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"An Accurate & Stable Scheme to Advection Equation for Weather Prediction Model"

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An Accurate and Stable Scheme to Advection Equation for Weather Prediction Model

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OUTLINE

- Advection Equations
- Numerical Methods
- Theoretical Predictions of Accuracy and Stability
- Numerical Results for Some Test Examples
- Applications

DEFINITION

- Advection Equation (1D)

$$\frac{\partial C}{\partial t} + u\frac{\partial C}{\partial x} = 0$$

- Advection Equation (2D)

$$\frac{\partial C}{\partial t} + u\frac{\partial C}{\partial x} + v\frac{\partial C}{\partial y} = 0$$

- Advection Equation (3D)

$$\frac{\partial C}{\partial t} + u\frac{\partial C}{\partial x} + v\frac{\partial C}{\partial y} + w\frac{\partial C}{\partial z} = 0$$

NUMERICAL METHODS

Temporal Differencing (Two level schemes)

$$C^* = C^n - u\frac{\partial C^n}{\partial x}\Delta t$$
$$C^{n+1} = C^n - u\frac{\partial(\alpha C^n + \beta C^*)}{\partial x}\Delta t$$
$$(\alpha + \beta = 1)$$

- **Euler FT**: $\alpha = 1, \beta = 0$, (1st-order)
- **Matsuno scheme**: $\alpha = 0, \beta = 1$, (1st-order)
- **Heun scheme**: $\alpha = \beta = 1/2$, (2nd-order)
- **Miller-Pearce scheme**: Combination (2nd-order)

NUMERICAL METHODS

Spatial Differencing

- **Upwinding difference**
- **Central differencing**

Time + Space Schemes

- **Standard schemes**: 1st-order upwind method; Lax-Wendroff method; Leap-Frog method; ...
- **Advanced schemes**: Wave Propagation Schemes; Semi-Lagrangian Schemes; ...

NUMERICAL METHODS

Heun Scheme +
Higher-Odd-Order upwinding

Motivations:

- Higher resolution with little computational cost
- 2nd-order accuracy in time, sufficiently stable
- Dissipation mechanism in upwinding scheme help to filter out spurious $2\Delta x$ waves
- Good amplitude properties and little phase error
- Vector supercomputer adaptable

STABILITY AND ACCURACY

$$A = 1 - C_{\Re}^2/2 + iC_{\Re}$$
$$|A| = (1 + \frac{C_{\Re}^4}{4})^{1/2}$$
$$P = \frac{1}{C_{\Re}} \tan^{-1}[\frac{C_{\Re}}{1 - C_{\Re}^2/2}]$$

- A — amplification factor
- P — relative phase error
- $C_{\Re} = u\frac{\Delta t}{\Delta x}$ — Courant number

$C_{\Re}$	$\|A\|$	P
0.1	1.00017	1.0004
0.3	1.0146	1.0001
0.5	1.0383	1.0008
0.7	1.0682	1.0029

Table 1: Amplification factors and relative phase errors for Heun scheme

TESTING

- #1: **Smooth initial data + constant coefficient + periodic boundary conditions**

$$C(x, y, 0) = \sin(2\pi x)\sin(2\pi y)$$

- #2: **Solid body rotation**

$$u = -(y - 1/2), v = (x - 1/2),$$
$$C(x, y, 0) = \frac{1}{4}(1 + \cos(\pi\rho(x, y)))$$

where

$$\rho(x, y) = \min(\sqrt{(x - x_0)^2 + (y - y_0)^2}, \rho_0)/\rho_0.$$

$(x_0 = 0.5, y_0 = 1.25, \rho_0 = 0.2)$

- #3: **Slotted disk + cone + hump**
- #4: **Swirling flow**

APPLICATIONS

- **Case #1**: Tropical cyclone evolution
- **Case #2**: Atmospheric dust transport

CONCLUSIONS

- Works well for both smooth and discontinuous data.
- ADVECTION-diffusion more suitable.
- Boundary condition non-sensitive.
- Computationally robust and effective.
- Parallel computing adaptable.
- Simplicity and flexibility.

Overall, the algorithm performs quite well for the test problems and is a very good scheme for the weather prediction model.

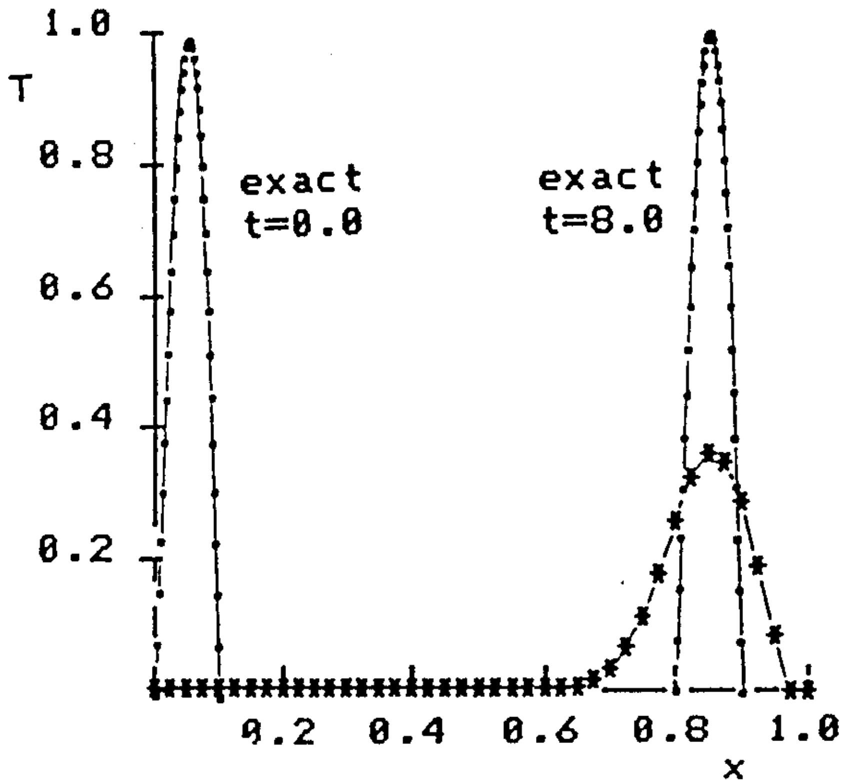


Fig. 9.2. Upwind solution for the convection equation with $C=0.8$

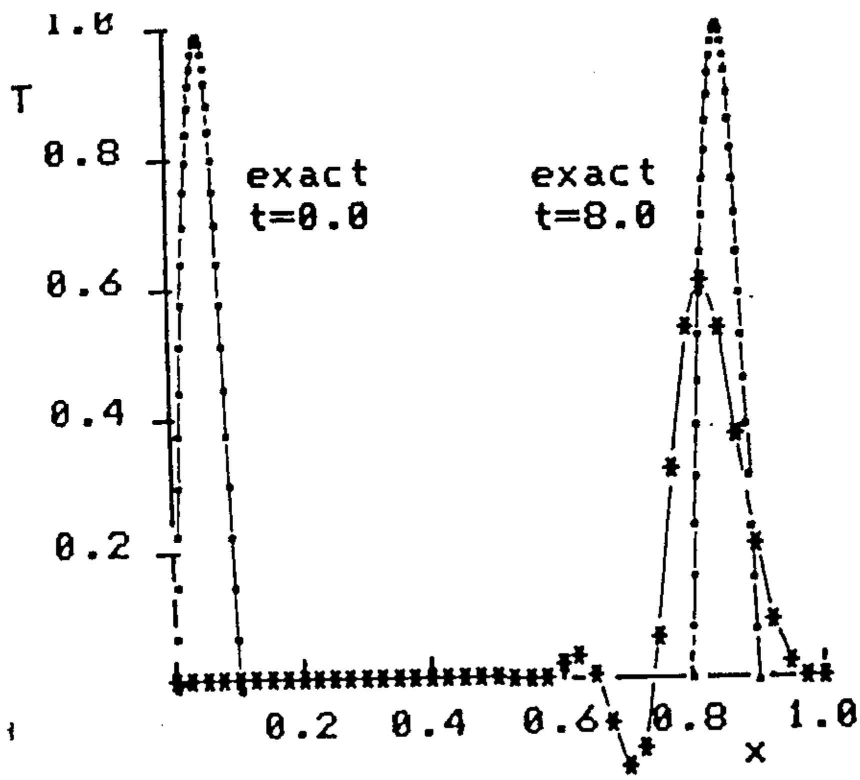
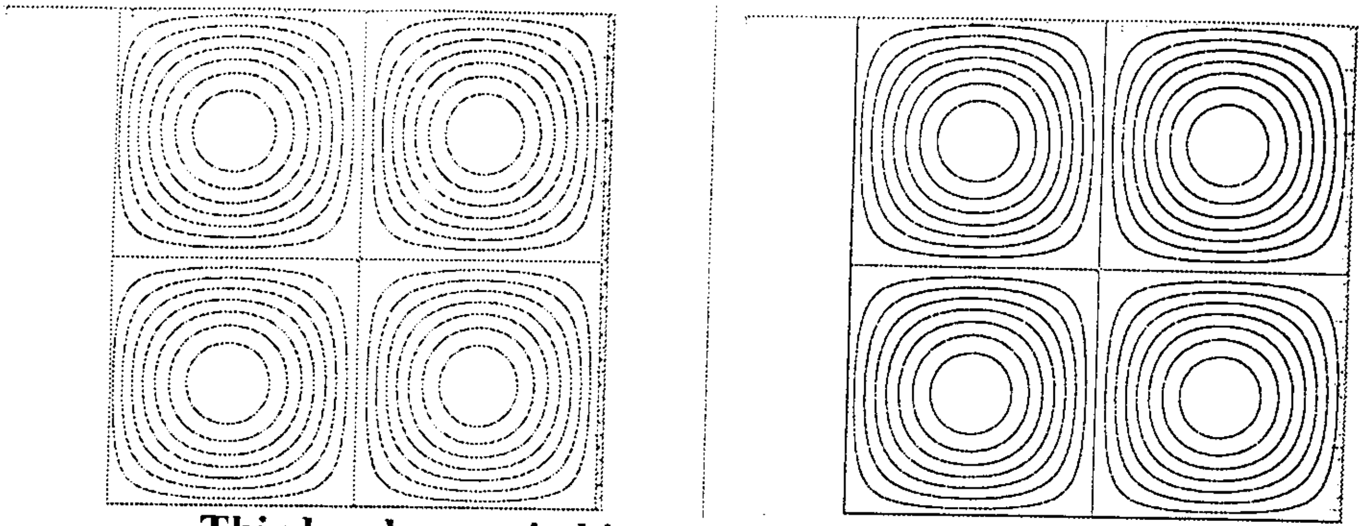
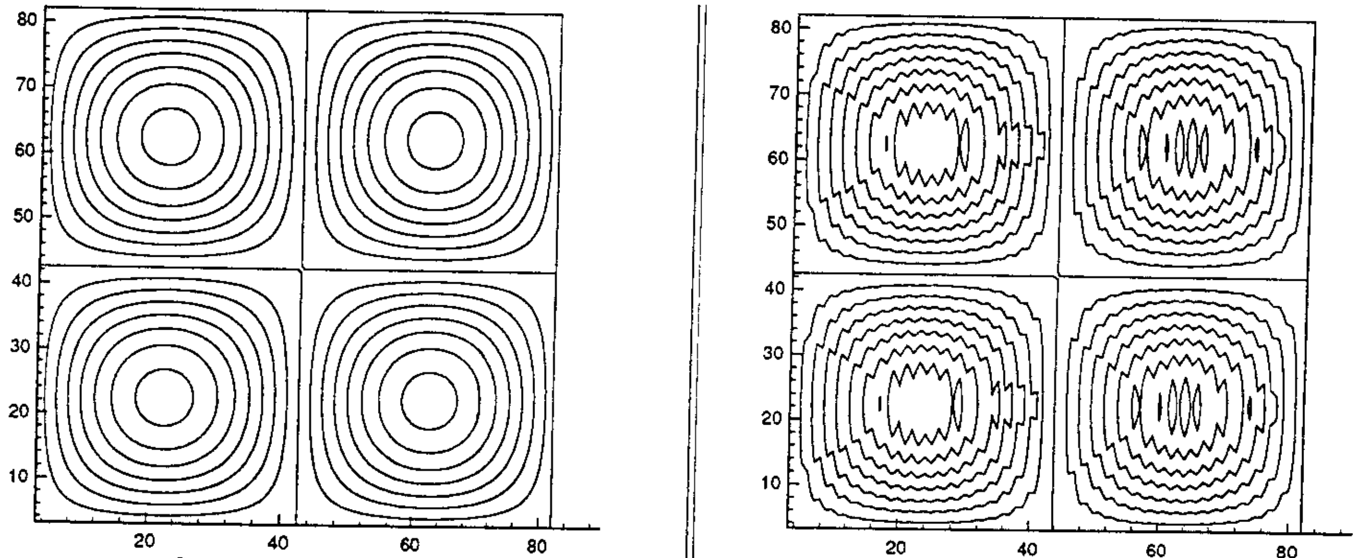


Fig. 9.3. Lax-Wendroff solution for the convection equation with $C=0.8$

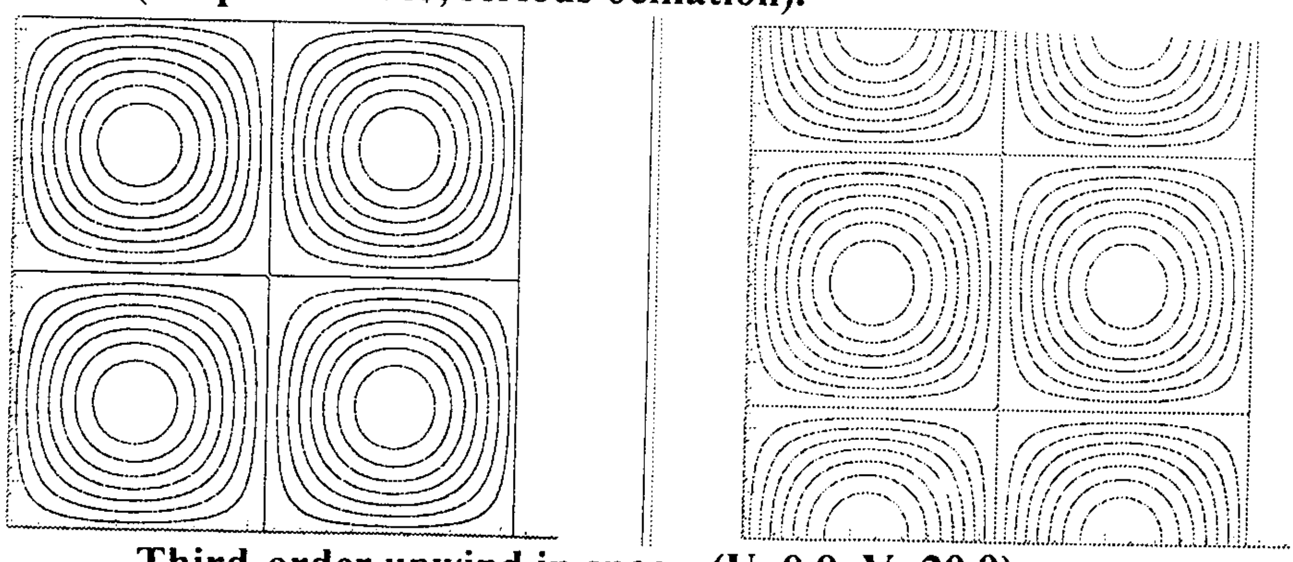
Example #1



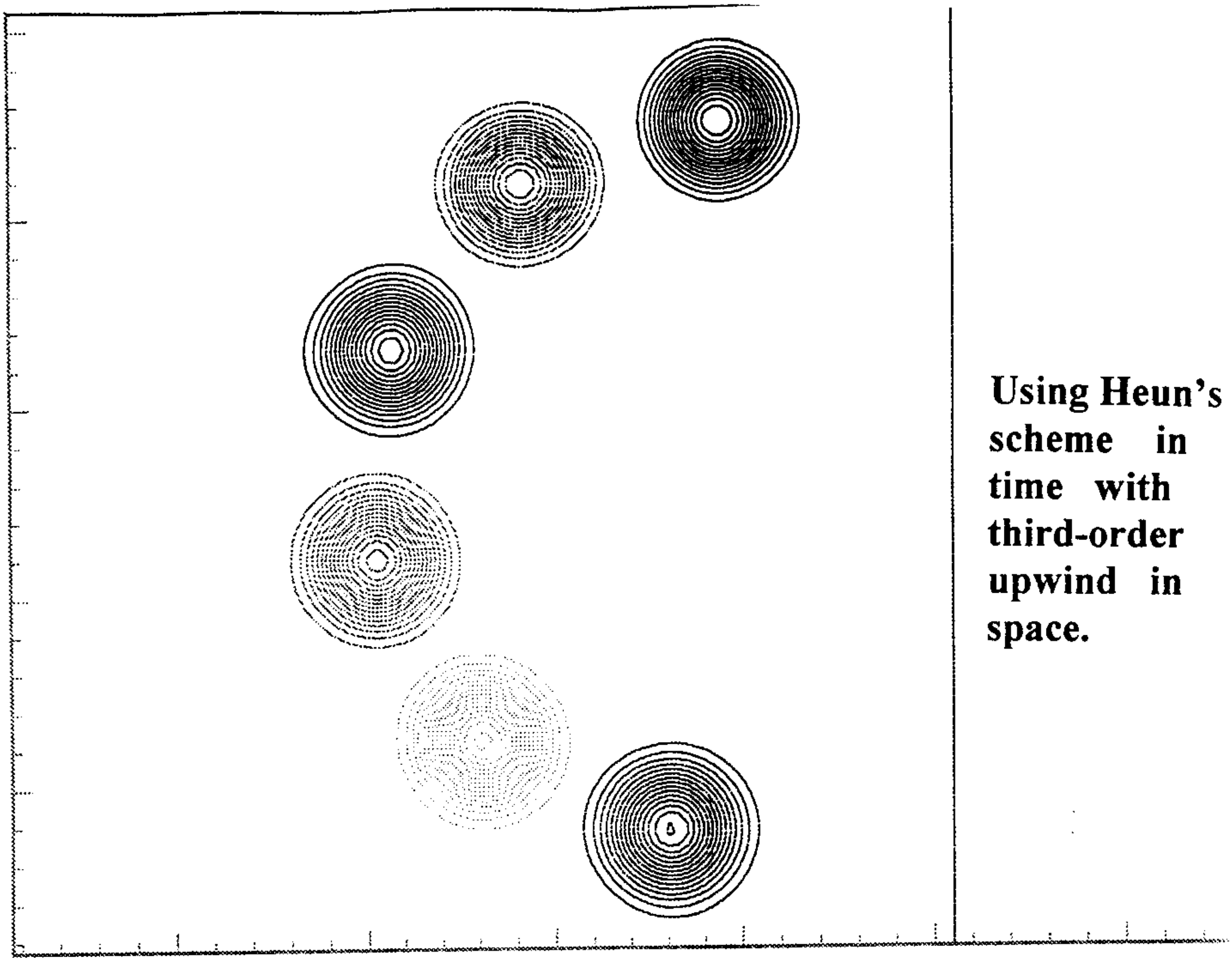
Third-order upwind in space. (U=1.0, V=1.0)
(No phase error, almost no damp).



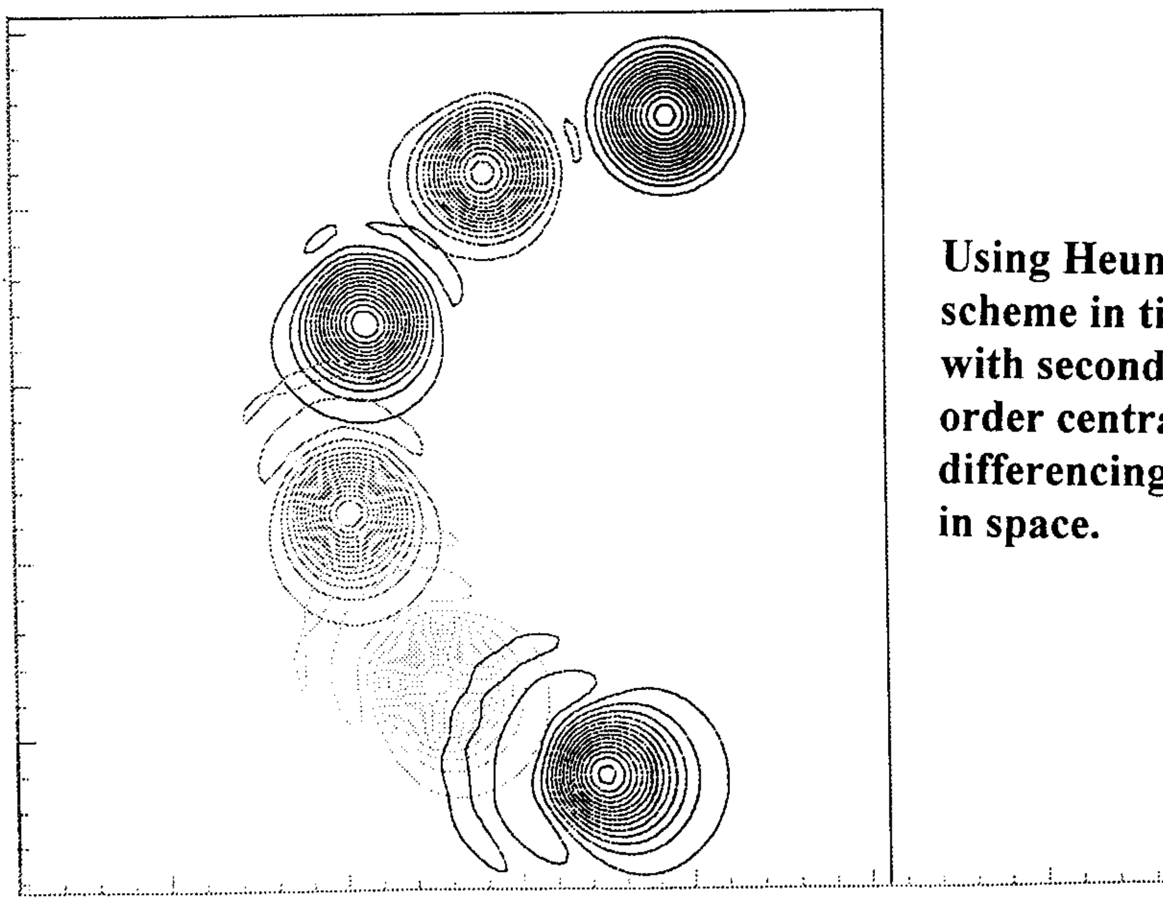
Second-order central diferencing in space. (U=1.0, V=1.0)
(No phase error, serious ocillation).



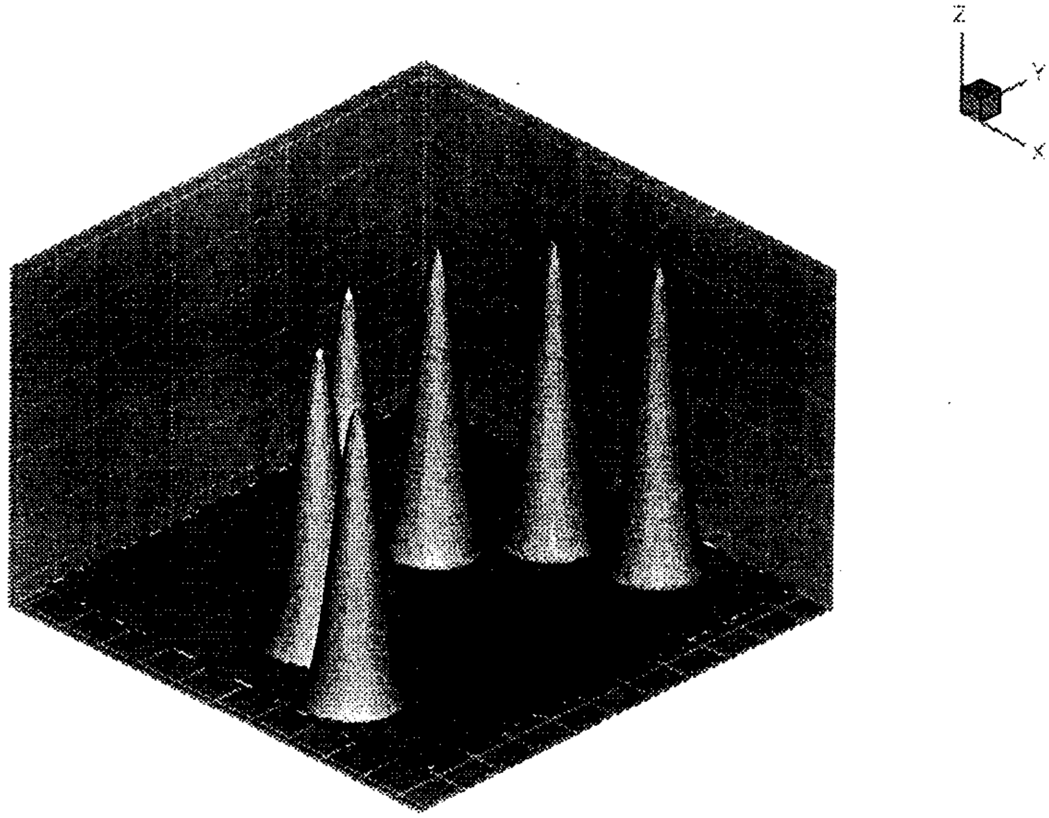
Third-order upwind in space. (U=0.0, V=20.0)
(Serious phase, Acceleration scheme when Velocity is too big).

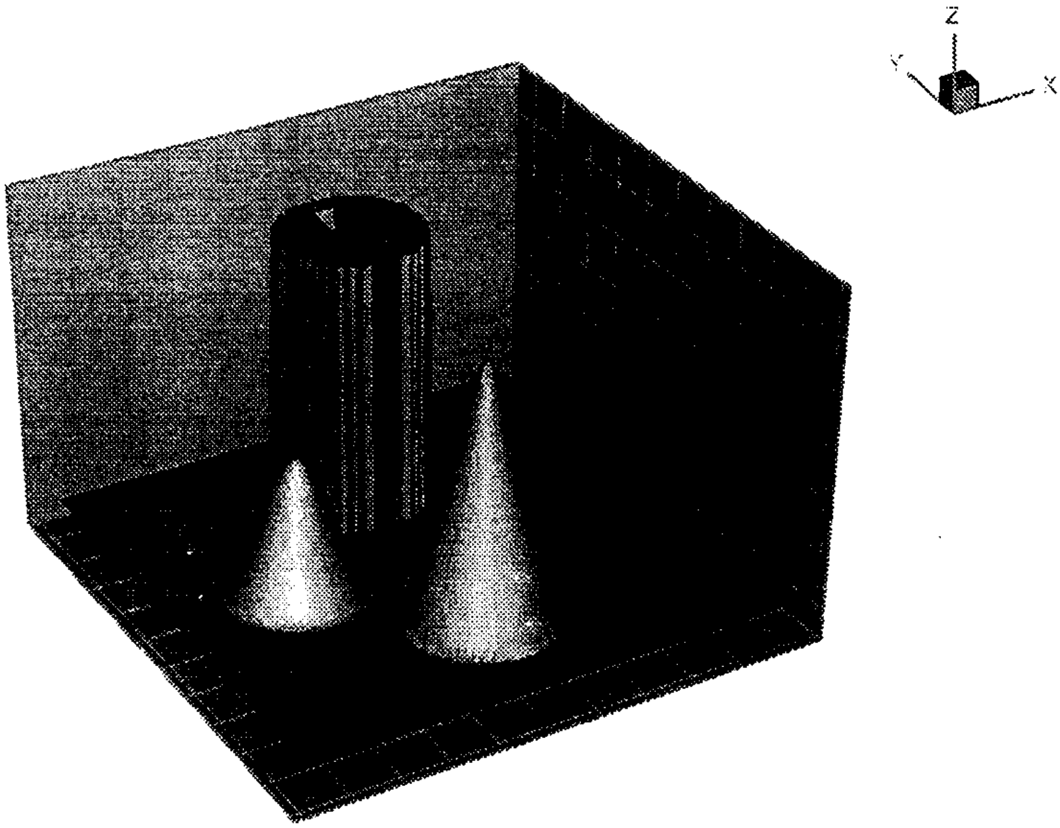


Using Heun's scheme in time with third-order upwind in space.

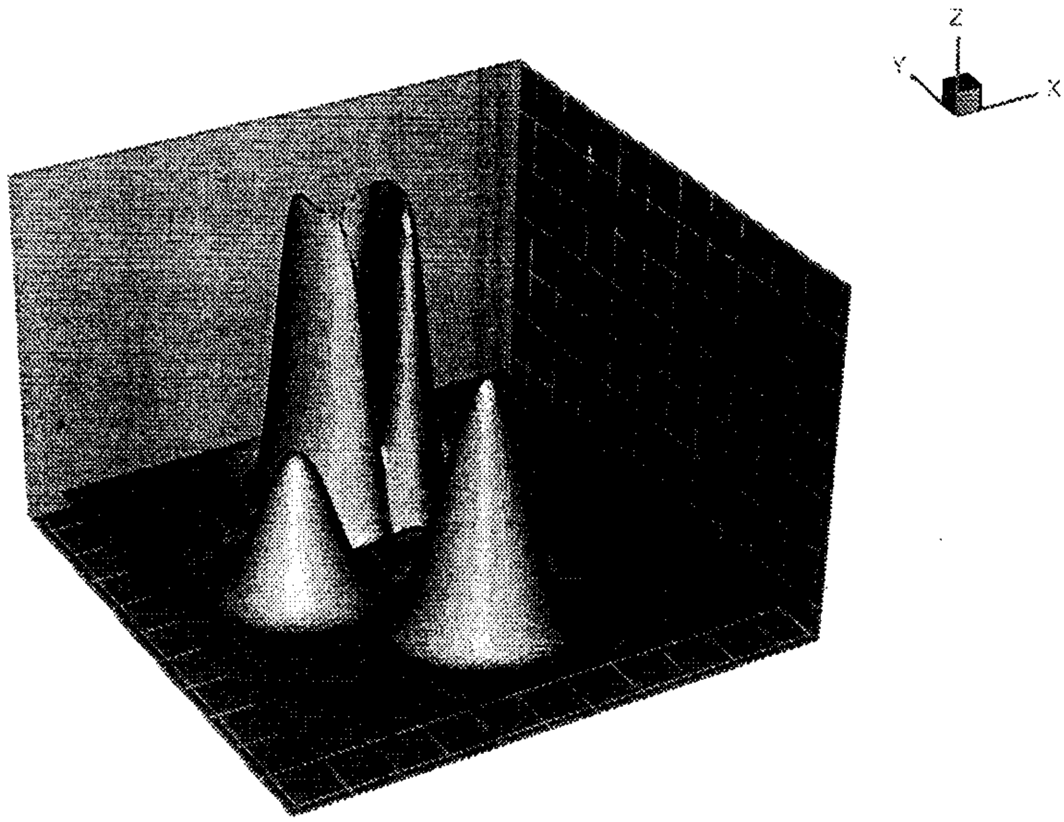


Using Heun's scheme in time with second-order central differencing in space.


Z
Y
X

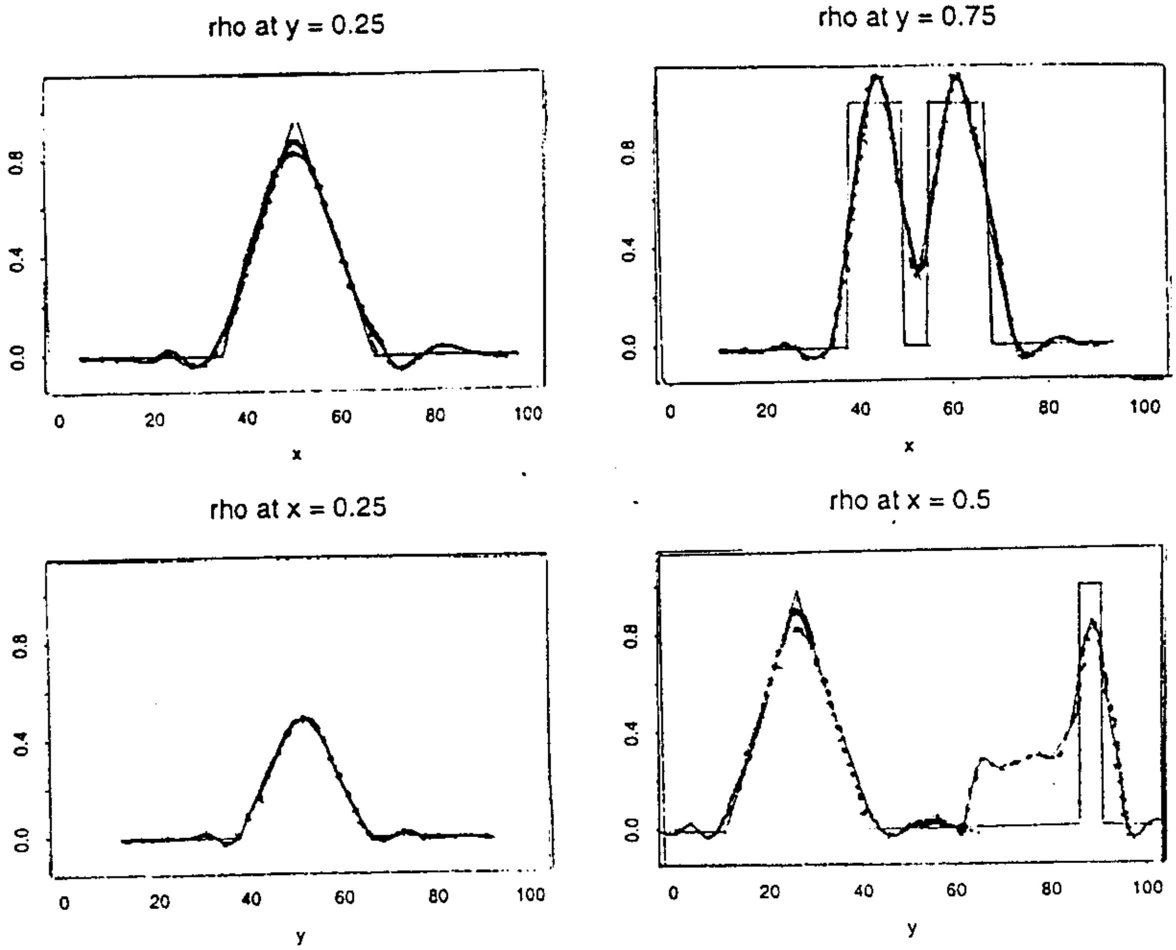


Initial condition of solid body before rotation.



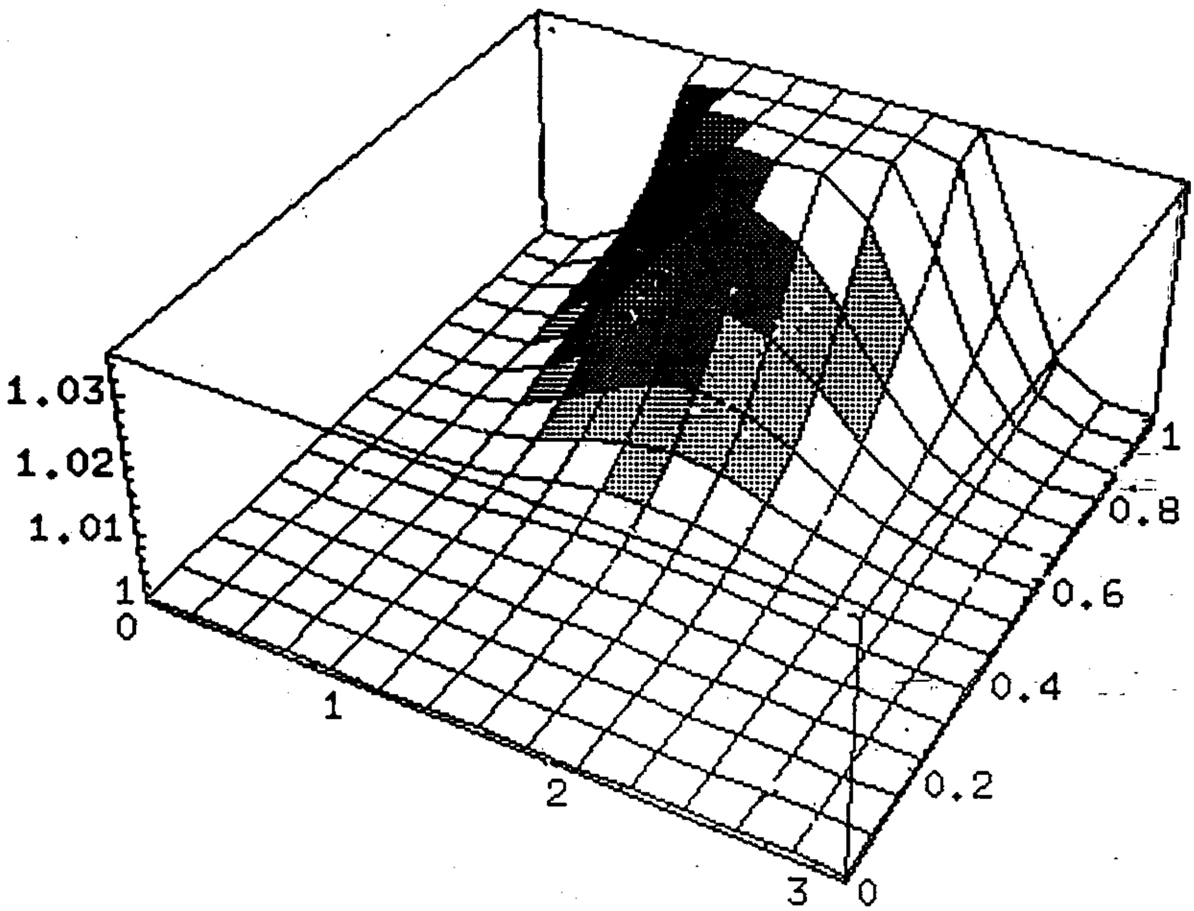
Numerical results after one revolution.

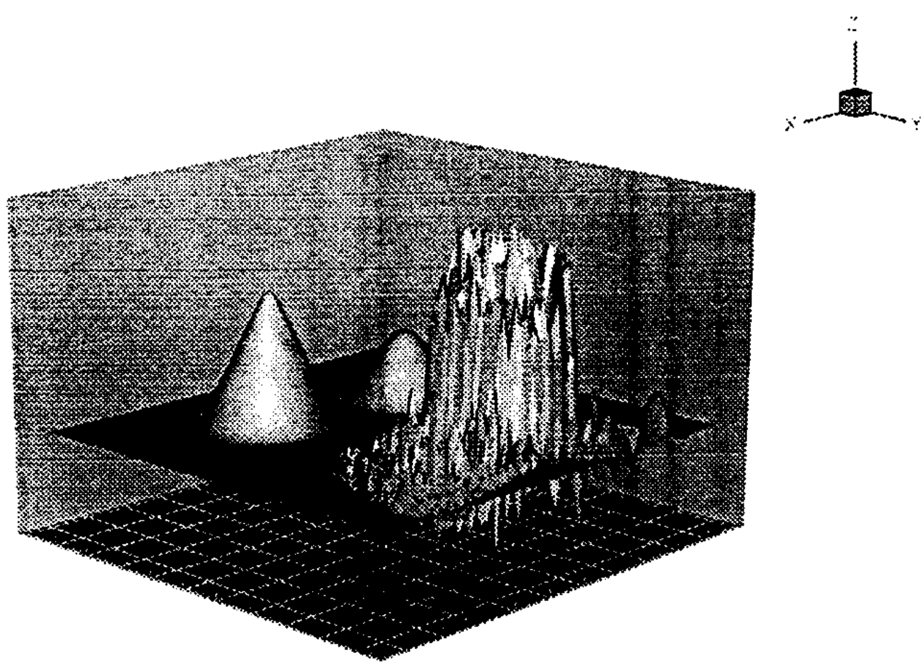
628 time steps. Mesh size: 100 * 100



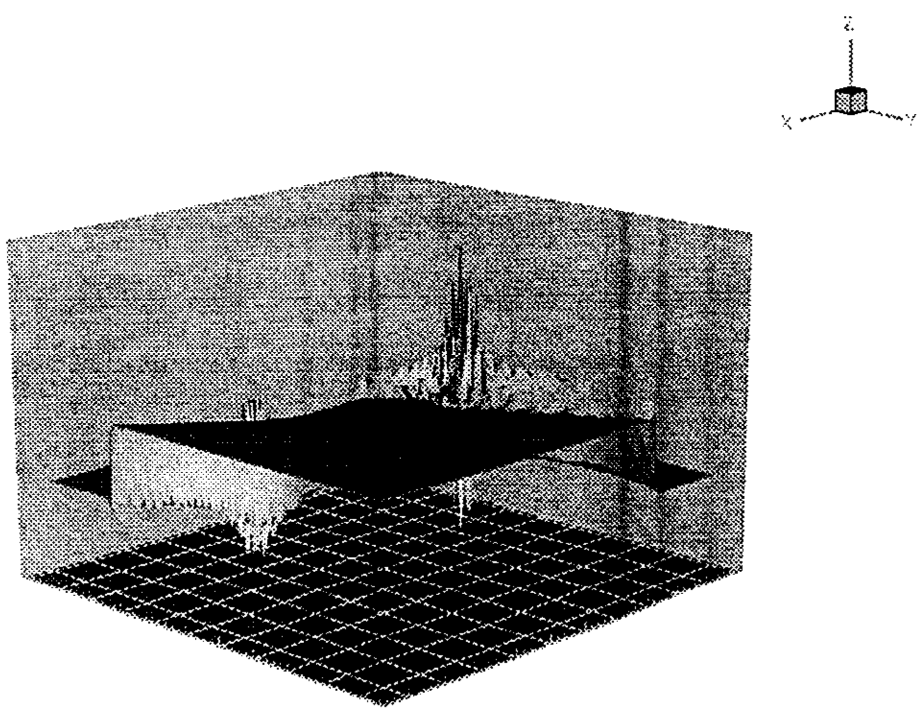
Numerical results for solid body rotation after one revolution (628 time steps) usingHeun's scheme in time and third-order upwind scheme in space. Four different cross sections of the solution are shown along with the perspective plot. The solid lines are the true solution.

Amp. Err. Heun/2-Central

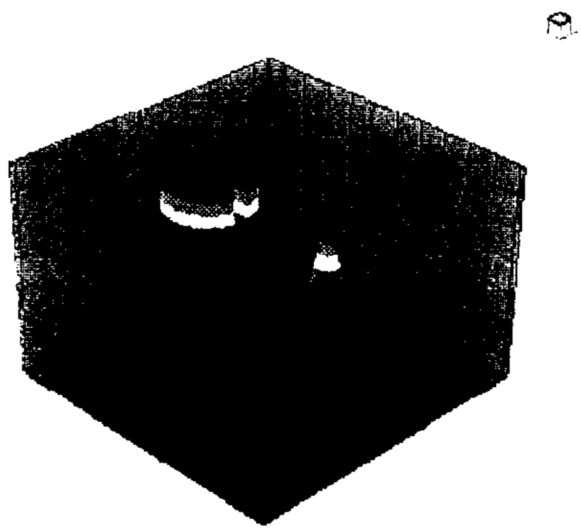




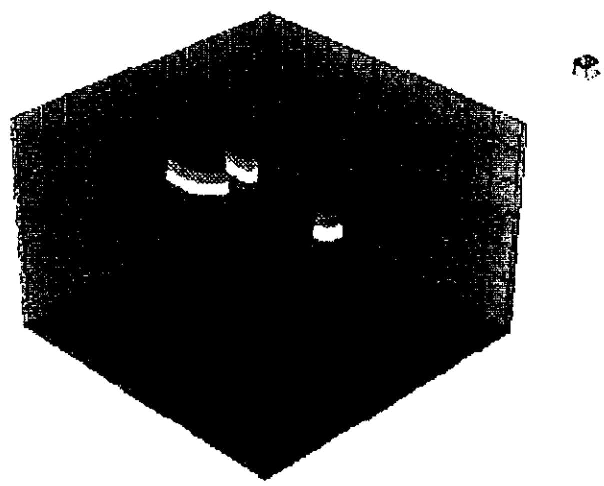
Numerical results by using Heun's in time
central differencing in space(40 time steps).
For one revolution, this scheme is not stable at all.



Numerical results for example #4 (15 time step).
Heun Scheme + Central differecing.
Strong ocillation, unstable.



Initial condition of solid body rotation.



Numerical results after one revolution (628 time steps).