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**Third Workshop on
3D Modelling of Seismic Waves Generation
Propagation and their Inversion**

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Elasticity Theory and the Seismic Equation of Motion

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1. Elasticity Theory and the Seismic Equation of Motion

1.1 Vector versus Indicial Notation

There are a variety of notations commonly used to represent vectors and tensors in Cartesian coordinates. We will try to be consistent and use indicial notation, but will probably fail. In any event, you need to be aware of the existence of the various notations and how they are related. Table 1 attempts to provide such a comparison.

We will represent the unit vectors in the 1, 2, and 3 or x, y, and z directions usually with either the triple $(\hat{x}_1, \hat{x}_2, \hat{x}_3)$ or $(\hat{x}, \hat{y}, \hat{z})$. In indicial notation, a vector is represented as a variable with a single subscript: e.g., x_i. In vector notation, I represent a vector as a bold-faced variable: e.g., $\mathbf{x}$. Matrices are represented usually with capital letters with two indices, A_{ij}, or as a capital letter in bold face, $\mathbf{A}$. There are a couple of problems with vector notation. The first is that it does not easily discriminate between second order and higher order tensors. For example, we will be using the Levi-Civita symbol, ϵ_{ijk}, which is a 3rd order tensor and the strain tensor, ϵ_{ij}, which is a 2nd order tensor. They denote very different things. Sometimes you might see a single line under vectors, or a double line under matrices, a triple line under 3rd order tensors, etc; or replacing the lines with arrows above the letter. I don't do that since it's too hard to text edit, but do resort to that notation sometimes in long hand. The second problem is that bold-faced Greek letters are hard to tell from non-bold faced Greek letters. For example, the stress tensor, a 2nd order tensor, in vector notation is denoted as $\boldsymbol{\sigma}$ and the standard deviation is a scalar, σ.

Finally, recall the Einstein summation Convention (ESC): repeated indices denote summation. Thus, the dot product between two vectors $\mathbf{x} \cdot \mathbf{y}$ is $x_i y_i$ which is just $\sum_i x_i y_i$. The sum goes over the dimensionality of the space considered. In our case that will almost always be 3. Recall also that a comma denotes differentiation with respect to the following index. For example, the gradient of the scalar variable ϕ, $\nabla\phi$, is written $\phi_{,i}$ and the divergence of a vector variable $\mathbf{u}$, $\nabla \cdot \mathbf{u}$, as $u_{i,i}$. This is pretty confusing, I think, and I will usually write the gradient as either $\partial\phi/\partial x_i$ or $\partial_i \phi$ and the divergence as $\partial u_i/\partial x_i$ or $\partial_i u_i$. In the latter case the ESC is still invoked. Please review this and the contents of Table 1 and make sure you are comfortable with it.

Table 1. Vector vs. Indicial Notation

Vector	*Indicial*	*Description*
$\mathbf{u}$	u_i	vector
$\|\mathbf{u}\|^2$	$u_i u_i$	magnitude of a vector
$\mathbf{u} \cdot \mathbf{v}$	$u_i v_i$	dot product
$\mathbf{uv}$	$u_i v_j$	dyadic tensor
$\nabla\phi$ or grad ϕ	$\phi_{,i}$ or $\partial_i \phi$ or $\partial\phi/\partial x_i$	gradient
$\nabla \cdot \mathbf{u}$ or div $\mathbf{u}$	$u_{i,i}$ or $\partial_i u_i$ or $\partial u_i/\partial x_i$	divergence
$\nabla^2\phi = \nabla \cdot (\nabla\phi)$ or div grad ϕ	$\phi_{,ii}$ or $\partial^2_{ii}\phi$ or $\partial^2\phi/\partial x_i \partial x_i$	laplacian
$\mathbf{I}$	δ_{ij}	identity matrix
$\mathbf{A}$	A_{ij}	matrix
$\mathbf{A}^T$	A_{ji}	matrix transpose
$\mathbf{Ax}$ or $\mathbf{A} \cdot \mathbf{x}$ or $\sum_j A_{ij} x_j$	$A_{ij} x_j$	matrix - vector product
$(\mathbf{Ax})^T = \mathbf{x}^T \mathbf{A}^T$	$A_{ij} x_i$	transpose of matrix - vector product
$\mathbf{x}^T \mathbf{Ax}$	$A_{ij} x_i x_j$	quadratic form
$\mathbf{AB} = \mathbf{A} \cdot \mathbf{B}$	$A_{ij} B_{jk}$	matrix - matrix product
$\mathbf{A} : \mathbf{B}$	$A_{ij} B_{ij}$	2nd order tensor - tensor contraction
$\nabla\mathbf{u}$	$u_{i,j} = \partial_j u_i = \partial u_i/\partial x_j$	related to strain tensor. see eqn. (1).
$(\nabla\mathbf{u})^T = \mathbf{u}^T \nabla^T$	$u_{j,i} = \partial_i u_j = \partial u_j/\partial x_i$	transpose of tensor in eqn. (1).
$\nabla^2\mathbf{u} = \nabla(\nabla \cdot \mathbf{u}) - (\nabla \times \nabla \times \mathbf{u})$	$x_{i,ij}$ or $\partial_j \partial_i x_i$ or $\frac{\partial}{\partial x_j}\frac{\partial u_i}{\partial x_i}$	laplacian of a vector

In Table 1, the row related to the strain tensor $\nabla\mathbf{u}$ can be written in matrix form as:

$$\nabla\mathbf{u} = \begin{bmatrix} \frac{\partial u_1}{\partial x_1} & \frac{\partial u_1}{\partial x_2} & \frac{\partial u_1}{\partial x_3} \\ \frac{\partial u_2}{\partial x_1} & \frac{\partial u_2}{\partial x_2} & \frac{\partial u_2}{\partial x_3} \\ \frac{\partial u_3}{\partial x_1} & \frac{\partial u_3}{\partial x_2} & \frac{\partial u_3}{\partial x_3} \end{bmatrix}. \tag{1}$$

This is, itself, a dyad.

1.2 Tensors

We will be considering equations that contain mathematical objects called *tensors*. First, what's a tensor? That's a tough question to answer succinctly. A first-order tensor is just a vector. There are several ways to define vectors. One is to present a list of properties that define a vector field, and state that a vector is a member of a vector field. Another is to specify how vectors transform under coordinate transformations. The latter is the way one usually sees tensors defined, and we will follow it here. More rigorous treatments of tensors follow the first type of definition, however.

One property of vectors is that they transform in a certain way under rotation. If $\mathbf{M}$ is an orthogonal (i.e., rotation) matrix, then $\mathbf{x}$ is a vector if and only if when operated on by $\mathbf{M}$ another vector $\mathbf{x}'$ emerges:

$$\mathbf{x}' = \mathbf{M}\mathbf{x}. \tag{2}$$

In discussions of tensors, equation (2) is in fact the definitive property of first-order tensors, or vectors. Tensors are objects that transform under coordinate transformations in certain ways. Why are they important in physics? Well, the way they transform means that their effect is independent of the coordinate system used to represent them. One wouldn't want to use mathematical objects that give different answers in Cartesian and spherical coordinates, say, or if you choose z positive up or positive down. But, this operational definition of tensors is precisely what makes them difficult for most people to get their brains around. A more rigorous definition, and one which is more rewarding in the long one, involves defining tensors as certain *multi-linear operators*. I will leave it to you to look up this definition in advanced linear algebra texts.

Moving forward with the operational definition of tensors, higher rank tensors are built from lower rank tensors, and like first-rank tensors are defined by their 'transformation properties'. For example, a representation theorem states that all second rank tensors can be represented as the outer product of a pair of vectors, as follows:

$$T_{ij} = A_i B_j, \tag{3}$$

where T_{ij} is a tensor and A_i and B_i are vectors. Thus, every second rank tensor can be represented as a *dyad*, in this case $\mathbf{T} = \mathbf{A}\mathbf{B}^T$. The order of the dyad is important because in general $\mathbf{A}\mathbf{B}^T \neq \mathbf{B}\mathbf{A}^T$. The components of T_{ij} transform when the coordinate system is rotated in a way following from equation (2):

$$\begin{aligned} \mathbf{T}' &= \mathbf{M}\mathbf{T}\mathbf{M}^T & (4) \\ \text{i.e.} \quad T'_{ij} &= \sum_{kl} M_{ik} M_{jl} T_{kl}. & (5) \end{aligned}$$

Any object that transforms like equations (4) and (5) is a second-rank tensor. For example, the Kronecker delta, δ_{ij}, is a tensor, it is defined to equal 1 when i = j and 0 otherwise in *all* coordinate systems. To see this, insert $\mathbf{I}$ in equation (4) to get:

$$\mathbf{I}' = \mathbf{M}\mathbf{I}\mathbf{M}^T = \mathbf{M}\mathbf{M}^T = \mathbf{I} \tag{6}$$

Tensors of third and higher rank are defined by an obvious extension of the definition for rank 2. A tensor of rank n can be defined as a set of n outer products of vectors:

$$T_{ij...l} = A_i B_j \dots G_l. \tag{7}$$

This is frequently called a *polyad* – for third-order tensors, it is a *triad*. The nth-order tensor $\mathbf{T}$ is a tensor if it transforms as follows:

$$T'_{ij...l} = \sum_{i'j'...l'} M_{ii'} M_{jj'} \dots M_{ll'} T_{i'j'...l'} \tag{8}$$

For example, consider the Levi-Civita symbol, ϵ_{ijk}, defined as

$$\epsilon_{ijk} = \left\{ \begin{array}{lc} 1 & \text{if} \quad ijk = 123, 231, 312 \\ -1 & \text{if} \quad ijk = 321, 213, 132 \\ 0 & \text{otherwise} \end{array} \right\}. \tag{9}$$

For ϵ_{ijk} to be a tensor, it must satisfy (9) in all coordinate systems. Applying equation (8):

$$\begin{aligned} \epsilon'_{ijk} &= \sum_{lmn} M_{il} M_{jm} M_{kn} \epsilon_{lmn} && (10) \\ &= \det(M) \epsilon_{ijk} = \epsilon_{ijk}, && (11) \end{aligned}$$

where we have used the fact that the determinant of an orthogonal or rotation matrix is unity. Thus, the Levi-Civita symbol is a third-order tensor.

Tensors can operate on other tensors. In doing so, if there are repeated indices, *contraction* occurs. For example, consider the following operations:

$$\begin{aligned} T_{ijk} &= M_{ij} x_k && (12) \\ x'_i &= M_{ij} x_j \qquad \text{(ESC)}. && (13) \end{aligned}$$

The second equation represents a second order - first order tensor product, 'contracting' the two tensor components results in a vector.

1.3 Stress

There are two types of forces we wish to consider acting on parcels of material through which seismic waves propagate. The first are body forces which are proportional to the volume over which the force acts. An example is gravity. The second is surface or contact forces which are proportional to the surface over which the forces acts. These forces are best expressed as a ratio of force per unit area and are known as stresses. Expressed as such, they are of greatest interest in seismology.

A word or two about units. The SI snit of pressures and stresses are the Pascal, 1 Pa = 1 N/m^2. More commonly you run into the following units: 1 atm $\approx$ 1 bar = 10^5 N/m^2 = 0.1 MPa. 1 kbar = 10^2 MPa. If density near the earth's surface is about 3.3 $\times 10^3$ kg/m^2, then confining pressure at a depth z near the earth's surface is approximately equivalent to $\rho g z$ which is about 1 kbar at 3 km. This yields a useful rule of thumb, the change in confining pressure with depth is about 1 kbar/3 km. Thus, the confining pressure at the Moho ($\approx$ 30 km) is about 10 kbar $\approx$ 1 GPa, a useful number to remember. More accurate estimates of the confining pressure at a depth z can be estimated by doing the following integral:

$$P(z) = \int_0^z g(z')\rho(z')dz'. \tag{14}$$

1.3a Stress Tensor

Consider a force ΔF acting on each point within a body as shown in Figure 1. If the body is sliced open and the normal direction $\hat{\mathbf{n}}$ is the 1-direction, then the small force, $\Delta\mathbf{F}$, acting on each point in the body is

$$\Delta\mathbf{F} = \Delta F_1\hat{x}_1 + \Delta F_2\hat{x}_2 + \Delta F_3\hat{x}_3. \tag{15}$$

The force ΔF_1 acts normal to the slice and ΔF_2 and ΔF_3 act in the plane of the slice.

The stress or traction vector is given by:

$$\mathbf{T}_1 = \frac{\Delta F_1}{A_1}\hat{\mathbf{x}}_1 + \frac{\Delta F_2}{A_1}\hat{\mathbf{x}}_2 + \frac{\Delta F_3}{A_1}\hat{\mathbf{x}}_3 \tag{16}$$

$$= \sigma_{11}\hat{\mathbf{x}}_1 + \sigma_{12}\hat{\mathbf{x}}_2 + \sigma_{13}\hat{\mathbf{x}}_3. \tag{17}$$

Slicing the body perpendicular to the 2- and 3-directions similarly yields $\sigma_{21}, \sigma_{22}, \sigma_{23}, \sigma_{31}, \sigma_{32}, \sigma_{33}$. These nine components of the three traction vectors, $\mathbf{T}_1$, $\mathbf{T}_2$, and $\mathbf{T}_3$ for the stress tensor:

$$\sigma = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix}. \tag{18}$$

The diagonal components of the stress tensor constitute the *normal stresses*. Positive normal stresses mean tension and negative normal stresses mean compression, generally. Off-diagonal components represent *shear stresses*. The non-hydrostatic or deviatoric stress tensor is usually defined as the stress tensor minus confining pressure: $D_{ij} = \sigma_{ij} - P\delta_{ij}$.

1.3b Equilibrium Conditions

A parcel of material is said to be in 'equilibrium' when two conditions are met:

- The net force on the parcel is zero. (Translational Equilibrium)
- The net torque on the parcel is zero. (Rotational Equilibrium)

These conditions derive from the conservation of linear and angular momentum, respectively. If a material is in equilibrium, the time rate of change of the linear and angular momenta are zero. Since the time rate of change of these momenta are simply force and torque ($\dot{p} = F; \dot{L} = \tau$), the net sum of the forces and torques must be zero if the material is in equilibrium.

Consider the parcel of material shown in Figure 2, and concentrate first on two forces acting in the 1-direction. Each face has a force-couple, either composed of forces normal or tangential to the face, in the 1-direction. Then the sum of the forces acting in the 1-direction is:

$$
\begin{aligned}
0 &= \sum_i F_{1i} = F_1 = \left(\frac{\partial \sigma_{11}}{\partial x_1}\right) \Delta x_2 \Delta x_3 + \left(\frac{\partial \sigma_{21}}{\partial x_2}\right) \Delta x_1 \Delta x_3 + \left(\frac{\partial \sigma_{31}}{\partial x_3}\right) \Delta x_1 \Delta x_2 && (19)\\
&= \frac{\partial \sigma_{11}}{\partial x_1} + \frac{\partial \sigma_{21}}{\partial x_2} + \frac{\partial \sigma_{31}}{\partial x_3} && (20)\\
&= \sum_j \frac{\partial \sigma_{j1}}{\partial x_j} && (21)\\
&= (\nabla \cdot \sigma)_1 . && (22)
\end{aligned}
$$

Thus, in general $F_i = (\nabla \cdot \sigma)_i$, and since $\sum F_i = 0$ by the first of the equilibrium conditions:

$$\nabla \cdot \sigma = 0. \qquad (23)$$

Equation (23) is called the *translational equilibrium condition.* If it is satisfied for a parcel of material, no net forces are acting on that parcel, thus no translational motion will occur.

Note that the divergence of the stress tensor is a force per unit volume.

1.3c Symmetry of the Stress Tensor

Now, let's use the second of the equilibrium conditions, the condition for rotational equilibrium. Consider Figure 3. Note that the shear stresses produce rotations. A moment, or a component of torque, is stress × lever arm × area.

Remark 1: The stress tensor σ is symmetric.

Proof: Consider the rotation of the rectangular unit of material in Figure 3 subjected to the four stresses listed in the figure. Let positive torque produce counter-clockwise rotation and negative torque produce clockwise rotation. For equilibrium to rule, the torques resulting from the four stresses must sum to zero:

$$0 = \left(2\sigma_{12} + \frac{\partial \sigma_{12}}{\partial x_1}\Delta x_1\right)\frac{\Delta x_1}{2}\Delta x_2 \Delta x_3 - \left(2\sigma_{21} + \frac{\partial \sigma_{21}}{\partial x_2}\Delta x_2\right)\frac{\Delta x_1}{2}\Delta x_1 \Delta x_3, \tag{24}$$

which means that

$$2\sigma_{12} + \frac{\partial \sigma_{12}}{\partial x_1}\Delta x_1 = 2\sigma_{21} + \frac{\partial \sigma_{21}}{\partial x_2}\Delta x_2. \tag{25}$$

In the limit, $\Delta x_1, \Delta x_2 \to 0$,

$$\sigma_{12} = \sigma_{21}. \tag{26}$$

Replacing the indices (1,2) everywhere with (i,j), generalizes the argument, so we have shown that the *stress tensor is symmetric*:

$$\sigma^T = \sigma. \tag{27}$$

1.3d Principal Stresses and Axes

Since the stress tensor is symmetric, it eigenvalues are real and its eigenvectors are orthogonal to one another:

$$\sigma = \mathbf{U}\sigma'\mathbf{U}^T \tag{28}$$

$$\sigma' = \mathbf{U}^T\sigma\mathbf{U} \tag{29}$$

The matrix σ' is diagonal, and contains what are known as the *principal stresses*. The matrix $\mathbf{U}$, composed of the eigenvectors of σ, is an orthogonal matrix, which means it is simply a rotation matrix and rotates σ into a coordinate system in which the stresses are all normal stresses – i.e., are diagonal only, there are no shear stresses. The columns of U define the *principal coordinate directions*.

1.4 Equation of Motion: Preamble

The equation of motion is simply Newton's 2nd Law, $m\mathbf{a} = m\ddot{\mathbf{x}} = \mathbf{f}$, but where both sides have been divided by volume so that mass → density, force → force density. We also want to add to the right hand side the expression that we found in Section 1.2b to be a force density $(\nabla \cdot \sigma)$, the divergence of the stress tensor. In this form the equation of motion is given by:

$$\rho \ddot{\mathbf{u}} = \mathbf{f} + \nabla \cdot \sigma, \tag{30}$$

where $\mathbf{u}$ now is vector displacement and $\mathbf{f}$ is an applied body force like an earthquake, gravity, etc.

Equation (30) is not in the form that we desire. In fact, stresses within a material result from displacements. The stresses represent the material's attempt to restore equilibrium. Since σ results from $\mathbf{u}$, we would like to somehow replace σ in equation (30) with some function of $\mathbf{u}$ to create a PDE in $\mathbf{u}$ alone. We will do this in two steps:

- Step 1. The *strain-displacement relation*: $\epsilon_{kl} = \frac{1}{2}\left(\frac{\partial u_k}{\partial x_l} + \frac{\partial u_l}{\partial x_k}\right)$.
- Step 2. The *stress-strain relation*: $\sigma_{ij} = c_{ijkl}\epsilon_{kl}$. (Generalized Hooke's Law)

Combining the results from these two steps allows us to relate stress to displacement and rewrite equation (30) in the desired form. Along the way we will need to introduce and discuss the *strain tensor*, ϵ_{kl}, and the elastic tensor, c_{ijkl}. These relations will be used Section 1.8 to derive the equation of motion including gravity.

1.5 Strain, Stress-Strain, and Strain-Displacement

1.5a Internal Deformations and the Strain-Displacement Relation

Consider two nearby points in a material, P and Q, which, prior to deformation, are separated by a vector y_i. Let z_i and $z_i + y_i$ represent the initial locations of these two points, respectively. Subject the medium to a deformation u_i at point P and $u_i + (\partial u_i/\partial x_j)y_j$ at the nearby point Q. After the deformation but before the response of the material to the deformation, points P and Q at $z_i + u_i$ and $(z_i + y_i) + (u_i + \partial u_i/\partial x_j)y_j$ (ESC), respectively. The following summarizes this state of affairs:

Point	*Before Deformation*	*After Deformation*
P	z_i	$z_i + u_i$
Q	$z_i + y_i$	$(z_i + y_i) + (u_i + \partial u_i/\partial x_j)y_j$

The change in the distance between points P and Q caused by the deformation is prescribed by the tensor $\partial u_i/\partial x_j$. Any tensor can be represented as a sum of symmetric and anit-symmetric tensors:

$$\frac{\partial u_i}{\partial x_j} = \text{Symmetric Part} + \text{Anti-Symmetric Part} \quad (31)$$

$$= \epsilon_{ij} - \xi_{ij} \quad (32)$$

$$= \frac{1}{2}\left(\frac{\partial u_j}{\partial x_i} + \frac{\partial u_i}{\partial x_j}\right) - \frac{1}{2}\left(\frac{\partial u_j}{\partial x_i} - \frac{\partial u_i}{\partial x_j}\right) \quad (33)$$

The point of rewriting $\partial u_i/\partial x_y$ in this way is that the symmetric part, ϵ_{ij}, represents the internal deformation and the anti-symmetric part, ξ_{ij}, represents a rigid body rotation as proven in Section 1.5b. We are interested in internal deformations, not rigid body rotations and can, therefore, ignore ξ_{ij}.

Internal deformations are represented by the symmetric part of $\partial u_i/\partial x_y$, which is just the *strain tensor*:

$$\epsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_1}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right). \quad (34)$$

This equation is a statement of the *strain-displacement relation.*

1.5b ξ_{ij} is a Rigid Body Rotation

Remark 2: The anti-symmetric tensor ξ represents a rigid body rotation.

Proof: Consider a pair of elements of ξ, ξ_{23} and ξ_{32}, and the deformation $\tilde{\mathbf{y}}$ that results from them relative to point P:

$$\tilde{\mathbf{y}} = \mathbf{y} - \xi_{23} y_3 \hat{\mathbf{x}}_2 - \xi_{32} y_2 \hat{\mathbf{x}}_3 \quad (35)$$

$$= \mathbf{y} - \xi_{23} y_3 \hat{\mathbf{x}}_2 + \xi_{23} y_2 \hat{\mathbf{x}}_3, \quad (36)$$

where the second equality follows by the anti-symmetry of ξ ($\xi_{32} = -\xi_{23}$). For $\mathbf{y}$ and $\tilde{\mathbf{y}}$ to be related by a rotation, they should be of the same length. The change in length squared is just:

$$|\mathbf{y}|^2 - |\tilde{\mathbf{y}}|^2 = (y_1^2 + y_2^2 + y_3^2) - \left(y_1^2 + (y_2 - \xi_{23} y_3)^2 + (y_3 + \xi_{23} y_2)^2\right) \quad (37)$$

$$= 0 + O(\xi_{23}^2) \approx 0 \qquad \text{to first order.} \quad (38)$$

Thus, the deformation produced by ξ_{23} and ξ_{32} does not change the distance between points P and Q.

If the initial angle in the 2-3 plane (see Figure 4) between the 2-axis and the vector, $\mathbf{y}$, linking P and Q is θ, then $\theta = \tan^{-1}(y_3/y_2) = \tan^{-1}(q)$, where $q = y_3/y_2$. The tangent of the new angle,

θ', after deformation is:

$$\tan\theta' = \frac{y_3 + \xi_{23}y_2}{y_2 - \xi_{23}y_3} = \frac{q + \xi_{23}}{1 - q\xi_{23}} \approx (q + \xi_{23})(1 + q\xi_{23}) \approx q + (1 + q^2)\xi_{23}, \tag{39}$$

$$\theta' = \tan^{-1}\left(q + (1 + q^2)\xi_{23}\right) \approx \theta + \xi_{23}, \tag{40}$$

where we retain only first order terms and the latter equality follows by Taylor expanding the $\tan^{-1}$ since $1 >> (1 + q^2)\xi_{23}$. Therefore, ξ_{23} is the angle of rotation, and the proof is complete.

1.5c Geometrical Interpretation of the Strain Tensor

The diagonal elements of ϵ produce length changes in the coordinate directions. The relative volumetric change ($\Delta V/V_0$) is given by the *cubic dilatation*: $\Theta = \epsilon_{ii}$ (ESC). The off-diagonal elements produce internal rotations which characterize internal deformation. These rotations represent the change in internal angles linking points in the medium.

1.5d Stress-Strain Relation

Probably the greatest difference between doing seismology in greater than 1D as opposed to 1D, is that applied forces or stresses produce displacements or strains in directions other than the direction of the applied force. This relation between stress and strain in 3D then, has to be generalized to allow for the response of the medium in arbitrary directions. Similarly, deformations or strains produce stresses in directions other than the strains have been applied.

In 1D, the stress-strain relation for small displacements was simply Hooke's Law:

$$F = kx, \tag{41}$$

as in Figure 5. In 3D, Hooke's Law is generalized and stresses are some function of all the strains:

$$\sigma_{ij} = F_{ij}(\epsilon_{11}, \epsilon_{12}, \ldots, \epsilon_{33}). \tag{42}$$

The assumption is that there is an analytical 1-1 relationship between σ and ϵ, and as $\epsilon \to 0$, $\sigma \to 0$. Expanding F_{ij} in a Taylor Series in ϵ_{kl} in equation (42) and retaining only linear terms yields:

$$\sigma_{ij} = c_{ijkl}\epsilon_{kl}. \tag{43}$$

c_{ijkl} is a fourth order tensor, related to the derive of F_{ij} with respect to ϵ_{kl}, and analogous to the spring constant in equation (41). It is a constant at each point in space, but may differ from place to place. Equation (43) is, therefore, a straightforward generalization of Hooke's Law. The tensor c_{ijkl} is called the *elastic tensor*, and represents how the medium responds to deformation. The response of the medium to deformation is the reason why seismic waves propagate.

Recall that for Hooke's Law the displacement potential energy or work is $kx^2/2$. Similarly, the strain energy is simply a quadratic function of ϵ_{ij}:

$$W = \frac{1}{2} c_{ijkl} \epsilon_{ij} \epsilon_{kl}. \tag{44}$$

1.6 The Elastic Tensor, Isotropy, and the Lame Parameters

The total number of elements in c_{ijkl} is $3^4 = 81$. Symmetries in the stress - strain equation (eq. (43)) can be used to reduce the number of *independent* elements of c_{ijkl}. Note that equation (44) is not altered if we simply relabel i as j and j as i in c_{ijkl}:

$$U = \frac{1}{2} c_{jikl} \epsilon_{ij} \epsilon_{kl}. \tag{45}$$

or $c_{ijkl} = c_{jikl}$. Similarly, by flipping k and l, $c_{ijkl} = c_{ijlk}$. In this way, we reduce the number of independent pairs of (i, j) from 9 to 6, and hence the number independent elements of c_{ijkl} from $9^2 = 81$ to $6^2 = 36$. Alternately, we could have argued for this reduction from the symmetries in σ_{ij} and then ϵ_{kl} in equation (43).

Further consideration of the strain-energy function and the stress-strain relation reveals that:

$$\begin{aligned} W &= \frac{1}{2} c_{ijkl} \epsilon_{kl} \epsilon_{ij} & (46) \\ &= \frac{1}{2} \sigma_{ij} \epsilon_{ij} & (47) \\ &= \frac{1}{2} \sigma_{kl} \epsilon_{kl} & (48) \\ &= \frac{1}{2} c_{klij} \epsilon_{ij} \epsilon_{kl}, & (49) \end{aligned}$$

or $c_{ijkl} = c_{klij}$. This reduces the number of independent elements from 36 to 21. For a general anisotropic material on 21 of the 81 individual elements are independent: there are 21 elastic moduli! Without proof we will note that in a transversely isotropic material, the number of independent elements reduces to 5.

In an isotropic medium, the number of independent elements is only 2. These two elements are called the *Lame parameters*, usually referred to as λ and μ. In this case:

$$c_{ijkl} = \lambda \delta_{ij} \delta_{kl} + \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}). \tag{50}$$

Substituting this expression for the elastic tensor in an isotropic medium into the stress-strain relation:

$$\begin{aligned}
\sigma_{ij} &= c_{ijkl}\epsilon_{kl} && (51)\\
&= [\lambda\delta_{ij}\delta_{kl} + \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})]\,\epsilon_{kl} && (52)\\
&= \lambda\epsilon_{kk}\delta_{ij} + \mu(\epsilon_{ij} + \epsilon_{ji}) && (53)\\
&= \lambda\epsilon_{kk}\delta_{ij} + 2\mu\epsilon_{ij} && (54)\\
&= \lambda\Theta\delta_{ij} + 2\mu\epsilon_{ij}, && (55)
\end{aligned}$$

where Θ is cubic dilatation. This is Hooke's Law for an isotropic material.

First, note in passing that a *Poisson solid* is a material in which $\lambda = \mu$. In this case, $\sigma_{ij} = \lambda(\Theta\delta_{ij} + 2\epsilon_{ij})$.

1.7 Other Moduli for Isotropic Solids

Let's start by considering what μ is. By equation (55):

$$\mu = \frac{\sigma_{ij}}{2\epsilon_{ij}} \quad i \neq j. \tag{56}$$

This modulus is seen to be related to the stress required to produce a unit shear ($i \neq j$) strain, and is, therefore, called the *shear modulus*. It represents resistance to shear. For many earth materials, at STP the shear modulus is about 200 kbars. Beyond this point, most terrestrial materials suffer shear failure – they break.

The modulus λ has no such simple interpretation and is usually replaced by another modulus, the *compressional modulus or bulk modulus*, κ, which represents the resistance to compression:

$$\kappa^{-1} = -\frac{1}{V}\left(\frac{\partial V}{\partial P}\right)_S, \tag{57}$$

where the subscript S means that the partial derivative is taken at constant entropy. So defined,

$$\kappa = \lambda + \frac{2}{3}\mu. \tag{58}$$

Substituting equation (58) into equation (55) produces the stress-strain relation in terms of κ and μ.

As we will see in the next section, the equation of motion for an isotropic solid involves waves propagating at two characteristic speeds, the compressional or P-wave velocity (v_p or α) and the shear or S-wave velocity (v_s or β). In terms of these moduli:

$$v_s^2 = \frac{\mu}{\rho} \tag{59}$$

$$v_p^2 = \frac{\lambda + 2\mu}{\rho} = \frac{\kappa + \frac{4}{3}\mu}{\rho} \tag{60}$$

Another modulus commonly encountered is *Young's modulus*, E, which is defined as the ratio of uniaxial stress and strain:

$$E_1 = \frac{\sigma_{11}}{\epsilon_{11}} = \frac{\mu(3\lambda + 2\mu)}{\lambda + \mu}. \tag{61}$$

Finally, *Poisson's ration*, μ, is another modulus you might run into. It is defined as the ratio of the radial and axial strains under a uniaxial stress as depicted in Figure 6. That is, if $\sigma_{11} \neq 0$ but $\sigma_{22} = \sigma_{33} = 0$, then

$$\nu = -\frac{\epsilon_{22}}{\epsilon_{11}} \tag{62}$$

$$= \frac{\lambda}{2(\lambda + \mu)} < .5. \tag{63}$$

For crustal rocks, ν normally runs between about .24 and .32, although values as low a .22 and as high as .35 have been measured for some rocks. Average crustal values are about .27 - .28. For a Poisson solid ($\lambda = \mu$), $\nu = .25$. You can see then, that a Poisson solid is not too bad of a 0th-order approximation for crustal rocks. From equation (60), $v_p = \sqrt{3}v_s$ for a Poisson solid. If an idealized material possesses infinite shear resistance, $\nu = 0$.

1.8 Derivation of the Equation of Motion

From Sections 1.1 - 1.7 we have compiled the following information:

$\rho\ddot{\mathbf{u}} = \nabla \cdot \boldsymbol{\sigma} + \mathbf{f}$	General Equation of Motion,	(64)
$\boldsymbol{\epsilon} = \frac{1}{2}\left(\nabla\mathbf{u} + \mathbf{u}^T\nabla^T\right)$	Strain - Displacement Relation,	(65)
$\sigma_{ij} = c_{ijkl}\epsilon_{kl}$	General Hooke's Law (Stress - Strain),	(66)
$c_{ijkl} = \lambda\delta_{ij}\delta_{kl} + \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})$	Elastic Tensor (Isotropic Solid),	(67)
$\sigma_{ij} = \lambda\theta\delta_{ij} + 2\mu\epsilon_{ij}$	Hooke's Law (Isotropic Solid).	(68)

The vector $\mathbf{u}$ is displacement, overdot represents a time derivative, $\boldsymbol{\sigma}$ is the second-order stress tensor, $\boldsymbol{\epsilon}$ is the second order strain tensor, $\mathbf{c}$ is the fourth-order elastic tensor, λ and μ are the Lame parameters, and

$$\theta = \epsilon_{kk} = \frac{\partial u_k}{\partial x_k} = \nabla \cdot \mathbf{u} \tag{69}$$

is cubical dilatation.

We want to eliminate $\boldsymbol{\sigma}$ from equation (64). In doing this we will concentrate on isotropic solids and use the stress-strain and strain-displacement relations given by equations (68) and (65). After doing this we will consider the effect of gravitational body forces on the equation of motion.

1.8.1 Elastic Terms ($\nabla \cdot \boldsymbol{\sigma}$)

We want to take the divergence of $\boldsymbol{\sigma}$. For simplicity, consider only the first component of the vector $\nabla \cdot \boldsymbol{\sigma}$:

$$
\begin{aligned}
(\nabla \cdot \boldsymbol{\sigma})_1 &= \frac{\partial \sigma_{11}}{\partial x_1} + \frac{\partial \sigma_{12}}{\partial x_2} + \frac{\partial \sigma_{13}}{\partial x_3}, && (70)\\
&= \lambda \frac{\partial}{\partial x_1}\left(\frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_3}\right) + 2\mu \frac{\partial}{\partial x_1}\left(\frac{\partial u_1}{\partial x_1}\right) + \mu \frac{\partial}{\partial x_2}\left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}\right) + \mu \frac{\partial}{\partial x_3}\left(\frac{\partial u_1}{\partial x_3} + \frac{\partial u_3}{\partial x_1}\right) && (71)\\
&= \lambda \frac{\partial \theta}{\partial x_1} + \mu \frac{\partial}{\partial x_1}\left(\frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_3}\right) + \mu\left(\frac{\partial^2 u_1}{\partial x_1^2} + \frac{\partial^2 u_1}{\partial x_2^2} + \frac{\partial^2 u_1}{\partial x_3^2}\right) && (72)\\
&= (\lambda + \mu)\frac{\partial \theta}{\partial x_1} + \mu(\nabla^2 \mathbf{u})_1. && (73)
\end{aligned}
$$

In deriving equation (71) we have used equations (68) and (69) and for simplicity have assumed that λ and μ do not vary with position so that they come outside of the derivative. We will add terms in the gradient of the elastic moduli back into the equations of motion later. In deriving equation (72) we have merely rearranged terms. In deriving equation (73) we have used the definition of the Laplacian and equation (69). The same procedure can be gone through for the 2- and 3- components and we get

$$\nabla \cdot \boldsymbol{\sigma} = (\lambda + \mu)\,\nabla(\nabla \cdot \mathbf{u}) + \mu \nabla^2 \mathbf{u}, \tag{74}$$

where we have rewritten θ using equation (69).

It is useful to modify equation (74) by using the vector identity

$$\nabla^2 \mathbf{u} = \nabla(\nabla \cdot \mathbf{u}) - (\nabla \times \nabla \times \mathbf{u}). \tag{75}$$

With this identity, equation (74) can be rewritten as

$$\nabla \cdot \boldsymbol{\sigma} = (\lambda + 2\mu)\,\nabla(\nabla \cdot \mathbf{u}) - \mu(\nabla \times \nabla \times \mathbf{u}), \tag{76}$$

so that the equation of motion becomes

$$\rho \ddot{\mathbf{u}} = \mathbf{f} + (\lambda + 2\mu)\,\nabla(\nabla \cdot \mathbf{u}) - \mu(\nabla \times \nabla \times \mathbf{u}) \qquad \text{Homogeneous Media.} \tag{77}$$

Without specifying body forces, this is the final form of the equation of motion for a homogeneous body, i.e. if the elastic moduli are constant. If the elastic moduli vary spatially then, if the right-hand-side of equation (77) is represented as $\mathbf{H}(\mathbf{u}, \lambda, \mu)$, the appropriate equation would be:

$$\rho \ddot{\mathbf{u}} = \mathbf{f} + \mathbf{H}(\mathbf{u}, \lambda, \mu) + \nabla\lambda(\nabla \cdot \mathbf{u}) + \nabla\mu \cdot \left(\nabla\mathbf{u} + \mathbf{u}^T\nabla^T\right) \qquad \text{Inhomogeneous Media,} \tag{78}$$

where we have simply added gradient terms in the elastic moduli.

1.8.2 Gravitational Body Forces (f)

When seismic disturbances propagate through a region, they compress and displace material. These disturbances (1) perturb the local density field which modifies the acceleration of gravity at the surface and (2) perturb convective equilibrium by bringing material to new non-equilibrium radial levels which induces a buoyancy restoring force. Both of these effects introduce gravitational body forces into the equation of motion. Density perturbations resulting from seismic compressions lead to *self-gravitational* terms in the equation of motion. Radial seismic displacements lead to *buoyancy* terms. We will derive the forms of both kinds of terms in the equation of motion here.

Gravity

Consider the force of gravity, $\mathbf{f}(\mathbf{r})$, at a radius $\mathbf{r}$ in the Earth:

$$\mathbf{f}(\mathbf{r}) = \rho(r)\mathbf{g}(\mathbf{r}) = -\rho(r)\nabla\phi(\mathbf{r}), \tag{79}$$

$$\mathbf{g}(\mathbf{r}) = -\nabla\phi(\mathbf{r}), \tag{80}$$

where ϕ is the graviatational potential:

$$\phi(\mathbf{r}) = -G\int \frac{\rho(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|}dV, \tag{81}$$

and $G = 6.6732 \times 10^{-11}$ $\mathrm{Nm^2/kg^2}$ is the universal gravitational constant.

Spherical Symmetry

Assume for the moment that the density distribution in the Earth is spherically symmetric; i.e., $\rho(\mathbf{r}) = \rho(r) = \rho_0(r)$ and $\phi(\mathbf{r}) = \phi_0(r)$. Then at rest, the equation of motion (64) becomes

$$\rho_0\nabla\phi_0 = \nabla \cdot \boldsymbol{\sigma}_0. \tag{82}$$

Break spherically symmetric stress into hydrostatic, P_0, and deviatoric, $\boldsymbol{\tau}$, components:

$$\boldsymbol{\sigma}_0 = -P_0\mathbf{I} + \boldsymbol{\tau}. \tag{83}$$

Over geological time, the Earth responds to deviatoric stresses by flowing and thereby reduces equilibrium deviatoric stresses. They are, therefore, very small ($|D_{ij}| << P_0$), so that equation (83) can be rewritten:

$$\rho_0 \nabla \phi_0 + \nabla P_0 = 0. \tag{84}$$

This is the *equilibrium condition* with gravity. When we added gravity, the divergence of the stress tensor was perturbed to no longer equal zero. This equation can be simplified further by noting that if $\rho(r)$ is spherically symmetric, then so will $P(r)$ and $\phi(r)$. Indeed, ρ_0, P_0, and ϕ_0 will be constant on the same spherical surfaces. Then

$$\nabla P_0 = \partial_r P_0 \hat{\mathbf{r}}, \tag{85}$$

$$\mathbf{g}_0 = -g_0 \hat{\mathbf{r}} = -\partial_r \phi_0 \hat{\mathbf{r}}, \tag{86}$$

although $g_0 = \partial_r \phi_0$. Finally, then

$$\rho_0 g_0 + \partial_r P_0 = 0, \tag{87}$$

which is the *final form of the equilibrium condition* that we seek.

Poisson's Equation

In relating density, gravity and pressure, we need, in addition to the equilibrium condition, Poisson's equation:

$$\nabla^2 \phi = 4\pi G \rho, \tag{88}$$

$$\frac{1}{r^2}\partial_r \left(r^2 \partial_r \phi_0\right) = 4\pi G \rho_0, \tag{89}$$

$$\frac{1}{r^2}\partial_r \left(r^2 g_0\right) = \left(\partial_r + \frac{2}{r}\right) g_0 = 4\pi G \rho_0. \tag{90}$$

Solving the final equation for g_0 yields:

$$g_0(r) = \frac{4\pi G}{r^2} \int_0^r \rho_0(r') r'^2 dr'. \tag{91}$$

Thus, given ρ_0 we can use equation (91) to calculate g_0 and then use the equilibrium conditions in equation (87) to calculate P_0:

$$P_0(r) = \int_a^r \rho_0(r') g_0(r') dr'. \tag{92}$$

Perturbations Caused by Seismic Motions

Seismic motions perturb density and the gravitational potential. Let's write these perturbations as first-order perturbation expansions:

$$\rho(r,\theta,\phi) = \rho_0(r) + \rho_1(r,\theta,\phi), \quad (93)$$

$$\phi(r,\theta,\phi) = \phi_0(r) + \phi_1(r,\theta,\phi), \quad (94)$$

where ρ_1 and ϕ_1 are perturbations in density and the gravitational potential caused by seismic disturbances. Note that the seismic disturbances will lift the spherical symmetry in density, gravity, and pressure. Then

$$\mathbf{g} = -\nabla\phi = -\nabla(\phi_0 + \phi_1). \quad (95)$$

Substituting this perturbed expression for g into equation equation (64) and using equations (79) and (83) we get

$$(\rho_0 + \rho_1)\ddot{\mathbf{u}} = \nabla\cdot\tau - \nabla(P_0 + P_1) - (\rho_0 + \rho_1)\nabla(\phi_0 + \phi_1). \quad (96)$$

Now, subtract the equilibrium condition given by equation (84), drop $\rho_1\ddot{\mathbf{u}}$ since it's second order in small quantities, and use equation (86) to get

$$\rho_0\ddot{\mathbf{u}} = \nabla\cdot\tau - \nabla P_1 - \rho_0\nabla\phi_1 + \rho_1 g_0\hat{\mathbf{r}}. \quad (97)$$

What we want to do to equation (97) is to eliminate the unknown perturbed quantities ρ_1 and P_1 and to replace them with the known quantities ρ_0 and g_0 and the single unknown $\mathbf{u}$.

Eliminating ρ_1

Consider a volume V. Pass a seismic wave through the volume, and the mass in V changes:

$$\text{Mass in V} = \int_V \rho dV = \text{Equilibrium Mass - Change in Mass due to Seismic Motion} \quad (98)$$

$$= \int_V \rho_0 dV - \int_{\partial V} \rho_0 \left[\mathbf{u}\cdot\hat{\mathbf{n}}ds\right], \quad (99)$$

$$= \int_V \rho_0 dV - \int_V \nabla\cdot(\rho_0\mathbf{u})\, dV. \quad (100)$$

where $\mathbf{u}\cdot\hat{\mathbf{n}}ds$ is the perturbed volume element and the final equation results from the Divergence Theorem. From this follows:

$$\rho = \rho_0 - \nabla\cdot(\rho_0\mathbf{u}_0) \quad (101)$$

$$= \rho_0 - (\mathbf{u}\cdot\nabla\rho_0 + \rho_0\nabla\cdot\mathbf{u}) \quad (102)$$

$$= \rho_0 + \rho_1. \quad (103)$$

Thus,

$$\rho_1 = \mathbf{u} \cdot \nabla \rho_0 + \rho_0 \nabla \cdot \mathbf{u}. \tag{104}$$

Half way there.

Eliminating P_1

Expand P in a Taylor Series in $\mathbf{u}$:

$$\begin{aligned} P &= P_0 + \nabla P_0 \cdot \mathbf{u} + O(\mathbf{u} \cdot \mathbf{u}) && (105)\\ &\approx P_0 + \nabla P_0 \cdot \mathbf{u} && (106)\\ &= P_0 - (\rho_0 \nabla \phi_0) \cdot \mathbf{u} && (107)\\ &= P_0 + \rho_0 g_0 (\hat{\mathbf{r}} \cdot \mathbf{u}) && (108)\\ &= P_0 + P_1, && (109) \end{aligned}$$

where the middle two equations follow from equations (84) and (86), respectively. Thus,

$$\nabla P_1 = \nabla \left(\rho_0 g_0 \hat{\mathbf{r}} \cdot \mathbf{u} \right). \tag{110}$$

1.8.3 Final Form of the Equation of Motion ($\nabla \cdot \sigma + \mathbf{f}$)

Finally, combining results from equations (77), (78), (97), (104), and (110), the equation of motion can be written in terms of known quantities and one unknown quantity, $\mathbf{u}$:

$$\begin{aligned} \rho_0 \ddot{\mathbf{u}} &= \mathbf{f} + \mathbf{H}(\mathbf{u}, \lambda, \mu) + \mathbf{I}(\mathbf{u}, \lambda, \mu) + \mathbf{S}(\mathbf{u}, \rho_0, \phi_1) + \mathbf{B}(\mathbf{u}, \rho_0, g_0), && (111)\\ \mathbf{H}(\mathbf{u}, \lambda, \mu) &= (\lambda + 2\mu) \nabla(\nabla \cdot \mathbf{u}) - \mu(\nabla \times \nabla \times \mathbf{u}) \quad \text{Homogeneous Elastic,} && (112)\\ \mathbf{I}(\mathbf{u}, \lambda, \mu) &= \nabla \lambda (\nabla \cdot \mathbf{u}) + \nabla \mu \cdot \left(\nabla \mathbf{u} + \mathbf{u}^T \nabla^T \right) \quad \text{Inhomogeneous Elastic,} && (113)\\ \mathbf{S}(\mathbf{u}, \rho_0, \phi_1) &= -\rho_0 \nabla \phi_1 + (\mathbf{u} \cdot \nabla \rho_0 + \rho_0 \nabla \cdot \mathbf{u}) \, g_0 \hat{\mathbf{r}} \quad \text{Self-Gravitation,} && (114)\\ \mathbf{B}(\mathbf{u}, \rho_0, g_0) &= -\nabla \left(\rho_0 g_0 \hat{\mathbf{r}} \cdot \mathbf{u} \right) \quad \text{Buoyancy.} && (115) \end{aligned}$$

where $\mathbf{f}$ now represents all body forces other than self-gravitation and buoyancy, and ϕ_1 is given in terms of known quantities by Poisson's equation:

$$\nabla^2 \phi_1 = -4\pi G \rho_1 = -4\pi G \left(\mathbf{u} \cdot \nabla \rho_0 + \rho_0 \nabla \cdot \mathbf{u} \right). \tag{116}$$

The coupled pair of equations (111) and (116) represent the result we want. Written as they have been here the equations are not totally consistent in that the elastic moduli are allowed unconstrained variations, but density has been constrained to vary only radially. Typically this is overcome by assuming that λ and μ are themselves spherically symmetric in the early stages of the solution of these equations. The gradients of the elastic moduli then reduce to derives in r. Such a model is spherically symmetric, nonrotating, elastic, and isotropic (SNREI). So modified, and given $\rho_0(r), \lambda_0(r)$, and $\mu_0(r)$, equations (111) and (116) can then be solved simultaneously for $\mathbf{u}(\mathbf{r}, t)$ subject to the boundary conditions we will discuss in the next section. Asphericities in ρ, λ, and μ are are then usually represented as expansions in the eigenfunctions of the SNREI model and are then added as structural perturbations. This is exactly the same way as we dealt with variations in wave speed in the perturbation theoretic treatment of the inhomogeneous string.

The reason why the expression $\mathbf{S}(\mathbf{u}, \rho_0, \phi_1)$ is called *self-gravitation* is easy to see, it represents forces that result from perturbations in density and the gravitational potential caused by volume changes associated with seismic disturbances. The expression $\mathbf{B}(\mathbf{u}, \rho_0, g_0)$ is called *buoyancy* since it is a pressure gradient force (∇P_1) resulting from radial displacements of material ($\hat{\mathbf{r}} \cdot \mathbf{u}$).

1.9 Boundary Conditions

Elastic constants can change abruptly across an interface, therefore displacements and stresses might as well. The equations of motion govern wave propagation on either side of an interface and boundary conditions must match the solutions at the boundary. Boundary conditions both on displacement (kinematic conditions) and stress (dynamic conditions) must be introduced.

Kinematic Conditions

The condition on displacement is the following: media initially in contact must remain in contact and not separate or interpenetrate. If a boundary is welded or locked, then in addition no slip can occur along that boundary. Let $\mathbf{u}^{(1)}$ and $\mathbf{u}^{(2)}$ represent displacement on either side of an interface d, then if $\hat{\mathbf{n}}_d$ is the normal to the interface:

B.C. 1: (All boundaries) The normal component of displacement is continuous across the boundary; i.e. $\mathbf{u}^{(1)} \cdot \hat{\mathbf{n}}_d = \mathbf{u}^{(2)} \cdot \hat{\mathbf{n}}_d$

B.C. 1': (Solid-Solid or other welded boundaries) Displacement is continuous across the boundary; i.e., $\mathbf{u}^{(1)} = \mathbf{u}^{(2)}$.

Typically, in the Earth's interior welded interfaces are assumed and B.C. 1' is employed. At the ocean-solid Earth interface and the the core-mantle boundary, B.C. 1 is normally used. It is sometimes assumed that waves propagate in a layer over a rigid half-space, at this displacement would then go to zero trapping all waves in the layer: $\mathbf{u} = 0$.

Dynamic Conditions

The internal force acting across an interface d is the surface integral of the difference in stresses acting at the boundary:

$$\mathbf{f} = \int_d \left(\boldsymbol{\tau}^{(1)} - \boldsymbol{\tau}^{(2)}\right) \cdot \hat{\mathbf{n}}_d ds. \tag{117}$$

This internal body force cannot lead to a net force on the whole body, so it must be equal to zero. Hence, the integrand is zero and

$$\boldsymbol{\tau}^{(1)} \cdot \hat{\mathbf{n}}_d = \boldsymbol{\tau}^{(2)} \cdot \hat{\mathbf{n}}_d, \tag{118}$$

the normal component of stress is continuous across the interface. The normal component of stress is called traction and is commonly written as $\mathbf{T}(\hat{\mathbf{n}}_d) = \boldsymbol{\tau} \cdot \hat{\mathbf{n}}_d$. At the free surface, it is normally assumed that no stresses are imposed on the boundary from outside (i.e., $\mathbf{T}^{(1)}(\hat{\mathbf{n}}_d) = 0$). The dynamical boundary condition can, therefore, be written as follows:

B.C. 2: (All boundaries) Traction is continuous across the boundary; i.e., $\mathbf{T}^{(1)}(\hat{\mathbf{n}}_d) = \mathbf{T}^{(2)}(\hat{\mathbf{n}}_d)$.

B.C. 2': (Free surface) Traction vanishes at the free surface; i.e., $\mathbf{T}(\hat{\mathbf{n}}_d) = 0$.

1.10 The Wave Equation in 3D Homogeneous Media: P- and S-Waves

The equations of motion and boundary conditions derived for a SNREI model form the foundation for seismic wave propagation theory. Much of this class will be concerned with discussing solutions to these equations. In this, the closing, section of these lecture notes, we will consider a particular class of solutions known as *body waves.*

Wave equations can be obtained easily from the equation of motion for a homogeneous medium ignoring gravity by using Helmholtz's Theorem and introducing potential functions. Helmholtz's Theorem states that any finite, continuous vector field $\mathbf{u}$ that vanishes at infinity can be represented as the sum of the gradient of a scalar potential and the curl of a divergence-free vector potential. Let's expand both $\mathbf{u}$ and $\mathbf{f}$ this way by introducing the scalar potentials ϕ and Φ and the vector potentials $\boldsymbol{\psi}$ and $\boldsymbol{\Psi}$

$$\mathbf{u} = \nabla\phi + \nabla \times \boldsymbol{\psi}, \tag{119}$$

$$\mathbf{f} = \nabla\Phi + \nabla \times \boldsymbol{\Psi}. \tag{120}$$

Substituting these potentials into equation (111) and dropping terms related to gravity and inhomogeneity yields:

$$0 = \nabla\left((\lambda + 2\mu)\nabla^2\phi + \Phi - \rho\ddot{\phi}\right) + \nabla \times \left(\mu\nabla^2\boldsymbol{\psi} + \boldsymbol{\Psi} - \rho\ddot{\boldsymbol{\psi}}\right), \tag{121}$$

where we have used the fact that curl grad and div curl are both zero; e.g.,

$$\nabla \times (\nabla \phi) = 0, \tag{122}$$

$$\nabla \cdot (\nabla \times \boldsymbol{\psi}) = 0. \tag{123}$$

The only way these functions can generally equal zero is if the terms in parentheses are themselves separately zero:

$$\alpha^2 \nabla^2 \phi + \frac{\Phi}{\rho} = \ddot{\phi}, \tag{124}$$

$$\beta^2 \nabla^2 \boldsymbol{\psi} + \frac{\boldsymbol{\Psi}}{\rho} = \ddot{\boldsymbol{\psi}}, \tag{125}$$

where

$$\alpha = \left(\frac{\lambda + 2\mu}{\rho}\right)^{1/2} = \left(\frac{\kappa + \frac{4}{3}\mu}{\rho}\right)^{1/2}, \tag{126}$$

$$\beta = \left(\frac{\mu}{\rho}\right)^{1/2}. \tag{127}$$

Note that for a Poisson solid, $\lambda = \mu$, so $\alpha = \sqrt{3}\beta$. Therefore, the vectors $\nabla\phi$ and $\nabla \times \boldsymbol{\psi}$ are called the *P-wave* (for Primus meaning first) and *S-wave* (for secundus) components of displacement **u**. Thus, equations (124) and (125) are the equations for P- and S- waves that travel, respectively, with speeds given by equations (126) and (127).

The proof of Helmholtz's theorem is not completely trivial. However, Arken (1985) does it. In fact, it was used for more than 100 years before it was proved rigorously. To begin to understand its validity, let's consider a vector field $\mathbf{u}(\mathbf{r})$ that's finite, continuous and goes to zero as $r \to \infty$. The proof of Helmholtz's theorem can be reduced to showing that for any such vector field, there exists another vector field **W** such that

$$\nabla^2 \mathbf{W} = -\mathbf{u}, \tag{128}$$

which may not be as hard to swallow as Helmholtz's theorem itself. But if we take this as a fact we can get Helmholtz's theorem simply enough. Using the vector identity (3-12) we can rewrite equation (128) as:

$$\mathbf{u} = -\nabla^2 W = -\nabla(\nabla \cdot \mathbf{W}) + \nabla \times (\nabla \times \mathbf{W}) \tag{129}$$

$$= \nabla\phi + \nabla \times \boldsymbol{\psi}. \tag{130}$$

Furthermore, since $\nabla \cdot (\nabla \times \mathbf{W}) = 0$, we can require that $\nabla \cdot \boldsymbol{\psi} = 0$.

P-wave Propagating the the x-Direction

Consider a P-wave propagating in the x-direction, $\phi(x,t) = \phi(x - \alpha t)$, which is a solution of the 1-D wave equation

$$\alpha^2 \partial_{xx}\phi = \ddot{\phi}. \tag{131}$$

Because $\mathbf{u} = \nabla\phi$, $\mathbf{u} = u_x\hat{x} = \partial_x\phi\hat{x}$, displacement occurs only in the direction of motion. P-waves are 'longitudinal' waves. The dilatation $\theta = \nabla \cdot \mathbf{u} = \nabla \cdot \nabla\phi = \nabla^2\phi \neq 0$, so P-waves introduce volume changes in the elastic medium as they propagate. They are, therefore, frequently called 'compressional' or 'dilatational' waves. Also, since $\nabla \times \nabla\phi = 0$, P-wave motion is 'irrotational'.

S-wave Propagating the the x-Direction

Consideration of an S-wave propagating in the x-direction is somewhat more complicated since the S-wave potential is a vector and we have to investigate each of its components: $\mathbf{u} = \nabla \times \boldsymbol{\psi}$, where in general:

$$\boldsymbol{\psi} = \psi_x(x - \beta t)\hat{x} + \psi_y(x - \beta t)\hat{y} + \psi_z(x - \beta t)\hat{z}. \tag{132}$$

Consider the curl of $\boldsymbol{\psi}$:

$$(\nabla \times \boldsymbol{\psi})_i = \epsilon_{ijk}\partial_j\psi_k, \tag{133}$$

$$(\nabla \times \boldsymbol{\psi})_1 = \epsilon_{1jk}\partial_j\psi_k = \partial_y\psi_z - \partial_z\psi_y = 0, \tag{134}$$

$$(\nabla \times \boldsymbol{\psi})_2 = \epsilon_{2jk}\partial_j\psi_k = \partial_z\psi_x - \partial_x\psi_z = -\partial_x\psi_z, \tag{135}$$

$$(\nabla \times \boldsymbol{\psi})_3 = \epsilon_{3jk}\partial_j\psi_k = \partial_x\psi_y - \partial_y\psi_x = \partial_x\psi_y, \tag{136}$$

$$\tag{137}$$

where ϵ_{ijk} is the permutation tensor or the Levi-Civita tensor which is defined to be 1 if ϵ_{ijk} is any even permutation of ϵ_{123}, -1 if its an odd permutation, and 0 if any of the subscripts are repeated. Thus,

$$\mathbf{u} = \nabla \times \boldsymbol{\psi} = -\partial_x\psi_z\hat{y} + \partial_x\psi_y\hat{z}, \tag{138}$$

$$\psi_x = 0, \tag{139}$$

from which we see that displacement is in the (y, z)-plane and, therefore, is perpendicular to the direction of motion. Also, $\nabla \cdot \mathbf{u} = 0$, so the motion is dilatationless; i.e., there are no volume changes during propagation of an S-wave. If we consider an element of the strain tensor:

$$\epsilon_{xy} = \frac{1}{2}\left(\partial_y u_x + \partial_x u_y\right) \tag{140}$$

$$= -\frac{1}{2}\partial_x\left(\partial_x\psi_z\right), \tag{141}$$

where the latter equality follows from equation (138). Thus, shear strains occur without volume changes for S-waves, and for this reason they are called 'shear' waves.

To get a scalar wave equation for S-waves propagating in the x-direction analogous to equation (131) for P-waves, we need to consider the Laplacian of the vector potential $\boldsymbol{\psi}$:

$$\nabla^2\boldsymbol{\psi} = = -\nabla \times \nabla \times \boldsymbol{\psi}, \tag{142}$$

$$(\nabla \times \nabla \times \boldsymbol{\psi})_i = \epsilon_{ijk}\partial_j (\nabla \times \boldsymbol{\psi})_k = \epsilon_{ijk}\partial_j \mathbf{u}_k, \tag{143}$$

$$(\nabla \times \nabla \times \boldsymbol{\psi})_x = \partial_y \partial_x \psi_y + \partial_z \partial_x \psi_z = 0, \tag{144}$$

$$(\nabla \times \nabla \times \boldsymbol{\psi})_y = \partial_z(0) - \partial_{xx}\psi_y = -\partial_{xx}\psi_y, \tag{145}$$

$$(\nabla \times \nabla \times \boldsymbol{\psi})_z = -\partial_{xx}\psi_z - \partial_y(0) = -\partial_{xx}\psi_z, \tag{146}$$

where equation (138) has been used to evaluate the cross products. Thus,

$$\nabla^2\boldsymbol{\psi} = \partial_{xx}\psi_y\hat{y} + \partial_{xx}\psi_z\hat{z}. \tag{147}$$

We need to substitute this into the equation of motion for S-waves, equation (125). Ignoring the forcing terms:

$$\beta^2\left[\partial_{xx}\psi_y\hat{y} + \partial_{xx}\psi_z\hat{z}\right] = \ddot{\psi}_y\hat{y} + \ddot{\psi}_z\hat{z}, \tag{148}$$

which yields two separate uncoupled scalar equations for S-wave displacements in the y- and z-directions:

$$\ddot{\psi}_y = \beta^2\partial_{xx}\psi_y, \tag{149}$$

$$\ddot{\psi}_z = \beta^2\partial_{xx}\psi_z, \tag{150}$$

which are the S-wave analogues of equation (131).

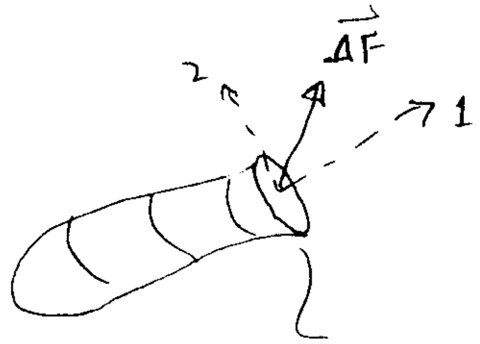

Figure 1

Slice ⊥ to 1-axis

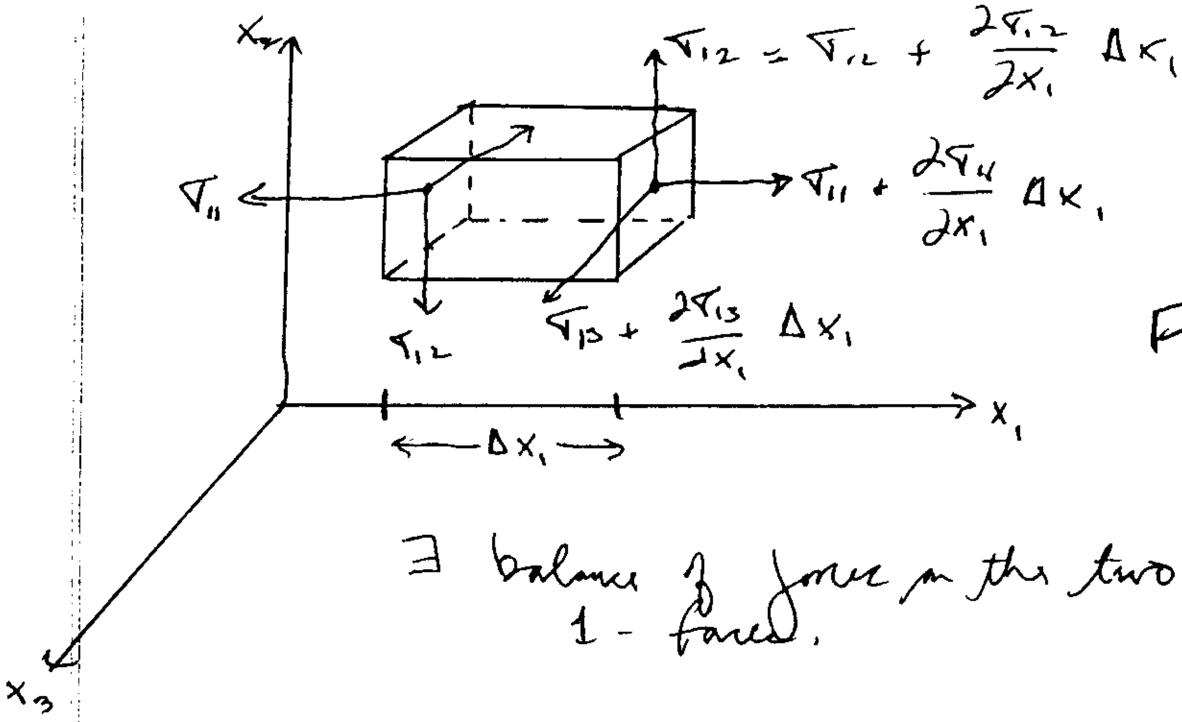

Figure 2

∃ balance of forces on the two 1-faces.

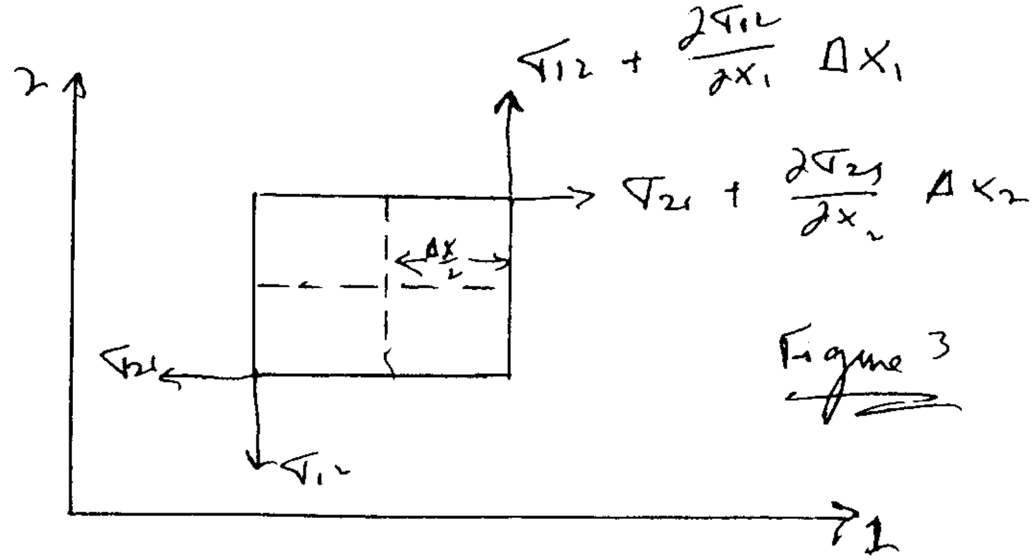

Figure 3

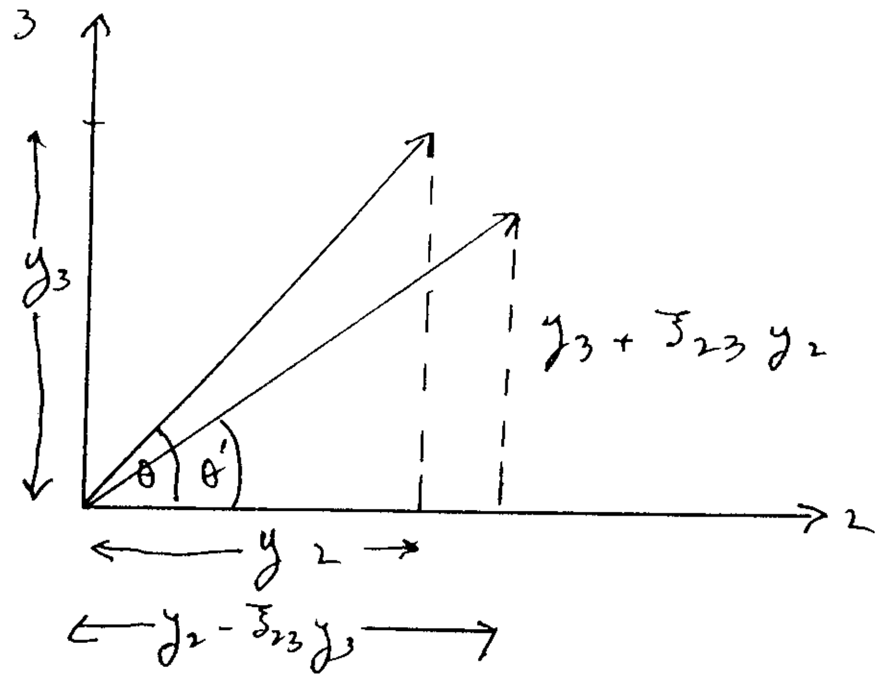

Fig. 4

$\theta' \simeq \theta + \gamma_{23}$

$\gamma_{23} < 0$ here.

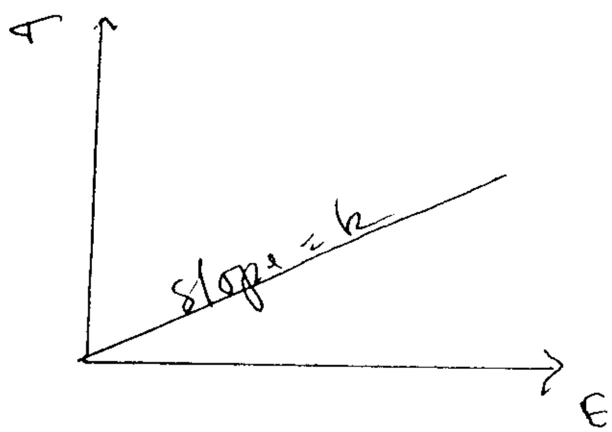

Figure 5

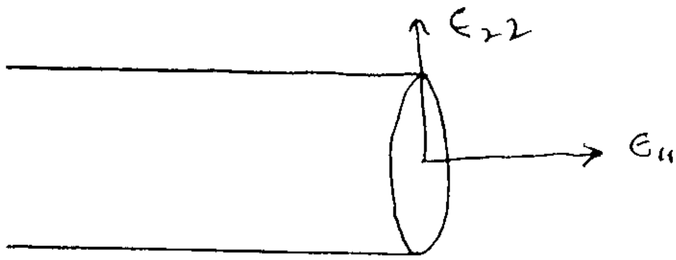

Figure 6