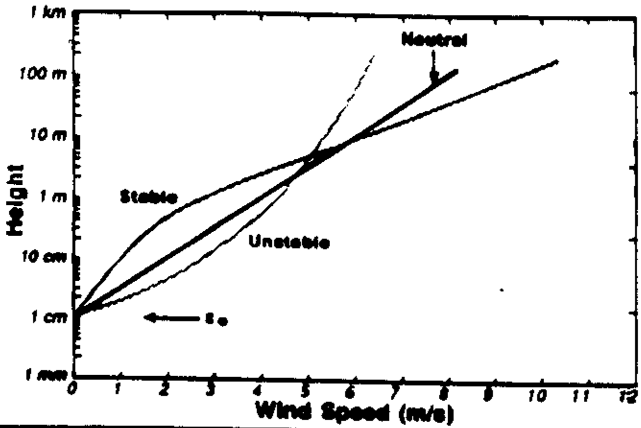


Fig. 9.6 Aerodynamic roughness lengths for typical terrain types. (After Garratt 1977, Smedman-Högström & Högström 1978, Kondo & Yamazawa 1986, Thompson 1978, Napo 1977, and Hicks 1975).

Fig. 9.5 Typical wind speed profiles vs. static stability in the surface layer.



$$\phi_M = \frac{kz}{u_*}\frac{\partial \bar{V}}{\partial z}.$$

$$R_f \phi_M = -g\overline{w''\theta''}\, kz/\theta_0 u_*^3 = z/L.$$

$$\frac{kz}{u_*}\frac{\partial \bar{V}}{\partial z} = 1 - (1 - \phi_M).$$

$$\frac{\partial \bar{V}}{\partial z} = \frac{u_*}{kz} - \frac{(1 - \phi_M)}{kz} u_*.$$

$$\bar{V} = \frac{u_*}{k} \ln \frac{z}{z_0} - \frac{u_*}{k} \int_{z_0/L}^{z/L} (1 - \phi_M)\, d \ln \frac{z}{L},$$

$$\bar{V}(z) = \frac{u_*}{k}\left[\ln \frac{z}{z_0} - \psi_M\left(\frac{z}{L}\right)\right],$$

$$\psi_M = \int_0^{z/L} \frac{(1 - \phi_M)}{z/L}\, d\left(\frac{z}{L}\right)$$

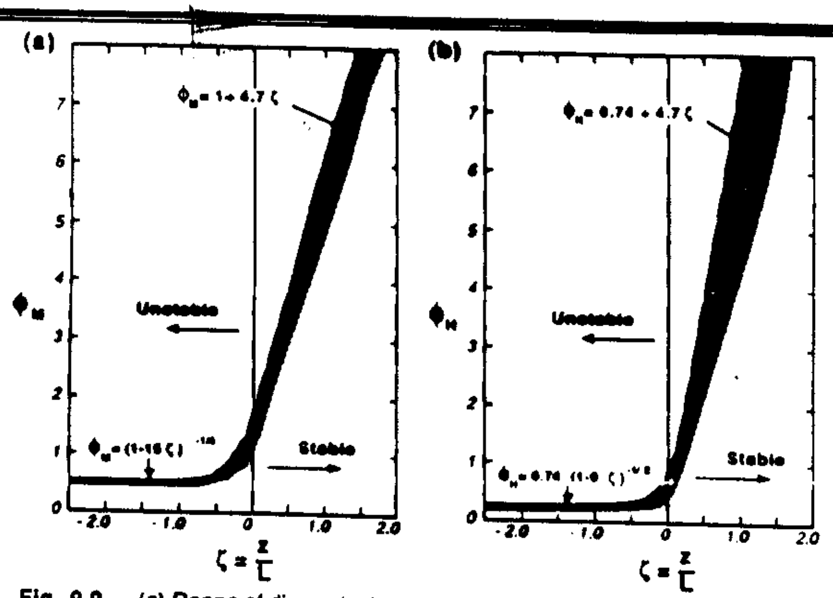


Fig. 9.9 (a) Range of dimensionless wind shear observations in the surface layer, plotted with interpolation formulas. (b) Range of dimensionless temperature gradient observations in the surface layer, plotted with interpolation formulas. After Businger, et al. (1971).

$$\psi_M(z/L) = \begin{cases} 2\ln[(1+\phi_M^{-1})/2] + \ln[(1+\phi_M^{-2})/2] - 2\tan^{-1}\phi_M^{-1} + \pi/2, & z/L \le 0 \\ -4.7z/L, & z/L > 0 \end{cases}$$

$$\psi_H(z/L) = \begin{cases} 2\ln[(1+0.74\phi_H^{-1})/2], & z/L \le 0 \\ -6.35z/L, & z/L > 0 \end{cases}$$

$$\phi_M = \frac{kz}{u_*}\frac{\partial \bar{V}}{\partial z} \simeq \begin{cases} (1-15z/L)^{-1/4} & z/L \le 0 \\ 1+4.7z/L & z/L \le 0 \end{cases}$$

$$\phi_H = \frac{kz}{\theta_*}\frac{\partial \bar{\theta}}{\partial z} = \frac{kz}{q_*}\frac{\partial \bar{q}_3}{\partial z} = \frac{kz}{\chi_{*m}}\frac{\partial \bar{\chi}_m}{\partial z} \simeq \begin{cases} 0.74(1-9z/L)^{-1/2} & z/L \le 0 \\ 0.74+4.7z/L & z/L > 0 \end{cases}$$

$\overline{w'\theta'} = K_h \frac{\partial \overline{\theta}}{\partial z}$

$=$

$$\frac{\beta k z}{\theta_*}\frac{\partial \bar{\theta}}{\partial z} = \frac{\beta k z}{q_{n_*}}\frac{\partial \bar{q}_n}{\partial z} = \frac{\beta k z}{\chi_{m_*}}\frac{\partial \bar{\chi}_m}{\partial z} = \beta \phi_H = \hat{\phi}_H,$$

$$\bar{\theta}(z) = \bar{\theta}(z_0) + \frac{\theta_*}{\beta k}\left[\ln\frac{z}{z_0} - \psi_H\left(\frac{z}{L}\right)\right]$$

$$\bar{q}_n(z) = \bar{q}_n(z_0) + \frac{q_{n_*}}{\beta k}\left[\ln\frac{z}{z_0} - \psi_H\left(\frac{z}{L}\right)\right]$$

$$\bar{\chi}_m(z) = \bar{\chi}_m(z_0) + \frac{\chi_{m_*}}{\beta k}\left[\ln\frac{z}{z_0} - \psi_H\left(\frac{z}{L}\right)\right].$$

$$\psi_H = \int_0^{z/L} \frac{1-\hat{\phi}_H}{z/L}\, d(z/L),$$

$$\bar{\theta}_{z_0} = \theta_G + 0.0962(\theta_*/k)(u_* z_0/\nu)^{0.45}$$

$$\bar{q}_{z_0} = q_G + 0.0962(q_*/k)(u_* z_0/\nu)^{0.45}.$$

$$\bar{\chi}_{z_0} = \chi_G + 0.0962(\chi_*/k)(u_* z_0/\nu)^{0.45}$$

$$u_* = k\bar{V}/[\ln(z/z_0) - \psi_M(z/L)],$$
$$\theta_* = k(\bar{\theta}(z) - \bar{\theta}_{z_0})/0.74[\ln(z/z_0) - \psi_H(z/L)],$$
$$q_* = k(\bar{q}_3(z) - \bar{q}_{z_n})/0.74[\ln(z/z_0) - \psi_H(z/L)],$$
$$\chi_{*m} = k(\bar{\chi}_m(z) - \bar{\chi}_{z_{0_m}})/0.74[\ln(z/z_0) - \psi_H(z/L)],$$

$$\overline{w''\theta''} = -K_\theta \frac{\partial \bar{\theta}}{\partial z} = -u_*\theta_*,$$

$$\overline{w''q_n''} = -K_q \frac{\partial \bar{q}_n}{\partial z} = -u_* q_{n_*},$$

$$\overline{w''\chi_m''} = -K_\chi \frac{\partial \bar{\chi}_m}{\partial z} = -u_*\chi_{m_*}.$$

$$K_m = \frac{k u_* z}{\phi_M}, \qquad K_\theta = \frac{\beta k u_* z}{\hat{\phi}_H}.$$

The vertical distribution of the surface fluxes within the lowest tropospheric levels requires an estimation of the height of the planetary boundary layer. The stable and unstable planetary boundary layers are treated differently, and the definition of the stability is based on the bulk Richardson's number.

Relevant references for this section may be found in Deardorff (1972), Smeda (1977) and the various reports on the planetary boundary layer published by the European Center for Medium Range Weather Forecast (ECMWF). In the unstable case, the Monin-Obukhov length, L, the bulk Richardson number, RiB, and the modified bulk Richardson number are negative. The surface heat flux is upward, i.e. $-w\theta > 0$ and $\theta^* > 0$. Let h denote the height of the planetary boundary layer. Air is unstable below h and is stable above it. Across the interface ($z = h$), two infinitesimal layers are defined. The upper and lower infinitesimal layers potential temperatures are denoted by θ^+ and θ^-, respectively. Let $\Delta\theta = \theta^+ - \theta^-$ and require that $\frac{\partial \Delta\theta}{\partial t} = 0$. This assures the finiteness of the interfacial gradient, i.e.,

$$\frac{\partial \theta^+}{\partial t} = \frac{\partial \theta^-}{\partial t} \tag{8.46}$$

In the well-mixed layer the governing equation for the change of potential temperature is

$$\frac{\partial \theta^-}{\partial t} = -\frac{\partial}{\partial z}\overline{w'\theta'} \tag{8.47}$$

It is assumed that the flux $\overline{w'\theta'}$ decreases linearly away from the surface, z_1, and vanishes at $z = h$. Thus,

$$\overline{w'\theta'}(z) = \overline{w'\theta'})z_1\,(h - z)/h$$
$$\frac{\partial}{\partial z}\overline{w'\theta'} = \frac{-\overline{w'\theta'})z_1}{h} \tag{8.48}$$

In the stable layer ($z \geq h$) the change of θ^+ arises due to the large scale convection and the changes in the interfacial height h. If w^+ is the large scale vertical velocity, then

$$\frac{\partial \theta^+}{\partial t} = -\left(w^+\frac{\partial \theta^+}{\partial z} - \frac{dh}{dt}\frac{\partial \theta^+}{\partial z}\right) \tag{8.49}$$

where $\frac{dh}{dt}$ is the rate of change of the height of the interface. Equating (8.47) and (8.49) and using (8.48) yields

$$\frac{dh}{dt} - w^+ = \frac{\overline{w'\theta'})z_1}{h\,\frac{\partial \theta^+}{\partial z}} \tag{8.50}$$

It should be noted that the heat flux at the lower boundary is obtained from the surface flux calculations for the unstable stratification (this was the similarity solution discussed earlier). $\frac{\partial \theta^+}{\partial z}$ is the stability above the planetary boundary layer and is a known quantity since it can be determined from large scale data. W^+ is the large scale vertical velocity at the top of the planetary boundary layer and is again defined over the large scale grid. Thus, the only unknown in the above equation is h as a function of time t, for a given height z. Since the horizontal advection of h is small, $\frac{dh}{dt}$ can be replaced by its local variation.

$$\frac{\partial h}{\partial t} = w^{+} + \frac{\overline{w'\theta'})z_1}{h\ \frac{\partial \theta^{+}}{\partial z}} \tag{8.51}$$

It can be shown that (8.51) has a solution of the form

$$h = t(w^{+} + \frac{\overline{(w'\theta')}z_1}{w^{+}\ \frac{\partial \theta^{+}}{\partial z}} \ln h + \frac{\overline{(w'\theta'})z_1}{w^{+}\ \frac{\partial \theta^{+}}{\partial z}} + Cw^{+} \tag{8.52}$$

where C is an integration constant. However, this solution is transcendental in h and has a limited value in this form. Since the effect of the large scale vertical velocity is smaller than that of the vertical heat flux in unstable conditions, a simplified solution for (8.51) may be obtained as

$$h^2 = \frac{\overline{(w'\theta'})z_1}{\frac{\partial \theta^{+}}{\partial z}} t + C \tag{8.53}$$

or

$$h = \sqrt{2at + h_o^2} \tag{8.54}$$

where $a = \overline{(w'\theta'})z_1 / (\partial\theta^{+}/\partial z)$ and h_o is an initial height of the planetary boundary layer. Thus, h increases slowly as the heat flux remains upward, and larger stability of the air above level h slows its growth. Separating the growth of h due to large scale vertical velocity from that due to the eddy heat flux, an approximate solution may be written in the form.

$$h = wt + \sqrt{2at + h_o^2} \tag{8.55}$$

In practical application this requires a knowledge of $h_o(x,y)$ at each time step.

5. Height of the Planetary Boundary Layer in a Stable Situation

The formulation of the stable case is a difficult problem since the rate of change of the height of the planetary boundary layer is not easily definable. No theories on the behavior of the so-called nocturnal boundary layer exist for large scale numerical weather prediction. Although several rate equations have appeared in recent literature, their applications have been of limited value. According to Deardorff (1972) the time rate of change of the height of the planetary boundary layer may be expressed by

$$\frac{\partial h}{\partial t} = \frac{2u_*^2}{\beta h^2 \Upsilon^2 + 7u_*^2} \left[1 - 10 \frac{fh^2}{u_*} \right] - Cu_* \frac{h}{L} \tag{8.56}$$

where β is the buoyancy parameter and L denotes the Monin Obukhov length. The parameter f is the Coriolis parameter and Υ represents the stability $\frac{\partial \theta^+}{\partial z}$ of a layer immediately above the planetary boundary layer h. C is an empirical constant whose value is around 5×10^{-3}. That is a best fit value based on ***WANGARA*** field experiment. The principal idea here is that the growth of the planetary boundary layer depends on the stress induced by the near surface wind. The last term provides a somewhat reasonable transition from an unstable to a stable planetary boundary layer.

Table 6-4. Examples of parameterizations for the eddy viscosity, K, in the boundary layer.

Neutral Surface Layer:

$K = \text{constant}$	not the best parameterization
$K = u_*^2 T_o$	where u_* is the friction velocity
$K = U^2 T_o$	where T_o is a timescale
$K = k z u_*$	where k is von Karman's constant
$K = k^2 z^2 [(\partial\bar{U}/\partial z)^2 + (\partial\bar{V}/\partial z)^2]^{1/2}$	from mixing-length theory
$K = l^2 (\partial\bar{U}/\partial z)^2$	where $l = k(z+z_o)/(1+[k(z+z_o)/\Lambda])$. Λ=length scale

Diabatic Surface Layer (generally, $K_{\text{statically unstable}} > K_{\text{neutral}} > K_{\text{statically stable}}$)

$K = k z u_* / \phi_M(z/L)$ where ϕ_M a dimensionless shear (see appendix A), and L is the Obukhov length (appendix A)

$K = k^2 z^2 [(\partial\bar{U}/\partial z) + \{(g/\bar{\theta}_v)\cdot|\partial\bar{\theta}_v/\partial z|\}^{1/2}]$ for statically unstable conditions

$K = k^2 z^2 [(\partial\bar{U}/\partial z) - (L_*/z)^{1/6}\{(15g/\bar{\theta}_v)\cdot|\partial\bar{\theta}_v/\partial z|\}^{1/2}]$ for statically stable conditions, where

$L_* = -\theta u_*^2/(15\, k\, g\, \theta_*)$

Neutral or Stable Boundary Layer

$K = \text{constant}$ see Ekman Spiral derivation in next subsection

$K = K(h) + [(h-z)/(h-z_{SL})]^2 \{K(z_{SL}) - K(h) + (z-z_{SL})[\partial K/\partial z|_{z_{SL}} + 2(K(z_{SL})-K(h))/(h-z_{SL})]\}$

this is known as the O'Brien cubic polynomial approximation (O'Brien, 1970), see Fig 6-2, where z_{SL} represents the surface layer depth.

Unstable (Convective) Boundary Layer:

$K = 1.1\, [(R_c - Ri)\, l^2 / Ri\,]\, |\partial\bar{U}/\partial z|$	for $\partial\bar{\theta}_v/\partial z > 0$	where $l = kz$ for $z < 200$ m and
$K = (1 - 18\, Ri)^{-1/2}\, l^2\, |\partial\bar{U}/\partial z|$	for $\partial\bar{\theta}_v/\partial z < 0$	$l = 70$ m for $z > 200$ m.

Numerical Model Approximation for Anelastic 3-D Flow:

$K = (0.25\,\Delta)^2 \cdot |0.5 \Sigma_i \Sigma_j [\partial\bar{U}_i/\partial x_j + \partial\bar{U}_j/\partial x_i - (2/3)\delta_{ij}\Sigma_k(\partial\bar{U}_k/\partial x_k)]^2|^{1/2}$ where Δ=grid size

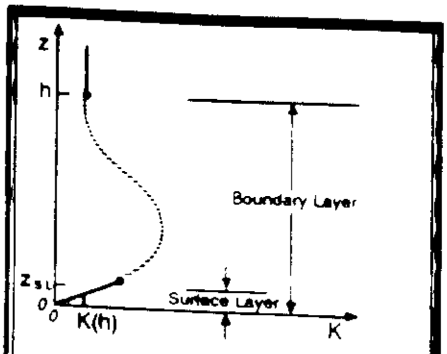
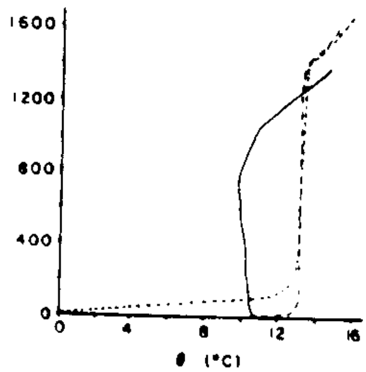
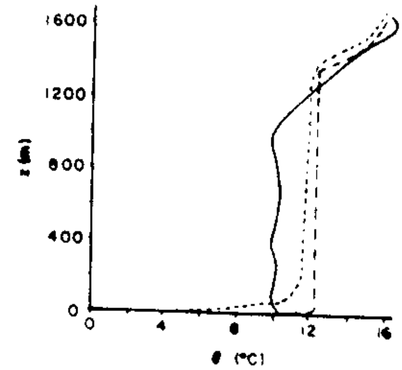
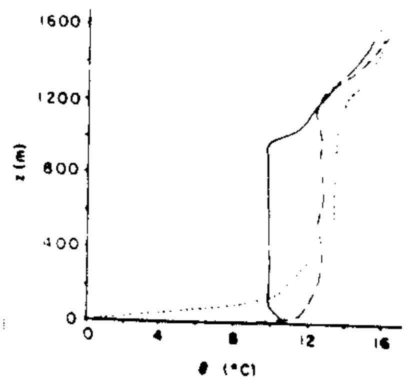
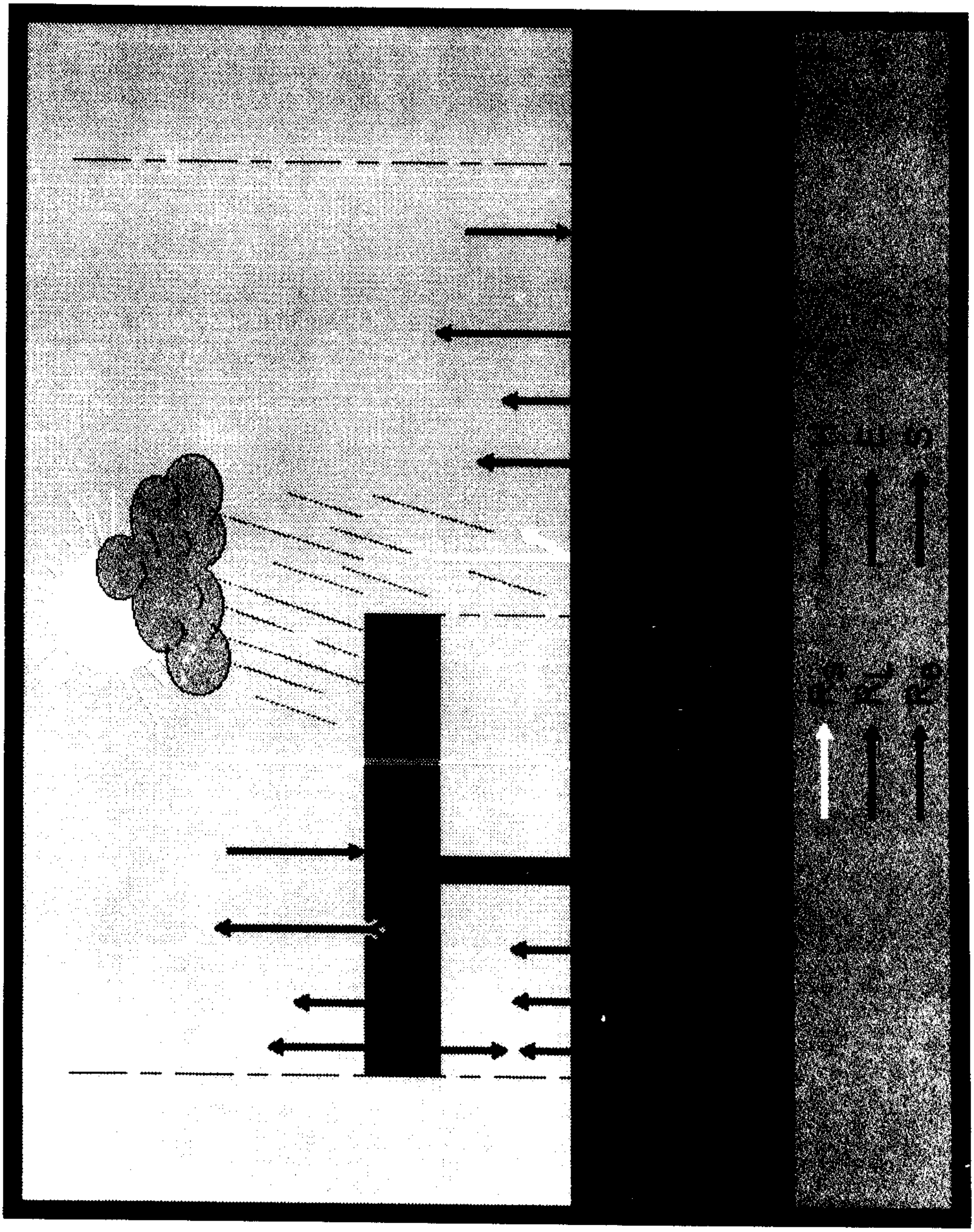


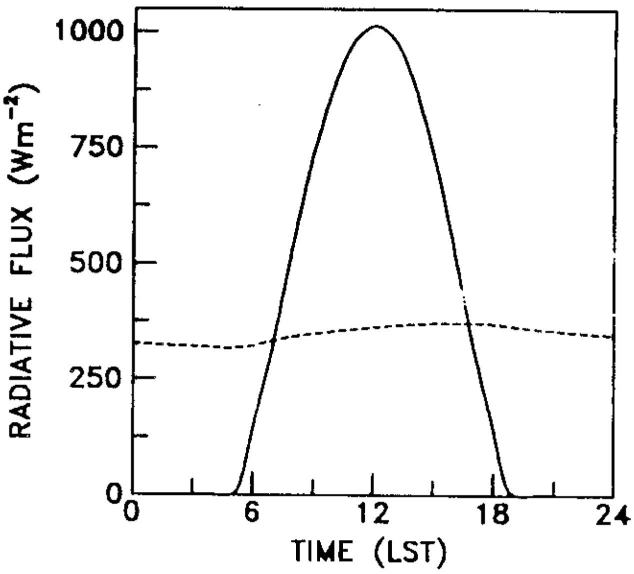
Fig. 6.2 Typical variation of eddy viscosity, K, with height in the boundary layer. After O'Brien (1970).


1600
1200
800
400
0
z (m)
0
4
8
12
16
θ (°C)
(a)

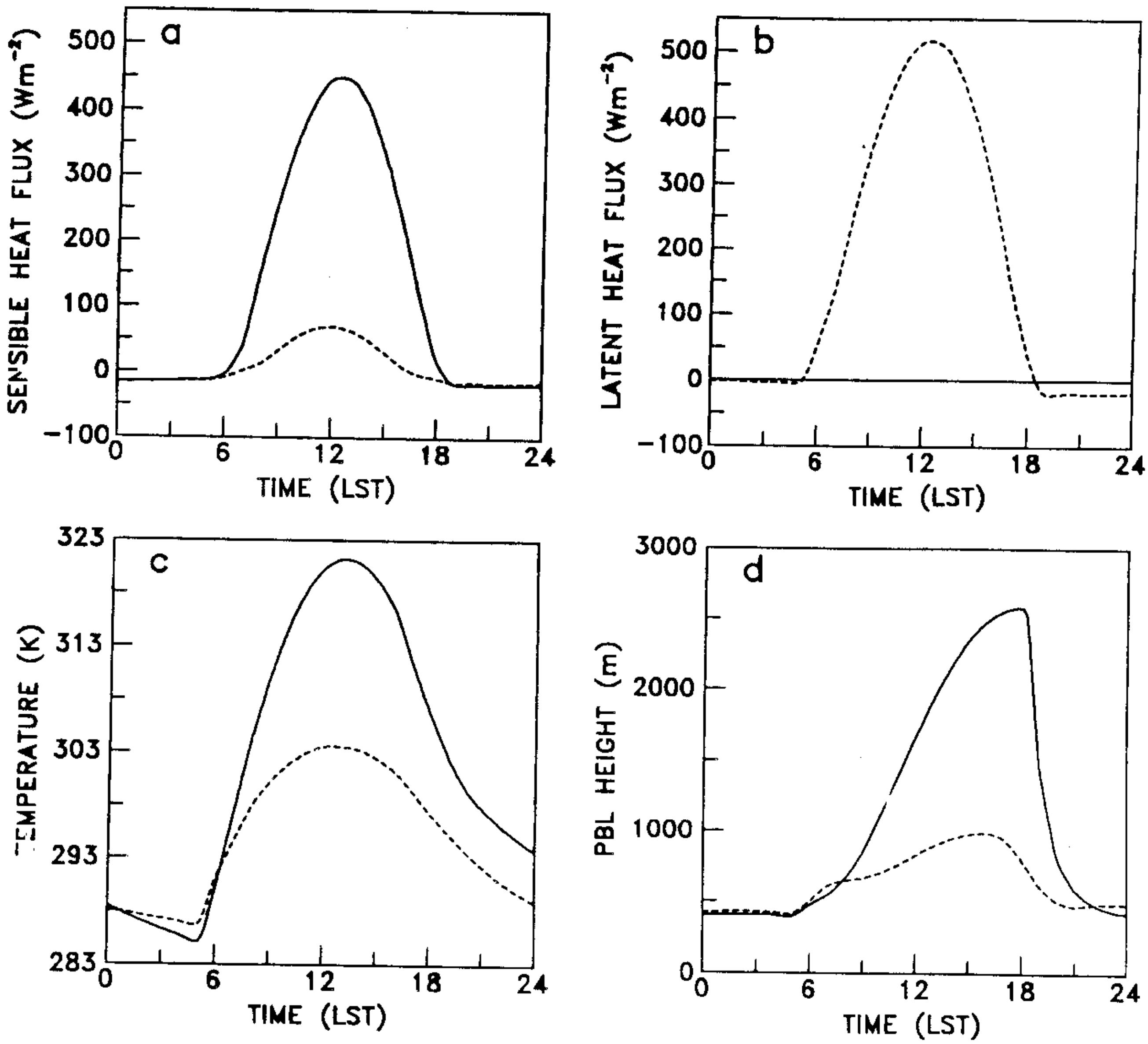

1600
1200
800
400
0
z (m)
0
4
8
12
16
θ (°C)
(b)


1600
1200
800
400
0
z (m)
0
4
8
12
16
θ (°C)
(c)

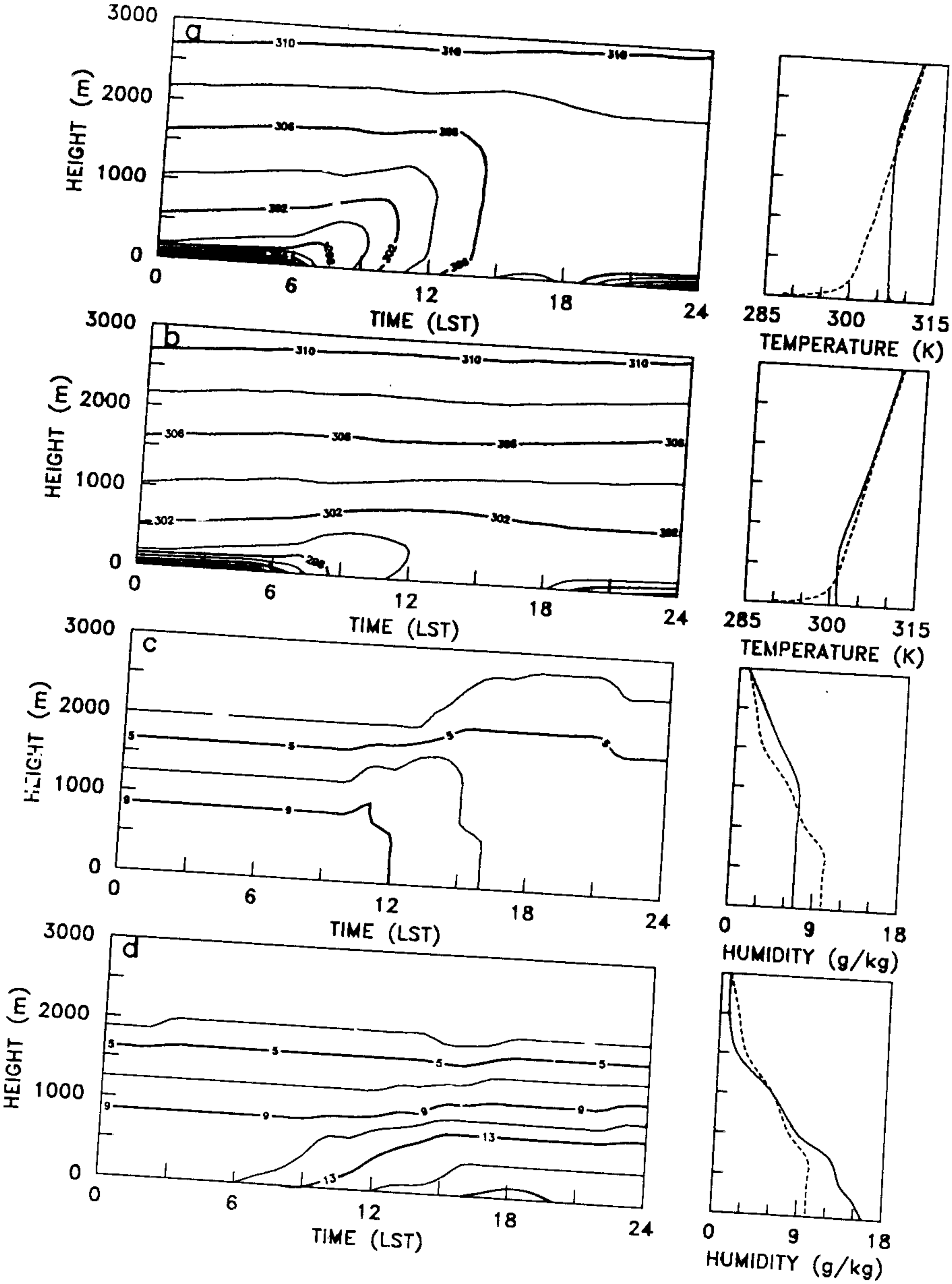


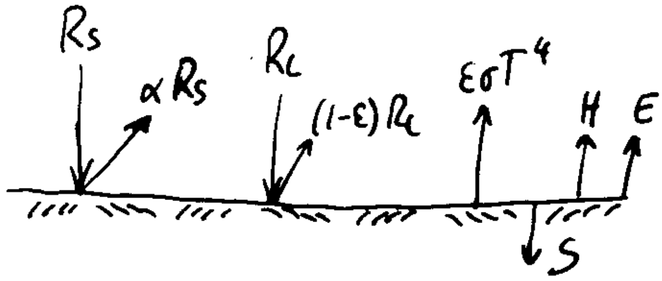


1000
750
500
250
0
RADIATIVE FLUX (Wm^{-2})
0
6
12
18
24
TIME (LST)



a
SENSIBLE HEAT FLUX (Wm⁻²)
TIME (LST)
b
LATENT HEAT FLUX (Wm⁻²)
TIME (LST)
c
TEMPERATURE (K)
TIME (LST)
d
PBL HEIGHT (m)
TIME (LST)


a
b
c
d
3000
2000
1000
0
HEIGHT (m)
0
6
12
18
24
TIME (LST)
310
306
302
5
9
13
285
300
315
TEMPERATURE (K)
0
9
18
HUMIDITY (g/kg)



$$(1-\alpha) R_s + \varepsilon R_L - \varepsilon \sigma T^4 + \lambda\left(\frac{T_s - T}{\Delta z}\right) + \rho c_p u_* \theta_* + \rho L u_* q_* = 0$$

$$u_* = \frac{u(z)\, k}{\ln\left(\frac{z-d}{z_0}\right)} ; \quad \theta_* = \frac{k\,(\theta_a - \theta_{z0})}{\ln\left(\frac{z-d}{z_0}\right)} ; \quad q_* = \frac{k\,(q_a - q_{z0})}{\ln\left(\frac{z-d}{z_0}\right)}$$

$$\gamma = k / \ln\left(\frac{z-d}{z_0}\right)$$

$$\theta = T \quad (\text{i.e., } P_{atm} = 100{,}000 \text{ Pa})$$

$$\theta_{z0} = T + 0.0962 \, \frac{\theta_*}{k} \left(\frac{u_* z_0}{\nu}\right)^{.45}$$

$$q_{z0} = q + 0.0962 \, \frac{q_*}{k} \left(\frac{u_* z_0}{\nu}\right)^{.45}$$

$$f = 0.0962 \, \frac{1}{k} \left(\frac{u_* z_0}{\nu}\right)^{.45}$$

$$\rightarrow \theta_{z0} = T + f\left[\gamma\,(T_a - \theta_{z0})\right] = T + f\gamma T_a - f\gamma\theta_{z0}$$

$$\rightarrow \theta_{z0} = \frac{T + f\gamma T_a}{1 + f\gamma} ; \quad q_{z0} = \frac{q + f\gamma q_a}{1 + f\gamma}$$

$$(1-\alpha) R_s + \varepsilon R_L - \varepsilon\sigma T^4 + \lambda\left(\frac{T_s - T}{\Delta z}\right) + \rho c_p \gamma^2 u(z)\left[T_a - (T + f\gamma T_a)/(1+f\gamma)\right]$$
$$+ \rho L \gamma^2 u(z)\left[q_a - (q + f\gamma q_a)/(1 + f\gamma)\right] = 0$$

$$(1-\alpha) R_s + \varepsilon R_L - \varepsilon\sigma \textcircled{T}^4 + \lambda\left(\frac{T_s - \textcircled{T}}{\Delta z}\right) +$$
$$\rho c_p \gamma^2 u(z)\left[T_a\left(1 - \frac{f\gamma}{1+f\gamma}\right) - \textcircled{T}\left(\frac{1}{1+f\gamma}\right)\right] + \rho L \gamma^2 u(z)\left[q_a\left(1 - \frac{f\gamma}{1+f\gamma}\right) - \textcircled{q}\left(\frac{1}{1+f\gamma}\right)\right] = 0$$

$$q = \eta \; q_s$$

$$\eta = \exp\left[\frac{-g \, |\psi|}{R_v \; T}\right]$$

$$q_s = \frac{0.622 \, e_s}{P_{atm} - .378 \, e_s}$$

$$e_s = 610.78 \exp\left[\frac{17.269 \, (T - 273.15)}{T - 35.85}\right]$$

1) $$a T^4 + b T + c e^{f(T)} + d = 0$$

$$a = -\varepsilon \sigma$$

$$b = -\lambda / \Delta z - \rho c_p \gamma^2 u(z) / (1 + f\gamma)$$

$$c = -\rho L \gamma^2 u(z) / (1 + f\gamma)$$

$$d = (1-\alpha) R_s + \varepsilon R_L + \lambda T_s / \Delta z + \rho c_p \gamma^2 u(z) T_a \left(1 - \frac{f\gamma}{1+f\gamma}\right) + \rho L \gamma^2 u(z) q_a \left(1 - \frac{f\gamma}{1+f\gamma}\right)$$

5) Newton - Raphson solution

```
    T = 300.
    n = 0
10  e = 610.78 * EXP(17.269*(T-273.15)/(T-35.85))
    q = eta * .622 * e/(patm - .378*e)
    IF q < qa THEN q = qa
    ff = a*T^4 + b*T + c*q + d
    ffp = 4.*a*T^3 + b + c*q*4097.934/(T-35.85)^2
    TNEW = T - ff/ffp
    PRINT TNEW
    IF ABS(TNEW - T) < .01 GOTO 20
    T = TNEW
    n = n + 1
    IF n < 25 GOTO 10
    PRINT "no convergence for T !!!"
    STOP
0   RE = a*T^4
```

$$T_{z0} = (T + f\gamma T_a)/(1 + f\gamma)$$

$$q_{z0} = (q + f\gamma q_a)/(1 + f\gamma)$$

$$H = \rho c_p \gamma u(z)(T_a - T_{z0})$$

$$E = \rho L \gamma u(z)(q_a - q_{z0})$$

$$S = \lambda\left(\frac{T_s - T}{\Delta z}\right)$$

6) Constants in MKS

$\rho = 1.2$

$C_p = 1013$

$L = 2\,500\,000$

$g = 9.81$

$R_v = 461$

$k = 0.4$

$\phi = 65$

$P_{atm} = 100\,000$

$\sigma = 5.67\ 10^{-8}$

$\nu = 0.15 \times 10^{-4}$

7) Vapor pressure from T_D and T_W

$$e = e_s - \phi\,(T_D - T_W)$$

8) Additional values

$0.8 < \lambda < 2$

$10^{-2} < z_0 < 1$

$-10^{80} < \psi < 0$

$0.05 < \alpha < 0.8$

$0.8 < \varepsilon < 1.0$