INTERNATIONAL ATOMIC ENERGY AGENCY
UNITED NATIONS EDUCATIONAL, SCIENTIFIC AND CULTURAL ORGANIZATION
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H4.SMR/942-17

Third Workshop on
3D Modelling of Seismic Waves Generation Propagation and their Inversion

4 - 15 November 1996

Ray-Tracing

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RAY-TRACING

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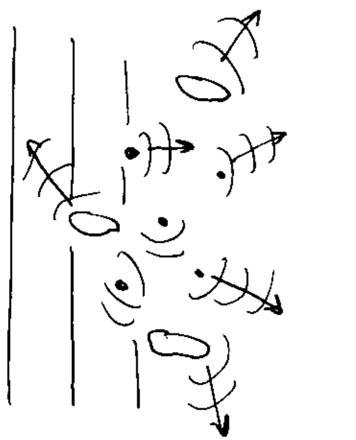
NETHERLANDS

HELMHOLTZ EQ: $\nabla^2 p + \frac{\omega^2}{c^2} p = 0$

ACOUSTIC WAVE EQ: $\boxed{\nabla \cdot \left(\frac{1}{\rho} \nabla p \right) + \frac{\omega^2}{\kappa} p = 0}$

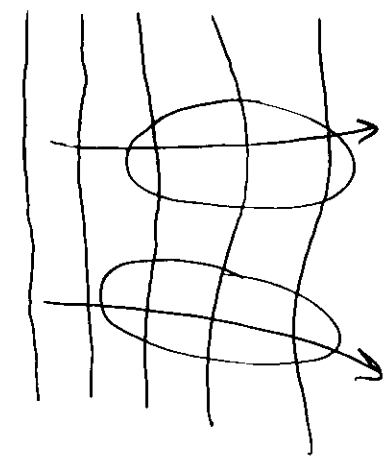
ELASTIC WAVE EQ: $\partial_j \left(c_{ijkl} \partial_k u_l \right) + \rho \omega^2 u_i = 0$

$\lambda \sim L$ OR $\lambda \gg L$



RAY-THEORY

$\lambda \ll L$



ACOUSTIC WAVE EQ: $\nabla \cdot \left(\frac{1}{\rho} \nabla p \right) + \frac{\omega^2}{\kappa} p = 0$ (2.1)

SET $p(\vec{r}, \omega) = A(\vec{r}, \omega)\, e^{i\psi(\vec{r}, \omega)}$ (2.2)

USE $\nabla p = \nabla A\, e^{i\psi} + i A \nabla\psi\, e^{i\psi}$ ETC.

AND INSERT:

$$\left\{ -\frac{1}{\rho^2} (\nabla\rho \cdot \nabla A) + \frac{1}{\rho} \nabla^2 A - \frac{A}{\rho} |\nabla\psi|^2 \right\} e^{i\psi}$$
$$+ i \left\{ \frac{2}{\rho} (\nabla A \cdot \nabla\psi) - \frac{A}{\rho^2} (\nabla\rho \cdot \nabla\psi) + \frac{A}{\rho} \nabla^2\psi \right\} e^{i\psi}$$
$$+ \frac{\omega^2}{\kappa} A e^{i\psi} = 0$$

DIVIDE BY $e^{i\psi}$ AND EQUATE REAL AND IMAGINARY PARTS:

$$\boxed{-\frac{1}{\rho}(\nabla\rho \cdot \nabla A) + \nabla^2 A - A|\nabla\psi|^2 + \frac{\omega^2}{c^2} A = 0} \quad (2.3)$$

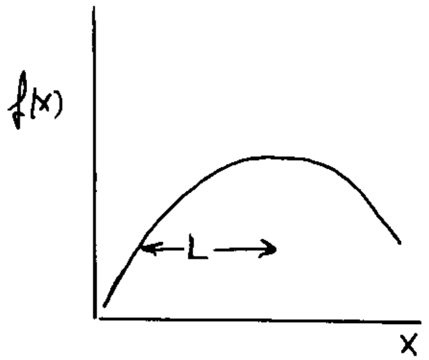
$$\boxed{2(\nabla A \cdot \nabla\psi) - \frac{A}{\rho}(\nabla\rho \cdot \nabla\psi) + A \nabla^2\psi = 0} \quad (2.4)$$

WITH $c^2 = \kappa / \rho$ (2.5)

UP TO THIS POINT THE ANALYSIS IS EXACT BUT WE HAVE NOT GAINED MUCH BECAUSE (2.3) AND (2.4) ARE TOO DIFFICULT TO SOLVE

INTERMEZZO: ESTIMATING A DERIVATIVE

LET A FUNCTION $f(x)$ VARY OVER A LENGTH-SCALE L



$$\boxed{\frac{df}{dx} = \text{SLOPE} \sim O\left(f/L\right)} \qquad (3.1)$$

EXAMPLE: $f(x) = \cos kx$

$$\frac{df}{dx} = -k \sin kx = -\frac{2\pi}{L} \sin kx$$

$$\rightarrow \quad \frac{df}{dx} = O\left(f/L\right)$$

$$(2.3): \quad \underset{1}{-\frac{1}{\rho}(\nabla\rho\cdot\nabla A)} + \underset{2}{\nabla^2 A} - \underset{3}{A|\nabla\psi|^2} + \underset{4}{\frac{\omega^2}{c^2}A} = 0$$

LET A VARY ON A LENGTH-SCALE L_A

" ρ " " " " " L_ρ

" ψ " " " " " λ

λ IS THE WAVELENGTH, LOCALLY ψ BEHAVES LIKE A PLANE WAVE: $\psi = \vec{k}\cdot\vec{r}$

HENCE $\nabla\psi = \vec{k}$ AND $|\nabla\psi| = k = \frac{2\pi}{\lambda} = O(\lambda)$ (4.1)

1: $\frac{1}{\rho}(\nabla\rho\cdot\nabla A) \sim \frac{1}{\rho}\cdot\frac{\rho}{L_\rho}\cdot\frac{A}{L_A} = A/L_\rho L_A$

2: $\nabla^2 A \sim A/L_A^2$

3: $A|\nabla\psi|^2 \sim A/\lambda^2$

4: $\frac{\omega^2}{c^2}A \sim k^2 A \sim A/\lambda^2$

SHORT WAVELENGTH LIMIT (= HIGH FREQUENCY LIMIT)

$$\boxed{\lambda \ll L_A} \quad \text{AND} \quad \boxed{\lambda \ll L_\rho} \qquad (4.2)$$

THE TERMS 1 AND 2 CAN BE IGNORED

BALANCE TERMS ③ AND ④ : $|\nabla\psi|^2 = \frac{\omega^2}{c^2}$ (5.1)

(INDEED $|\nabla\psi| = \frac{\omega}{c(\vec{r})} = |\vec{k}(\vec{r})|$ AS USED EARLIER)

SET $\psi = \omega\theta$ THEN

$$\boxed{|\nabla\theta|^2 = \frac{1}{c^2}}$$ EIKONAL EQUATION (5.2)

$\theta(\vec{r})$ DOES NOT DEPEND ON FREQUENCY ←

(2.4) AGAIN:

$$\boxed{2(\nabla A\cdot\nabla\theta) - \frac{A}{\rho}(\nabla\rho\cdot\nabla\theta) + A\nabla^2\theta = 0} \quad (5.3)$$

TRANSPORT EQUATION

$A(\vec{r})$ DOES NOT DEPEND ON FREQUENCY ←

$$p(\vec{r},\omega) = A(\vec{r})\, e^{i\omega\theta(\vec{r})} S(\omega) \quad (6.1)$$

WITH $S(\omega)$ THE SOURCE SPECTRUM:

$$S(t) = \int S(\omega)\, e^{-i\omega t}\, d\omega \quad (6.2)$$

IN THE TIME DOMAIN:

$$p(\vec{r},t) = \int p(\vec{r},\omega)\, e^{-i\omega t}\, d\omega$$

$$= \int A(\vec{r})\, e^{i\omega\theta(\vec{r})}\, e^{-i\omega t}\, S(\omega)\, d\omega$$

$$= A(\vec{r}) \int S(\omega)\, e^{-i\omega(t-\theta(\vec{r}))}\, d\omega$$

$$\boxed{p(\vec{r},t) \underset{(6.2)}{=} A(\vec{r})\, S(t-\theta(\vec{r}))} \quad (6.3)$$

<u>HENCE:</u>

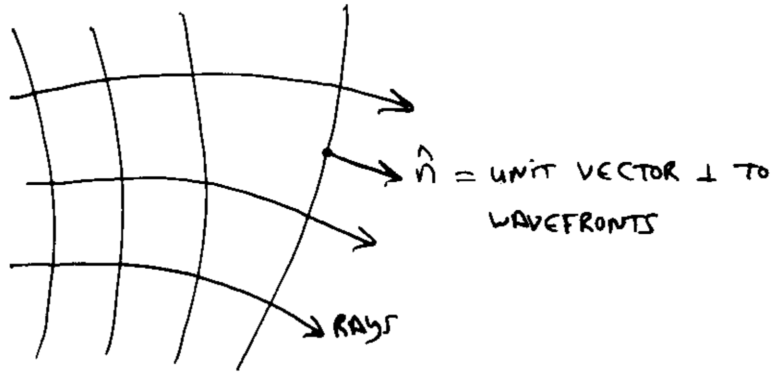
- IN THE RAY-GEOMETRICAL LIMIT THE WAVEFORM DOES NOT CHANGE
- THERE ARE NO FREQUENCY-DEPENDENT EFFECTS
- $\boxed{\theta(\vec{r}) = \text{TRAVEL TIME}}$ (6.4)

THE TRAVEL TIME IS GOVERNED BY THE EIKONAL EQ:

$$\boxed{|\nabla\theta|^2 = \frac{1}{c^2}} \qquad (7.1)$$

THE LINES <u>$\theta(\vec r)$ = CONSTANT</u> ARE <u>WAVEFRONTS</u>

<u>RAYS</u> ARE CURVES $\perp$ TO THE <u>WAVEFRONTS</u>



LET $\vec r(s)$ DENOTE A RAY, THEN

$$\boxed{\hat n = \frac{d\vec r}{ds}} \qquad (7.2)$$

SET $\quad \nabla\theta = \vec p \quad$ (= SLOWNESS VECTOR) $\qquad (7.3)$

$p^2 = 1/c^2$ (FROM EIKONAL) $\qquad (7.4)$

AND $\quad \boxed{\nabla\theta = p\hat n} \qquad (7.5)$

<u>WHICH EQUATION GOVERNS RAYS?</u>

EIKONAL EQ: $\quad \frac{1}{c^2} = |\nabla\theta|^2 = \sum_j (\partial_j\theta)(\partial_j\theta)$

TAKE THE x_i DERIVATIVE $\left(\partial_i \equiv \frac{\partial}{\partial x_i}\right)$:

$$\partial_i\left(\frac{1}{c^2}\right) = \partial_i \sum_j (\partial_j\theta)(\partial_j\theta)$$

$$= 2\sum_j (\partial_j\theta)(\partial_i\partial_j\theta) = 2\sum_j (\partial_j\theta)(\partial_j\partial_i\theta)$$

OR $\quad \boxed{\nabla\frac{1}{c^2} = 2(\nabla\theta\cdot\nabla\nabla\theta)} \qquad (8.1)$

USE THAT $\quad \nabla\frac{1}{c^2} = -\frac{2}{c^3}\nabla c = \frac{2}{c}\nabla\frac{1}{c} \qquad (8.2)$

$$\nabla\theta = \vec p$$

$$\longrightarrow \quad (\vec p\cdot\nabla\vec p) = \frac{1}{c}\nabla\frac{1}{c} \qquad (8.3)$$

USE THAT $\quad \vec p = \frac{1}{c}\hat n$

$$\hat n\cdot\nabla F(\vec r) = \frac{dF}{ds}$$

$$\longrightarrow \quad \frac{1}{c}\hat n\cdot\nabla\left(\frac{1}{c}\hat n\right) = \frac{1}{c}\nabla\frac{1}{c} \longrightarrow \frac{d}{ds}\left(\frac{1}{c}\hat n\right) = \nabla\frac{1}{c}$$

BUT $\quad \hat n = d\vec r/ds$

$$\longrightarrow \quad \boxed{\frac{d}{ds}\left(\frac{1}{c}\frac{d\vec r}{ds}\right) = \nabla\frac{1}{c} \quad \text{EQ. OF KINEMATIC RAY TRACING}}$$

$$\frac{d}{ds}\left(\frac{1}{c}\frac{d\vec{r}}{ds}\right) = \nabla\frac{1}{c}$$

WHAT DOES THIS MEAN?

USE THAT $\frac{d}{ds}\left(\frac{1}{c}\frac{d\vec{r}}{ds}\right) = \frac{d}{ds}\left(\frac{1}{c}\hat{n}\right) = \frac{d}{ds}\left(\frac{1}{c}\right)\hat{n} + \underbrace{\frac{1}{c}\frac{d\hat{n}}{ds}}_{\perp \hat{n}}$ (9.1)

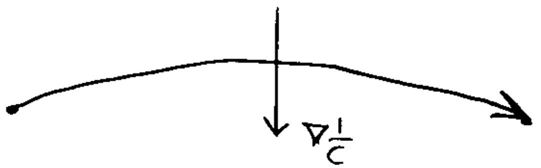
$$\nabla\frac{1}{c} = \hat{n}\frac{d}{ds}\left(\frac{1}{c}\right) + \nabla_{\perp}\frac{1}{c} \quad (9.2)$$

HENCE: $\cancel{\frac{d}{ds}\left(\frac{1}{c}\right)\hat{n}} + \frac{1}{c}\frac{d\hat{n}}{ds} = \cancel{\hat{n}\frac{d}{ds}\left(\frac{1}{c}\right)} + \nabla_{\perp}\frac{1}{c}$

SO THE EQUATION OF KINEMATIC RAY TRACING IS EQUIVALENT WITH:

$$\boxed{\frac{d\hat{n}}{ds} = c\,\nabla_{\perp}\frac{1}{c}} \quad (9.3)$$

FAST ($\frac{1}{c}$ LOW)



SLOW ($\frac{1}{c}$ HIGH)

THE RELATION WITH NEWTON'S LAW

$$\frac{d}{dt}(m\vec{v}) = \vec{F}$$

OR: $\boxed{\frac{d}{dt}\left(m\frac{d\vec{r}}{dt}\right) = \vec{F}}$ (10.1)

RAY-TRACING $\boxed{\frac{d}{ds}\left(\frac{1}{c}\frac{d\vec{r}}{ds}\right) = \nabla\frac{1}{c}}$ (10.2)

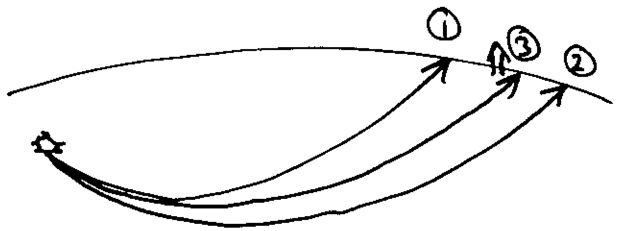
CLASSICAL MECHANICS		RAY-TRACING
t	$\longleftrightarrow$	s
m	$\longleftrightarrow$	$1/c$
$\vec{F}$	$\longleftrightarrow$	$\nabla\frac{1}{c}$

- METHODS FROM CLASSICAL MECHANICS CAN BE USED FOR RAY-TRACING AS WELL!
- CLASSICAL MECHANICS IS THE RAY-GEOMETRICAL LIMIT OF QUANTUM MECHANICS

SHOOTING

- GIVEN INITIAL CONDITIONS $\frac{d}{ds}\left(\frac{1}{c}\frac{d\vec{r}}{ds}\right) = \nabla\frac{1}{c}$ CAN BE INTEGRATED

- ADJUST THE INITIAL CONDITIONS SO THAT A RAY HITS THE RECEIVER



TRAVEL TIME

$$\nabla\theta = p\hat{n} \longrightarrow \frac{d\theta}{ds} = \hat{n}\cdot\nabla\theta = \hat{n}\cdot p\hat{n} = p(\hat{n}\cdot\hat{n}) = \frac{1}{c}\cdot 1 = \frac{1}{c}$$

$$T = \int \frac{d\theta}{ds}\, ds \rightarrow \boxed{T = \int_{\vec{r}(s)} \frac{1}{c}\, ds} \qquad (11.1)$$

(i.e. TIME IS DISTANCE DIVIDED BY VELOCITY)

HOW ABOUT THE AMPLITUDE A?

TAKE THE TRANSPORT EQ. (5.3) WITH $\nabla\theta = \vec{p}$

$$\boxed{2(\nabla A\cdot\vec{p}) - \frac{A}{\rho}(\nabla\rho\cdot\vec{p}) + A(\nabla\cdot\vec{p}) = 0} \qquad (12.1)$$

$$(\vec{p}\cdot\nabla A) = \frac{1}{c}\hat{n}\cdot\nabla A = \frac{1}{c}\frac{dA}{ds}$$

$$(\vec{p}\cdot\nabla\rho) = - - - - - = \frac{1}{c}\frac{d\rho}{ds}$$

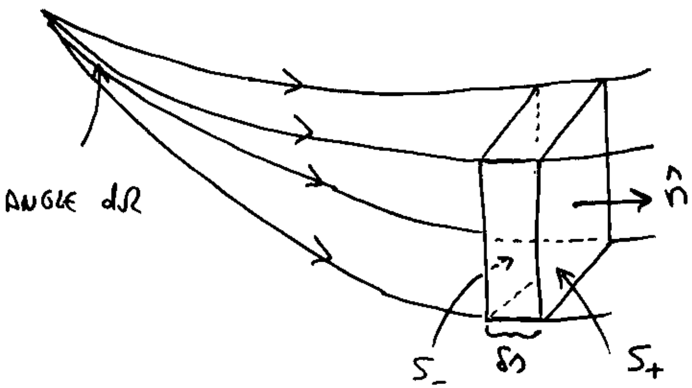
$$(\nabla\cdot\vec{p}) = \nabla\cdot\left(\frac{1}{c}\hat{n}\right) = \hat{n}\cdot\nabla\frac{1}{c} + \frac{1}{c}(\nabla\cdot\hat{n}) = \frac{d}{ds}\left(\frac{1}{c}\right) + \frac{1}{c}(\nabla\cdot\hat{n})$$

SO THAT

$$\boxed{\frac{2}{c}\frac{dA}{ds} - \frac{A}{\rho c}\frac{d\rho}{ds} + A\frac{d}{ds}\left(\frac{1}{c}\right) + \frac{A}{c}(\nabla\cdot\hat{n}) = 0} \qquad (12.2)$$

BUT WHAT IS $(\nabla\cdot\hat{n})$?

CALCULATION OF $(\nabla\cdot\hat{n})$



$$S_+ - S_- = \int_{S_+} dS - \int_{S_-} dS = \int_{S_+} \hat{n}\cdot\vec{dS} - \int_{S_-} \hat{n}\cdot\vec{dS}$$

$$= \oint \hat{n}\cdot\vec{dS} \underset{\uparrow\ \text{GAUSS}}{=} \int(\nabla\cdot\hat{n})\,dV = (\nabla\cdot\hat{n})\,\delta s\,.\,S_+ \qquad (13.1)$$

SO THAT

$$(\nabla\cdot\hat{n}) = \frac{S_+ - S_-}{\delta s\,.\,S_+} = \frac{1}{S_+}\frac{S^+ - S^-}{\delta s} = \frac{1}{S}\frac{dS}{ds} \qquad (13.2)$$

SET $\quad S = \mathcal{J}\,d\Omega \quad , \ \mathcal{J}$ = GEOMETRICAL SPREADING

$$\boxed{(\nabla\cdot\hat{n}) = \frac{1}{\mathcal{J}}\frac{d\mathcal{J}}{ds}} \qquad (13.3)$$

INSERT (13.3) IN (12.2) AND MULTIPLY WITH c/A:

$$\frac{2}{A}\frac{dA}{ds} - \frac{1}{\rho}\frac{d\rho}{ds} - \frac{1}{c}\frac{dc}{ds} + \frac{1}{\mathcal{J}}\frac{d\mathcal{J}}{ds} = 0$$

$$\frac{d}{ds}\ln A^2 + \frac{d}{ds}\ln\frac{1}{\rho} + \frac{d}{ds}\ln\frac{1}{c} + \frac{d}{ds}\ln\mathcal{J} = 0$$

$$\frac{d}{ds}\ln\left(\frac{A^2\mathcal{J}}{\rho c}\right) = 0$$

$$\frac{A^2\mathcal{J}}{\rho c} = \text{constant} \quad \text{or}$$

$$\boxed{A = \text{constant}\ \frac{\sqrt{\rho c}}{\sqrt{\mathcal{J}}}} \qquad (14.1)$$

- THE constant FOLLOWS BY CONSIDERING A SOURCE IN A HOMOGENOUS MEDIUM (?)

- (14.1) IS FOR THE PRESSURE AMPLITUDE, FOR DISPLACEMENT:

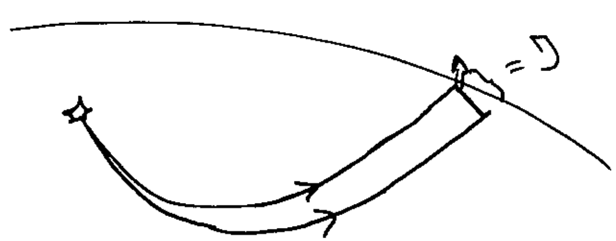
$$-\rho\omega^2\vec{u} = -\nabla p \;\rightarrow\; \vec{u} = +\frac{1}{\rho\omega^2}\nabla p = +\frac{1}{\rho\omega^2}\left(\nabla A + iA\nabla\psi\right)e^{i\psi}$$

$$\approx \frac{+iA\nabla\psi}{\rho\omega^2}\,e^{i\psi} \qquad (14.2)$$

BUT $\quad \nabla\psi = \omega\nabla\theta = \frac{\omega}{c}\nabla\theta$

$$\vec{u} = +\frac{iA\hat{n}}{\rho c\omega}\,e^{i\psi} \;\Longrightarrow\; \boxed{\vec{u} = \frac{+i\ \text{constant}\ \hat{n}}{\sqrt{\omega}\sqrt{\rho c}\sqrt{\mathcal{J}}}\,e^{i\psi}} \qquad (14.3)$$

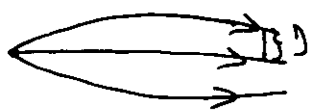
HOW TO COMPUTE J?



- THIS IS LIKE SHOOTING
- THIS IS NOT THE WAY TO DO THIS

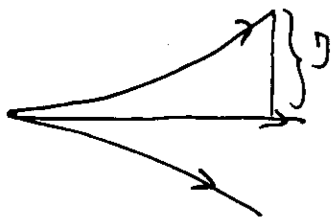
THE FACT THAT $A \sim 1/\sqrt{J}$ IS DUE TO ENERGY CONSERVATION

FOCUSING



J SMALL
LARGE AMPLITUDE

DEFOCUSING



J LARGE
SMALL AMPLITUDE

THE THEORY BREAKS DOWN WHEN $J=0$ $(A \to \infty)$, BUT THERE THE CONDITION $\lambda \ll L_A$ IS NOT JUSTIFIED (CAUSTICS)

FERMAT'S PRINCIPLE

TRAVEL TIME: $$T = \int_{\vec{r}(s)} \frac{1}{c}\, ds \qquad (16.1)$$

BUT THE RAY POSITION $\vec{r}(s)$ DEPENDS ON THE VELOCITY

SUPPOSE WE KNOW THE RAY $\vec{r}_0(s_0)$ FOR A REFERENCE VELOCITY c_0

EXPAND: SLOWNESS $= u \equiv u_0 + \varepsilon u_1$

$$\boxed{\begin{aligned} \vec{r} &= \vec{r}_0 + \varepsilon \vec{r}_1 + \varepsilon^2 \vec{r}_2 + \cdots \\ T &= T_0 + \varepsilon T_1 + \varepsilon^2 T_2 + \cdots \\ \theta &= \theta_0 + \varepsilon \theta_1 + \varepsilon^2 \theta_2 + \cdots \end{aligned}} \qquad (16.2)$$

INSERT IN EIKONAL EQ. $|\nabla\theta|^2 = u^2$

$O(1)$ TERMS: $$|\nabla\theta_0|^2 = u_0^2 \qquad (16.3)$$

$O(\varepsilon)$ TERMS: $$(\nabla\theta_0 \cdot \nabla\theta_1) = u_0 u_1 \qquad (16.4)$$

HERE WE USED $$|\nabla\theta|^2 = (\nabla\theta \cdot \nabla\theta) = (\nabla\theta_0 + \varepsilon\nabla\theta_1 + \cdots)\cdot(\nabla\theta_0 + \varepsilon\nabla\theta_1 + \cdots)$$
$$= (\nabla\theta_0 \cdot \nabla\theta_0) + 2\varepsilon(\nabla\theta_0 \cdot \nabla\theta_1) + \cdots$$

ALSO $$T = \int \nabla\theta_0 \cdot \hat{n}\, ds_0 + \varepsilon \int \frac{d\theta_1}{ds_0}\, ds_0 + \cdots$$

(16.3) $\rightarrow \quad |\nabla\theta_0|^2 = u_0^2 \quad \rightarrow \quad \nabla\theta_0 = u_0 \hat{n}_0$

$$T_0 = \int \frac{d\theta_0}{ds_0}\, ds_0 = \int \hat{n}\cdot\nabla\theta_0 \, ds_0 = \int u_0 (\hat{n}_0\cdot\hat{n}_0)\, ds_0$$

$$\boxed{T_0 = \int_{\vec{r}_0} u_0 \, ds_0} \qquad (17.1)$$

(16.4) $\rightarrow \quad (\nabla\theta_0\cdot\nabla\theta_1) = u_0 u_1$

USE THAT $\nabla\theta_0 = u_0\hat{n}_0 \rightarrow u_0(\hat{n}_0\cdot\nabla\theta_1) = u_0 u_1$

$$\frac{d\theta_1}{ds_0} = u_1 \qquad (17.2)$$

ALONG REFERENCE RAY!

$$\boxed{T_1 = \int_{\vec{r}_0} u_1 \, ds_0} \qquad (17.3)$$

<u>THE TRAVELTIME PERTURBATION IS THE INTEGRAL OF THE SLOWNESS PERTURBATION ALONG THE REFERENCE RAY</u>

THIS IS THE CRUX OF THE MATTER IN SEISMIC TOMOGRAPHY!

RAY PERTURBATION THEORY

- DETERMINE BEHAVIOUR OF NEIGHBOURING RAYS TO COMPUTE AMPLITUDE
  → EQUATION OF DYNAMIC RAY TRACING
- DETERMINE EFFECT OF VELOCITY PERTURBATION ON RAY POSITION AND TRAVEL TIME
- ASSES HOW CHANGES IN ENDPOINTS OF THE RAYS AFFECT THE RAY POSITION
- DETERMINE HOW INITIAL CONDITIONS OF A RAY MUST BE CHANGED IN ORDER TO 'HIT' A RECEI (IN SHOOTING APPLICATIONS)
- IF ONLY AN ESTIMATE OF A RAY IS KNOWN, DEFORM THIS CURVE TOWARDS THE TRUE RAY (IS NORMALLY KNOWN AS 'BENDING')

Starting point

Equation of kinematic ray tracing

$$\frac{d}{ds}\left(u(\mathbf{r}) \frac{d\mathbf{r}}{ds} \right) = \nabla u(\mathbf{r})$$

Travel time integral

$$T = \int_{\mathbf{r}(s)} u(\mathbf{r}(s))\, ds$$

Three perturbations

1) Slowness perturbation:

$$u(\mathbf{r}) = u_0(\mathbf{r}) + \epsilon u_1(\mathbf{r})$$

2) Reference ray need not be a true ray:

$$\nabla u_0 - \frac{d}{ds_0}\left(u_0 \frac{d\mathbf{r}_0}{ds_0} \right) \equiv \epsilon \mathbf{R}_b$$

3) Endpoint perturbations:

$$\mathbf{r}(0) = \mathbf{r}_0(0) + \epsilon \mathbf{a} \quad , \quad \mathbf{r}(S_0) = \mathbf{r}_0(S_0) + \epsilon \mathbf{a}$$

Then

$$br(s_0) = \mathbf{r}_0(s_0) + \epsilon \mathbf{r}_1(s_0) + \epsilon^2 \mathbf{r}_2(s_0) + \ldots..$$

$$T = T_0 + \epsilon T_1 + \epsilon T^2 + \ldots.$$

Role of the stretch factor

$$\left(\frac{d}{ds_0}(u_0\dot{\mathbf{r}}_1) \right)_{\perp} + \mathbf{M}'\cdot\dot{\mathbf{r}}_1 + \mathbf{N}'\cdot\mathbf{r}_1 = \mathbf{R}_b + \mathbf{R}_1$$

- First order system for component // reference ray

$$\frac{\partial s}{\partial s_0} = 1 + \epsilon(\dot{\mathbf{r}}_0\cdot\dot{\mathbf{r}}_1)$$

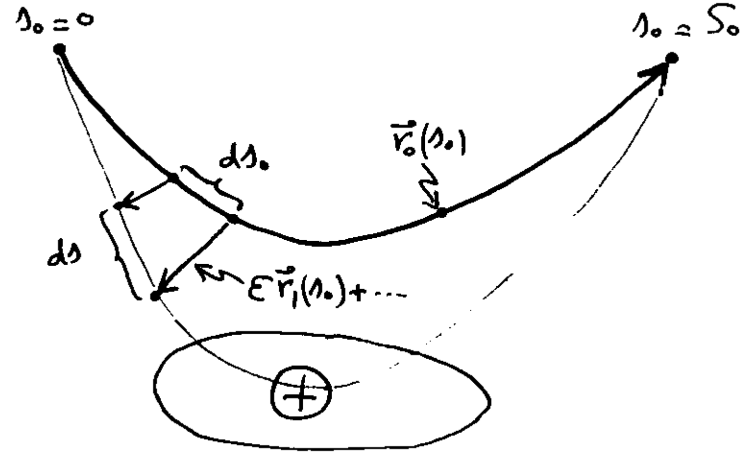
$$u_0(\dot{\mathbf{r}}_0\cdot\dot{\mathbf{r}}_1) = (\mathbf{g}\cdot\mathbf{r}_1) + F$$

$= \textit{momentum along reference curve}$

$$\frac{d}{ds_0}(u_0\dot{\mathbf{r}}_1) + \mathbf{M}\cdot\dot{\mathbf{r}}_1 + \mathbf{N}\cdot\mathbf{r}_1 = \mathbf{R}_b + \mathbf{R}_1$$

$+$ *terms dependent on F.bp* **Exa**

Two effects (physics)



Two effects (mathematics)

1) Change in path length

$$\frac{d}{ds} = \frac{\partial s_0}{\partial s}\,\frac{d}{ds_0}$$

$$\frac{\partial s_0}{\partial s} = 1 - \epsilon(\dot{\mathbf{r}}_0 \cdot \dot{\mathbf{r}}_1) + \epsilon^2 \ldots\ldots$$

2) Change in slowness sampling

$$u(\mathbf{r}) = u_0(\mathbf{r}_0) + \epsilon\left(u_1(\mathbf{r}_0) + \mathbf{r}_1 \cdot \nabla u_0(\mathbf{r}_0)\right) + \epsilon^2 \ldots\ldots$$

N.B. $\dot{} \equiv \dfrac{d}{ds_0}$

Travel time

$$T_1 = \int_0^{S_0} u_1\, ds_0 + [u_0(\dot{\mathbf{r}}_0 \cdot \mathbf{r}_1)]_0^{S_0}$$

$$T_2 = \frac{1}{2}\int_0^{S_0} \mathbf{r}_1 \cdot (\mathbf{R}_b + \mathbf{R}_1)\, ds_0$$

+ *terms quadratic in endpoint perturbations* and

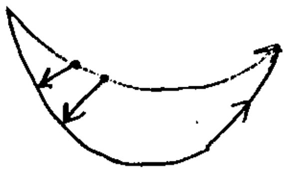
Note that T_2:

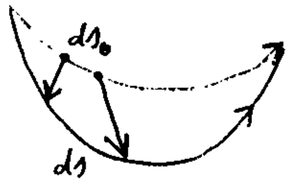
- Does not depend on $\mathbf{r}_2$
- Can easily be computed once $\mathbf{r}_1$ is known
- Contains nonlinear source relocation effects such as $[\,u_1\,(\dot{\mathbf{r}}_0 \cdot \mathbf{r}_1)\,]_0^{S_0}$

Ray perturbation

$$\frac{d}{ds_0}(u_0\dot{\mathbf{r}}_1) + \mathbf{M}\cdot\dot{\mathbf{r}}_1 + \mathbf{N}\cdot\mathbf{r}_1 = \mathbf{R}_b + \mathbf{R}_1$$

$$\mathbf{R}_1 = u_0\nabla_{\perp}\left(\frac{u_1}{u_0}\right)$$

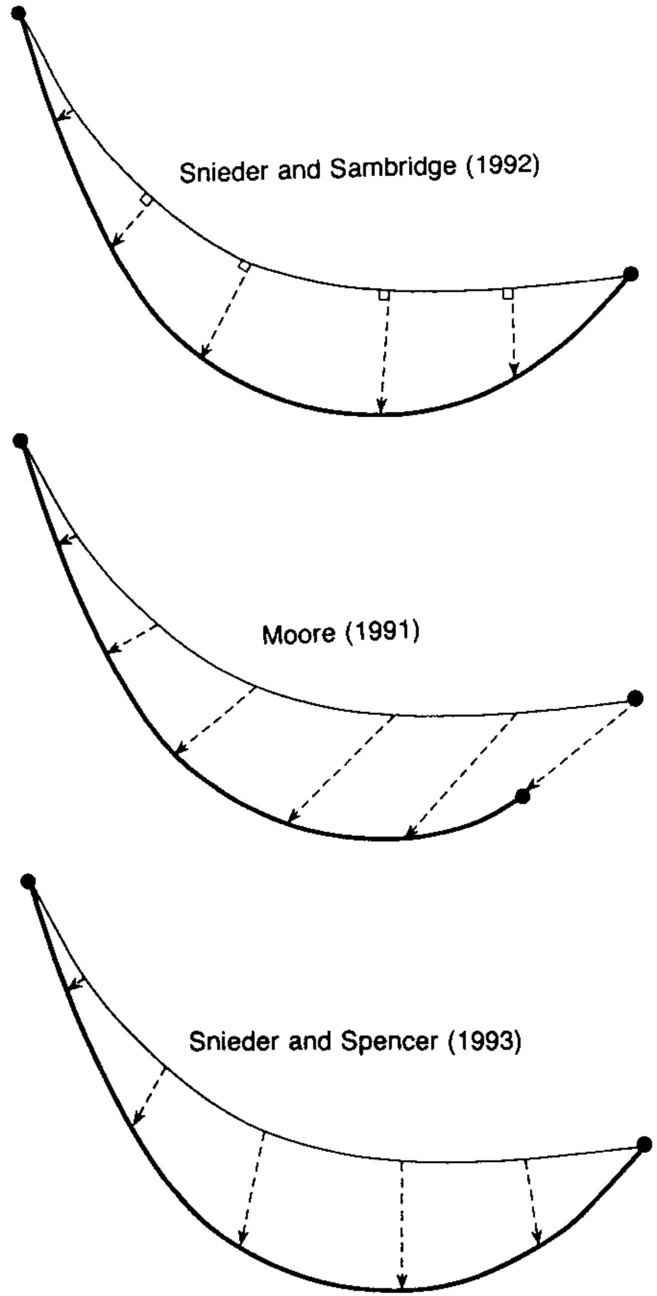




For $\dfrac{\partial s}{\partial s_0}$ = *constant*

$$\mathbf{M} \equiv \dot{\mathbf{r}}_0\nabla u_0 - 2\nabla u_0\dot{\mathbf{r}}_0$$

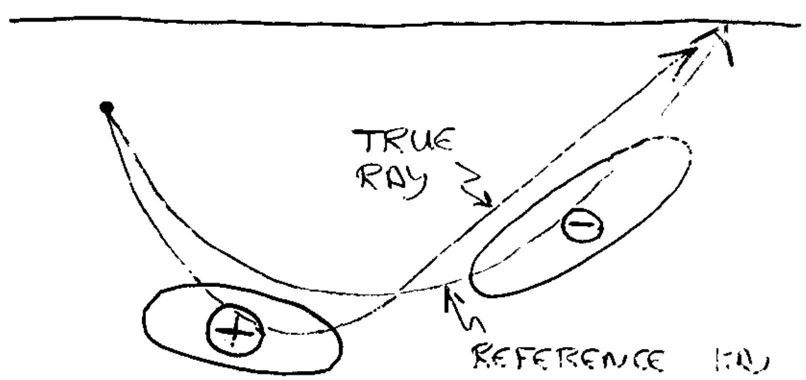
$$\mathbf{N} \equiv \ddot{\mathbf{r}}_0\nabla u_0 + \dot{\mathbf{r}}_0\frac{d}{ds_0}\nabla u_0 - \nabla\nabla u_0$$



Example: $\mathbf{g} = \nabla u_0,\ F = u_0 C$

$$\frac{d}{ds_0}(u_0\dot{\mathbf{r}}_1) - \frac{1}{2u_0}\nabla\nabla u_0^2 = \mathbf{R}_b + \mathbf{R}_1 + 2C\nabla u_0$$

TRAVEL TIME BIAS



$$T_{TRUE} \leqslant T_{REF}$$

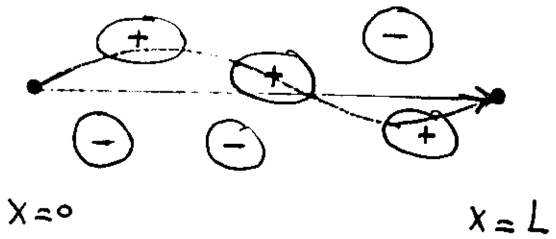
$$\boxed{\delta c_{TRUE} \leqslant \delta c_{REF}}$$

BIAS!

Velocity bias in a random medium

- Reference medium homogeneous (u_0 = *constant*)

- Gaussian autocorrelation function, correlation length a



$$c_{app} = c_0 \left(1 + \frac{\sqrt{\pi}}{3} \mu_0^2 \frac{L}{a} \right)$$

The apparent velocity is not an intensive property of the medium!

Ray perturbation

For $\frac{\partial s}{\partial s_0}$ = *constant*

$$\frac{d}{ds_0}(u_0 \dot{\mathbf{r}}_1) + \mathbf{M} \cdot \dot{\mathbf{r}}_1 + \mathbf{N} \cdot \mathbf{r}_1 = \mathbf{R}_b + \mathbf{R}_1$$

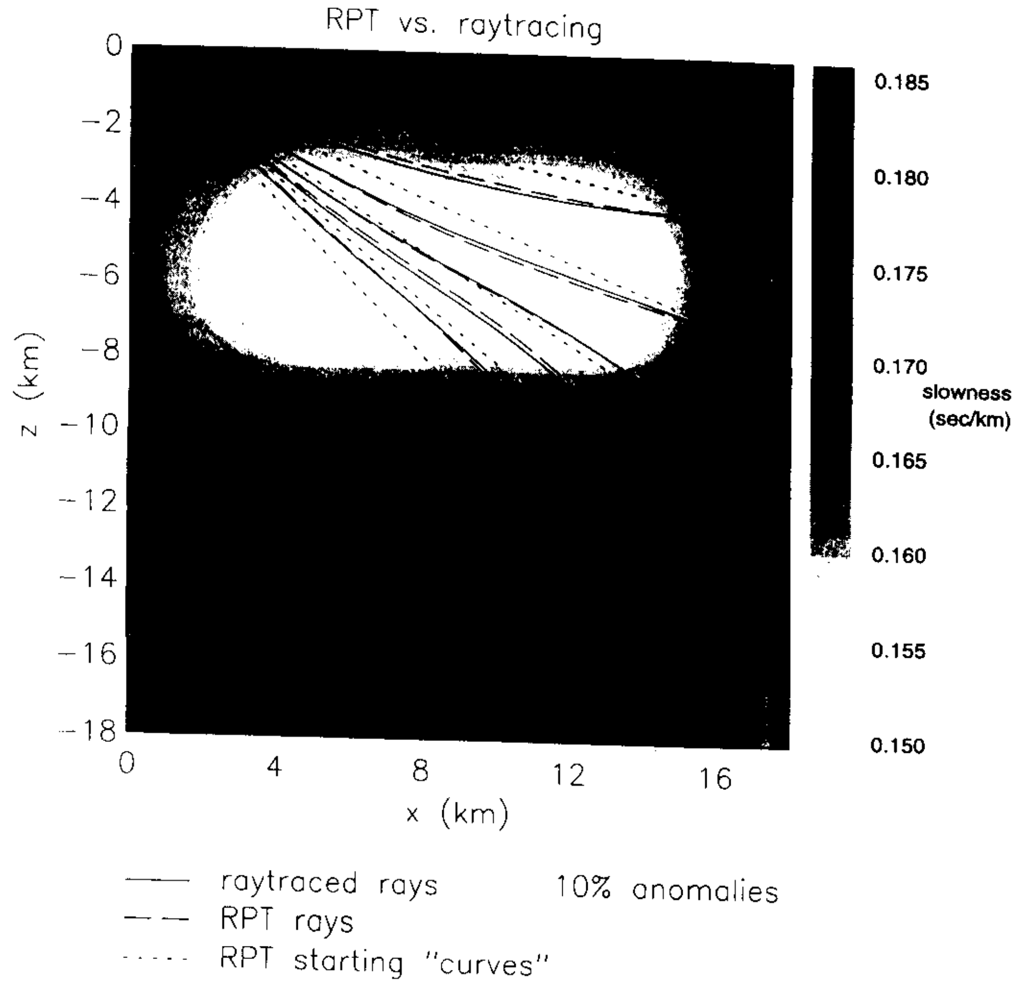
$$\mathbf{R}_1 = u_0 \nabla_{\perp} \left(\frac{u_1}{u_0} \right)$$

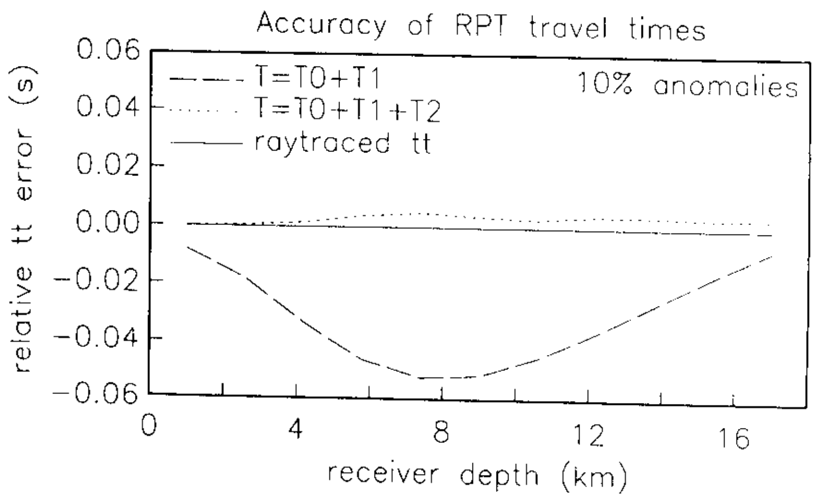
$$\mathbf{M} \equiv \dot{\mathbf{r}}_0 \nabla u_0 - 2 \nabla u_0 \dot{\mathbf{r}}_0$$

$$\mathbf{N} \equiv \ddot{\mathbf{r}}_0 \nabla u_0 + \dot{\mathbf{r}}_0 \frac{d}{ds_0} \nabla u_0 - \nabla \nabla u_0$$

fig 7

fig 7


RPT vs. raytracing
0
-2
-4
-6
-8
-10
-12
-14
-16
-18
z (km)
0
4
8
12
16
x (km)
0.185
0.180
0.175
0.170
0.165
0.160
0.155
0.150
slowness (sec/km)
raytraced rays
RPT rays
RPT starting "curves"
10% anomalies


Accuracy of RPT travel times
0.06
0.04
0.02
0.00
-0.02
-0.04
-0.06
relative tt error (s)
T=T0+T1
T=T0+T1+T2
raytraced tt
10% anomalies
0
4
8
12
16
receiver depth (km)

Advantages of combining

1) Slowness perturbations
2) True bending ($\mathbf{R}_b \neq 0$)
3) Endpoint perturbations

in seismic tomography

I) In tomographic inversions one can **simultaneously** update the slowness model, the source location and the ray position.

II) One can explicitly account for the fact that the 'reference ray' is not a true ray. (**Tomography without true rays!**)

Furthermore

By a suitable choice of the stretch factor ($\mathbf{g}$ and F) one can obtain solutions that can be solved analytically for a variety of parameterizations.

Travel time

$$T_1 = \int_0^{S_0} u_1 \, ds_0 + [u_0(\dot{\mathbf{r}}_0 \cdot \mathbf{r}_1)]_0^{S_0}$$

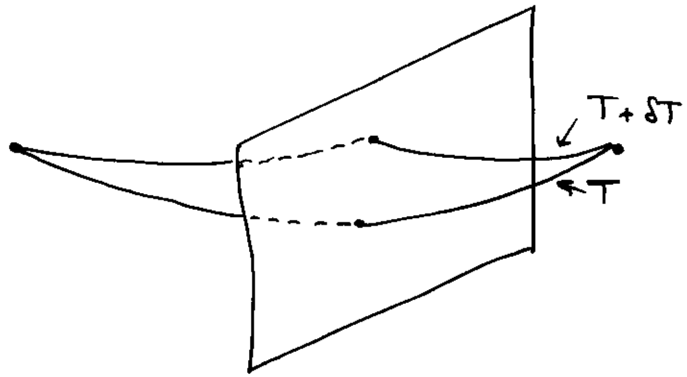
$$T_2 = \frac{1}{2} \int_0^{S_0} \mathbf{r}_1 \cdot (\mathbf{R}_b + \mathbf{R}_1) \, ds_0$$

$+$ *endpoint terms*

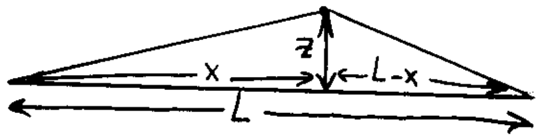
Note that T_2:

- Does not depend on $\mathbf{r}_2$
- Can easily be computed once $\mathbf{r}_1$ is known
- Contains nonlinear source relocation effects such as $[\, u_1 \, (\dot{\mathbf{r}}_0 \cdot \mathbf{r}_1) \,]_0^{S_0}$

HOW WIDE IS A RAY?



FOR A STRAIGHT REFERENCE RAY:



$$\delta T = \frac{1}{c}\left\{\sqrt{x^2+z^2} + \sqrt{(L-x)^2+z^2} - L\right\} = \text{DETOUR TIME}$$

TAYLOR EXPAND: $\sqrt{x^2+z^2} = x\sqrt{1+\frac{z^2}{x^2}} \approx x\left(1+\frac{z^2}{2x^2}\right) = x + \frac{z^2}{2x}$ (19.1)

$$\boxed{\delta T \approx \frac{1}{c}\left\{\frac{1}{2x} + \frac{1}{2(L-x)}\right\} z^2} \quad (19.2)$$

DEFINITION OF FIRST FRESNEL ZONE: SET OF POINTS FOR WHICH DETOUR TIME $< \frac{1}{4}$ · PERIOD

$$\delta T < \frac{1}{4}\frac{\lambda}{c} \quad (20.1)$$

OR

$$\left\{\frac{1}{2x} + \frac{1}{2(L-x)}\right\} z^2 < \frac{\lambda}{4} \quad (20.2)$$

OR

$$z < \sqrt{\frac{\lambda x (L-x)}{2L}} \quad (20.3)$$

THIS HALF-WIDTH IS MAXIMAL FOR $x = L/2$

WIDTH OF FRESNEL ZONE:

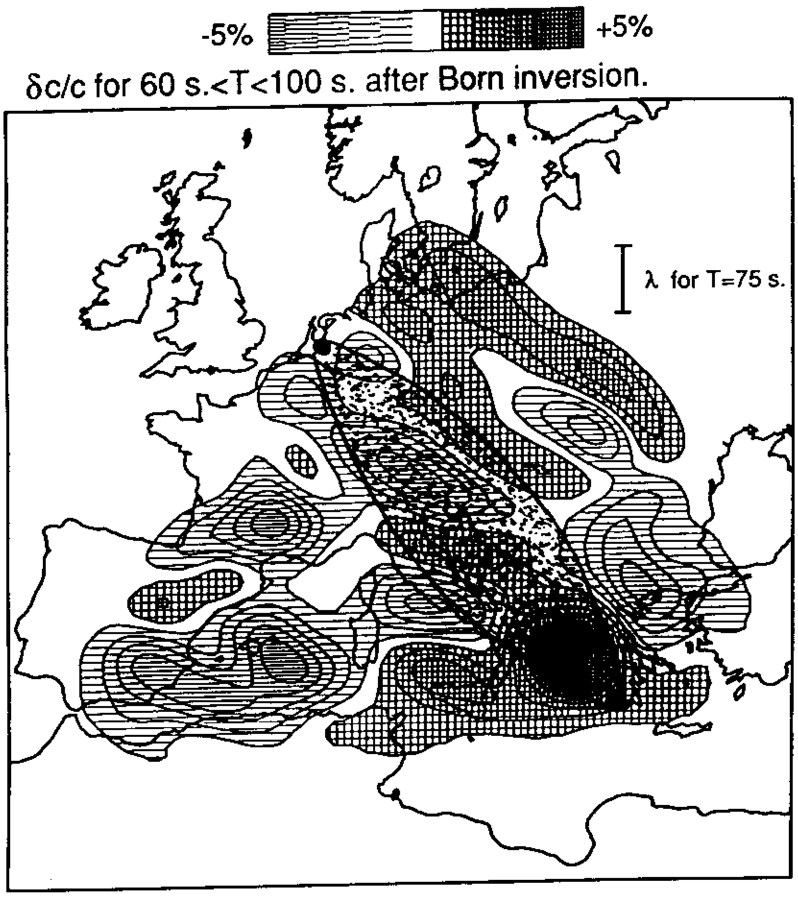
$$\boxed{L_F = \sqrt{\frac{\lambda L}{8}}} \quad (20.4)$$

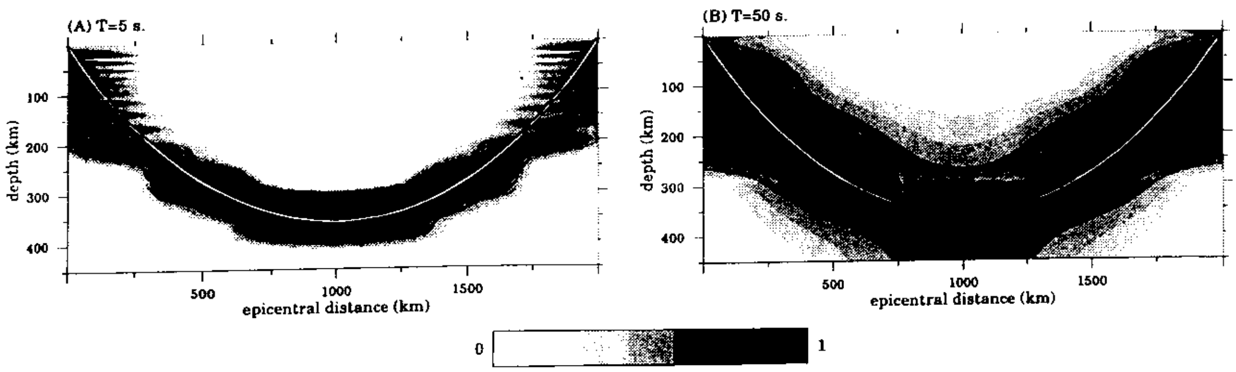
- SINCE $L \gg \lambda$ IN GENERAL $L_F \gg \lambda$
- FOR A P-WAVE OF 1 SEC. IN THE MANTLE AT TELESEISMIC DISTANCE:

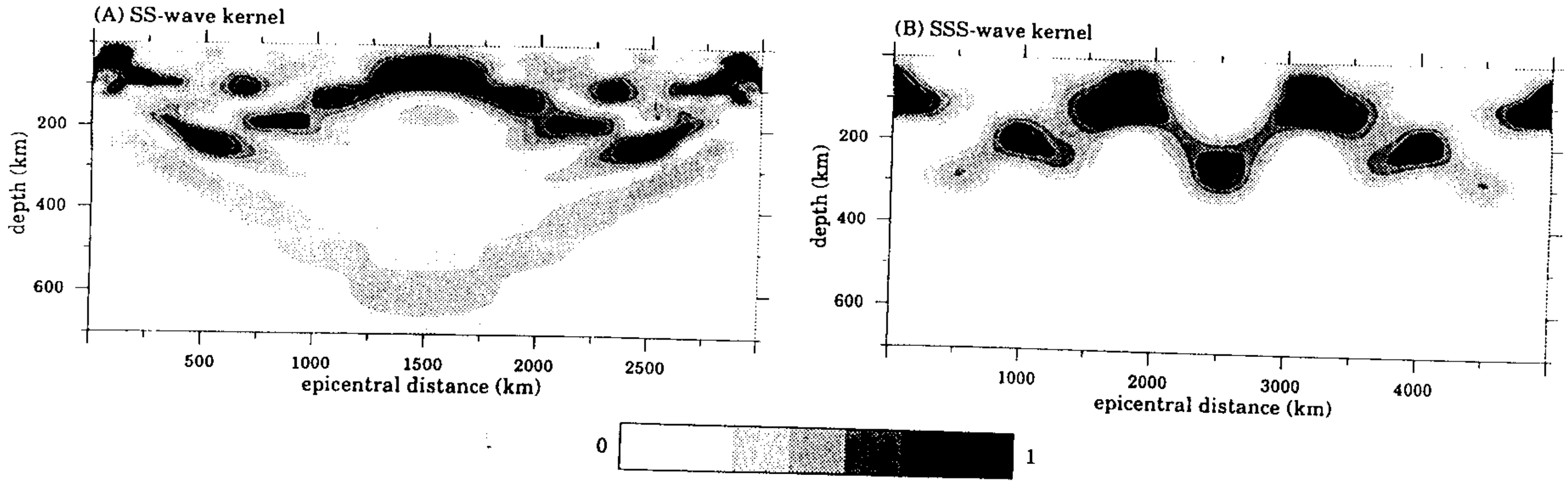
$\lambda = c \times \text{period} = 10\ \text{km/sec} \times 1\ \text{sec} = 10\ \text{km}$

$L = 10^4\ \text{km}$

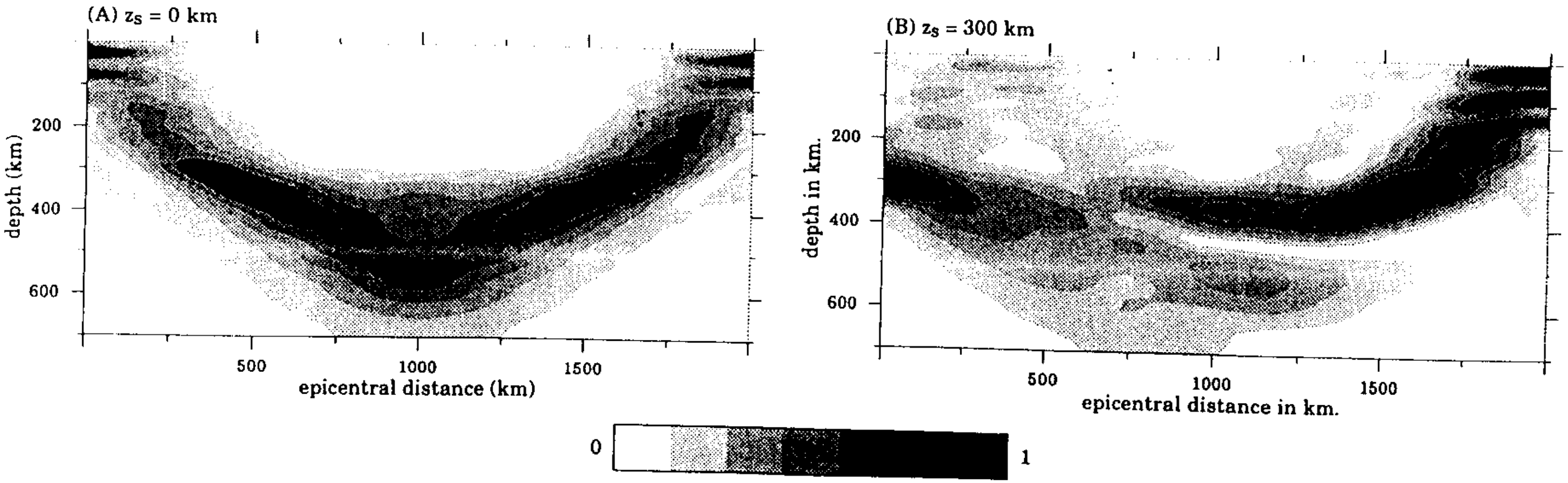
$\boxed{L_F = 150\ \text{km}}$ = HALF WIDTH


-5%
+5%
δc/c for 60 s.<T<100 s. after Born inversion.
λ for T=75 s.

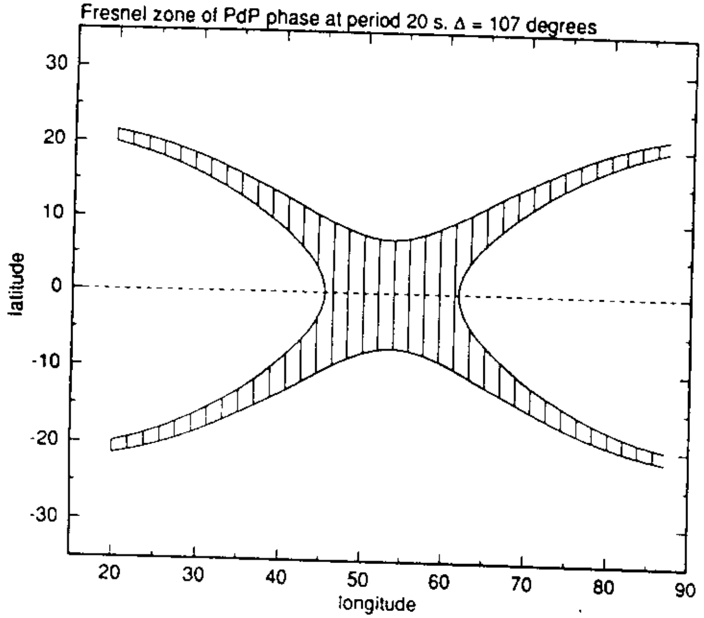

(A) T=5 s.
(B) T=50 s.
depth (km)
100
200
300
400
500
1000
1500
epicentral distance (km)
0
1



(A) SS-wave kernel
depth (km)
200
400
600
500
1000
1500
2000
2500
epicentral distance (km)
(B) SSS-wave kernel
depth (km)
200
400
600
1000
2000
3000
4000
epicentral distance (km)
0
1

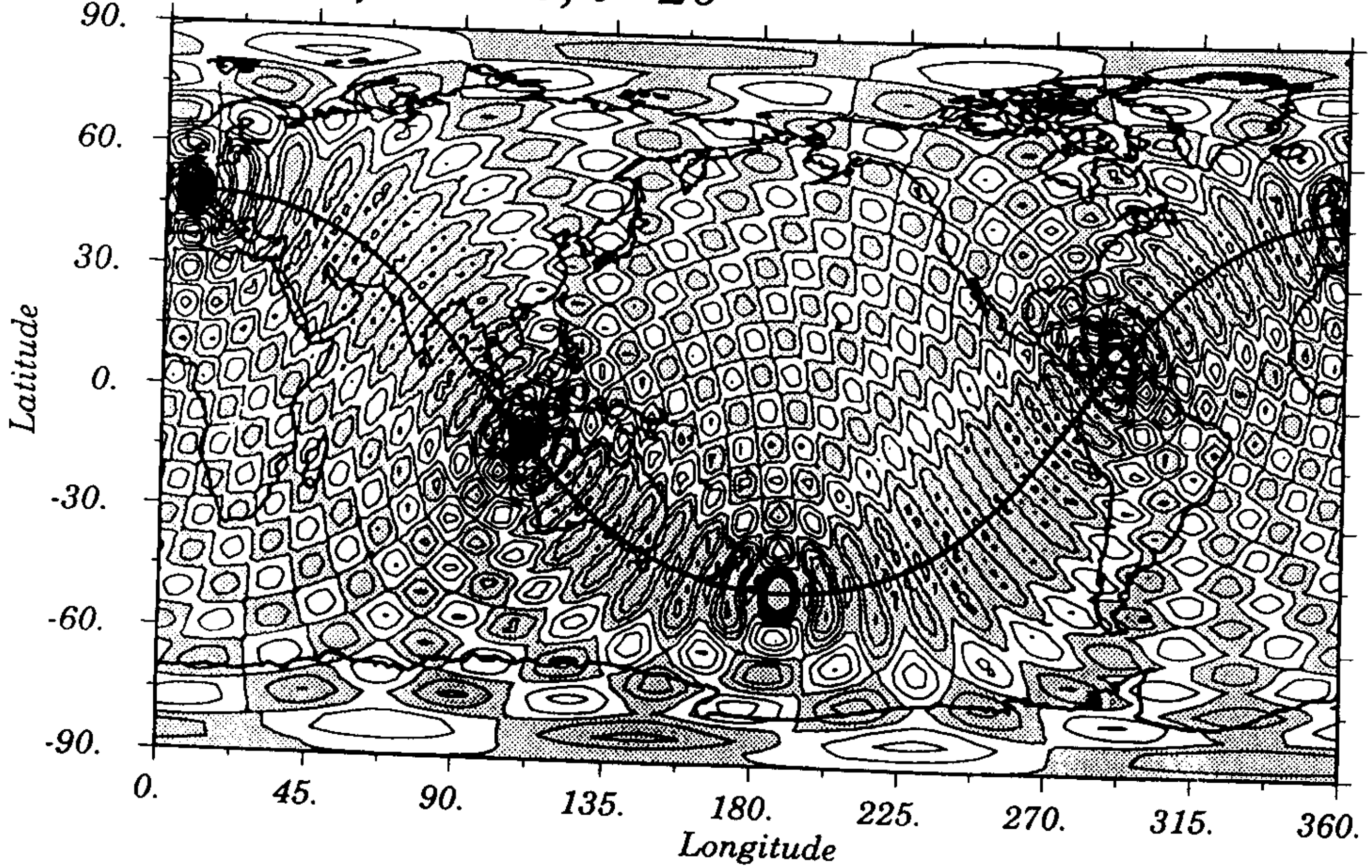


(A) z_S = 0 km
depth (km)
200
400
600
500
1000
1500
epicentral distance (km)
(B) z_S = 300 km
depth in km.
200
400
600
500
1000
1500
epicentral distance in km.
0
1



FOR 1 Hz BODY WAVE
(ISC DATA)
$L_F \sim$ SEVERAL HUNDRED KM.

Kernel for l=20, l'=20



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