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Third Workshop on 3D Modelling of Seismic Waves Generation Propagation and their Inversion

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1D Modes and Waves in Homogeneous and Inhomogeneous Media

M. Ritzwoller

Dept. of Physics
University of Colorado
Boulder, CO, U.S.A.

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Michael H. Ritzwoller
Department of Physics
University of Colorado, Boulder
Boulder, CO 80309 USA
ritzwoller@lemond.colorado.edu

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2. 1D Modes and Waves in Homogeneous and Inhomogeneous Media

A clear understanding of simple, linear 1D problems such as the damped harmonic oscillator or waves on a string should be the foundation of your intuition into seismology. We will begin to discuss the physical and seismological significance of these simple systems here and will also review a little more sophisticated mathematics that should come in useful later on in the class. Simple harmonic oscillators (SHO's) will be laid to rest quickly. Their importance derives from the fact that SHO ODE's fall out from certain wave propagation PDE's by separation of variables. However, the string is a much more useful 1D analogue of the Earth and it will have a longer life in these lecture notes. Its relation to seismology is the same as the relation between the 1D Schrodinger equation and quantum mechanics, it is a toy system on which to rest intuition and to continue to develop new mathematics in order to guide analyses of the more complicated multidimensional equations of motion. More on this in the weeks to come!

2.1 Unforced, Damped Simple Harmonic Oscillator (SHO)

The basis of the SHO is Hooke's Law, a linearized force-displacement relation, which holds only approximately in the limit of small displacements. Such an oscillator oscillates with a characteristic natural frequency which is dependent on the spring constant, $\omega_0 = \sqrt{k/m}$. When damping is included the natural frequency is modified. If damping is sufficiently small ($b^2 < \omega_0^2$) the system will oscillate, but its amplitude will decay with time. In this case, the natural frequency is reduced to $\sqrt{\omega_0^2 - b^2}$. This is a very powerful result that is extremely important and finds an analogue when considering the frequencies and wave velocities of real materials: damped systems oscillate with characteristic frequencies that are reduced relative to the undamped states. Scientists and engineers usually represent the strength of damping with the quality factor $Q = \omega_0/2b$. Because Q is inversely related to damping strength, researchers frequently use $q = Q^{-1}$ and call it 'little q', 'attenuation', 'dissipation', 'friction strength' and many other names.

Oscillating systems are often discussed in the frequency domain. The frequency domain representation of an unforced, damped oscillation ($y(t) = Ae^{-bt}\sin(\omega_0 t + \phi)$) is called a Lorentzian whose amplitude in the neighborhood of the natural frequency in the frequency domain is approximately the following:

$$|y(\omega)| \approx A/\sqrt{(\omega - \omega_0)^2 + b^2}. \tag{1}$$

This is derived by taking the Fourier Transform of $y(t)$:

$$\mathcal{F}[y] = y(\omega) = \int_{\infty}^{\infty} y(t) e^{-i\omega t} dt, \tag{2}$$

$$\mathcal{F}^{-1}[\mathcal{F}[y]] = y(t) \quad = \quad \frac{1}{2\pi}\int_{-\infty}^{\infty} y(\omega)e^{i\omega t}d\omega. \tag{3}$$

In the absence of damping, the Lorentzian is just a delta function, $\delta(\omega - \omega_0)$, but with damping the width of the Lorentzian is dependent upon attenuation. Seismic signals that tend to be spread out in the time domain (e.g., a sinusoid) are compressed in the frequency domain. This is one of the reasons that Fourier Transforms are so useful. For real oscillating systems measured in the time domain, Fourier Transforms are discrete and are estimated over a finite time interval. This is the basis of time series analysis and forms a completely separate and rich field of study that's, unfortunately, largely beyond the scope of these lecture notes. Suffice it to say here that the ways in which one samples the function whose spectrum one wishes to estimate have significant ramifications on the spectral (frequency domain) parameters that are estimated. In particular, time series length and temporal window affect the Fourier Transform of $y(t)$.

A system defined as a set of (potentially coupled) oscillators will in general exhibit multiple periodicities. This leads to multiple, perhaps overlapping, peaks in the frequency domain, and beating in the time domain. Coupled oscillators will be discussed below.

2.2 Forced, Damped SHO

When a force is applied to an oscillating system, the system will ultimately oscillate with the forcing frequency. On application of the force, a transient is generated (the solution of the homogeneous equation) that decays with time leaving the steady-state solution. If the system is damped there is a phase lag, between the application of the force and the response of the medium, which is dependent on the strength of damping. The amplitude of the steady-state motion is a function of the difference in frequency between the natural and forcing frequencies as well as damping strength. In the absence of damping, the amplitude of the oscillation is inversely proportion to this difference and pure resonance (the indefinite increase of amplitudes with time) occurs when a system is forced at or near its natural frequency. The addition of damping prohibits pure resonance, and since all real systems are at least weakly attenuative, pure resonance never occurs in nature. Weakly damped systems can display resonant-like phenomena, but as amplitudes increase many systems no longer behave linearly. This also inhibits resonance. The canonical geophysical example of a forced, damped system is the earth's tides.

There are many nonlinear phenomena in nature (in fact everything is really nonlinear). A simple example is the response of a medium to increasingly larger strains. Initially, a material will deform under small strains nearly elastically in rough agreement with Hooke's Law. Upon continued

deformation, the linear stress-strain relation will break down and the material will either begin to deform plastically or will undergo brittle fracture (it'll break). Another example is when waves, say in water or a plasma, increase in amplitude. Eventually the waves will break and higher order (nonlinear) terms in the equation of motion are needed to begin to deal with this remarkly common by theoretically intractable problem. Unlike in the Earth, acoustic waves near the surface of the Sun have very large amplitudes. To model solar acoustic mode amplitudes also requires retaining nonlinear terms in the equation of motion and the theoretical study of what's called higher-mode coupling. All of this is beyond the scope of these lecture notes which will investigate the solution to purely linear problems, and this becomes complicated enough as the medium in which the waves propagate becomes increasingly complex. But, as you can see, nonlinear dynamics should also be a required subject of a well rounded graduate education.

Multiple forces can be applied simultaneously, and for a linear system the response will simply be a sum of the individual responses.

2.3 1D Homogeneous String and Modes of Oscillation

A string is useful to study for a variety of reasons, principally because a string is one dimensional but complications can be compounded to a degree that it can be thought of as a 1D analogue of the earth. For this reason, we will be using the string as a model throughout the semester to develop mathematical tools and intuition about the ways waves propagate in the earth.

The simple homogeneous string with fixed ends is interesting since it demonstrates the effect of the application of boundary conditions. For a string with fixed ends, the requirement that $X(L) = 0$ means that $\sin kL = 0$ which requires that $kL = n\pi \rightarrow k_n = n\pi/L$. Thus, the application of the boundary conditions requires that the number of admissible solutions be greatly decreased from a continuum to a discrete set, the modes of oscillation of the string. In particular, wavenumber is quantized and since $k = \omega/c$, this also requires that frequency is discrete, $\omega_n = nc\pi/L$. The length of the string and its wave velocity determine the discrete frequencies of each mode of oscillation of the string (each n). The relation between k and ω is called a 'dispersion relation'. Since they are linearly related the string is nondispersive, every wavelength travels with the same group velocity, $U = d\omega/dk = c$. Most media are more complex than this and dispersion relations are commonly much more complicated. It is not generally the case that phase and group velocities are equal.

Derivation

The common derivation of the 1D string equation is for transverse oscillations. You can find this derivation in any freshman physics text or ODE text. Less common is the derivation for longitudinal

waves in a string which will be presented here due to its relation to the derivation of the seismic equation of motion. The derivation is based on two physical principals: (1) Hooke's Law and (2) Newton's 2nd Law.

Let $y(x)$ be displacement from equilibrium. We define the 'strain' in the string as the extension per unit length which varies from point to point in the string. The portion of the string initially between x and $x + \Delta x$ is stretched by an amount $y(x + \Delta x) - y(x)$ so that the strain is $\partial y/\partial x$ and y is longitudinal displacement. If stress τ is defined as the excess over its equilibrium value of tension, then the analogue of Hooke's Law is a linear stress-strain relation:

$$\tau = k\frac{\partial y}{\partial x}, \tag{4}$$

where k is Young's modulus. The net force acting on the element Δx is the excess of the stress at $x + \Delta x$ over that at x. Therefore, the force per unit length F is:

$$F = \frac{\partial}{\partial x}\left(k\frac{\partial y}{\partial x}\right). \tag{5}$$

Of course, Young's Modulus may be variable if the string is not homogeneous. By Newton's 2nd Law, mass per unit length (ρ) times acceleration equals force per unit length:

$$\rho\frac{\partial^2 y}{\partial t^2} = \frac{\partial}{\partial x}\left(k\frac{\partial y}{\partial x}\right). \tag{6}$$

Since this derivation of the string equation depends on the use of Hooke's Law (eqn. (4)), the wave equation can be seen to be an approximation exactly like the SHO equation is. The retention of nonlinear terms in equation (4) produces associated nonlinear terms in equation (6), which then becomes much harder to solve.

Assume that the string is homogeneous so that in equation (6) Young's modulus is a constant and can be removed from the parentheses yielding:

$$\frac{\partial^2 y}{\partial t^2} = c^2\frac{\partial^2 y}{\partial x^2}, \tag{7}$$

where here $c = \sqrt{k/\rho}$. The velocity of the wavefront is controlled by the material properties, in this case k and ρ. A stiff string, with high Young's modulus, produces faster traveling waves. Increasing density alone would tend to reduce the velocity, but in most solids k generally increases faster than ρ does, causing a compensation that usually results in a net increase in velocity. So, for example, if you increase temperature and hold everything else fixed in the earth, density will decrease but the elastic moduli will decrease faster and the resultant wave speed is reduced with increase in

temperature. This isn't true in all materials. For example, in water increasing temperature speeds up waves.

As an example of a traveling wave solution, consider the following:

$$y(x,t) = Ce^{i(\omega t - kx)} + C^* e^{-i(\omega t - kx)}, \tag{8}$$

where $*$ denotes complex conjugation, which is necessary to ensure a real function. This is called a plane wave solution, and is useful since an arbitrary function with only a finite number of discontinuities can be expressed as a sum of such plane waves. This is just a 2D Fourier Series.

Substituting equation (8) into (7) shows that the solution is acceptable provided that ω and k satisfy $\omega = \pm ck$ - the dispersion relation again. For sinusoidal traveling waves such as in equation (8), displacement patterns repeat. Points at which the displacement amplitudes are equal have equal phases; i.e. $\omega t - kx = \omega(t + \Delta t) - k(x + \Delta x)$. This is true if $\omega \Delta t - k\Delta x = 0 \rightarrow c = dx/dt = \omega/k$. This is the velocity of a phase, so is called the phase velocity. Wave groups are more complicated and we'll get back to them later, but they are constructed out of a set of component waves such as plane waves and satisfy the condition of constructive interference; that is each component must have the same value of the phase angle $\omega t - kx + \phi$ although the individual values of ω, k and ϕ may be different. Thus, the quantity $\omega t - kx + \phi$ must be independent of frequency if evaluated at a characteristic frequency, ω_0, of the group: $d(\omega t - kx + \phi)/d\omega|_{\omega_0} = 0$. Carrying out the differential we find that the constructive interference condition will be met for a wave traveling with the group velocity $U = d\omega/dk|_{\omega_0}$. Thus, the group velocity is just the slope of the dispersion relation. For a nondispersive system, the dispersion relation is linear and the group velocity is constant with frequency. For the 1D homogeneous string, the slope of the dispersion relation is just c, and therefore $U = c$. It is not true for all nondispersive systems that group and phase velocities are equal.

Normal Mode Solution

Taking a separation of variables and, therefore, a normal mode approach to solving equation (7) yields:

$$y(x,t) = \sum_n \sin(\frac{n\pi x}{L}) \left(A_n \cos(\frac{n\pi ct}{L}) + B_n \sin(\frac{n\pi ct}{L}) \right). \tag{9}$$

Assuming, for sake of simplicity, that the initial displacement $y(x,0)$ and velocity $v(x,0)$ of the string are odd functions with wavelength $2L$, they can be expressed as a Fourier sine series on that interval:

$$y(x,0) = \sum_{n=1}^{\infty} a_n \sin \frac{n\pi x}{L}, \tag{10}$$

$$v(x,0) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L}, \tag{11}$$

where

$$a_n = \frac{2}{L}\int_0^L y(x,0)\sin\frac{n\pi x}{L}dx, \tag{12}$$

$$b_n = \frac{2}{L}\int_0^L v(x,0)\sin\frac{n\pi x}{L}dx. \tag{13}$$

Then, equation (9) can be rewritten as follows:

$$y(x,t) = \sum \sin(\frac{n\pi x}{L})\left(a_n\cos(\frac{n\pi ct}{L}) + b_n\frac{L}{n\pi c}\sin(\frac{n\pi ct}{L})\right), \tag{14}$$

$$= \sum_{n=0}^{\infty} c_n \sin(\frac{n\pi x}{L})\sin(\frac{n\pi ct}{L} - \phi_n), \tag{15}$$

where ϕ_n is a phase factor, $\phi_n = \arctan(Lb_n/n\pi ca_n)$.

Traveling Wave Solution

An alternative is to take a wave approach to solving (7). This approach turns out to be less powerful mathematically in the long run (we will go back to the modal approach in dealing with more complicated string and seismology problems) but is useful for purposes of intuition. It can be immediately verified that $y_1(x,t) = f_1(x-ct)$ and $y_2(x,t) = f_2(x+ct)$ are solutions to (7) where f_1 and f_2 are arbitrary functions that satisfy the initial conditions associated with a particular source that excites the stress imbalance, giving rise to propagating disturbances. These functions represent traveling waves in the $+x$ and $-x$ directions, respectively. The whole pattern of disturbance moves with velocity c. The general solution is the sum:

$$y(x,t) = f_1(x-ct) + f_2(x+ct), \tag{16}$$

constants of superposition are not necessary since f_1 and f_2 are arbitrary functions.

Equation (16) is called D'Alembert's solution. The arguments $(x \pm ct)$ are called the phase of the wave solution. For a given value of phase, the translating functional shape is called a wavefront. A seismogram would correspond to a recording of $y(x,t)$ at a fixed position x_0. This would take the form $y(x_0,t) = f_1(x_0,t) + f_2(x_0,t)$, a function of time at x_0 that records the passage of two wave groups past position x_0. If the string is fixed on either end, then these disturbances would be recorded again and again as they reflect off the end points and reverse their path of travel. More on this later.

If we apply the initial conditions that $y(x,0) = y_0(x)$ and $\partial y(x,0)/\partial t = v_0(x)$ to equation (16), then f_1 and f_2 can by expressed in terms of $y_0(x)$ and $v_0(x)$ and the resulting equations solved for

f_1 and f_2 to yield:

$$f_1(x) = \frac{1}{2}\left[y_0(x) - c^{-1}\int_{x_1}^{x} v_0(x')dx'\right], \tag{17}$$

$$f_2(x) = \frac{1}{2}\left[y_0(x) - c^{-1}\int_{x_1}^{x} v_0(x')dx'\right]. \tag{18}$$

The lower limit x_1 is arbitrary, but in any case does not appear in the complete solution:

$$y(x,t) = \frac{1}{2}\left[y_0(x-ct) + y_0(x+ct) + c^{-1}\int_{x-ct}^{x+ct} v_0(x')dx'\right]. \tag{19}$$

This equation will be more explicitly derived by Fourier and Laplace techniques in Section 2.7 below.

Since the traveling wave and the normal mode solutions are equivalent, we see that the superposition of reflecting waves produces the modes of oscillation. Conversely, the modes of oscillation superpose to produce waves. In a string, we see (and hear) the modes of oscillation and the waves themselves are obscure. However, in the Earth we see individual packets of energy arriving on a seismogram. Although we will frequently use modal techniques to solve difficult problems, we frequently see and identify waves in data and, therefore, think of modes superposing to produce the waves. A facile geophysicist must be able to think in both modes and waves as he or she must be able to think both in the time and frequency domains.

Energy

Speaking of energy, expressions for the kinetic and potential energy density are required for Lagrangian and Hamiltonian dynamics, and we will consider them briefly here. The energy density of a string is its energy per unit length. Let $\mathcal{K}$ denote kinetic energy density, then:

$$\mathcal{K} = \frac{1}{2}\rho v^2 = \frac{1}{2}\rho\left(\frac{\partial y}{\partial t}\right)^2, \tag{20}$$

and the total kinetic energy, K, is:

$$K = \frac{1}{2}\int_0^L \rho\left(\frac{\partial y}{\partial t}\right)^2 dx. \tag{21}$$

The derivation of the appropriate expression for potential energy is a little more complicated. Consider an increment of the string, dx, stretched to a new length ds. The change in length is just $ds - dx$. This derivation is a little easier if we consider transverse rather than longitudinal oscillations. The final equations are the same if tension, T, is replaced with Young's modulus, k. So imagine a transverse perturbation, dy. Then, $ds^2 = dx^2 + dy^2$ and factoring out a dx:

$$ds - dx = dx\left[\sqrt{1 + (dy/dx)^2} - 1\right]. \tag{22}$$

Under the small oscillation approximation, $dy/dx << 1$, so we can Taylor expand the square root in equation (22) which yields:

$$ds - dx = \frac{1}{2}\left(\frac{dy}{dx}\right)^2 dx. \tag{23}$$

Stretching takes place against tension, and the work against tension is just tension times stretch:

$$W = \frac{1}{2}T\left(\frac{dy}{dx}\right)^2 dx, \tag{24}$$

and the potential energy density, $\mathcal{V}$, is:

$$\mathcal{V} = \frac{1}{2}T\left(\frac{dy}{dx}\right)^2, \tag{25}$$

so that the total potential energy becomes:

$$V = \frac{1}{2}\int_0^L T\left(\frac{dy}{dx}\right)^2 dx. \tag{26}$$

Total energy density $\mathcal{H} = \mathcal{K} + \mathcal{V}$, so that

$$\mathcal{H} = \frac{1}{2}\rho\left(\frac{\partial y}{\partial t}\right)^2 + \frac{1}{2}T\left(\frac{dy}{dx}\right)^2. \tag{27}$$

and the total energy can be written

$$H = \int_0^L \mathcal{H} dx \tag{28}$$

$$= \frac{1}{2}\rho\int_0^L \left[\left(\frac{\partial y}{\partial t}\right)^2 + c^2\left(\frac{dy}{dx}\right)^2\right] dx. \tag{29}$$

It should be noted that this equation only holds either when the endpoints of the string are fixed or when the spatial gradient of displacement at the endpoints goes to zero. The former is the the situation we are considering here so equation (29) is fine, but it should not be considered a general formula which is applicable in any situation. (For further discussion see Morse and Feshbach ch. 2.1.)

The kinetic, potential, and total energies in terms of the modes of oscillation from equation (15) is simply:

$$K = \frac{1}{4}\rho L\sum_n \left(\frac{n\pi c}{L}\right)^2 c_n^2 \cos^2\left(\frac{n\pi ct}{L} - \phi_n\right), \tag{30}$$

$$V = \frac{1}{4}\rho L\sum_n \left(\frac{n\pi c}{L}\right)^2 c_n^2 \sin^2\left(\frac{n\pi ct}{L} - \phi_n\right), \tag{31}$$

$$H = \frac{1}{4}\rho L\sum_n \left(\frac{n\pi c}{L}\right)^2 c_n^2. \tag{32}$$

Thus, the kinetic and potential energies are out of phase by 90 degrees and sum to give a constant of motion. In each mode, the energy oscillates between kinetic and potential forms as the string itself oscillates. The periods of the oscillations of the string are $2L/nc$, while those of energy are half that great, corresponding to successive realizations of a given phase of the motion.

Equations (30) - (32) demonstrate a striking result, that the individual terms in the Fourier series solution, equation (15), are independent in that each carries a fixed amount of energy and this energy cannot be exchanged with the energies of any other mode, much as the traveling waves in the string do not exchange energy when they pass one another but simply superpose. We say these modes are, therefore, uncoupled.

Complexities added to the string can produce modal coupling as we will see as the course progresses. Later on we will also consider reflection of string waves off fixed masses, the solution to the inhomogeneous string problem, and perturbative and approximate methods for finding frequencies and shapes of oscillations.

Conservation of Energy

Although the total energy, H, is stationary for the entire string, energy can flow along the string. Thus, an energy flux, J, should be defined. The energy flux, $\mathcal{J}$, and the energy density are related by the conservation of energy which states that the time rate of change of energy density at a point is related to the net amount of energy flowing into or out of a region. For example, if more energy flows across the point $x + dx$ than flows across x, then the energy contained in the length dx of the string must diminish:

$$\mathcal{J}(x + dx) - \mathcal{J}(x) = -dx\frac{\partial \mathcal{H}}{\partial t}, \tag{33}$$

which upon rewriting becomes

$$\frac{\partial \mathcal{J}}{\partial x} + \frac{\partial \mathcal{H}}{\partial t} = 0. \tag{34}$$

Therefore, a closed form expression for energy flux can be found as follows:

$$\begin{aligned}
\mathcal{J} &= -\int \frac{\partial \mathcal{H}}{\partial t} dx & (35)\\
&= -\frac{1}{2}\rho \int \frac{\partial}{\partial t}\left[\left(\frac{\partial y}{\partial t}\right)^2 + c^2\left(\frac{dy}{dx}\right)^2\right] dx & (36)\\
&= -\rho \int \left[\frac{\partial y}{\partial t}\frac{\partial^2 y}{\partial t^2} + \frac{T}{\rho}\frac{\partial y}{\partial x}\frac{\partial^2 y}{\partial x \partial t}\right] dx & (37)\\
&= -T \int \left[\frac{\partial y}{\partial t}\frac{\partial^2 y}{\partial x^2} + \frac{\partial y}{\partial x}\frac{\partial^2 y}{\partial x \partial t}\right] dx & (38)
\end{aligned}$$

$$= -T \int \frac{\partial}{\partial t} \left[\frac{\partial y}{\partial t} \frac{\partial y}{\partial x} \right] dx \tag{39}$$

$$= -T \frac{\partial y}{\partial t} \frac{\partial y}{\partial x}. \tag{40}$$

Using this expression for energy flux and the expression for total energy density given by equation (27), from the conservation of energy an alternative derivation of the wave equation for the string results:

$$0 = \frac{\partial \mathcal{J}}{\partial x} + \frac{\partial \mathcal{H}}{\partial t} \tag{41}$$

$$= -T \frac{\partial^2 y}{\partial x \partial t} \frac{\partial y}{\partial x} - T \frac{\partial y}{\partial t} \frac{\partial^2 y}{\partial x^2} + \rho \frac{\partial y}{\partial t} \frac{\partial^2 y}{\partial t^2} + T \frac{\partial y}{\partial x} \frac{\partial^2 y}{\partial t \partial x} \tag{42}$$

$$= \frac{\partial y}{\partial t} \left(-T \frac{\partial^2 y}{\partial x^2} + \rho \frac{\partial^2 y}{\partial t^2} \right). \tag{43}$$

Since $\partial y / \partial t$ is an arbitrary function, equation (43) implies (7) where $c^2 = T/\rho$. Recall the previous derivation of the string equation came from the application of Hooke's Law and Newton's Second Law.

2.4 Relevance to Seismology

Armed with an understanding of SHO's and strings one can understand a lot of seismology. The earth can be thought of as a huge set of coupled oscillators, where the spring constants and damping factors vary from place to place. It is easier in the long run to assume that there is an infinite number, actually a continuum, of oscillators, and use results from continuum mechanics to guide seismic investigations. The analogue of the spring constant is the elastic moduli. A complication arises since different waves or modes sample the earth differently, and the elastic moduli have to be integrated over the region of the earth that a given mode of oscillation or a wave samples to determine the frequency or wave speed. The same holds for Q in which one has to integrate attenuation over the appropriate region. The elastic frequencies of the earth then have to be modified for the effect of 'physical dispersion', the perturbation to the elastic moduli and therefore the frequencies caused by finite Q. This correction was not fully appreciated until the 1960's and still caused some confusion as late as the 1970's when it was first discovered that long period earth models were systematically slower than high frequency earth models (e.g., Kanamori and Anderson, 1977). This was called at the time the baseline shift' and was caused almost entirely by the dependence of the frequencies of oscillation and wave velocities of the earth on its intrinsic attenuation.

The displacement shapes from multiple oscillators superpose to produce waves. The details of the dispersion relations determine the nature of these waves.

As with a string, boundary conditions and shape affect the frequencies of oscillation. The earth's size (analogue of L) and spherical shape are important as are the location and nature of internal discontinuities. In analogy with the string, it might be thought that each new boundary within the earth and the associated boundary conditions on the displacement and tractions across the boundary should reduce the number of acceptable solutions. But, it's more complicated than this. If you add a boundary in the earth new waves come into existence (reflections from the boundary, head waves along the boundary) that were completely missing before the boundary was added.

Terrestrial oscillations are forced, but the duration of the forcing is short compared to the lifetime of seismic waves. However, to simulate seismic displacements for large earthquakes accurately the earthquake time-history must be convolved in the time domain with the natural response of the earth (estimated as if the source was instantaneous). This result can be seen in the next section.

2.5 Laplace Transforms and Convolution

I assume you are all familiar with Laplace and Fourier Transforms and their use in solving differential equations. Define the Laplace transform of a time function $f(t)$ as follows:

$$L[f] = F(s) = \int_0^\infty f(t)e^{-st}dt \tag{44}$$

Consider the differential equation

$$A\ddot{y} + B\dot{y} + Cy = f(t), \tag{45}$$

given initial conditions $y(0)$ and $\dot{y}(0)$. As you should know,

$$L[\dot{y}] = sL[y] - y(0), \tag{46}$$

$$L[\ddot{y}] = s^2L[y] - sy(0) - \dot{y}(0). \tag{47}$$

Assuming for simplicity that $y(0) = \dot{y}(0) = 0$ (we will break this assumption in the next subsection of these notes), then Laplace Transforming both sides of the ODE in equation (45) gives:

$$(As^2 + Bs + C)L[y] = F(s). \tag{48}$$

where $F(s) = L[f]$ is just the Laplace Transform of the right hand side of the ODE. Solving (48) for L[y] gives the product of two functions:

$$L[y] = Y(s) = \frac{1}{As^2 + Bs + C}F(s) \tag{49}$$

$$= \tau(s)F(s), \tag{50}$$

where $\tau(s)$ is called a transfer function - in the s-domain it transfers the forcing function into the response of the system. It represents all the physics in the ODE. The solution of equation (45) is the inverse Laplace Transform of $Y(s) = \tau(s)F(s)$.

The inverse Laplace Transform of the function $Y(s)$ is defined as follows:

$$y(t) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} Y(z)e^{zt}dz \tag{51}$$

where z is complex and $t > 0$. The integral is performed along any vertical line $x = c$ in the z-plane, such that $c > \mathrm{Re}(s)$. This definition is at first sight a little more daunting than the inverse Fourier Transform, but if $Y(s)$ can be written as the ratio of polynomials, $P(z)/Q(z)$, such that $Q(z)$ is of at least one degree higher than $P(s)$ (the number of poles outnumbers the number of zeroes by at least one) then the evaluation of (51) is particularly simple. Restricting ourselves to such a function, we can solve the integral in (51) explicitly:

$$y(t) = \sum \text{ residues at all poles of } Y(z)e^{zt}. \tag{52}$$

We must include all poles in (52) since c is largely arbitrary.

It can be easily seen here, however, that each of the two poles of the $\tau(s)$ will produce a 'wave' in the time domain - this is just the solution of the homogeneous equation. Added to these will be the 'waves' resulting from the poles of the forcing function, simply the particular solution. If in addition, an instrument response of some kind had been multiplied into $Y(s)$, the poles of the instrument would also produce additional time functions. You can see then that the solution of an ODE has been reduced to performing a contour integral, which itself can simply be reduced to a simple sum in which *each term of the sum (both homogeneous and particular solutions) is related to a pole of the function $Y(s)$*. This is a beautiful and remarkable result and is foundation for a great deal of theoretical seismology.

Yet another way of writing the inverse Laplace Transform of (49) is as follows:

$$\begin{aligned}
\tau(s)F(s) &= \int_0^\infty e^{-st''}\tau(t'')dt'' \cdot \int_0^\infty e^{-st}F(t')dt', && (53)\\
&= \int_0^\infty\int_0^\infty e^{-s(t'+t'')}\tau(t'')f(t')dt''dt, && (54)\\
&= \int_{t'=0}^\infty\int_{t=t'}^\infty e^{-st}\tau(t-t')f(t')dtdt', && (55)\\
&= \int_{t=0}^\infty\int_{t'=0}^t e^{-st}\tau(t-t')f(t')dt'dt, && (56)\\
&= \int_0^\infty e^{-st}\left[\int_0^t \tau(t-t')f(t')dt'\right]dt, && (57)\\
&= L\left[\int_0^t \tau(t-t')f(t')dt'\right], && (58)
\end{aligned}$$

where in equation (55) we have changed variables ($t = t' + t''$) and in equation (56) we have changed the order of integration. Equation (58) is just the Laplace Transform of a convolution where a convolution is defined as follows:

$$f(t) * g(t) = \int_0^t f(t - t')g(t')dt'. \tag{59}$$

Therefore,

$$y(t) \;=\; L^{-1}[\tau(s)F(s)] = L^{-1}[L[\tau(t) * f(t)]] = \tau(t) * f(t). \tag{60}$$

In conclusion, then, the solution to equation (45) is simply the time-domain convolution $y(t) = \tau(t) * f(t)$; the solution of the homogeneous equation, $\tau(t)$, convolved with the forcing function, $f(t)$. More discussion is contained in Section 2.6.

This is, again, a very powerful result. If the forcing function is a delta function, then it is seen immediately that the resultant solution is just the solution to the homogeneous equation. Indeed, if the forcing function is any time limited function, the solution is simply a locally time-filtered version of the solution to the homogeneous equation. The local time-filter is a detailed function of the source time history of the forcing function – in seismology, that's the energy release of seismic waves for an earthquake or an explosion.

2.6 Application of Laplace Transforms to a Forced, Damped SHO with an Arbitrary Forcing Function

Consider the damped, forced SHO equation:

$$\ddot{y} + 2b\dot{y} + \omega_0^2 y = f(t), \tag{61}$$

subject to general initial conditions

$$y(0) \;=\; x_0, \tag{62}$$

$$\dot{y}(0) = v_0. \tag{63}$$

One of the reasons that Laplace Transforms are so useful theoretically is that we can leave the forcing function $f(t)$ completely general, and we will derive solutions to equation (61) for a general forcing function and then look at a specific $f(t)$ as an example. Consistent with the notation in the previous section, the Laplace Transform will be denoted $L[f(t)] = F(s)$, so that the transform of (61) can be written:

$$Y(s) = \tau(s)\left[v_0 + (s + 2b)x_0 + F(s)\right]. \tag{64}$$

This is a generalization of equation (49) in that we have generalized the initial conditions and now have several terms in the Laplace Transform, but have specified here that $A = 1$, $B = 2b$, and $C = \omega_0^2$. The first two terms in braces on the right hand side of equation (64) represent the response to initial conditions, and therefore has a transient character. The last term represents the response to the external force. The final solution will be the sum of the two contributions: $y(t) = y_t(t) + y_f(t)$.

The only poles of the transient terms come from $\tau(s)$ which are, assuming an oscillatory solution ($\omega_0^2 > b^2$):

$$s_\pm = -b \pm i\sqrt{\omega_0^2 - b^2}. \tag{65}$$

Denoting these roots s_1 and s_2 (where $s_2 = s_1^*$), upon inverse Laplace Transforming the transient terms in the solution are seen to be:

$$\begin{aligned} y_t(t) &= (s_2 - s_1)^{-1}\left[(v_0 + (s_2 + 2b)x_0)e^{s_2 t} - (v_0 + (s_1 + 2b)x_0)e^{s_1 t}\right], && (66)\\ &= (v_0 + 2bx_0)\frac{e^{s_2 t} - e^{s_1 t}}{s_2 - s_1} + x_0\frac{s_2 e^{s_2 t} - s_1 e^{s_1 t}}{s_2 - s_1}, && (67)\\ &= e^{-bt}\left[(v_0 + bx_0)\frac{\sin\omega t}{\omega} + x_0\cos\omega t\right], && (68) \end{aligned}$$

where we have substituted the the values of s_1 and s_2 in the last equation and have defined $\omega = \sqrt{\omega_0^2 - b^2}$.

The forcing term of the solution, $y_f(t)$, results from the final term in equation (64) and comes from the poles of both $F(s)$ and $\tau(s)$. Let the poles of $F(s)$ comprise $N - 2$ complex conjugate pairs, s_n and s_n^* ($n = 3, 4, 5, \ldots, N$), and the residues be b_n and b_n^*. Then

$$\begin{aligned} y_f(t) &= y_{f_1}(t) + y_{f_2}(t), && (69)\\ &= \left[(s_2 - s_1)(F(s_2)e^{s_2 t} - F(s_1)e^{s_1 t})\right] + \sum_{n=3}^{N}\left[\frac{b_n}{s_n^2 + 2bs_n + \omega_0^2}e^{s_n t} + \frac{b_n^*}{s_n^{*2} + 2bs_n^* + \omega_0^2}e^{s_n^* t}\right], && (70)\\ &= \left[(s_2 - s_1)(F(s_2)e^{s_2 t} - F(s_1)e^{s_1 t})\right] + \sum_{n=3}^{N}\left[\frac{b_n}{(s_n - s_1)(s_n - s_1^*)}e^{s_n t} + \frac{b_n^*}{(s_n^* - s_1)(s_n^* - s_1^*)}e^{s_n^* t}\right], && (71) \end{aligned}$$

where the latter equation results from the fact that $s^2 + 2bs + \omega_0^2 = (s - s_1)(s - s_2) = (s - s_1)(s - s_1^*)$. While the first term in brackets on the right-hand-side of (71), $y_{f_1}(t)$, is transient, if $f(t)$ has a periodic component (i.e., if the s_n are purely imaginary), then the term under the sum, $y_{f_2}(t)$, gives a non-transient contribution. If we let the residues of $F(s)$ be $b_n = \beta_n e^{i\delta_n}$, and define $s_n = i\omega_n$, then

$$y_{f_2}(t) = \sum_{n=3}^{N}\left[\frac{\beta_n e^{i\delta_n}}{(i\omega_n - s_1)(i\omega_n - s_1^*)}e^{i\omega_n t} + \frac{\beta_n e^{-i\delta_n}}{(-i\omega_n - s_1)(-i\omega_n - s_1^*)}e^{-i\omega_n t}\right]. \tag{72}$$

$$= \sum_{n=3}^{N} \beta_n \left[\frac{1}{\omega^2 + b^2 - \omega_n^2 + i2b\omega_n} e^{i(\omega_n t + \delta_n)} + \frac{1}{\omega^2 + b^2 - \omega_n^2 - i2b\omega_n} e^{-i(\omega_n t + \delta_n)} \right], \quad (73)$$

$$= \sum_{n=3}^{N} \frac{2\beta_n}{(\omega_0^2 - \omega_n^2)^2 + 4b^2\omega_n^2} \left[(\omega_0^2 - \omega_n^2)\cos(\omega_n t + \delta_n) + 2b\omega_n \sin(\omega_n t + \delta_n) \right]. \quad (74)$$

where some algebra has been done to get to equations (73) and (74). Therefore, the final solution is $y(t) = y_t(t) + y_{f_1}(t) + y_{f_2}(t)$ where the terms in this equation are given by equations and (68), (71), and (74).

As an example, consider the case in which $f(t) = F\cos(\omega_1 t)$. Then $F(s) = Fs/(s^2 + \omega_1^2)$ which has poles at $s_1 = i\omega_1$ and $s_2 = s_1^* = -i\omega_i$ and residues $b_1 = F/2 = b_1^*$. From (74), the solution for $y_{f_2}(t)$ is, therefore:

$$y_{f_2}(t) = \frac{F}{(\omega_0^2 - \omega_1^2)^2 + 4b^2\omega_n^2} \left[(\omega_0^2 - \omega_1^2)\cos(\omega_1 t) + 2b\omega_1 \sin(\omega_1 t) \right], \quad (75)$$

$$= \frac{F}{\sqrt{(\omega_0^2 - \omega_1^2)^2 + 4b^2\omega_1^2}} \cos(\omega_1 t - \gamma_1), \quad (76)$$

where $\gamma_1 = \arctan(2b\omega_1/(\omega_0^2 - \omega_1^2))$.

2.7 Application of Fourier and Laplace Transforms to the Simple String

Consider again the vibrating homogeneous string equation. We seek a solution $y(x,t)$ that satisfies equation (7) subject to the initial conditions (62). Let $Y(x,s)$ be the temporal Laplace transform of $y(x,t)$, transforming equation (7) yields:

$$\frac{\partial^2 Y}{\partial x^2} = c^{-2}\left(-sy_0 - v_0 + s^2 Y\right). \quad (77)$$

Now, we also want to transform the spatial variable x. Typically this is done with a Fourier Transform if the string is infinitely long and with Fourier Series if the string is finite. Let's do both in turn.

Infinite String

Assuming that the string is infinitely long, define the spatial Fourier Transform of $Y(x,s)$ as

$$\xi(k,s) = \mathcal{F}[Y(x,s)] = \int_{-\infty}^{\infty} Y(x,s)e^{-ikx}dx, \quad (78)$$

$$(79)$$

where the spatial Fourier Transform variable is k (rather than the temporal variable ω). Thus, equation (77) becomes:

$$k^2\xi = c^{-2}\left[s\hat{y}_0 + \hat{v}_0 - s^2\xi\right], \quad (80)$$

where $\hat{y}_0(k)$ and $\hat{v}_0(k)$ are the spatial Fourier Transforms of $y_0(x)$ and $v_0(x)$, respectively. Therefore,

$$\xi(k,s) = \frac{s\hat{y}_0 + \hat{v}_0}{s^2 + c^2k^2}. \tag{81}$$

Equation (81) is the solution to (7) subject to the initial conditions (62), but it's in the transform domain; it's in (k,s) rather than (x,t) space. So we have to inverse transform back, first into the temporal domain with an inverse Laplace Transform and then into the spatial domain with an inverse Fourier Transform. The poles of (81) are $\pm ick$, and if $\xi(k,t)$ denotes the inverse Laplace Transform of $\xi(k,s)$ then

$$\begin{aligned}
\xi(k,t) &= \xi_1(k,t) + \xi_2(k,t), && (82)\\
&= \hat{y}_0 \cos kct + \hat{v}_0 \frac{\sin kct}{kc}, && (83)\\
&= \hat{y}_0 \cos kct + \hat{v}_0 \int_0^t \cos(kct')dt', && (84)
\end{aligned}$$

where we have rewritten the sinc function in equation (83) as an integral over a cosine for use directly below. The inverse Fourier Transform of the first term in (84) is straightforward:

$$\begin{aligned}
\mathcal{F}^{-1}(\xi_1(k,t)) &= y_1(x,t) && (85)\\
&= \frac{1}{2}\int_{-\infty}^{\infty} \hat{y}_0(k)(e^{ikct} + e^{-ikct})e^{ikx}dk && (86)\\
&= \frac{1}{2}\left(\int_{-\infty}^{\infty} \hat{y}_0(k)e^{ik(x+ct)}dk + \int_{-\infty}^{\infty} \hat{y}_0(k)e^{ik(x-ct)}dk\right) && (87)\\
&= \frac{1}{2}(y_0(x+ct) + y_0(x-ct)). && (88)
\end{aligned}$$

The inverse Laplace Transform of the second term in equation (84) is only slightly more complicated if you recognize that in the second term the $\hat{y}_0$ of the first term has simply been replaced by $\hat{v}_0$ and an integral over time has been added. With this observation we can write the inverse Laplace Transform of the second term as:

$$\begin{aligned}
\mathcal{F}^{-1}(\xi_2(k,t)) &= y_2(x,t) && (89)\\
&= \frac{1}{2}\int_0^t \int_{-\infty}^{\infty} \hat{v}_0(k)(e^{ikct'} + e^{-ikct'})e^{ikx}dkdt' && (90)\\
&= \frac{1}{2}\left(\int_0^t v_0(x+ct')dt' + \int_0^t v_0(x-ct')dt'\right) && (91)\\
&= \frac{1}{2c}\left(\int_0^{x+ct} v_0(x')dx' - \int_0^{x-ct} v_0(x')\right)dx' && (92)\\
&= \frac{1}{2c}\int_{x-ct}^{x+ct} v_0(x')dx', && (93)
\end{aligned}$$

where we have changed variables in equation (92) by setting $x' = x + ct'$ in the first term and $x' = x - ct'$ in the second term.

Combining equations (88) and (93) yields the solution to equation (7) subject to the initial conditions (62):

$$y(x,t) = \frac{1}{2}\left[y_0(x+ct) + y_0(x-ct) + c^{-1}\int_{x-ct}^{x+ct} v_0(x')dx'\right]. \tag{94}$$

Finite String

If the string is finite and of length L, we must take the Fourier Series of equation (77) rather than the Fourier Transform. Assuming that the ends of the string are fixed:

$$Y(x,s) = \sum_{n=1}^{\infty} \hat{c}_n(s)\sin\frac{n\pi x}{L}, \tag{95}$$

where

$$\hat{c}_n(s) = \frac{2}{L}\int_0^L Y(x',s)\sin\frac{n\pi x'}{L}dx', \tag{96}$$

$$\hat{a}_n = \frac{2}{L}\int_0^L y_0\sin\frac{n\pi x'}{L}dx', \tag{97}$$

$$\hat{b}_n = \frac{2}{L}\int_0^L v_0\sin\frac{n\pi x'}{L}dx'. \tag{98}$$

We use a hat on the Fourier coefficients in analogy with the hatting of functions in the continuous Fourier domain in our discussion of the infinite string. Substituting these series into equation (77) yields

$$\left(\frac{n\pi}{L}\right)^2 \hat{c}_n = c^{-2}\left[s\hat{a}_n + \hat{b}_n - s^2\hat{c}_n\right], \tag{99}$$

and upon simplifying it is found that

$$\hat{c}_n = \frac{s\hat{a}_n + b_n}{s^2 + (n\pi c/L)^2}. \tag{100}$$

Taking the inverse Laplace Transform of $c_n(s)$, which we denote as $c_n(t)$, gives:

$$c_n(t) = a_n\cos\frac{n\pi ct}{L} + \frac{L}{n\pi c}b_n\sin\frac{n\pi ct}{L}, \tag{101}$$

so that

$$y(x,t) = \sum_{n=1}^{\infty} c_n(t)\sin\frac{n\pi x}{L} \tag{102}$$

$$= \sum_{n=1}^{\infty}\left(a_n\cos\frac{n\pi ct}{L} + \frac{L}{n\pi c}b_n\sin\frac{n\pi ct}{L}\right)\sin\frac{n\pi x}{L}. \tag{103}$$

(See equation (14).)

2.8 Eigenfunction Expansions and Sturm-Liouville Problems

Eigenfunction Expansions

In seismology in general, as in the simple oscillating systems discussed above, we are concerned with finding the solutions of homogeneous differential equations with the ultimate object of treating inhomogeneous equations. In one dimension, the homogeneous equations are of the form $\mathcal{L}(y) = 0$ and the inhomogeneous equations are

$$\mathcal{L}(y) = f(x), \tag{104}$$

where $f(x)$ is a prescribed or general function and the boundary conditions to be satisfied by the solution at the end points 0 and L are given. The expression $\mathcal{L}$ represents here a general linear differential operator and not the Laplace Transform! For example, for the damped SHO $\mathcal{L} = m\frac{d^2}{dx^2} + 2b\frac{d}{dx} + \omega_0^2$. In general, unless $f(x)$ is particularly simple, one cannot simply integrate the inhomogeneous equation to solve for $y(x)$. The idea has been in our discussions above and is, in fact, generally true, that one seeks to exploit the linearity of the operator $\mathcal{L}$ by building up the required solution as a superposition of, generally, an infinite number of terms. This method is particularly efficient if we can find suitable functions which, when acted upon by $\mathcal{L}$, somehow eliminate the derivatives. The investigation of this choice and its consequences is the subject of this subsection.

Suppose that we can find a set of functions $\{y_n(x)\}$ $(n = 0, 1, 2, \ldots)$ such that

$$\text{Property One:} \qquad \mathcal{L}(y_n) = -\lambda_n y_n, \tag{105}$$

so that derivatives would be eliminated. Then, as a possible solution to (104) try the superposition

$$y(x) = \sum_n a_n y_n(x), \tag{106}$$

which when substituted into equation (104) yields:

$$f(x) = \mathcal{L}(y) = \mathcal{L}\left(\sum_n a_n y_n\right) = \sum_n a_n \mathcal{L}(y_n) = -\sum_n a_n \lambda_n y_n. \tag{107}$$

This has resulted in a purely algebraic equation, but at the price of introducing the set of unknowns $\{a_n\}$. This can be put right if, in addition, the set $\{y_n(x)\}$ is in some sense mutually orthogonal:

$$\text{Property Two:} \qquad \int_0^L y_m^*(z) y_n(z) dz = 0, \qquad \text{m} \neq \text{n}, \tag{108}$$

where $*$ denotes complex conjugate. The orthogonality expressed by this equation is called Hermitian orthogonality due to the complex conjugate. Multiplying both sides of equation (107) by y_m^* and integrating yields:

$$a_m = -\frac{1}{\lambda_m} \frac{\int_0^L y_m^*(z) f(z) dz}{\int_0^L y_m^*(z) y_m(z) dz}. \tag{109}$$

Equations (106) and (109) form a complete solution as long as the set of functions $\{y_n(x)\}$ exist that satisfy the Properties One and Two given by equations (105) and (108). Functions which satisfy equation (105) are called *eigenfunctions* of the operator $\mathcal{L}$ and, hence, equation (106) is known as an *eigenfunction expansion*. The quantities λ_n are the corresponding *eigenvalues*. The general idea of expansion in terms of a set of orthogonal eigenfunctions is the basis of Fourier Series solutions to differential equations, with which you are all familiar. You should be able to see why Fourier Series work so well, sines and cosines are solutions to the SHO equation and are, therefore, eigenfunctions of the SHO differential operator. They are also orthogonal. Thus, the Fourier coefficients can be computed. However, the ideas presented here are much more broadly based than that as we will now see.

The question remains, however, whether for a given operator $\mathcal{L}$ a suitable set of functions can be found. We cannot deal with this problem in general but it is worth pointing out that at least for linear operators of a particular form, such suitable sets of functions can be found and that fairly broad types of boundary conditions can be accommodated.

Sturm-Liouville Theory

Confine attention to second-order linear differential equations that are so common in wave propagation problems in which $\mathcal{L}$ has the form:

$$\mathcal{L}(y) \equiv p(x)y'' + r(x)y' - q(x)y, \qquad \text{with } r(x) = p'(x), \tag{110}$$

where p, q, and r are real functions of x. The class of differential equations of the form:

$$\mathcal{L}(y) = -\lambda \rho(x) y, \tag{111}$$

were first studied intensively in the 1830s by Sturm and Liouville. Writing equations (110) and (111) together yields

$$(py')' - qy + \lambda \rho y = 0 \tag{112}$$

This is known as the *Sturm-Liouville (S-L) equation* and linear differential operators of the forms given by equation (111) clearly satisfy Property One (eqn (105)), although equation (111) has been

slightly generalized to include a weighting function $\rho(x)$. The only conditions on the weighting function are that it is real valued and does not change sign. This latter requirement means that we can assume that it is everywhere positive without loss of generality. Its introduction also requires a generalized definition of orthogonality and the expansion coefficients:

$$\int_0^L \rho(z) y_m^*(z) y_n(z) dz = 0, \qquad \mathrm{m} \neq \mathrm{n}, \tag{113}$$

$$a_m = -\frac{1}{\lambda_m} \frac{\int_0^L \rho(z) y_m^*(z) f(z) dz}{\int_0^L \rho(z) y_m^*(z) y_m(z) dz}. \tag{114}$$

The operation in equation (113) defines the inner product in a function space which is actually an infinite vector space. An infinite vector space with an inner product is called a a Hilbert space in which the eigenfunctions reside. We will say very little more about this.

The satisfaction of Condition One is one of the reasons why S-L equations have been studied so intensively. Another reason is that, although the form looks very restrictive, many of the most important equations in mathematical physics are S-L equations. For example,

$$(1-x^2)y'' - 2xy' + l(l+1)y = 0 \qquad \text{Legendre's equation,} \tag{115}$$

$$((1-x^2)y')' + \left[l(l+1) - \frac{m^2}{1-x^2}\right] y = 0 \qquad \text{Associated Legendre equation,} \tag{116}$$

$$y'' - 2xy' + 2\alpha y = 0 \qquad \text{Hermite's equation,} \tag{117}$$

$$xy'' + (1-x)y' + \alpha y = 0 \qquad \text{Laguerre's equation,} \tag{118}$$

$$(1-x^2)y'' - xy' + n^2 y = 0 \qquad \text{Chebyshev equation,} \tag{119}$$

$$y'' + 2by' + \omega_0^2 y = 0 \qquad \text{Simple Harmonic Oscillator equation.} \tag{120}$$

Bessel's equation $(x^2y'' + xy' + (x^2 - n^2)y = 0)$ is also an S-L equation with an appropriate change of variables $(\xi = x/a)$. It should be noted that any second-order linear differential equation

$$p(x)y'' + r(x)y' + q(x)y + \lambda\rho(x)y = 0, \tag{121}$$

can be converted to the required type by multiplying through by the factor

$$F(x) = \exp\left[\int^x \frac{r(z) - p'(z)}{p(z)} dz\right], \tag{122}$$

provided that the indefinite integral is defined. It then takes on the S-L form

$$(F(x)p(x)y')' - (-F(x)q(x)y) + \lambda F(x)\rho(x)y = 0, \tag{123}$$

but clearly with a different, but still non-negative, weighting function $(F(x)\rho(x))$.

Second order linear differential operators, $\mathcal{L}$, for which $\mathcal{L}(y)$ can be written in the form

$$\mathcal{L}(y) = (py')' - qy, \tag{124}$$

where p and q are real functions of x are known as *self-adjoint* operators. This is a bit of a simplification but will suffice here. Therefore, the study of the eigenfunctions and eigenvalues of self-adjoint operators is synonymous with the study of S-L equations. Another useful definition is the following: $\mathcal{L}$ is said to be *Hermitian* if

$$\int_0^L y_m^*(x)\mathcal{L}y_n(x)dx = \left(\int_0^L y_n^*(x)\mathcal{L}y_m(x)dx\right)^* \tag{125}$$

where y_m and y_n are arbitrary functions satisfying the boundary conditions. The quantity on the left side of this equation is called the m,n matrix element of $\mathcal{L}$, or $\mathcal{L}_{mn}$, or $< y_m|\mathcal{L}y_n >$. The final notation, of course, is the bra-ket notation of Dirac, but is common in normal mode seismology. In this notation, Hermiticity is stated as $< y_m|\mathcal{L}y_n >=< \mathcal{L}y_m|y_n >$.

S-L equations satisfy Condition One since we seek solutions to the eigenvalue problem given by equation (111). It remains to show that such equations also satisfy Condition Two, that is that the eigenfunctions of self-adjoint operators are orthogonal in the generalized sense of equation (113). To do this it is necessary to show that

$$(\lambda_m^* - \lambda_n)\int_0^L y_m^*\rho y_n dx = 0, \tag{126}$$

from which the reality of the eigenvalues and the orthogonality of the eigenfunctions follow almost immediately. The derivation of equation (126) requires the specification of boundary conditions at both ends of the range of the free variable (i.e., at both 0 and L). The boundary condition required is the following:

$$[y_m^* p y_n']_{x=a} = [y_m^* p y_n']_{x=b}, \qquad \text{for all m,n .} \tag{127}$$

where $y_m(x)$ and $y_n(x)$ are any two solutions of the S-L equations. Again, this appears to be pretty restrictive, but is actually a pretty mild assumption that is met by many commonly ocuring cases. e.g., $y(0) = y(L) = 0, y(0) = y'(L) = 0, p(0) = p(L) = 0$, and many more.

A last consideration is the normalization of the eigenfunctions. Equation (113) only places a constraint on the eigenfunctions when $m \neq n$. When $m = n$, because of the linearity of $\mathcal{L}$, the normalization is arbitrary. We will assume for definiteness that they are normalized so that $\int y_n^2 \rho dx = 1$. In this case, equations (113) and (114) can be rewritten as

$$< y_m|\rho y_n > \;=\; \int_0^L \rho(z)y_m^*(z)y_n(z)dz = \delta_{mn}, \tag{128}$$

$$a_m = -\lambda_m^{-1} < y_m|\rho f > \;=\; -\lambda_m^{-1}\int_0^L \rho(z)y_m^*(z)f(z)dz. \tag{129}$$

Eigenfunctions corresponding to equal eigenvalues are said to be *degenerate*.

Summarizing then, equations of the form (104) admit the following solution

$$y(x) = -\sum_m \lambda_m^{-1} y_m(x) \int_0^L \rho(z) y_m^*(z) f(z) dz = -\sum_m y_m(x) \lambda_m{}^1 < y_m | \rho f >, \tag{130}$$

if the differential operator is linear and self-adjoint (i.e., results in a differential equation that is a Sturm-Liouville equation); that is, if equations (111), (128), and (127) hold. In addition, the eigenvalues λ_m are real and the eigenfunctions y_n are orthonormal. In addtion, they form a complete set. By a *complete set* we mean that any function satisfying the boundary conditions can be represented as a (potentially infinite) sum of the eigenfunctions. We will not attempt to prove this here. Resulting from this is the so-called *completeness relation* or *closure property* of the eigenfunctions:

$$\rho(x) \sum_n y_n^*(x) y_n(z) = \delta(z - x). \tag{131}$$

2.9 Green's (or Green) Functions

Starting with equation (130), assume that we can interchange the order of summation and integration:

$$y(x) = \int_0^L \left\{ \rho(z) \sum_m [-\lambda_m^{-1} y_m(x) y_m^*(z)] \right\} f(z) dz, \tag{132}$$

$$= \int_0^L G(x, z) f(z) dz. \tag{133}$$

In this form, the solution to S-L problems has clearer properties, it is integral of two factors, of which (1) the first is determined entirely by the boundary conditions and the eigenfunctions y_m, and hence by $\mathcal{L}$ itself, and (2) the second, $f(z)$, depends purely on the right-hand side of equation (104). Thus, there is the possibility of finding, once and for all, for any given function $\mathcal{L}$, a function $G(x, z)$ which will enable us to solve equation (104) for any right-hand side; that is any forcing function. The solution will be in the form of an integral which, at worst, can be evaluated numerically. This function, $G(x, z)$, is called the *Green's function* for the operator $\mathcal{L}$. This approach is somewhat similar to the use of Laplace Transforms in that we have reduced the problem to quadrature as the British say, but once $G(x, z)$ is found the remaining work to produce a solution is remarkably simple. In addition, Green's functions lend themselves to generalization to multiple dimensions and direct application to partial differential equations. For these reasons, Green's function methods are of greater practical significance than Laplace Transform methods which are mostly used theoretically.

One expression for Green's functions has already been given and can be seen by comparing equations (132) and (133),

$$G(x,z) = -\sum_m \lambda_m^{-1}\rho(z)y_m(x)y_m^*(z). \tag{134}$$

Alternately, we note that equation (133) is, by construction, a solution to equation (104). Hence

$$\mathcal{L}(y) = \int_0^L \mathcal{L}[G(x,z)]f(z)dz = f(x). \tag{135}$$

Now, recall that a delta function is defined as a function that satisfies the following properties:

$$\delta(x-x_0) = 0 \qquad \text{if } x \neq x_0, \tag{136}$$

$$\delta(x-x_0) = \infty \qquad \text{if } x = x_0, \tag{137}$$

$$\int_0^L \delta(x-x_0)dx = 1 \qquad \text{if } 0 \leq x_0 \leq L, \tag{138}$$

$$\int_0^L \delta(x-x_0)f(x)dx = f(x_0) \quad \text{if } 0 \leq x_0 \leq L. \tag{139}$$

A delta function can be *represented* as limits of a number of different functions; e.g., as an infinitesimally thick Gaussian, a sinc function with infinite frequency, the inverse Fourier Transform of an exponential, and a number of others. The only essential requirements are knowledge of the area under such a curve and that undefined operations such as differentiation are not carried out on the delta-function while some nonexplicit representation is being employed.

Using the property of the delta-function given by equation (139), equation (135) can be rewritten as

$$\mathcal{L}(y) - f(x) = \int_0^L \{\mathcal{L}[G(x,z)] - \delta(z-x)\}\, f(z)dz = 0. \tag{140}$$

For this to hold for any function f, it must be the case that

$$\mathcal{L}[G(x,z)] = \delta(z-x). \tag{141}$$

Note that in this equation, z is only a parameter and all the differential operations implicit in $\mathcal{L}$ act on the variable x.

Putting equation (141) in words, *the Green's function G is the solution of the differential equation obtained by replacing the right-hand side of equation (104) by a delta function. Thus, the solution to equation (104) given by equation (132) is the superposition of the effects of isolated 'impulses' of size $f(z)dz$ occurring at positions $x = z$.* Since each impulse has effects at locations other than where it acts, the total result at any position x must by obtained by integrating over all z:

$$f(x) = \int_0^\infty f(z)\delta(z-x)dz. \tag{142}$$

That is, $f(x)$ is a limiting case of a whole set of impulses.

2.10 Applications of Superposition and Green's Functions

Superposition

Consider, as an example, the spatial part of the string problem where we have included a forcing term

$$y'' + k^2 y = f(x) \tag{143}$$

to be solved on the interval $[0, L]$ with initial conditions

$$y(0) = y(L) = 0. \tag{144}$$

Temporarily, let the string length $L = \pi$ to simply the calculations, we won't have to carry around a π/L term. We'll translate back to a string of length L by simply replacing every π with an L and every n with an $n\pi/L$. The weight function ρ is unity. In this case, $\mathcal{L} = d^2/dx^2 + k^2$. We seek eigenfunctions satisfying the S-L equation

$$y'' + k^2 y + \lambda y = 0, \tag{145}$$

$$\mathcal{L}y = -\lambda y. \tag{146}$$

These are obviously $y_n = A_n \sin nx + B_n \cos nx$ corresponding to eigenvalues λ_n given by $n^2 = \lambda_n + k^2$. The boundary conditions require that n be a positive integer and that $B_n = 0$. Thus, the eigenfunctions are $y_n = A_n \sin nx$ and the normalization condition, equation (128), requires that $A_n = \sqrt{2/\pi}$. Using equation (129),

$$a_n = -(n^2 - k^2)^{-1} \int_0^\pi \left(\frac{2}{\pi}\right)^{1/2} f(z) \sin nz dz, \tag{147}$$

and finally that the solution in terms of the given function $f(x)$ is

$$y(x) = \sum_n a_n y_n = \sum_n a_n A_n \sin nx = -\frac{2}{\pi} \sum_{n=1}^{\infty} \frac{\sin nx}{n^2 - k^2} \int_0^\pi f(z) \sin nz dz. \tag{148}$$

Upon transforming back to a string of length L we get:

$$y(x) = -\frac{2}{L} \sum_{n=1}^{\infty} \frac{\sin(n\pi x/L)}{(n\pi/L)^2 - k^2} \int_0^\pi f(z) \sin(n\pi z/L) dz. \tag{149}$$

Note that this solution is also the Fourier Series form because of the particular form of the linear operator involved. However the above method is a general model for all equations involving S-L-like operators.

Green's Functions

From (149), the Green's function for the undamped SHO is immediately apparent:

$$G(x,z) = -\frac{2}{L}\sum_{n=1}^{\infty}\frac{\sin(n\pi x/L)}{(n\pi/L)^2 - k^2}\sin(n\pi z/L). \tag{150}$$

We call this the *discrete form of the Green's Function* and illustrates a symmetry relation for Green's functions also apparent in equation (134), that $G(x,z) = (G(z,x))^*$.

This form is not very convenient for computation since it involves an infinite sum. A more useful form of this Green's function can be obtained by using the fact that the Green's function is the solution of the differential equation (143) in which the forcing function has been replaced by the delta function, $\delta(z-x)$:

$$\frac{d^2G(x,z)}{dx^2} + k^2G(x,z) = \delta(z-x) \tag{151}$$

The Green's function will depend on the initial conditions. The solution of equation (151) subject to the initial conditions that $y(0) = y'(0) = 0$ is simply:

$$G(x,z) = \frac{1}{k}\sin k(x-z), \tag{152}$$

if $0 < z < x$ and 0 otherwise. Using this approach produces Green's functions that act as integral kernels and are called the *continuous form of the Green's Function*. Using this expression for the Green's function the solution to equation (143) subject to the different initial conditions listed directly above can be written:

$$y(x) = \int_0^x \frac{1}{k}\sin k(x-z)f(z)dz. \tag{153}$$

Finding the Green's function for the boundary conditions given in equation (144) is somewhat more complicated, but it is instructive, so let's do so but keep the string length equal to L (rather than π) here. $G(x,z)$ still satisfies equation (151). For x equal to anything but z we have

$$G(x,z) = A\sin kx \qquad (0 < x < z), \tag{154}$$

$$= B\sin k(x-L) \qquad (z < x < L). \tag{155}$$

To determine the constants A and B we need to apply conditions on G and its first derivative at $x = z$. To find the appropriate conditions, we integrate equation (151) from $x = x - \epsilon$ to $x = z + \epsilon$ and then let $\epsilon \to 0$. Since $\int d^2G/dx^2 = dG/dx$, we find that

$$\frac{dG}{dx}\Big|_{z-\epsilon}^{z+\epsilon} + \int_{z-\epsilon}^{z+\epsilon} G(x,z)dx = \int_{z-\epsilon}^{z+\epsilon}\delta(x-z)dx = 1, \tag{156}$$

so that letting $\epsilon \to 0$ the second term on the right-hand-side goes to zero and the change in slope at $x = z$ is 1. Integrating again gives

$$G|_{z-\epsilon}^{z+\epsilon} = 0. \tag{157}$$

which implies that G is continuous at $x = z$. This yields the pair of simultaneous equations

$$A \sin kz = B \sin k(z - L) \tag{158}$$

$$kA \cos kz + 1 = kB \cos k(z - L), \tag{159}$$

which upon solution give

$$A = \frac{\sin k(z - L)}{k \sin kL} \qquad B = \frac{\sin kz}{k \sin kL} \tag{160}$$

and the Green's function is

$$G(x, z) = (k \sin kL)^{-1} \sin kx \sin k(z - L) \qquad 0 < x < z \tag{161}$$

$$= (k \sin kL)^{-1} \sin kz \sin k(x - L) \qquad z < x < L \tag{162}$$

from which it is immediately apparent that

$$y(x) = \frac{\sin kx}{k \sin kL} \int_0^x f(z) \sin k(z - L) dz + \frac{\sin k(x - L)}{k \sin kL} \int_x^L f(z) \sin(kz) dz. \tag{163}$$

Following the same method it is possible to show that a solution to the differential equation

$$y'' + p(x)y' + q(x)y = f(x) \tag{164}$$

with $y(0) = y(L) = 0$ is given by

$$y(x) = y_2(x) \int_0^x \frac{y_1(x')f(x')}{W(x')} dx' + y_1(x) \int_x^L \frac{y_2(x')f(x')}{W(x')} dx', \tag{165}$$

where $y_1(x)$ and $y_2(x)$ are solutions of the homogeneous equation with $y_1(0) = y_2(0) = 0$, and W is the Wronskian of $y_1(x)$ and $y_2(x)$: $W(x) = y_1(x)y_2'(x) - y_2(x)y_1'(x)$. Recall that if $W \neq 0$, y_1 and y_2 are linearly independent. Also as in the above, we can find that a particular solution y_p of equation (164) is

$$y_p(x) = y_2(x) \int \frac{y_1(x)f(x)}{W(x)} dx + y_1(x) \int \frac{y_2(x)f(x)}{W(x)} dx. \tag{166}$$

This particular solution is exactly the same as that obtained by variation of parameters, but may seem somewhat less arbitrary.

2.11 The Inhomogeneous String

But what does all of this have to do with seismology? The earth is neither homogeneous nor one-dimensional, both key assumptions up to this point. In this section, we'll break with the first of these assumptions and consider waves of an inhomogeneous string, where density and/or Young's modulus vary with position along the string. The transition to higher dimensions will be made in a later subsection. The inhomogeneous string problem cannot be solved for a general perturbation in density or Young's modulus ($\rho(x)$ or $\kappa(x)$), but there are a number of approximate solutions commonly used in seismology that hold for variously constrained forms of these functions. We will consider three of these: the JWKB method, the Eikonal equation, and first order perturbation theory. The first two of these methods define geometrical ray theory, a high frequency approximation that holds in regions of smooth spatial variation of wave velocity.

The development of ray methods begins by referring back to equation (6)

$$\rho(x)\frac{\partial^2 y(x,t)}{\partial t^2} = \frac{\partial}{\partial x}\left(\kappa(x)\frac{\partial y(x,t)}{\partial x}\right) \tag{167}$$

$$= \kappa(x)\frac{\partial^2 y(x,t)}{\partial x^2} + \frac{\partial \kappa(x)}{\partial x}\frac{\partial y(x,t)}{\partial x} \tag{168}$$

where κ is Young's Modulus or tension for longitudinal or transverse vibrations, respectively. Generally the next step invokes the following assumption.

Ray Assumption One: The spatial gradient of Young's modulus (κ'/κ) is sufficiently small so that the rightmost term in equation (168) can be ignored.

We are then left with the equation

$$\rho(x)\frac{\partial^2 y(x,t)}{\partial t^2} = \kappa(x)\frac{\partial^2 y(x,t)}{\partial x^2}. \tag{169}$$

A more precise condition for dropping this term is that *the change in the velocity gradient over a single wavelength is small compared to the change in velocity itself.* This will be discussed further below.

Letting $c^2(x) = \kappa(x)/\rho(x)$, the spatial and temporal equations still separate ($y(x,t) = X(x)T(t)$) and yield:

$$\frac{d^2T(t)}{dt^2} + \omega_0^2 T(t) = 0, \tag{170}$$

$$\frac{d^2X(x)}{dx^2} + \frac{\omega_0^2}{c^2(x)}X(x) = 0. \tag{171}$$

The temporal equation is exactly the same as in the homogeneous string. This is the first observable consequence of Ray Assumption One and implies that frequency, unlike displacement, is stationary relative to small changes in seismic structure: under this approximation frequency is not changed by perturbing Young's modulus and/or density. Equation (171) is different than the spatial equation for the homogeneous string in that wave velocity is now an unspecified function of x. As a consequence, we cannot simply set $X(x) = A\exp(\pm iSx)$, where S is a constant. Ray methods diverge on how equation (171) is treated, as we shall now see.

Ray Method One: JWKB

Under the assumption that $\frac{\omega_0^2}{c^2(x)}$ is a slowly varying function of x, we assume that

$$X(x) = A(x)e^{iS(x)}. \tag{172}$$

Note that S plays the role of kx in the homogeneous string problem. We will discuss what 'slowly varying' means shortly. Upon substituting equation (172) into equation (171) we receive

$$A'' + 2iS'A' + iS''A - S'^2A + \frac{\omega_0^2}{c^2}A = 0 \tag{173}$$

The real and imaginary parts must separately equal zero, thus:

$$\text{Imag Part} \qquad 2S'A' + S''A = 0 \tag{174}$$

$$\text{Real Part} \qquad A'' + \frac{\omega_0^2}{c^2}A - S'^2A = 0. \tag{175}$$

Another consequence of Ray Assumption One is that the spatial gradients of both S and A will be small in comparison with the functions themselves. In particular, if we make a second assumption:

$$\text{Ray Assumption Two} \qquad |A''/A| << \omega_0^2/c^2, \tag{176}$$

$$\tag{177}$$

the real part equation can be rewritten as

$$\text{Real Part} \qquad S'^2(x) = \frac{\omega_0^2}{c^2(x)}, \tag{178}$$

$$S'(x) = \pm\frac{\omega_0}{c(x)}, \tag{179}$$

which can be integrated to yield

$$S(x) = \pm\omega_0\int^x \frac{dx'}{c(x')} = \pm\omega_0 T(x), \tag{180}$$

$$T(x) = \int^x \frac{dx'}{c(x')}. \tag{181}$$

The quantity $T(x)$ is called the *travel time* and is range-dependent, and the quantity $1/c(x)$ is commonly called the *ray slowness*. The imaginary-part equation can be solved without modificaton. We simply note that:

$$\frac{d}{dx}\left(A^2S'\right) = (2S'A' + S''A)A = 0, \tag{182}$$

where, by equation (174), we find

$$A^2(x) = \frac{C}{S'(x)} = C\frac{c(x)}{\omega_0}, \tag{183}$$

and C is an arbitrary constant.

Finally, then, the approximate solution to equation (171) is, after substituting from equations (180) and (183) and retaining the real part

$$X(x) = \pm\text{constant}\ \sqrt{c(x)}\sin\left(\omega_0 T(x)\right). \tag{184}$$

Combining solutions to equations (170) and (171) and writing in complex form yields

$$y(x,t) = X(x)T(t) \approx D(x)e^{\pm i\omega_0(t\pm T(x))}. \tag{185}$$

Please don't confuse $T(t)$, the solution to the temporal equation, with $T(x)$, the travel time here.

Solutions of the type given by equation (185) lead to ray theory as discussed in the next subsection on the Eikonal Equation and which will be discussed in much greater detail later in these notes. It is important to note that equation (185) still has a D'Alembert-type form, where the phase function $t \pm \int^x dx'/c(x')$ gives the travel time of the wave through the medium from the source at $x = 0$.

If the other end of the string is fixed at $x = L$, the eigenvalues ω_0 can be determined from equation (184) by the boundary condition $X(L) = 0$ which will only hold if the argument of the sin is an integral multiple of π:

$$\omega_0 = \frac{n\pi}{T(L)}. \tag{186}$$

This is the string analogue of the Bohr-Sommerfeld quantization condition of pre-1925 quantum mechanics.

Ray Assumption Two is not a common form of this assumption. We can derive the more common expression of the underlying assumption of ray theory by, instead of breaking equation (173) into real and imaginary parts, breaking it into parts dependent on and independent of A. If we do so, we get the A-independent equation:

$$iS'' - S'^2 + \frac{\omega_0^2}{c^2} = 0. \tag{187}$$

We can get the next higher order estimate of $S(x)$ than that given by equation (180) by substituting from equation (179) to yield

$$S'^2(x) = \frac{\omega_0^2}{c^2(x)} \pm i\frac{\omega_0 c'}{c^2(x)}, \tag{188}$$

We see that the lower-order solutions for $S(x)$ and $A(x)$ given by equations (180) and (183) have assumed that

$$\lambda_0 \frac{c'}{c} << \lambda_0 \frac{\omega_0}{c} \approx 2\pi, \tag{189}$$

where we have multiplied both sides of the equation through by the average wavelength $\lambda_0 = 2\pi c/\omega_0$. This condition can be rewritten as

$$\text{Ray Assumption Two'} \qquad c'\lambda_0 << 2\pi c, \tag{190}$$

which in words is translated as *the change in wave speed over a single wavelength must be much less than the wave speed itself.* Thus, we have seen that ray theory holds for media with properties that vary slowly (Ray Assumptions 1 and 2') and is also a high frequency approximation (Ray Assumption 2).

The JWKB method is named after a British applied mathematician who did most of his work in seismology, Sir Harold Jeffreys, and two German physicists and one French physicist who studied quantum mechanical scattering: Wentzel, Kramers, and Brillouin. The Brits call it JWKB and most US physicists leave Jeffreys out entirely and call the method WKB, although Jeffreys was the first to completely solve this problem. In fact, the main ideas date back to Green and Liouville in the 19th century and Rayleigh used them in his treatise on wave propagation in a stratified medium. Most seismologists call it WKBJ, but let's go with the Brits in loyalty to Jeffreys as a seismologist. In fact, the JWKB method is considerably more complicated than our discussion here since we have assumed that the solution is oscillatory which is not necessarily the case if there exists turning rays as in the earth. For nice discussions see Mathews and Walker and/or Aki and Richards, Ch. 9.

Ray Method Two: The Eikonal Equation

The Eikonal equation can be derived from equation (175) with the variable change

$$S(x) = \omega_0 \xi(x)/c_0, \tag{191}$$

where c_0 is meant to be the average wave velocity in the string. With this substitution we can rewrite equation (175) as

$$\xi'^2(x) - \frac{c_0^2}{c^2(x)} = \frac{c_0^2}{\omega_0^2}\frac{A''(x)}{A(x)}. \tag{192}$$

By Ray Assumption Two, the right hand side can be approximated by zero and we get the *Eikonal equation*

$$\xi'^2(x) = \frac{c_0^2}{c^2(x)}. \tag{193}$$

Recall that $S(x) = \xi\omega_0/c_0$ was just kx for the homogeneous string equation. In multiple dimensions we will see that the wavenumber k forms a vector $\mathbf{k}$ that points normal to the wavefront, or along a ray. *The Eikonal equation is, therefore, a PDE that relates rays to the seismic velocity distribution.*

It appears that the Eikonal equation (eq. (193)) is complicated and that it is not easier to deal with than the wave equation. However, we will see later on that very simple equations can be obtained from it for rays which form the foundation for body wave seismology.

First-Order Perturbation Theory

First-order perturbation theory is a very common technique used in seismology to compute the effect of a small perturbation in some material property on the frequencies and displacements of the earth. Let's consider first the general outlines of the nondegenerate theory and then specify the problem for the inhomogeneous string.

Consider again the differential operator $\mathcal{L}$. We wish to compute the eigenvalues λ_n and the eigenfunctions y_n of $\mathcal{L}$:

$$\mathcal{L}y_n = \lambda_n y_n. \tag{194}$$

Suppose that the operator $\mathcal{L}$ is nearly equal to another operator $\mathcal{L}^{(0)}$:

$$\mathcal{L} \approx \mathcal{L}^{(0)} + \delta\mathcal{L}, \tag{195}$$

whose eigenvalues, $\lambda_n^{(0)}$, and eigenfunctions, $y_n^{(0)}$, we already know:

$$\mathcal{L}^{(0)}y_n^{(0)} = \lambda_n^{(0)}y_n^{(0)}. \tag{196}$$

The perturbation operator $\delta\mathcal{L}$ is small in some sense. Assume that the zeroth-order eigenfunctions compose a complete, orthonormal set and that the first order perturbation in the operator causes first-order perturbations in the eigenvalues and the eigenfunctions:

$$\lambda_n = \lambda_n^{(0)} + \lambda_n^{(1)}, \tag{197}$$

$$y_n = y_n^{(0)} + \sum_m a_{mn}^{(1)} y_m^{(0)}. \tag{198}$$

Substitute the trial forms for λ_n and y_n into the original eigenvalue problem of equation (194) and separate zeroth and first order terms:

$$\text{Zeroth Order} \qquad \mathcal{L}^{(0)} y_n^{(0)} = \lambda_n^{(0)} y_n^{(0)}. \tag{199}$$

$$\text{First Order} \qquad \delta\mathcal{L} y_n^{(0)} + \sum_m a_{mn}^{(1)} \lambda_m^{(0)} y_m^{(0)} = \lambda_n^{(1)} y_n^{(0)} + \lambda_n^{(0)} \sum_m a_{mn}^{(1)} y_m^{(0)}. \tag{200}$$

Multiplying both sides of equation (200) by $y_n^{(0)*}$ and integrating over all allowable values of x we get the perturbation to the eigenvalue:

$$\lambda_n^{(1)} = \left\langle y_n^{(0)} | \delta\mathcal{L} y_n^{(0)} \right\rangle, \tag{201}$$

where the Dirac braket notation was defined around equation (125). Thus, the corrected eigenvalue is

$$\lambda_n = \lambda_n^{(0)} + \left\langle y_n^{(0)} | \delta\mathcal{L} y_n^{(0)} \right\rangle + O((\delta\mathcal{L})^2), \tag{202}$$

where $O((\delta\mathcal{L})^2)$ is read 'terms of order delta L-squared'. To obtain the expansion coefficients for the perturbed eigenfunctions multiply both sides of equation (200) by $y_p^{(0)*}$ $(p \neq n)$, integrate over x as before, and solve for $a_{pn}^{(1)}$:

$$a_{pn}^{(1)} = \frac{\left\langle y_p^{(0)} | \delta\mathcal{L} y_n^{(0)} \right\rangle}{\lambda_n^{(0)} - \lambda_p^{(0)}} \qquad (\mathrm{p} \neq \mathrm{n}), \tag{203}$$

where we have assumed that $\lambda_n^{(0)} \neq \lambda_p^{(0)}$ (nondegenerate perturbation theory). The corrected eigenfunction is then:

$$y_n = y_n^{(0)} + \sum_{m \neq n} \frac{\left\langle y_m^{(0)} | \delta\mathcal{L} y_n^{(0)} \right\rangle}{\lambda_n^{(0)} - \lambda_m^{(0)}} y_m^{(0)}. \tag{204}$$

From equation (203) we see that the perturbation $\delta\mathcal{L}$ may be considered small if $\left|\left\langle y_p^{(0)} | \delta\mathcal{L} y_n^{(0)} \right\rangle\right| << \left|\lambda_n^{(0)} - \lambda_p^{(0)}\right|$.

You can see in equation (203) that the diagonal of the matrix $a_{pn}^{(1)}$ is not determined $(p = n)$. However, if the eigenfunctions y_n are orthonormal then using equation (198):

$$\langle y_m | y_n \rangle = \delta_{mn} + a_{nm}^{(1)*} + a_{nm}^{(1)} + \ldots, \tag{205}$$

and if $m = n$ then $\langle y_m | y_n \rangle = 1$ which implies that $\mathrm{Re}(a_{nn}^{(1)}) = 0$ but $\mathrm{Im}(a_{nn}^{(1)})$ is completely unconstrained and can be chosen to be zero without any loss of generality, thus:

$$a_{nn}^{(1)} = 0. \tag{206}$$

This is the reason why in equation (204) the sum is over all m except when $m = n$. In this case the denominator goes to zero, but so does the numerator and the term does not contribute to the sum.

In summary, the recipe for the use of nondegenerate perturbation theory is:

Step 1. Solve the unperturbed problem (eq. (196)) for the eigenvalues $\lambda_n^{(0)}$ and eigenfunctions $y_n^{(0)}$.

Step 2. Determine the perturbation operator $\delta\mathcal{L}$.

Step 3. Do the integrals $\left\langle y_n^{(0)}|\delta\mathcal{L}y_n^{(0)}\right\rangle$ and $\left\langle y_p^{(0)}|\delta\mathcal{L}y_n^{(0)}\right\rangle$.

Step 4. Compute the corrected eigenvalue and eigenfunction by using equations (202) and (204).

Application of First-Order Perturbaton Theory.

Now, let's apply this theory to the inhomogeneous string problem (eq. (171)):

$$c^2(x)\frac{d^2X(x)}{dx^2} + \omega^2 X(x) = 0 \tag{207}$$

$$\mathcal{L}(X) = -\omega^2 X = \lambda X, \tag{208}$$

with fixed boundaries ($X(0) = X(L) = 0$) by following these four steps. Note that we have kept c^2 on the left-hand side of the equation, this means that the eigenvalues are squared-frequencies rather than squared-wavenumbers.

Step 1. Solution of the Unperturbed Problem

We already have solved the unperturbed problem where $c(x) = c_0$ for the the eigenvalues, $\lambda_n^{(0)}$, and eigenfunctions, $X_n^{(0)}(x)$, of the homogeneous string. The notation $X(x)$ is kind of klunky with all those x's running around, so let's change notation and let $y_n(x) = X_n(x)$. The zeroth-order eigenfunctions and eigenvalues are:

$$y_n^{(0)}(x) = \sqrt{\frac{2}{L}}\sin\left(\frac{n\pi x}{L}\right), \tag{209}$$

$$\omega_n^{(0)2} = -\lambda_n^{(0)} = c_0^2\left(\frac{n\pi}{L}\right)^2, \tag{210}$$

where the eigenfunctions have been normalized ($< y_n^{(0)}|y_m^{(0)} >= \delta_{mn}$) and c_0 is defined directly below.

Step 2. Determine the Perturbation Operator $\delta\mathcal{L}$

From equations (207) and (208), if we write $c(x) = c_0 + \delta c(x)$ and discard terms of $O((\delta c)^2)$:

$$\mathcal{L} = c^2(x)\frac{d^2}{dx^2} \tag{211}$$

$$= (c_0 + \delta c(x))^2 \frac{d^2}{dx^2} \tag{212}$$

$$= c_0^2 \frac{d^2}{dx^2} + 2c_0 \delta c(x) \frac{d^2}{dx^2} + O((\delta c)^2) \tag{213}$$

$$= \mathcal{L}_0 + \delta\mathcal{L} + \delta\mathcal{L}^2. \tag{214}$$

Thus,

$$\delta\mathcal{L} = 2c_0 \delta c(x) \frac{d^2}{dx^2}. \tag{215}$$

Step 3. Compute Matrix Elements

Applying $\delta\mathcal{L}$ to the unperturbed eigenfunction yields:

$$\delta\mathcal{L} y_n^{(0)}(x) = -2c_0 \left(\frac{n\pi}{L}\right)^2 \delta c(x) y_n^{(0)}(x) = -2\frac{\omega_n^{(0)2}}{c_0} \delta c(x) y_n^{(0)}(x) \tag{216}$$

so that

$$\left\langle y_n^{(0)} | \delta\mathcal{L} y_n^{(0)} \right\rangle = -2\frac{\omega_n^{(0)2}}{c_0} \int_0^L \delta c(x) y_n^{(0)2}(x) dx = -\frac{4\omega_n^{(0)2}}{Lc_0} \int_0^L \delta c(x) \sin^2(n\pi x/L) dx. \tag{217}$$

$$\left\langle y_m^{(0)} | \delta\mathcal{L} y_n^{(0)} \right\rangle = -2\frac{\omega_n^{(0)2}}{c_0} \int_0^L \delta c(x) y_n^{(0)}(x) y_m^{(0)}(x) dx = -\frac{4\omega_n^{(0)2}}{Lc_0} \int_0^L \delta c(x) \sin(n\pi x/L) \sin(m\pi x/L) dx \tag{218}$$

Step 4. Compute Perturbed Frequency and Eigenfunction

The perturbed eigenvalues and eigenfunctions can be determined by inserting equations (217) and (218) into equations (202) and (204).

$$\lambda_n = \lambda_n^{(0)} + \left\langle y_n^{(0)} | \delta\mathcal{L} y_n^{(0)} \right\rangle, \tag{219}$$

$$\omega_n^2 = \omega_n^{(0)2} + \omega_n^{(1)2} = -\lambda_n = c_0^2 \left(\frac{n\pi}{L}\right)^2 + \frac{4\omega_n^{(0)2}}{Lc_0} \int_0^L \delta c(x) \sin^2(n\pi x/L) dx, \tag{220}$$

where we have substituted from equation (210) and

$$y_n(x) = y_n^{(0)} + y_n^{(1)} = y_n^{(0)} + \sum_{m \neq n} \frac{\left\langle y_m^{(0)} | \delta\mathcal{L} y_n^{(0)} \right\rangle}{\lambda_n^{(0)} - \lambda_m^{(0)}} y_m^{(0)} \tag{221}$$

$$= \sqrt{\frac{2}{L}} \sin\left(\frac{n\pi x}{L}\right) + \frac{4\omega_n^{(0)2}}{Lc_0^3} \sqrt{\frac{2}{L}} \sum_{m \neq n} \left[\frac{\int_0^L \delta c(x') \sin(n\pi x'/L) \sin(m\pi x'/L) dx'}{\left(\frac{m\pi}{L}\right)^2 - \left(\frac{n\pi}{L}\right)^2} \right] \sin(m\pi x/L) \tag{222}$$

$$= \sqrt{\frac{2}{L}} \sin\left(\frac{n\pi x}{L}\right) + \frac{4n^2}{Lc_0} \sqrt{\frac{2}{L}} \sum_{m \neq n} \left[\frac{\int_0^L \delta c(x') \sin(n\pi x'/L) \sin(m\pi x'/L) dx'}{m^2 - n^2} \right] \sin(m\pi x/L). \tag{223}$$

where in equation (222) we have changed the sign of the perturbation term by swapping the order of the m and n terms in the denominator and have factored a c_0^2 out into the front from that denominator and have then used equation (210) to get the final equation.

It is common to proceed further in this derivation by expanding the wave speed distribution in the zeroth-order eigenfunctions, in this case sines. This reduces the integral in equation (223) to a form that can be calculated analytically. (The importance of this is discussed by Ritzwoller and Lavely, 1991.) We will do this as an example below. It is important to note that although the zeroth order eigenfunctions form a complete set for any displacement of the string, they do not form a complete set for any wave speed perturbation. For example, although all displacements are zero at the end points, the wave speed perturbation does not have to be. Therefore, commonly wave speed perturbations are expanded in the eigenfunctions (sines) and their first derivatives (cosines). We will see the result of this below, but the terms of the expansion in cosines integrate to zero so they can be neglected. Nevertheless, this is an important general point. it is not always the case that the eigenfunctions form a complete set for a different quantity – wave speed here.

Equation (223) demonstrates the phenomenon of *coupling*. The new eigenfunctions are linear combinations of the old eigenfunctions. Consequently, if you would estimate the energy density as we did in Section 2.3 above (which see), cross-terms no longer cancel perfectly and you would find that the energy of every perturbed mode of oscillation depends on the energy of most of the unperturbed modes of oscillation. By *mode*, we mean a displacement pattern (or eigenfunction) that has a single frequency. By *unperturbed mode*, we mean a mode of oscillation of the homogeneous string - that is, a mode whose eigenfunction is a single sine. By *perturbed mode*, we mean a mode of the inhomogeneous string, which is a sum of many sines, in fact it is a sum over all the unperturbed modes. Modes are usually denoted by their index (or quantum number) n. We say that the mode n is coupled to the mode m if the expansion coefficient in equation (223) is non-zero for those values of n and m. This will depend on whether the integral in this equation is non-zero. We will discuss below what are called *selection rules* which provide information about whether the integral is zero and, therefore, about which modes couple.

In summary, the perturbed squared-frequencies and eigenfunctions are given by equations (220) and (223), respectively. The use of these equations requires the integral of a product of sines and the velocity perturbation. Let's now consider two examples of velocity perturbations: (1) a delta function perturbation and (2) a general Fourier series representation of the perturbation.

Example 1: Delta Function Velocity Perturbation

Let $\delta c(x) = \delta c \delta(x - x_0)$. In this case,

$$\omega_n^{(1)2} = \left[\frac{4\delta c \sin^2(n\pi x_0/L)}{Lc_0}\right] \omega_n^{(0)2}. \tag{224}$$

$$y_n^{(1)} = \frac{4n^2\delta c}{Lc_0}\sqrt{\frac{2}{L}}\sum_{m\neq n}\left[\frac{\sin(n\pi x_0/L)\sin(m\pi x_0/L)}{m^2-n^2}\right]\sin(m\pi x/L). \quad (225)$$

Example 2: Fourier Series Velocity Perturbation

Expand $\delta c(x)$ in a Fourier Sine series:

$$\delta c(x) = \sum_p \delta c_p \sin(p\pi x/L). \quad (226)$$

This is not a general perturbation in wave speed. A general perturbation would include a cosine term. But when substituted into equation (220), the cosine part integrates to zero which is easy to see by consideration of eveness and oddness. Physically , this is consistent with the observation that a sound speed perturbation at a displacement node has no impact on the oscillation pattern or frequency. The wave does not know it's there. So we simply ignore it here. Evaluation of equations (220) and (223) will then require that we compute the following integral:

$$\int_0^L \sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{n'\pi x}{L}\right)\sin\left(\frac{n''\pi x}{L}\right)dx \equiv (nn'n''). \quad (227)$$

The right hand side introduces some new notation. Let's call this a $3-n$ symbol. With this notation we can then rewrite equations (220) and (223):

$$\omega_n^{(1)2} = \left[\frac{4}{Lc_0}\sum_p \delta c_p\,(pnn)\right]\omega_n^{(0)2}, \quad (228)$$

$$y_n^{(1)} = \frac{4n^2}{Lc_0}\sqrt{\frac{2}{L}}\sum_{m\neq n}\left[\frac{1}{m^2-n^2}\sum_p \delta c_p\,(pnm)\right]\sin\left(\frac{m\pi x}{L}\right). \quad (229)$$

The evaluation of both of these equations reduces to computing the $3-n$ symbol. Doing so we find that

$$(pnm) = \frac{-4Lmnp}{\pi(m-n-p)(m+n-p)(m-n+p)(m+n+p)} \quad \text{if } m+n+p \text{ is odd} \quad (230)$$

$$= 0 \quad \text{if } m+n+p \text{ is even.} \quad (231)$$

and

$$(pnn) = \frac{-4Ln^2}{\pi p(p^2-4n^2)} \quad \text{if } p \text{ is odd} \quad (232)$$

$$= 0 \quad \text{if } p \text{ is even.} \quad (233)$$

Thus, the eigenfunction and frequency perturbations are only non-zero for certain values of p, m, and n. These conditions are called *Selection Rules*. We will state two of them as well as an observation resulting from the Selection Rules.

Selection Rule 1: Contributions to the perturbed eigenfunctions will only result if $p + m + n$ is odd.

Selection Rule 2: Only odd p contributes to perturb the eigenfrequencies.

Observation 1: Wave speed perturbations that would be represented as cosines rather than sines, have no impact on either the eigenfunctions or eigenfrequencies.

Observation 2: Even-even or odd-odd coupling ($m + n$ even) occurs only through odd degree structure (p odd), even-odd coupling ($m + n$ odd) occurs only through even structure (p even).

2.12 References

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