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Perturbation Theory for Travel Times

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Perturbation theory for travel times

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In many applications such as travel time tomography, one needs to investigate the effect of perturbations of a slowness model on travel times. In order to carry this out efficiently, a perturbation theory is developed for the travel times along rays under perturbations of the slowness model. The derivation is based on a perturbation analysis of the eikonal equation and leads to simple expressions for the travel time perturbation to arbitrary order. An explicit proof is given that the second-order travel time perturbation, the travel time bias, is equivalent to the second-order travel perturbation that was derived previously. It is also shown that a bending scheme where the slowness model is fixed and where one iteratively updates the travel times can also be based on a perturbation treatment of the eikonal equation. © *1995 Acoustical Society of America.*

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INTRODUCTION

Ray perturbation theory describes how changes in the slowness field alter ray positions and travel times.[1-11] This theory is ideally suited for nonlinear travel time tomography since in this application one updates a slowness model iteratively in the course of the inversion process. Ray perturbation theory can then efficiently be used to determine the effect of the slowness updates on the ray positions. Because of Fermat's theorem, the travel time is to first order not affected by perturbations of the ray positions. This principle forms the basis for linearized travel time tomography.[12,13] The change of the ray positions due to the change in the slowness models leads to higher-order perturbations for the travel times. To leading order, this effect is described by the expressions for the second-order travel time perturbations derived in Refs. 8–10. These derivations for the second travel time perturbations are rather complex. This is in clear contrast with the extremely simple derivation of Fermat's principle of Aldridge.[12] The reason for this simplicity is that in this derivation of Fermat's principle, perturbation theory was applied to the eikonal, rather than to the ray positions, and in a second stage to the travel time integral.[8-10]

In this paper, higher-order perturbation theory is applied to the eikonal. This leads to simple expressions for the travel time perturbations that can recursively be computed to any order. The perturbation equations for the eikonal for a perturbation of the slowness is derived in Sec. I. In Sec. II, it is shown explicitly that the resulting second-order travel time perturbation is identical to the second-order travel time perturbation derived by Snieder and Sambridge.[8] Some details of this proof are given in two appendices. The perturbation theory of Sec. I can thus be regarded as a higher-order extension of the second-order perturbation theory of Snieder and Sambridge.[8] In Sec. IV, it is shown that perturbation theory for the eikonal can not only be used to study the effect of slowness perturbations on the travel times, but that it can also be used to iteratively update the estimate of the eikonal for a fixed slowness. This corresponds to a bending approach for rays.[14-17]

I. THE PERTURBATION EQUATIONS FOR THE EIKONAL

The starting point for perturbation theory for the travel time is the eikonal equation

$$|\nabla T|^2 = u^2, \tag{1}$$

where $u(\mathbf{r})$ is the slowness. The ray position follows from the eikonal through the relation

$$\frac{d\mathbf{r}}{ds} = \frac{1}{u}\,\nabla T. \tag{2}$$

Now assume that a reference slowness $u_0(\mathbf{r})$ is perturbed:

$$u(\mathbf{r}) = u_0(\mathbf{r}) + \epsilon u_1(\mathbf{r}). \tag{3}$$

The small parameter ϵ is used to facilitate a systematic perturbation approach. It is assumed that the resulting perturbation in the eikonal can be expressed in a perturbation series

$$T(\mathbf{r}) = T_0(\mathbf{r}) + \epsilon T_1(\mathbf{r}) + \epsilon^2 T_2(\mathbf{r}) + \cdots . \tag{4}$$

Note that it is tacitly assumed that the perturbation problem is regular. This assumption breaks down when the wave field has passed through caustics and multipathing occurs.

Inserting (3) and (4) in the eikonal equation and collecting the terms with equal powers of ϵ leads to the following equations for the perturbation of the eikonal:

$$|\nabla T_0|^2 = u_0^2, \tag{5a}$$

$$(\nabla T_0 \cdot \nabla T_1) = u_0 u_1, \tag{5b}$$

$$(\nabla T_0 \cdot \nabla T_2) = \tfrac{1}{2}(u_1^2 - |\nabla T_1|^2), \tag{5c}$$

$$(\nabla T_0 \cdot \nabla T_n) = -\sum_{m=1}^{\mathrm{int}(n/2)} (1 - \tfrac{1}{2}\delta_{m,n-m}) \times (\nabla T_m \cdot \nabla T_{n-m}) \quad (n \geq 3). \tag{5d}$$

In the last expression $\mathrm{int}(n/2)$ denotes the largest integer smaller or equal than $n/2$. For example, $(\nabla T_0 \cdot \nabla T_4) = -(\nabla T_1 \cdot \nabla T_3) - \frac{1}{2}(\nabla T_2 \cdot \nabla T_2)$.

Let the rays associated with the reference eikonal T_0 be denoted by $\mathbf{r}_0$. The unit vector tangent to the reference rays is denoted with $\hat{\mathbf{t}}_0$. This vector can be used to solve Eqs. (5a)–(5d). Since the reference rays are perpendicular to the wavefronts, the vector $\hat{\mathbf{t}}_0$ is parallel to the gradient of the reference eikonal T_0. With (5a) this implies that

$$\nabla T_0 = u_0 \hat{\mathbf{t}}_0. \tag{6}$$

The left-hand side of the perturbation equations (5b)–(5d) is of the form $(\nabla T_0 \cdot \nabla T_n)$. With (6) these terms can be written as

$$(\nabla T_0 \cdot \nabla T_n) = u_0 (\hat{\mathbf{t}}_0 \cdot \nabla T_n) = u_0 \frac{dT_n}{ds_0}. \tag{7}$$

In this expression we used the directional derivative $\hat{\mathbf{t}}_0 \cdot \nabla$ as the derivative d/ds_0 along the reference ray. The arclength along the reference ray is denoted with s_0 in order to distinguish it from the arclength s along the perturbed ray. As shown by Snieder and Sambridge[10] this distinction is crucial in ray perturbation theory for the ray displacement.

With (7), the perturbation equations (5b)–(5d) can be written as

$$\frac{dT_1}{ds_0} = u_1, \tag{8a}$$

$$\frac{dT_2}{ds_0} = \frac{1}{2u_0} (u_1^2 - |\nabla T_1|^2). \tag{8b}$$

$$\frac{dT_n}{ds_0} = \frac{-1}{u_0} \sum_{m=1}^{\mathrm{int}(n/2)} (1 - \tfrac{1}{2}\delta_{m,n-m})(\nabla T_m \cdot \nabla T_{n-m}) \quad (n \geq 3). \tag{8c}$$

These equations can be integrated along the reference ray $\mathbf{r}_0(s_0)$:

$$T_1 = \int_{\mathbf{r}_0} u_1 \, ds_0, \tag{9a}$$

$$T_2 = \int_{\mathbf{r}_0} \frac{1}{2u_0} (u_1^2 - |\nabla T_1|^2) ds_0, \tag{9b}$$

$$T_n = -\sum_{m=1}^{\mathrm{int}(n/2)} (1 - \tfrac{1}{2}\delta_{m,n-m}) \int_{\mathbf{r}_0} \frac{1}{u_0} (\nabla T_m \cdot \nabla T_{n-m}) ds_0 \tag{9c}$$

The first-order equation (9a) states that the first-order trav[el] time perturbation is the integral of the slowness perturbati[on] along the reference ray.[12] This expression forms the basis f[or] linearized travel time tomography. The second-order tra[vel] time perturbation (9b) describes the leading-order effect [of] the ray bending due to the slowness perturbations on [the] travel time. The higher-order perturbations of the eikonal [are] given by (9c). Once the reference rays $\mathbf{r}_0$ are known, the travel time perturbations can be computed recursively from expressions (9a)–(9c).

II. THE SECOND-ORDER TRAVEL TIME PERTURBATION

The second-order perturbation T_2 of the eikonal is of particular interest because it describes the leading-order effect of the ray bending due to slowness perturbations on the travel time. Since first arrival rays are curves that yield a minimum travel time, these ray bending effects always reduce the travel time of first arrivals. Ignoring this may result in a bias in velocity models obtained from linear travel time tomography.[18,19] Expressions for the second-order travel time perturbation were obtained from ray perturbation theory.[8-10] Although the final expressions for the second-order travel time perturbation in Refs. 8–10 are quite simple, the derivation of these expressions is rather complex. It will be shown in this section that the second-order perturbation (9b) of the eikonal is equivalent to the second-order travel time perturbation derived by Snieder and Sambridge.[8] Since the theory of Sec. I is for the eikonal at a given location $\mathbf{r}$, the comparison with the second-order theory of Snieder and Sambridge[8] is made for a ray with fixed end points.

The ray perturbation theory of Snieder and Sambridge[8] is based on ray-centered coordinates. In this formulation two mutually orthogonal unit vectors $\hat{\mathbf{q}}_1$ and $\hat{\mathbf{q}}_2$ are defined that are perpendicular to the reference ray. This means that the system $(\hat{\mathbf{t}}_0, \hat{\mathbf{q}}_1, \hat{\mathbf{q}}_2)$ is used as an orthonormal basis to describe the ray perturbation. As shown in Ref. 8, one is still free to define the rotation of the unit vectors $\hat{\mathbf{q}}_1$ and $\hat{\mathbf{q}}_2$ around the reference ray. Since expression (61) in Ref. 8 for the second-order travel time perturbation does not depend on this rate of rotation, the rate of rotation is set equal to zero, i.e., in the notation of Ref. 8 it is used that $\Omega = 0$.

Any vector $\mathbf{v}$ can be decomposed into a component along the reference ray and a perpendicular component $\mathbf{v}^P$:

$$\mathbf{v}^P \equiv \mathbf{v} - \hat{\mathbf{t}}_0 (\hat{\mathbf{t}}_0 \cdot \mathbf{v}). \tag{10}$$

Using this notation one can rewrite expression (9b) by decomposing ∇T_1 as

$$\nabla T_1 = \hat{\mathbf{t}}_0 \frac{dT_1}{ds_0} + \nabla^P T_1 = u_1 \hat{\mathbf{t}}_0 + \nabla^P T_1, \tag{11}$$

[...]plies that

(12)

(13)

[...]perturbation of [...]rder change in [...]ness perturba-

[...]e second-order [...] perturbation of [...]ion (3):

$$\mathbf{r}=\mathbf{r}_0+\epsilon\mathbf{r}_1+\epsilon^2\mathbf{r}_2+\cdots. \tag{14}$$

In order to establish the relation with these perturbations schemes, the relation between the ray perturbation and the perturbation of the eikonal is needed. According to expression (2) one needs to evaluate the derivative $d\mathbf{r}/ds$. The unit vector along the unperturbed ray is given by $d\mathbf{r}_0/ds_0$; hence

$$\hat{\mathbf{t}}_0=d\mathbf{r}_0/ds_0. \tag{15}$$

As pointed out by Snieder and Sambridge,[10] when deriving the perturbation in the ray position it is crucial to account for the fact that the arclength s_0 along the reference ray and the arclength s along the perturbed ray are different. This should be accounted for when taking the derivatives d/ds and d/ds_0. The increment ds can to first order be found from the relation

$$ds=\sqrt{d\mathbf{r}\cdot d\mathbf{r}}=(d\mathbf{r}_0\cdot d\mathbf{r}_0+2\epsilon\, d\mathbf{r}_0\cdot d\mathbf{r}_1)^{1/2}$$
$$=\left(1+2\epsilon\frac{d\mathbf{r}_0}{ds_0}\cdot\frac{d\mathbf{r}_1}{ds_0}\right)^{1/2}ds_0=\left(1+\epsilon\hat{\mathbf{t}}_0\cdot\frac{d\mathbf{r}_1}{ds_0}\right)ds_0, \tag{16}$$

where (14) has been used to expand $d\mathbf{r}$. From this relation it follows that to first order in ϵ,

$$\frac{\partial s_0}{\partial s}=\left(1+\epsilon\hat{\mathbf{t}}_0\cdot\frac{d\mathbf{r}_1}{ds_0}\right)^{-1}=\left(1-\epsilon\hat{\mathbf{t}}_0\cdot\frac{d\mathbf{r}_1}{ds_0}\right). \tag{17}$$

To first order the derivatives with respect to these quantities are thus related by

$$\frac{d}{ds}=\frac{\partial s_0}{\partial s}\frac{d}{ds_0}=(1-\epsilon(\hat{\mathbf{t}}_0\cdot\dot{\mathbf{r}}_1))\frac{d}{ds_0}. \tag{18}$$

In this paper the derivative with respect to the arclength along the unperturbed ray is denoted with a dot, e.g., $\dot{F}\equiv dF/ds_0$. By combining (14) and (18) one obtains

$$\frac{d\mathbf{r}}{ds}=\hat{\mathbf{t}}_0+\epsilon(\dot{\mathbf{r}}_1-\epsilon\hat{\mathbf{t}}_0(\hat{\mathbf{t}}_0\cdot\dot{\mathbf{r}}_1))=\hat{\mathbf{t}}_0+\epsilon\dot{\mathbf{r}}_1^P, \tag{19}$$

It is shown in Appendix A that

$$\dot{\mathbf{r}}_1^P=\frac{1}{u_0}\nabla^P T_1. \tag{20}$$

This expression can be used to rewrite the second-order perturbation (13) of the eikonal as

$$T_2=\frac{-1}{2}\int_{\mathbf{r}_0}(\dot{\mathbf{r}}_1^P\cdot\nabla^P T_1)ds_0=\frac{-1}{2}\int_{\mathbf{r}_0}(\dot{\mathbf{r}}_1\cdot\nabla^P T_1)ds_0, \tag{21}$$

where it is used in the last identity that the component of $\dot{\mathbf{r}}_1$ parallel to $\hat{\mathbf{t}}_0$ has a vanishing projection on $\nabla^P T_1$. Using an integration by parts and exploiting that the end points of the ray are assumed to be fixed ($\mathbf{r}_1=0$ at the end points), this expression can be rewritten as

$$T_2=\frac{1}{2}\int_{\mathbf{r}_0}\left(\mathbf{r}_1\cdot\frac{d}{ds_0}\nabla^P T_1\right)ds_0. \tag{22}$$

No choice has been made yet for a coordinate system. In order to establish the relation with the second-order travel time perturbation in Ref. 8, expression (22) is projected in ray-centered coordinates. Any vector perpendicular to the reference ray can be expanded in the unit vectors $\hat{\mathbf{q}}_1$ and $\hat{\mathbf{q}}_2$:

$$\mathbf{v}^P=\sum_{i=1}^{2}\hat{\mathbf{q}}_i(\hat{\mathbf{q}}_i\cdot\mathbf{v}). \tag{23}$$

In ray-centered coordinates, the ray perturbation is by definition perpendicular to the reference ray

$$\mathbf{r}_1=\sum_{i=1}^{2}q_i\hat{\mathbf{q}}_i; \tag{24}$$

the components q_1 and q_2 therefore describe the ray perturbation. It is shown in Appendix B that in this coordinate system

$$\frac{d}{ds_0}\nabla^P T_1=\frac{-1}{u_0}\sum_{i=1}^{2}(\hat{\mathbf{q}}_i\cdot\nabla u_0)(\hat{\mathbf{q}}_i\cdot\nabla T_1)\hat{\mathbf{t}}_0+u_0\nabla^P\left(\frac{u_1}{u_0}\right). \tag{25}$$

When this result is inserted in (22), the first term on the right-hand side of (25) does not contribute because of (24) since $\hat{\mathbf{t}}_0$ is perpendicular to $\hat{\mathbf{q}}_i$. This leads to

$$T_2=\frac{1}{2}\int_{\mathbf{r}_0}u_0\mathbf{r}_1\cdot\nabla^P\left(\frac{u_1}{u_0}\right)ds_0. \tag{26}$$

With (24) this gives

$$T_2=\frac{1}{2}\sum_{i=1}^{2}\int_{\mathbf{r}_0}u_0q_i\hat{\mathbf{q}}_i\cdot\nabla^P\left(\frac{u_1}{u_0}\right)ds_0. \tag{27}$$

This expression is identical to Eq. (61) of Snieder and Sambridge[8] for the second-order travel time perturbation for the case of fixed end points. This implies that the first- and second-order perturbations for the eikonal derived in Sec. I are identical to the previously derived expressions for the first- and second-order travel time perturbations. However, note that the theory of Sec. I can be used to compute the travel time perturbation to arbitrary order. The results of Sec. I are thus a higher-order generalization of the second-order perturbation theory of Snieder and Sambridge.[8] One should note that it is much simpler to use Eq. (9b) for the second-order travel time perturbation than Eq. (26) because in Eq. (9b) the perturbation of the ray location ($\mathbf{r}_1=\Sigma q_i\hat{\mathbf{q}}_i$) is not needed.

III. A BENDING EQUATION FOR THE EIKONAL

In the previous sections the slowness was perturbed, and it was assumed that the reference eikonal satisfied the eikonal equation (5a) for the reference slowness. This approach is similar to the one used in ray perturbation theory, e.g., Refs. 1–8. Independently, ray bending was developed.[14–17] In this approach the slowness is fixed, but the initial ray estimate does not satisfy the equation of kinematic ray tracing for the employed slowness. Using first-order perturbation theory one then iteratively updates the ray estimate until the ray estimate satisfies the equation of kinematic ray tracing within a prescribed tolerance. A similar algorithm is presented in this section for the eikonal. Analogous to the bending methods described above, the idea is to start with an estimate $T_0(\mathbf{r})$ of the eikonal that does not necessarily satisfy the eikonal equation. An update $\epsilon T_1(\mathbf{r})$ is derived so that the new eikonal

$T_{new}=T_0+\epsilon T_1$ satisfies the eikonal equation to first order. Although this constitutes a simple linear problem for the update T_1, this does in general not lead to the true eikonal since the new eikonal T_{new} need not satisfy the eikonal equation to higher order. However, this procedure can be repeated by taking the new eikonal T_{new} as a starting point for the next update. This can be repeated until the estimate of the eikonal satisfies the eikonal equation with a prescribed tolerance.

Assume that the slowness u is fixed and that the reference eikonal $T_0(\mathbf{r})$ does not satisfy the eikonal equation. Let the extent to which $T_0(\mathbf{r})$ violates the eikonal equation be given by

$$u^2(\mathbf{r})-|\nabla T_0(\mathbf{r})|^2=2\epsilon u(\mathbf{r})F(\mathbf{r}). \tag{28}$$

The factor $2u(\mathbf{r})$ is inserted for further notational convenience. Note that it is tacitly assumed here that the extent to which $T_0(\mathbf{r})$ violates the eikonal equation is sufficiently small so that it can be used as a starting point for a perturbation analysis. The term $F(\mathbf{r})$ in the right-hand side plays the same role as the term $\mathbf{R}_b$ in Refs. 9 and 10 which measures to what extent the reference ray violates the equation of kinematic ray tracing in the reference medium.

Inserting (28) and the perturbation series (4) for the eikonal in the eikonal equation (1), one obtains for the $O(\epsilon)$ contribution the following expression:

$$(\nabla T_0\cdot\nabla T_1)=uF. \tag{29}$$

It can be seen from (28) that $|\nabla T_0|^2=u^2+O(\epsilon)$, hence to leading order (7) can be used in (29). This gives

$$T_1=\int_{\mathbf{r}_0} F(\mathbf{r}_0)ds_0. \tag{30}$$

This implies that one can obtain an update for the eikonal using the simple integration (30) along the reference rays. The new eikonal estimate is given by a simple addition: $T_{new}=T_0+\epsilon T_1$. Taking T_{new} as the new estimate of the eikonal one can use Eqs. (29) and (27) recursively to iteratively update the eikonal in the same fashion as the ray positions are updated in bending equations for the ray position.[14–17]

IV. CONCLUSIONS

The results of Sec. I show that one can derive perturbation equations for the eikonal, and hence for the travel time, in a relatively simple fashion to arbitrary order. In general, the following two questions arise for perturbation problems. First, is the perturbation problem regular? Second, to what order must the perturbations be computed? Concerning the first problem, one must be aware of the fact that the perturbation problem of the eikonal is not regular when multipathing occurs. Suppose that in the unperturbed problem a single ray arrives at a given receiver, and that when the slowness perturbation is taken into account multiple rays arrive at a receiver; the slowness perturbation then induces multipathing. This implies that under such a perturbation a single-valued solution of the eikonal equation is perturbed to a multiple-valued solution. Since regular perturbation theory always leads to a single-valued result, it follows that when multipathing is induced by the slowness perturbation, the regular perturbation theory of this paper should be replaced by a formulation based on singular perturbation theory. In case the perturbation problem is indeed regular, the question of how many terms of the perturbation series (4) must be taken into account to achieve a prescribed accuracy arises. It is difficult to make general statements about the required order of perturbation theory. However, increasing the strength of the slowness perturbation, decreasing the length scale of the slowness perturbation, and increasing the path length will all render the perturbation problem more nonlinear. The reason is that all these effects lead to an increased ray bending, and hence to stronger nonlinear travel time perturbations.[8] The theory is therefore most suited for applications in which the travel time perturbation is only mildly nonlinear. This is, for example, the case in geophysical travel time tomography.[20]

The simplicity of the derivation of Sec. I for the travel time perturbation is in sharp contrast with earlier expressions for the second-order travel time perturbation.[8–10] There is a simple reason why the approach of this paper leads to the travel time perturbations in a much simpler way than in previous derivations. The reason is that in Refs. 8–10 a perturbation theory was first developed for the ray position. From this, the perturbation of the travel time integral was computed by integrating the slowness along the perturbed ray. This implies that in Refs. 8–10 the perturbation of the ray position is needed for computing the second-order travel time perturbation. Our derivation avoids this tortuous route by using the eikonal equation directly. Since the eikonal is the travel time, this implies that the perturbation of the travel time is derived without using the ray position at any point in the derivation. This approach has the additional advantage that ambiguities in the mapping from the unperturbed ray to the perturbed ray are avoided.[10] In addition, the theory of Sec. I extends the second-order perturbation theory for the travel times of Refs. 8–10 to any arbitrary higher order.

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APPENDIX A: DERIVATION OF EQ. (20)

In order to establish the relation between the perturbation of the eikonal and the perturbation of the ray position, Eq. (2) is needed. This expression is evaluated at a fixed location $\mathbf{r}$. From (3) one finds, using a Taylor expansion, that to first order

$$\frac{1}{u(\mathbf{r})}=\frac{1}{u_0(\mathbf{r})}\left\{1-\frac{\epsilon}{u_0(\mathbf{r})}u_1(\mathbf{r})\right\}. \tag{A1}$$

Similarly, it follows from (4) that

$$\nabla T(\mathbf{r})=\nabla T_0(\mathbf{r})+\epsilon\nabla T_1(\mathbf{r}). \tag{A2}$$

Using Eqs. (6) and (11) this expression can be written as

$$\nabla T(\mathbf{r}) = u_0\hat{\mathbf{t}}_0 + \epsilon(u_1\hat{\mathbf{t}}_0 + \nabla^P T_1). \tag{A3}$$

Inserting (A1), (A3), and (19) in (2) gives, after cancellation of terms,

$$\hat{\mathbf{t}}_0 + \epsilon\dot{\mathbf{r}}_1^P = \hat{\mathbf{t}}_0 + \frac{\epsilon}{u_0}\nabla^P T_1. \tag{A4}$$

The order ϵ contribution of this expression is given by (20).

APPENDIX B: DERIVATION OF EQ. (25)

In order to determine the derivative of $\nabla^P T_1$ along the reference ray one needs to expand this quantity in ray-centered coordinates. Following (23) this gives

$$\nabla^P T_1 = \sum_{i=1}^{2} \hat{\mathbf{q}}_i(\hat{\mathbf{q}}_i\cdot\nabla T_1). \tag{B1}$$

When the derivative of this expression along the reference ray is taken one needs to account for the fact that the unit vectors $\hat{\mathbf{q}}_i$ are not constant along the reference ray. Equation (43) of Snieder and Sambridge[8] for the special case $\Omega=0$ gives the change of the unit vectors $\hat{\mathbf{q}}_i$ along the reference ray:

$$\dot{\hat{\mathbf{q}}}_i = \frac{-1}{u_0}(\hat{\mathbf{q}}_i\cdot\nabla u_0)\hat{\mathbf{t}}_0. \tag{B2}$$

Using this result and (8a) to eliminate dT_1/ds_0 one obtains

$$\frac{d}{ds_0}\nabla^P T_1 = \frac{-1}{u_0}\sum_{i=1}^{2}(\hat{\mathbf{q}}_i\cdot\nabla u_0)(\hat{\mathbf{q}}_i\cdot\nabla T_1)\hat{\mathbf{t}}_0 - \frac{1}{u_0}\sum_{i=1}^{2}(\hat{\mathbf{q}}_i\cdot\nabla u_0)(\hat{\mathbf{t}}_0\cdot\nabla T_1)\hat{\mathbf{q}}_i + \sum_{i=1}^{2}\hat{\mathbf{q}}_i(\hat{\mathbf{q}}_i\cdot\nabla u_1). \tag{B3}$$

In the second term on the right-hand side one can use with (8a) that $(\hat{\mathbf{t}}_0\cdot\nabla T_1) = dT_1/ds_0 = u_1$. This leads to

$$\frac{d}{ds_0}\nabla^P T_1 = \frac{-1}{u_0}\sum_{i=1}^{2}(\hat{\mathbf{q}}_i\cdot\nabla u_0)(\hat{\mathbf{q}}_i\cdot\nabla T_1)\hat{\mathbf{t}}_0 + \sum_{i=1}^{2}\hat{\mathbf{q}}_i\left((\hat{\mathbf{q}}_i\cdot\nabla u_1) - \frac{u_1}{u_0}(\hat{\mathbf{q}}_i\cdot\nabla u_0)\right). \tag{B4}$$

Using (23) the last terms can be rewritten as

$$\sum_{i=1}^{2}\hat{\mathbf{q}}_i\left((\hat{\mathbf{q}}_i\cdot\nabla u_1) - \frac{u_1}{u_0}(\hat{\mathbf{q}}_i\cdot\nabla u_0)\right) = \nabla^P u_1 - \frac{u_1}{u_0}\nabla^P u_0 = u_0\nabla^P\left(\frac{u_1}{u_0}\right). \tag{B5}$$

Inserting this result in (B4) one arrives at the desired expression (25).

[1] C. H. Chapman, "Ray theory and its extensions: WKBJ and Maslov seismograms," J. Geophys. **58**, 27–43 (1985).
[2] F. A. Dahlen and I. H. Henson, "Asymptotic normal modes of a laterally heterogeneous earth," J. Geophys. Res. **90**, 12653–12681 (1985).
[3] V. Farra and R. Madariaga, "Seismic waveform modelling in heterogeneous media by ray perturbation theory," J. Geophys. Res. **92**, 2697–2712 (1987).
[4] J. Virieux, V. Farra, and R. Madariaga, "Ray tracing for earthquake location in laterally heterogeneous media," J. Geophys. Res. **93**, 6585–6599 (1988).
[5] J. Virieux, "Fast and accurate ray tracing by Hamiltonian perturbation," J. Geophys. Res. **96**, 579–594 (1991).
[6] B. J. Moore, "Seismic rays in media with slight variation in velocity," Geophys. J. Int. **105**, 213–227 (1991).
[7] B. J. Moore, "Analytical solutions for the ray perturbation in depth varying media," Geophys. J. Int. **114**, 281–288 (1993).
[8] R. Snieder and M. Sambridge, "Ray perturbation theory for traveltimes and ray paths in 3-D heterogeneous media," Geophys. J. Int. **109**, 294–322 (1992).
[9] R. Snieder and C. Spencer, "A unified approach to ray bending, ray perturbation and paraxial ray theories," Geophys. J. Int. **115**, 456–470 (1993).
[10] R. Snieder and M. Sambridge, "The ambiguity in ray perturbation theory," J. Geophys. Res. **98**, 22021–22034 (1993).
[11] M. Sambridge and R. Snieder, "Applicability of ray perturbation theory to mantle tomography," Geophys. Res. Lett. **20**, 73–76 (1993).
[12] D. Aldridge, "Linearization of the eikonal equation," Geophysics **59**, 1631–1632 (1994).
[13] G. Nolet, *Seismic Tomography: With Applications in Global Seismology and Exploration Geophysics* (Reidel, Dordrecht, 1987).
[14] B. R. Julian and D. Gubbins, "Three-dimensional seismic ray tracing," J. Geophys. **43**, 95–113 (1977).
[15] V. Pereyra, W. H. Lee, and H. B. Keller, "Solving two-point seismic ray tracing in a heterogeneous medium," Bull. Seismol. Soc. Am. **70**, 79–99 (1980).
[16] V. Farra, "Bending method revisited: A Hamiltonian approach," Geophys. J. Int. **109**, 138–150 (1992).
[17] T. J. Moser, G. Nolet, and R. Snieder, "Ray bending revisited," Bull. Seismol. Soc. Am. **82**, 259–288 (1992).
[18] G. Nolet and T. J. Moser, "Teleseismic delay times in a 3-D earth and a new look at the S discrepancy," Geophys. J. Int. **114**, 185–195 (1993).
[19] M. Roth, G. Muller, and R. Snieder, "Velocity shift in random media," Geophys. J. Int. **115**, 552–563 (1993).
[20] W. Spakman, "Iterative strategies for nonlinear travel-time tomography using global earthquake data," in *Seismic Tomography: Theory and Applications*, edited by H. M. Iyer and K. Hirahara (Chapman and Hall, London, 1993), pp. 190–226.