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"Data Assimilation"

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Please note: These are preliminary notes intended for internal distribution only.

GANDIN STATISTICAL INTERPOLATION

A = Analysed Value

T = True Value

F = Forecast Value

D = Data Value

non-→dimensional errors

$$\alpha(r) = \frac{A(r)-T(r)}{\sigma_f(r)}$$

$$f(r) = \frac{F(r) - T(r)}{\sigma_f(r)}$$

$$\delta(r) = \frac{D(r) - T(r)}{\sigma_f(r)}$$

INTERPOLATION FORMULA IF OBSERVATIONS AT r_i

$$\alpha(r) - f(r) = \sum_i W_i(r)\ (\delta_i - f_i)$$

$$\delta_i = \delta(r_i)$$
$$f_i = f(r_i)$$

CHOOSE $W_i(r)$ TO MINIMISE $<\alpha^2(r)>$

$$\alpha(r) = f(r) + \sum_i W_i(r)\ [\delta_i - f_i]$$

$$\alpha^2(r) = f^2(r) + \sum_i \sum_j W_i(r)\ W_j(r)\ [\delta_i - f_i]\ [\delta_j - f_j] + 2 \sum_i W_i(r)\ f(r)\ [\delta_i - f_i]$$

TAKE EXPECTATIONS, ASSUMING $\langle \delta_i f_j \rangle = \langle \delta_i f_i \rangle = 0$

$$\langle \alpha^2(r) \rangle = \langle f^2(r) \rangle + \sum_i \sum_j W_i W_j \{ \langle f_i f_j \rangle + \langle \delta_i \delta_j \rangle \} - 2 \sum_i W_i \langle f(r) f_i \rangle$$

Setting $\frac{\partial}{\partial W_i} \langle \alpha^2(r) \rangle = 0$ we find

$$[W_i(r)] = [\langle f(r), f_i \rangle]^T \cdot (\underline{\underline{P}} + \underline{\underline{D}})^{-1}$$

where $\underline{\underline{P}} = [\langle f_i , f_j \rangle]$ and $\underline{\underline{D}} = [\langle \delta_i , \delta_j \rangle]$ } and depend only on observation position and observation variable

THE RESULTING ANALYSIS EQUATION IS

$$\alpha(r) - f(r) = [\langle f(r), f_i \rangle]^T \, [\underline{\underline{P}}+\underline{\underline{D}}]^{-1} \, [\delta_i - f_i]$$

or $$a(r) = [\mathcal{F}_i(r)]^T \, [\underline{\underline{P}}+\underline{\underline{D}}]^{-1} \, [d_i]$$

where $a(r) = \alpha(r) - f(r), \quad d_i = \delta_i - f_i, \quad \mathcal{F}_i(r) = \langle f(r), f_i \rangle$

or $$a(r) = [\mathcal{F}_i(r)]^T \, \underline{\underline{P}}^{-1} \, \{\underline{\underline{P}} \, [\underline{\underline{P}}+\underline{\underline{D}}]^{-1}\} \, [d_i]$$

This is a 2-stage operation

1. The FILTER: Filter the observation increments $[d_i]$ to produce the analysed values at the OBSERVATION POINTS

$$[a_i] = \underline{\underline{P}} \, [\underline{\underline{P}}+\underline{\underline{D}}]^{-1} \, [d_i]$$

2. The INTERPOLATOR: Interpolate the filtered data to any desired position r

$$a(r) = \mathcal{F}_i[(r)]^T \, \underline{\underline{P}}^{-1} \, [a_i]$$

where the interpolating functions are the prediction error correlations between observation points and analysis points

INTERPOLATION WITH AN ARBITRARY SET OF FUNCTIONS

Let $\mathscr{F}(r) = \{\mathscr{F}_j(r)\}$ be <u>any</u> set of n functions, $j = 1, n$

Let $\underline{a} = \{a(r_i)\}$ be <u>any</u> set of n data points $i = 1, n$.

FIND AN INTERPOLATION FORMULA SO THAT f INTERPOLATES a

Let $a(r) = \sum \gamma_j \mathscr{F}_j(r) = \mathscr{F}^T(r).\underline{\gamma}$

Let $\underline{a} = [a(r_i)]$

Since $\mathscr{F}$ interpolates $\underline{a}$

$$\underline{a} = [\mathscr{F}^T(r_i)].\underline{\gamma}$$

so $\underline{\gamma} = [\mathscr{F}^T(r_i)]^{-1}.\underline{a}$ provided the inverse exists

Thus $a(r) = \mathscr{F}^T(r).[\mathscr{F}^T(r_i)]^{-1}.\underline{a}$

cf $a(r) = [<f(r)f_i>]^T.[\underline{\underline{P}}]^{-1}.[a_i]$

THE INTERPOLATED EIGENVECTORS

$\mathcal{F}_i(r)$ is the prediction error correlation between the analysis point (and variable) indexed by r and the observation point (and variable) indexed by i

Let $\quad \varepsilon_i(r) = [\mathcal{F}_i(r)]^T \underline{\underline{P}}^{-1} \underline{e}_i$

O/I Interpolation Formula:

$$\begin{aligned} a(r) &= [\mathcal{F}_j(r)]^T \underline{\underline{P}}^{-1} [\underline{a}_i] \\ &= [\mathcal{F}_j(r)]^T \underline{\underline{P}}^{-1} \sum d_i \frac{1}{1+\nu_i} \underline{e}_i \\ &= \sum_i d_i \frac{1}{1+\nu_i} [\mathcal{F}_j(r)]^T \underline{\underline{P}}^{-1} \underline{e}_i \\ &= \sum d_i \frac{1}{1+\nu_i} \varepsilon_i(r) \end{aligned}$$

RESPONSE PROPERTIES OF THE O/I FILTER

Simple case: Uniform un-correlated errors $\underline{\underline{D}} = \sigma^2 I$.

$$a = \underline{\underline{P}}(\underline{\underline{P}}+\underline{\underline{D}})^{-1}\, \underline{d}$$

Let $\underline{\underline{P}} = \underline{\underline{E}}^T[\lambda_i^2]\, \underline{\underline{E}}, \quad \underline{\underline{E}}^T \underline{\underline{E}} = I, \quad$ s.t. $\underline{\underline{D}} = \underline{\underline{E}}^T[\sigma^2]\, \underline{\underline{E}}$

$\underline{\underline{P}}$ and $\underline{\underline{D}}$ therefore commute

$$\underline{\underline{P}} + \underline{\underline{D}} = \underline{\underline{E}}^T\, [\lambda_i^2+\sigma^2]\, \underline{\underline{E}}$$

$$(\underline{\underline{P}}+\underline{\underline{D}})^{-1} = \underline{\underline{E}}^T \left[\frac{1}{\lambda_i^2+\sigma^2}\right] \underline{\underline{E}}$$

$$\underline{\underline{P}}(\underline{\underline{P}}+\underline{\underline{D}})^{-1} = \underline{\underline{E}}^T \left[\frac{\lambda_i^2}{\lambda_i^2+\sigma^2}\right] \underline{\underline{E}}$$

$$= \underline{\underline{E}}^T \left[\frac{1}{1+\nu_i}\right] \underline{\underline{E}}$$

$$\nu_i = \frac{\sigma^2}{\lambda_i^2}$$

noise to signal ratio for component i

Scale Dependent Response

$\lambda_i^2 >> \sigma^2, \quad \nu_i \sim 0, \quad R_i = \frac{q}{1+\nu_i} \sim 1$: BELIEVE THE OBSERVATIONS

$\lambda_i^2 << \sigma^2, \quad \nu_i \sim \alpha, \quad R_i \sim 0$: IGNORE THE OBSERVATIONS AND BELIEVE THE FIRST GUESS

Write $\underline{\underline{E}} = [\underline{e}_i]$ $\quad\quad$ $\underline{e}_i$ individual eigenvector

and let

$$\underline{d} = \sum_i d_i \, \underline{e}_i.$$

since $\quad \underline{a} = \underline{\underline{P}}(\underline{\underline{P}}+\underline{\underline{D}})^{-1} \, \underline{d}$

$$\underline{a} = \sum_i d_i \, \frac{1}{1+\nu_i} \, \underline{e}_i$$

THIS GENERALISES EASILY TO THE NON-COMMUTATIVE CASE

THUS THE O/I ALGORITHM HAS THE FOLLOWING STRUCTURE

DATA: $$\underline{d} = \sum d_i \, \underline{e}_i$$

FILTERED DATA: $$\underline{a} = \sum d_i \frac{1}{1+v_i} e_i$$

INTERPOLATED VALUE AT ANY POINT: $$a(r) = \sum d_i \frac{1}{1+v_i} e_i(r)$$