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SUMMER WORKSHOP ON FIBRE BUNDLES AND GEOMETRY

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DIFFERENTIAL GEOMETRY

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CONTENTS

(Based on lectures at CIMPA, Nice)

Smooth Manifolds

Let Ω be an open subset of $\mathbb{R}^n$. A function $f : \Omega \to \mathbb{R}$ is said to be smooth (or C^∞) if partial derivatives of all orders of f exist and are continuous. A map $\Phi : \Omega \to \mathbb{R}^m$ is said to be smooth if, for $i = 1,\ldots,m$, the function $p_i \circ \Phi : \Omega \to \mathbb{R}$ is smooth, where $p_i : \mathbb{R}^m \to \mathbb{R}$ denotes the i^{th} projection (i.e., if $\Phi = (f_1,\ldots,f_m)$ the functions f_i are smooth).

Let Ω_1 and Ω_2 be two open subsets of $\mathbb{R}^n$. A map $\Phi : \Omega_1 \to \Omega_2$ is said to be a diffeomorphism if Φ is bijective and both Φ and Φ^{-1} are smooth.

Let M be a Hausdorff topological space, which we shall assume to be paracompact. Suppose that we are given an open cover $\{U_\alpha\}_{\alpha\in I}$ of M and for each α a homeomorphism

$$\phi_\alpha : U_\alpha \to \phi_\alpha(U_\alpha)$$

of U_α onto an open subset $\phi_\alpha(U_\alpha)$ of $\mathbb{R}^n$ such that whenever $U_\alpha \cap U_\beta \neq \emptyset$, the map

$$\phi_\alpha \circ \phi_\beta^{-1} : \phi_\beta(U_\alpha \cap U_\beta) \to \phi_\alpha(U_\alpha \cap U_\beta)$$

is a diffeomorphism. We then say that M has a structure of a smooth (or differentiable or C^∞) manifold, or simply that M is a smooth manifold (of dimension n).

The pair (U_α, ϕ_α) is called a chart or a map and the family $\{(U_\alpha, \phi_\alpha)\}$ is called an atlas. We shall assume that the atlas we have is a maximal (or complete) atlas in the sense that we cannot add more maps to the atlas still preserving the compatibility conditions on the overlaps. Any atlas is contained in a unique complete atlas.

If (U_α, ϕ_α) is a map, the functions $x_i = p_i \circ \phi_\alpha$ are called coordinate functions on U_α.

Examples

1. The Euclidean n-space $\mathbb{R}^n$.

2. The n-dimensional sphere $S^n : \{(x_1,\ldots,x_{n+1}) \in \mathbb{R}^{n+1} \mid \sum_{i=1}^{n+1} x_i^2 = 1\}$.

Let $t = (t_1,\ldots,t_n) \in \mathbb{R}^n$ and $|t|^2 = t_1^2 + \cdots + t_n^2$.

Let $\phi_1^{-1} : \mathbb{R}^n \to S^n$ be the map $t \mapsto \left(\frac{2t}{|t|^2+1}, \frac{|t|^2-1}{|t|^2+1}\right)$ and $\phi_2^{-1} : \mathbb{R}^n \to S^n$ the map $t \mapsto \left(\frac{2t}{|t|^2+1}, \frac{1-|t|^2}{1+|t|^2}\right)$. Then $\phi_1^{-1}(\mathbb{R}^n) = S^n - \{(0,\ldots,0,1)\}$ and $\phi_2^{-1}(\mathbb{R}^n) = S^n - \{(0,0,\ldots,-1)\}$. ($\phi_1$, ϕ_2 are sterographic projections). We have $\phi_1 \circ \phi_2^{-1} : \mathbb{R}^n - (0) \to \mathbb{R}^n - (0)$ is the map $t \to \frac{t}{|t|^2}$ and hence a diffeomorphism. Thus S^n is a smooth n-manifold.

3. Product of two manifolds: If M is a m dimensional manifold and N a n-dimensional manifold then $M \times N$ is in a natural way an $(m+n)$-manifold. If $\{(U_\alpha, \phi_\alpha)\}$ (resp. $\{(V_\beta, \psi_\beta)\}$) is an atlas for M (resp. N), then the maps

$$\phi_\alpha \times \psi_\beta \; : U_\alpha \times V_\beta \to \phi_\alpha(U_\alpha) \times \psi_\beta(U_\beta) \subset \mathbb{R}^{m+n}$$

give an atlas for $M \times N$.

4. From 2) and 3) we see that the n dimensional torus $T^n \underset{def}{=} S^1 \times \cdots \times S^1$ (n-fold product) is a smooth n-manifold.

5. An open subset of a smooth manifold is a smooth manifold.

Smooth maps

Let M be a smooth manifold. A function $f : M \to \mathbb{R}$ is said to be smooth if for each α the function $f \circ \phi_\alpha^{-1} : \varphi_\alpha(U_\alpha) \to \mathbb{R}$ is smooth.

Let M and N be two smooth manifolds and $\Phi : M \to N$ a map. We say that Φ is a smooth map if Φ is continuous and the following condition is satisfied : let $\{(U_\alpha, \phi_\alpha)\}$ be an atlas for M and $\{(V_\beta, \phi_\beta)\}$ an atlas for N; then for each (α,β) with $W = U_\alpha \cap \Phi^{-1}(V_\beta) \neq \emptyset$, the map

$$\psi_\beta \circ \Phi \circ \phi_\alpha^{-1} : \phi_\alpha(U_\alpha) \to \psi_\beta(V_\beta)$$

is smooth.

If $\Phi : M \to N$ and $\psi : N \to P$ are smooth maps then the map $\psi \circ \Phi : M \to P$ is smooth.

A map $\Phi : M \to N$ is said to be a diffeomorphism if Φ is bijective and both Φ, Φ^{-1} are smooth.

Tangent Vectors

Let $m \in M$. A tangent vector L in M **assigns to each** smooth real valued function f defined in a neighbourhood of m a real number $L(f)$ satisfying the following conditions:

1) If $f = g$ in a neighbourhood of m, we have $L(f) = L(g)$

2) L is linear over $\mathbb{R}$: $L(\lambda f + \mu g) = \lambda L(f) + \mu L(g)$, for $\lambda, \mu \in \mathbb{R}$ (Note : f is defined in a neighbourhood U_1 of m, and g in U_2, $\lambda f + \mu g$ is defined in $U_1 \cap U_2$)

3) $L(f.g) = L(f).g(m) + f(m)\, L(g)$.

(Thus L is a linear map from the ring of germs of smooth functions at m, satisfying the Leibniz rule).

It is easily seen that tangent vectors in M form a vector space, denoted by $T_m(M)$ or simply T_m .

• Let $m \in \mathbb{R}^n$. The map $f \mapsto \frac{\partial f}{\partial x_i}(m)$, for f smooth in a neighbourhood of m, defines a tangent vector in m, denoted by $(\frac{\partial}{\partial x_i})_m$. If $m = (m_1,\ldots,m_n)$, a smooth function f with $f(m) = 0$ can be written, in a neighbourhood of m, in the form $f = \sum_{i=1}^{n} (x_i - m_i) g_i$, where g_i are smooth. Using this fact one sees that $\left\{(\frac{\partial}{\partial x_i})_m\right\}$, $i = 1,\ldots,n$ form a base for $T_m(\mathbb{R}^n)$, which is thus a vector space (over $\mathbb{R}$) of dimension n. It follows (for example using the considerations in the next section) that if M is an n-manifold, $T_m(M)$, $m \in M$, is an n-dimensional vector space.

The Tangent map

Let $\Phi : M \to N$ be a smooth map and $m \in M$. We now define a linear map

$$T_m(\Phi) : T_m(M) \to T_{\Phi(m)}(N)$$

called the tangent linear map (or differential) of Φ in m. Let $v \in T_m(M)$. $T_m(\Phi)(v)$ is defined by

$$\{T_m(\Phi)\ (v)\}(f) = v(f \circ \Phi)$$

where f is a smooth function defined in a neighbourhood of $\Phi(m)$, noting that $(f \circ \Phi)$ is a smooth function in a neighbourhood of m.

Let $\Phi : M \to N$ and $\psi : N \to P$ be smooth maps. Let $m \in M$. We then have the chain-rule :

$$T_m(\psi \circ \Phi) = T_{\Phi(m)}(\psi) \circ T_m(\Phi)$$

(The two sides are linear maps of $T_m(M)$ into $T_{(\psi \circ \Phi)(m)}(P)$) .
Using the chain rule we see that if $\Phi : M \to N$ is a diffeomorphism, the map $T_m(\Phi) : T_m(M) \to T_{\Phi(m)}(N)$ is an isomorphism of vector spaces.

Since the tangent space to $\mathbb{R}^n$ at a point is n-dimensional, we see, using a map at m, that the tangent space at a point m of an n-dimensional manifold is of dimension n. (Note that $\phi_\alpha : U_\alpha \to \phi_\alpha(U_\alpha)$ is a diffeomorphism) .

We shall put a structure of a smooth manifold (in fact that of a 'vector bundle') on the set of tangent vectors of a smooth manifold.

2. VECTOR BUNDLES

(Smooth) Vector bundles

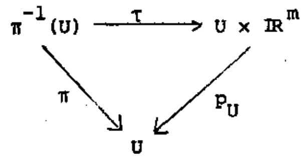
Let M be a smooth n-manifold. A smooth (real) vector bundle of rank m over M is a smooth (n+m) manifold E together with a smooth map $\pi : E \to M$ such that the following two conditions are satisfied:

(i) For each $x \in M$, $\pi^{-1}(x)$ has the structure of an m-dimensional vector space over $\mathbb{R}$.

(ii) For each $x \in M$, there exists an (open) neighbourhood U of m and a diffeomorphism

$$\tau : \pi^{-1}(U) \to U \times \mathbb{R}^m$$

such that the diagram



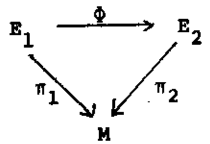
commutes and such that the induced map

$$\tau_x : \pi^{-1}(x) \to x \times \mathbb{R}^m = \mathbb{R}^m$$

is a bijective linear map for each $m \in U$.
($p_U : U \times \mathbb{R}^m \to U$ is the natural projection onto U).

If M is a smooth manifold, $M \times \mathbb{R}^m$, has a structure of a vector bundle over M. This bundle is called the trivial bundle of rank m over M.

If E_1 and E_2 are vector bundles over M. An isomorphism from E_1 onto E_2 is a diffeomorphism $\Phi : E_1 \to E_2$ such that the diagram



commutes and such that for each $x \in M$, the induced map $\Phi_x : \pi^{-1}(x) \to \pi_2^{-1}(x)$ is an isomorphism of vector spaces. By definition every vector bundle is locally (on M) isomorphic to the trivial bundle.

Let U be an open subset of M. A section of E over U is a map $\sigma : U \to E$ such that for $x \in U$ we have $(\pi o \sigma)(x) = x$. A section is said to be smooth if σ is smooth. A <u>frame</u> of E over U is a set of smooth sections $\{\sigma_1, \ldots, \sigma_m\}$ over U such that for each $x \in U$, $\{\sigma_1(x), \ldots, \sigma_m(x)\}$ form a base for $\pi^{-1}(x)$. If there is a frame over U, the restriction of E to U is trivial. If σ is a section over U, and if we write $\sigma(x) = \sum_{i=1}^{m} f_i(x)\sigma_i(x)$, the section σ is smooth if and only the real valued functions f_i are smooth in U. The trivial bundle has an obvious canonical frame given by $x \mapsto \{(x,e_1), \ldots, (x,e_m)\}$ where $(e_1, \ldots, e_m)$ is the canonical base of $\mathbb{R}^m$: $e_1 = (1,0,\ldots,0), \ldots, e_m = (0,0,\ldots,1)$. If σ is a section of the trivial bundle, $\sigma(x) = (x, f(x))$, $f(x) \in \mathbb{R}^m$. Thus a section of the trivial bundle is the same as a function on M with values in $\mathbb{R}^m$.

Hereafter we shall mean by a section a smooth section. Note that sections of E over U form a vector space.

<u>The tangent bundle</u>

Let M be a smooth manifold. Put $T(M) = \coprod_{x \in M} T_x(M)$ (disjoint union) and let $\pi : T(M) \to M$ the natural projection (if $v \in T_x(M)$, $\pi(v) = x$). Then $\pi : T(M) \to M$ has a natural structure of a vector bundle of rank n ($n = \dim M$).

First suppose that Ω is an open subset of $\mathbb{R}^n$. If $L \in T_a(\Omega)$, $a \in \Omega$, we can write

$$L = \sum_{i=1}^{n} \lambda_i \left(\frac{\partial}{\partial x_i}\right), \quad \lambda_i \in \mathbb{R} .$$

We thus have a map

$$\tau : T(\Omega) \to \Omega \times \mathbb{R}^n$$
$$\tau(a,L) = (a, \lambda_1, \ldots, \lambda_n)$$

with $p_\Omega \circ \tau = \pi$. The map τ is a bijection and we can transport the structure of the trivial vector bundle on $\Omega \times \mathbb{R}^n$ to get a structure of a vector bundle on $T(\Omega)$.

Let now (U_α, ϕ_α) be a chart. The tangent map associated to ϕ induces a bijection $T(M)|_{U_\alpha} \overset{\text{def}}{=} \pi^{-1}(U_\alpha) \to T(\phi_\alpha(U))$ and, as above, we have a bijection of $T(\phi_\alpha(U))$ with $\phi_\alpha(U) \times \mathbb{R}^n$. Composing these maps we have a bijection

$$\psi_\alpha : \pi^{-1}(U_\alpha) \to \phi_\alpha(U) \times \mathbb{R}^n .$$

We define a topology on $\pi^{-1}(U_\alpha)$ by requiring ψ_α to be a homeomorphism and then a topology of $T(M)$ by declaring that a subset of $T(M)$ is open if and only if its intersection with each $\pi^{-1}(U_\alpha)$ is open in $\pi^{-1}(U_\alpha)$. Take for charts on $T(M)$ the $(\pi^{-1}(U_\alpha), \psi_\alpha)$. If $U_\alpha \cap U_\beta \neq \emptyset$, the map $\psi_\beta \circ \psi_\alpha^{-1} : \phi_\alpha(U_\alpha \cap U_\beta) \times \mathbb{R}^n \to \phi_\beta(U_\alpha \cap U_\beta) \times \mathbb{R}^n$ is given by

$$(a, \lambda_1, \ldots, \lambda_n) \to (\varphi_\beta \circ \phi_\alpha^{-1}(a), \mu_1, \ldots, \mu_n)$$

where $\mu_i = \sum_{i=1}^{n} \lambda_j \frac{\partial f_i}{\partial x_j}(a)$, with $f_1, \ldots, f_n$ being the components of $\phi_\beta \circ \phi_\alpha^{-1}$ (i.e., $\phi_\beta \circ \phi_\alpha^{-1} = (f_1, \ldots, f_n)$). This shows the compatibility condition of the charts on overlaps. It is clear that $T(M)$, with this smooth structure is a vector bundle of rank n over M. We call $T(M)$ the <u>tangent bundle</u> of M.

A section of $T(M)$ is called a <u>vector field</u>.

<u>The dual vector bundle</u>

Let E be a vector bundle on M. For $x \in M$ we shall denote by E_x the fibre $\pi^{-1}(x)$ of E over x. Let $E^* = \coprod_{x \in M} E_x^*$, where E_x^* the dual of the vector space E_x. We have a natural projection $\tilde{\pi} : E^* \to M$. Let U be an open covering of M with

$$\begin{array}{ccc} \pi^{-1}(U_\alpha) & \xrightarrow{\ \tau\ } & U_\alpha \times \mathbb{R}^m \\ & {\scriptstyle\pi}\searrow \quad \swarrow & \\ & U_\alpha & \end{array}$$

Let $\tau_x^* : (\mathbb{R}^n)^* \to E_x^*$ be the transpose of $\tau_x : E_x \to \mathbb{R}^n$ and $\tau_x^{*-1} : E_x^* \to (\mathbb{R}^n)^*$ the inverse isomorphism. The τ_x^{*-1} give :

$$\begin{array}{ccc} \tilde{\tau}_\alpha : \tilde{\pi}^{-1}(U_\alpha) & \longrightarrow & U_\alpha \times (\mathbb{R}^n)^* \simeq U_\alpha \times \mathbb{R}^n \\ & \searrow \quad \swarrow & \\ & U_\alpha & \end{array}$$

using that $(\mathbb{R}^n)^*$ is naturally isomorphic to $\mathbb{R}^n$. Using the $\tilde{\tau}_\alpha$ we define a structure of a vector bundle on E^* with the property that for each α, $\tilde{\tau}_\alpha$ is an isomorphism of vector bundles. E^* is called the <u>dual bundle</u> of E.

The dual bundle of the tangent bundle $T(M)$ is called the <u>cotangent bundle</u>.

Suppose that $\{\sigma_1, \ldots, \sigma_m\}$ is a frame for E over an open set U. For $x \in U$, let $\{\sigma_1^*(x), \ldots, \sigma_m^*(x)\}$ be the dual base of E_x^* (dual to the base $\sigma_1(x), \ldots, \sigma_m(x)$ of E_x). Let σ_i^* be the section of E^* given by $x \mapsto \sigma_i^*(x)$. Then $\{\sigma_1^*, \ldots, \sigma_m^*\}$ is a frame for E^* over U.

If E and F are vector bundles over M, then $E \oplus F = \coprod_{x \in M} E_x \oplus F_x$ has a natural structure of a vector bundle called the direct sum of E and F. We shall also define the tensor product $E \otimes F$ of two vector bundles and the exterior products $\Lambda^p E$ of a vector bundle E.

3. <u>TENSOR AND EXTERIOR PRODUCTS</u>

<u>Tensor product</u>

Let E and F be finite dimensional vector spaces (over $\mathbb{R}$). The set of bilinear maps of $E \times F$ into $\mathbb{R}$ form a vector space denoted by $E^* \otimes F^*$; $E^* \otimes F^*$ is the tensor product of E^* and F^*. (E^* = dual of E).

If $\varphi \in E^*$ and $\psi \in F^*$, the map $(e,f) \mapsto \varphi(e)\psi(f)$, $e \in E$, $f \in F$ is a bilinear form on $E \times F$, denoted by $\varphi \otimes \psi$. If $\{\tilde{e}_1,\dots,\tilde{e}_m\}$ (resp $\{\tilde{f}_1,\dots,\tilde{f}_\ell\}$) a base for E^* (resp. for F^*) then the $\{\tilde{e}_i \otimes \tilde{f}_j\}$, $1 \leq i \leq m$, $1 \leq j \leq \ell$, form a base for $E^* \otimes F^*$.

We can define the tensor product of E and F, $E \otimes F$ as the vector space of bilinear forms on $E^* \times F^*$. If $e \in E$ and $f \in F$, $e \otimes f$ is defined as an element of $E \otimes F$.

Let now E and F be vector bundles over M. Let $E \otimes F = \coprod_{x \in M} E_x \otimes F_x$ with the natural projection $E \otimes F$ onto M. Then there exists a (unique) structure of a vector bundle on $E \otimes F$ with the following property :if $\{\sigma_1,\dots,\sigma_m\}$ (resp. $\{s_1,\dots,s_\ell\}$) is a frame for E (resp. for F) over an open set U then the set theoretic sections $\sigma_i \otimes s_j$ of $E \otimes F$ defined by $\sigma_i \otimes s_j(x) = \sigma_i(x) \otimes s_j(x)$ form a (smooth) frame for $E \otimes F$ over U ($1 \leq i \leq m$, $1 \leq j \leq \ell$). The bundle $E \otimes F$ is called the tensor product of E and F.

In a similar way the tensor product of a finite number vector bundles is defined. If E is a vector bundle we shall denote by $\overset{r}{\otimes} E$ the tensor product $E \otimes \cdots \otimes E$, r times.

The tensor bundles and tensors

Let $T(M)$ be the tangent bundle of M. The bundle $\{\overset{r}{\otimes} T(M)\} \otimes \{\overset{s}{\otimes} T^*(M)\}$ is called the tensor bundle of contravariant order r and covariant order s. A section of this bundle is called a tensor of contravariant order r and covariant order s. A section of

$\overset{r}{\otimes} T(M)$ { resp. $\overset{s}{\otimes} T^*(M)$ }is called a contravariant tensor of order r (resp. covariant tensor of order s).

Exterior product

Let E be a finite dimensional vector space over $\mathbb{R}$. We put $\overset{0}{\Lambda} E^* = \mathbb{R}$ and $\overset{1}{\Lambda} E^* = E^*$ and call these spaces respectively the space of alternating o-forms and 1 forms over E. Let $p \geq 2$ be an integer . An alternating p-form over E is a multilinear map

$$f : \underbrace{E \times \dots \times E}_{p \text{ times}} \longrightarrow \mathbb{R}$$

satisfying one of the following equivalent conditions :

i) If $x_1,\dots,x_p \in E$ with $x_i = x_{i+1}$ for some index i with $1 \leq i < p$ we have

$$f(x_1,\dots,x_p) = 0$$

ii) If $x_1,\dots,x_p \in E$ with $x_i = x_j$ for a pair of distinct indices (i,j) we have

$$f(x_1,\dots,x_p) = 0$$

iii) For each permutation σ of the set $\{1,\dots,p\}$ we have, for $x_1,\dots,x_p \in E$,

$$f(x_{\sigma(1)},\dots,x_{\sigma(p)}) = \varepsilon(\sigma)\, f(x_1,\dots,x_p)$$

where $\varepsilon(\sigma)$ is the signature of the permutation σ.

The set of alternating p-forms over E form a vector space denoted by $\overset{p}{\Lambda} E^*$ and called the p-th exterior product (or power) of E^*

If $f \in \Lambda^p E^*$ and $g \in \Lambda^q E^*$ consider the function $h : E^{(p+q)} \to \mathbb{R}$ defined by

$$h(x_1,\ldots,x_{p+q}) = \sum_{\sigma} \varepsilon(\sigma)\, f(x_{\sigma(1)},\ldots, x_{\sigma(p)})\, g(x_{\sigma(p+1)},\ldots,x_{\sigma(p+q)})$$

where σ runs through the set of permutations σ of $(1,\ldots,p+q)$ satisfying $\sigma(1) < \ldots < \sigma(p)$ and $\sigma(p+1) < \ldots < \sigma(p+q)$. Then h is an alternating $(p+q)$ form called the exterior product of f and g and is denoted by $f \wedge g$.

We have

<u>Proposition</u>

1) Exterior multiplication is associative i.e., if $f_i \in \Lambda^{p_i} E^*$ $i = 1,2,3$, we have $f_1 \wedge (f_2 \wedge f_3) = (f_1 \wedge f_2) \wedge f_3$.

2) If $f \in \Lambda^p E^*$ and $g \in \Lambda^q E^*$, we have

$$f \wedge g = (-1)^{pq}\, g \wedge f \,.$$

3) If $\{\tilde{e}_1,\ldots,\tilde{e}_m\}$ is a base of E^*, the set $\{\tilde{e}_{i_1} \wedge \quad \wedge \tilde{e}_{i_p}\}$ with $1 \leqslant i_1 < \quad < i_p \leqslant m$ forms a base of $\Lambda^p E^*$.

In particular if $m = \dim E$, $\Lambda^m E^*$ is one dimensional and $\Lambda^p E^* = 0$ for $p > m$.

If $T : E_1 \to E_2$ be a linear map between finite dimensional vector spaces. We then have the transpose linear map

$${}^tT^{(p)} : \Lambda^p E_2^* \to \Lambda^p E_1^* \quad \text{defined by :}$$

for $f \in \Lambda^p E_2^*$,

$${}^tT^{(p)}(f)\,(x_1,\ldots,x_p) = f(Tx_1,\ldots,Tx_p), \quad x_i \in E_1 \,.$$

We have $\{{}^tT^{(p)}(f)\} \wedge \{{}^tT^{(q)}(g)\} = {}^tT^{(p+q)}(f \wedge g)$ for $f \in \Lambda^p E_2^*$ and $g \in \Lambda^q E_2^*$.

4. <u>DIFFERENTIAL FORMS AND DE RHAM COHOMOLOGY</u>

<u>Differential Forms</u>

Let now E be a vector bundle over M. Put $\Lambda^p E^* = \coprod_{x \in M} \Lambda^p E_x^*$ and let $\pi_p : \Lambda^p E^* \to M$ the natural projection. Then $(\Lambda^p E^*, \pi_p)$ has a structure of vector bundle over M with the following property: if $\{\sigma_1,\ldots,\sigma_m\}$ is a frame for E^* over an open set U, the set theoretic sections $\{\sigma_{i_1} \wedge \ldots \wedge \sigma_{i_p}\}$ $(1 < i_1 < \cdots < i_p < m)$ of $\Lambda^p E^*$ over U defined by

$$\sigma_{i_1} \wedge \ldots \wedge \sigma_{i_p}(x) = \sigma_{i_1}(x) \wedge \ldots \wedge \sigma_{i_p}(x)$$

form a frame for $\Lambda^p E^*$ over U.

The bundle $\Lambda^p T^*(M)$ is called the bundle of p-forms. (A section of $\Lambda^o T^*(M)$ is a smooth real-valued function on M). A section of $\Lambda^p T^*(M)$ is called a <u>differential form of degree p</u> or a p-form. If w_1 is a p-form and w_2 a q-form then $a \mapsto w_1(a) \wedge w_2(a)$ defines a $(p+q)$ form denoted by $w_1 \wedge w_2$.

<u>The differential of a function</u>

Let f be a smooth function with values in $\mathbb{R}$, defined over an open subset U of M. Let $a \in U$. Define an element $w(a) \in T_a^*$ by $w(a)(v) = v(f)$, for $v \in T_a$. Then $a \mapsto w(a)$ is a differential form of degree 1 over U. We denote this differential form by df and call it the differential of f.

<u>Expression for a differential form in a chart</u>

Let (U, ϕ) be a chart. The map $m \mapsto T_{\phi(m)}(\phi^{-1})\left\{\left(\frac{\partial}{\partial x_i}\right)_{\phi(m)}\right\}$ is a vector field over U for $i = 1,\dots,n$. By abuse of notation we denote this vector field (over U) by $\frac{\partial}{\partial x_i}$. Then $\left\{\frac{\partial}{\partial x_1},\dots,\frac{\partial}{\partial x_n}\right\}$ is a frame for the tangent bundle over U. If $x_i = p_i \circ \phi$ are the coordinate functions over U, the 1-forms $\{dx_1,\dots,dx_n\}$ is the dual frame of $\left\{\frac{\partial}{\partial x_1},\dots,\frac{\partial}{\partial x_n}\right\}$. The differential forms (of degree p)

$$\left\{dx_{i_1} \wedge \quad \wedge dx_{i_p}\right\}_{1 \leq i_1 < \cdots < i_p \leq n}$$

form a frame for $\overset{p}{\Lambda} T^*$ over U. Thus if w is a p-form over U, then w can be written uniquely in the form

$$w = \sum_{i_1 < .. < i_p} f_{(i_1,\dots,i_p)} \, dx_{i_1} \wedge \dots \wedge dx_{i_p}$$

where $f_{i_1 \dots i_p}$ are smooth functions over U.

<u>The inverse image of a differential form</u>

Let $\Phi : M \to N$ be a smooth map between two smooth manifolds. Let $m \in M$ and $T_m(\Phi) : T_m(M) \to T_{\Phi(m)}(N)$ be the tangent map of Φ at m. Write T_m for $T_m(\Phi)$. Let

$$t_{T_m}(p) : \overset{p}{\Lambda} T^*_{\Phi(m)}(N) \to \overset{p}{\Lambda} T^*_m(M)$$

be the transpose of T_m.

Let now w be a p-form on N.

Then $\quad m \mapsto t_{T_m}(p)\,(w(\Phi(m)))$

defines a (smooth) differential form of degree p over M, denoted by $\Phi^*(w)$ and called the inverse of w by Φ.

If f is a real-valued smooth function on N we have

$$d(f \circ \Phi) = \Phi^*(df).$$

Moreover if w is a p-form and η a q-form over N we have

$$\Phi^*(w \wedge \eta) = \Phi^*(w) \wedge \Phi^*(\eta).$$

<u>Exterior differential.</u>

If U is an open subset of M we denote by $\overset{p}{\mathcal{E}}(U)$ the vector space (over $\mathbb{R}$) of p-forms on U.

<u>Proposition 4.1</u>.

There exists a unique collection of maps $d : \overset{p}{\mathcal{E}}(U) \to \overset{p+1}{\mathcal{E}}(U)$, p running through non-negative integers and U through open subsets of M, satisfying the following conditions.

1) $d : \overset{p}{\mathcal{E}}(U) \to \overset{p+1}{\mathcal{E}}(U)$ is a linear map

2) If V is an open subset of U the diagram

$$\begin{array}{ccc} \overset{p}{\mathcal{E}}(U) & \xrightarrow{d} & \overset{p+1}{\mathcal{E}}(U) \\ \downarrow & & \downarrow \\ \overset{p}{\mathcal{E}}(V) & \xrightarrow{d} & \overset{p+1}{\mathcal{E}}(V) \end{array}$$

is commutative, where $\overset{p}{\mathcal{E}}(U) \to \overset{p}{\mathcal{E}}(V)$ is the restriction map. (This condition expresses the local character of d).

3) If $w \in \overset{p}{\mathcal{E}}(U)$ and $\eta \in \overset{q}{\mathcal{E}}(U)$, we have

$$d(w \wedge \eta) = dw \wedge \eta + (-1)^p w \wedge d\eta .$$

4) For a real valued function f over U, df coincides with the differential of the function, as already defined.

5) $d^2 = 0$ (i.e., for $w \in \overset{p}{\mathcal{E}}(U)$, we have $d^2 w = 0$).

If U is contained in the domain of a chart and

$$w = \sum_{i_1 < \ldots < i_p} f_{i_1, \ldots, i_p} \, dx_{i_1} \wedge \quad \wedge dx_{i_p}$$

is a p-form on U, one proves that the above conditions imply that

$$dw = \sum_{i_1 < \ldots < i_p} df_{i_1 \ldots i_p} \wedge dx_{i_1} \wedge \quad \wedge dx_{i_p} .$$

This shows the uniqueness of d and also how one should define d to prove the existence. For open sets U contained in the domain of a fixed chart (Ω, ϕ) and a differential w on U, we define dw by the above formula and verify that conditions 1 to 5 are satisfied. If U is an arbitrary open set, we cover U by domains U_i of maps with $U_i \subset U$ and define $d(w|_{U_i})$ by the above formula; by uniqueness dw is well defined globally on U.

The operator d is called the operator of exterior differentiation and dw is called the <u>exterior differential</u> of w.

If $\Phi : M \to N$ is a smooth map and w a p-form on N, we have

$$d\,\Phi^*(w) = \Phi^*(dw) .$$

To prove this it is enough to show that if w is a form of the type $f\, dg_1 \wedge \ldots \wedge dg_p$ where f, g_i are function in an open set V of N, then $d\Phi^*(w) = \Phi^* dw$ in $\Phi^{-1}(V)$. Now $d(f\, dg_1 \wedge \ldots \wedge dg_p) = df \wedge dg_1 \wedge \ldots \wedge dg_p$ and $\Phi^*(dw) = \Phi^*(df \wedge \ldots \wedge dg_p) = \Phi^* df \wedge \quad \wedge \Phi^* dg_p$. On the other hand $\Phi^*(w) = (f \circ \Phi)\, \Phi^*(dg_1) \wedge \ldots \wedge \Phi^*(dg_p)$. Now it is easy to check that if g is a function on V we have $\Phi^*(dg) = d(g \circ \Phi)$. Hence $\Phi^*(w) = (f \circ \Phi)\, d(g_1 \circ \Phi) \wedge \quad \wedge d(g_p \circ \Lambda)$ so that

$$
\begin{aligned}
d\Phi^*(w) &= d(f \circ \Phi) \wedge d(g_1 \circ \Phi) \wedge \ldots \wedge d(g_p \circ \Phi) \\
&= \Phi^* df \wedge \Phi^* dg_1 \wedge \quad \wedge \Phi^*(dg_p) \\
&= \Phi^* dw .
\end{aligned}
$$

<u>de Rham Cohomology</u>

Let M be a smooth manifold. Consider the complex of vector spaces

$$0 \longrightarrow \overset{0}{\mathcal{E}}(M) \xrightarrow{d} \overset{1}{\mathcal{E}}(M) \to \cdots \to \overset{p}{\mathcal{E}}(M) \xrightarrow{d} \overset{p+1}{\mathcal{E}} \longrightarrow \cdots$$

Let Z^p = kernel of $d : \overset{p}{\mathcal{E}}(M) \to \overset{p+1}{\mathcal{E}}(M)$ and B^p = Image of $d : \overset{p-1}{\mathcal{E}}(M) \to \overset{p}{\mathcal{E}}(M)$. Since $d^2 = 0$ we have $B^p \subset Z^p$. We define

$$H^p_{DR}(M) = Z^p / B^p .$$

($H^0_{DR}(M) = \ker d : \overset{0}{\mathcal{E}}(M) \to \overset{1}{\mathcal{E}}(M)$) .

The vector space $H^p_{DR}(M)$ is called the p^{th} de Rham Cohomology space (or group) of M. Note that for $p > \dim M$, $H^p_{DR}(M) = 0$.

The space Z^p is called the space of p-cocycles and B^p the space of p-coboundaries. A form w with $dw = 0$ is said to be closed. A p-form w of the form $w = d\eta$, η a (p-1)-form, is called a coboundary.

Induced map on the Cohomology

Let $\Phi : M \to N$ be a smooth map and $\Phi^* : \mathcal{E}^p(N) \to \mathcal{E}^p(M)$ the induced map. Since $d\Phi^* = \Phi^* d$ we see that Φ^* maps $Z^p(N)$ into $Z^p(M)$ and $B^p(N)$ into $B^p(M)$. Hence Φ^* induces a map, still denoted by Φ^*,

$$\Phi^* : H^p_{DR}(N) \longrightarrow H^p_{DR}(M)$$

Homotopic maps.

Definition: Let M and N be smooth manifolds and φ_1, φ_2 smooth maps from M to N. We say that φ_1 and φ_2 are homotopic if there exists a smooth map Φ: $\mathbb{R} \times M \to N$ such that

$$\Phi(1,x) = \varphi_1(x) \quad \text{and} \quad \Phi(0, x) = \varphi_2(x)$$

for every $x \in M$.

Theorem 4.2. Let φ_1 and φ_2 be two smooth homotopic maps from M to N. Then φ_1 and φ_2 induce the same map from $H^p_{D(R)}(N)$ to $H^p_{DR}(M)$ (i.e. $\varphi_1^* = \varphi_2^* : H^p_{DR}(N) \to H^p_{DR(M)}$).

Sketch of proof.

We construct maps $h : \mathcal{E}^p(N) \to \mathcal{E}^{p-1}(M)$ satisfying $dhw + hdw = \varphi_1^*(w) - \varphi_2^*(w)$ for every smooth form w ($h = 0$ on $\overset{o}{\mathcal{E}}(N)$). This will prove the theorem for if w is a closed form on N we will

have $dh(w) = \varphi_1^*(w) - \varphi_2^*(w)$ so that $\varphi_1^*(w)$ and $\varphi_2^*(w)$ define the same element in $H_{DR}(M)$.

To construct h, we consider the p-form $\Phi^*(w)$ on $\mathbb{R} \times M$, for a p-form w on N. If U is a chart on M we write $\Phi^*(w)$ on $\mathbb{R} \times U$ in the form

$$\Phi^*(w) = \sum_I f_I(t,x)dx_{i_1} \wedge \dots \wedge dx_{i_p}$$

$$+ \sum_J g_J(t,x)\, dt \wedge dx_{j_1} \wedge \quad \wedge dx_{j_{p-1}}$$

where $I = (i_1,\dots,i_p)$, $i_1 < \dots < i_p$ and $J = (j_1,\dots,j_{p-1})$, $j_1 < \dots < j_{p-1}$. (t is the coordinate function on $\mathbb{R}$). Define

$$h(w) = \sum_J \left\{ \int_0^1 g_J(t,x)dt \right\} dx_{j_1} \wedge \quad \wedge dx_{j_{p-1}}$$

on U. One can check that $h(w)$ is well defined globally on M (This method of obtaining a (p-1) form on M from a p-form on $\mathbb{R} \times M$ is called 'integration along the fibres'). One verifies that

$$dhw + hdw = \varphi_1^*(w) - \varphi_2^*(w)$$

5. LIE GROUPS

Lie groups.

Definition. A Lie group G is a smooth manifold G which is also endowed with a structure of a group such that the map $G \times G \to G$ defined by $(x,y) \to xy^{-1}$ is smooth.

Examples: 1) The additive group $\mathbb{R}^n$.

2) Let $m \geqslant 1$ an integer and let $GL(m,\mathbb{R})$ denote the group of $m \times m$ real invertible matrices. $GL(m,\mathbb{R})$ is an open subset of the (m^2-dimensional) vector space of all $m \times m$ real matrices and hence has a structure of a smooth manifold. One verifies that with these structures $GL(m,\mathbb{R})$ is a Lie group. Similarly, $GL(m,\mathbb{C})$ is a Lie group.

Action of a Lie group on a smooth manifold.

Let G be a Lie group and M a smooth manifold A (smooth) action of G on M on the left is a map $\phi : G \times M \to M$ satisfying the conditions:

1) ϕ is smooth

2) If e is the identity element of e, then $\varphi(e,x) = x$ for every $x \in M$.

3) If $g_1, g_2 \in G$ then

$$\phi(g_1, \phi(g_2, x)) = \phi(g_1 g_2, x) \quad \text{for} \quad x \in M.$$

We shall write gx for $\phi(g,x)$ so that 2) reads $ex = x$ and 3) reads $(g_1 g_2)m = g_1(g_2 m)$.

For $g \in G$, let $\phi_g : M \to M$ be the map $x \mapsto gx$, $x \in M$. Then ϕ_g is a diffeomorphism and condition 3) may also be written as $\phi_{g_1 g_2} = \phi_{g_1} \circ \phi_{g_2}$.

Similarly we have the notion of a right action of G on M.

6. FLOWS AND LIE DERIVATIVES

Flows and local flows.

A smooth action of the additive group $\mathbb{R}$ on a smooth manifold M is called a flow (or a one-parameter group of diffeomorphisms) on M. If $\varphi_t : M \to M$ is defined by $\phi_t(x) = \varphi(t,x)$, for $t \in \mathbb{R}$, we have $\varphi_{t+s}(x) = \phi_t \circ \phi_s(x)$, for $t, s \in \mathbb{R}$ and $x \in M$.

A local flow on M is a smooth map $\phi : I \times U \to M$, where I is an open interval in $\mathbb{R}$ containing the origin and U an open subset of M, satisfying the following conditions.

1) For any $t \in I$, the map $\phi_t : U \to M$ defined by $x \mapsto \varphi(t,x)$, $x \in U$, is a diffeomorphism of U onto an open subset of M.

2) $\phi_o = Id_U$, where Id_U is the identity map of U.

3) If s,t, $s+t \in I$ and x, $\phi_t(x) \in U$, then we have $\phi_{s+t}(x) = \phi_s(\phi_t(x))$.

(Note that since $\phi_t(x) \in U$, $\phi_s(\phi_t(x))$ is defined).

A local flow is also called local one parameter group of (local) diffeomorphisms.

Lie bracket of vector fields.

Let X and Y be two vector fields defined in an open set U of M. If X is a vector field (over U) and $a \in U$, we denote the value of X at a, which is an element of $T_a(M)$, by $X(a)$ or X_a .

We now define a new vector field, $[X,Y]$, on U by setting, for $a \in U$,

$$[X,Y]_a(f) = X_a(Yf) - Y_a(Xf)$$

where f is a smooth function defined in a neighbourhood of $\underline{a}$ and Yf (resp. Xf) is the smooth function in a neighbourhood of a defined by $(Yf)(x) = Y_x f$ (resp. $(Xf)(x) = X_x f$). The vector field $[X,Y]$ is called the (Lie) bracket of X and Y.

The vector fields over M form a Lie algebra over $\mathbb{R}$, under the bracket operation : i.e.,

1) $(X,Y) \mapsto [X,Y]$ is bilinear over $\mathbb{R}$

e.g. $[\lambda_1 X_1 + \lambda_2 X_2, Y] = \lambda_1 [X_1, Y] + \lambda_2 [X_2, Y]$, $\lambda_1, \lambda_2 \in \mathbb{R}$

2) $[X,X] = 0$ (and $[X,Y] = -[Y,X]$)

3) Jacobi identity :

$$[X,[Y,Z]] + [Y,[Z,X]] + [Z,[X,Y]] = 0 ,$$

for any three vector fields X,Y,Z.

<u>The vector field associated to a flow.</u>

Let $\phi : \mathbb{R} \times M \to M$ be a flow on M.
Let $a \in M$ and define $\chi_a : \mathbb{R} \to M$ by $\chi_a(t) = \phi(t,a)$. Let $X(a) = T_o(\chi_a)((\frac{d}{dt})_o)$ where $(\frac{d}{dt})_o$ is the value of the vector field $\frac{d}{dt}$ (on $\mathbb{R}$) at $0 \in \mathbb{R}$. (Sometimes we write $T_s(\chi_a)(\frac{d}{dt})_s = (\frac{d\chi(t)}{dt})_s$ for $s \in \mathbb{R}$). Then $a \mapsto X(a)$ is a vector field on M, denoted by X. This vector field is the vector field associated to the flow, or the infinitesimal transformation of the flow.

<u>Lemma 6.1</u>: Let $s \in \mathbb{R}$. We have

$$\left(\frac{d}{dt} \chi_a(t)\right)_s = X(\chi_a(s)) .$$

Let X be a vector field on X and $\gamma : I \to M$ a smooth map, where I is an open interval in $\mathbb{R}$. We say that γ is an integral curve of X, if for each $s \in \mathbb{R}$, we have

$$\left(\frac{d}{dt} \gamma(t)\right)_s = X(\gamma(s)) .$$

By the above lemma, each of the 'orbit maps' χ_a are integral curves for the vector field associated with the flow.

<u>Local flows associated to a vector field.</u>

<u>Theorem 6.2</u>. Let X be a vector field over M and U a relatively compact open subset of M. Then there exists a local flow $\phi : I \times U \to M$ such that for $a \in U$, the map $t \mapsto \phi(t,a)$ from I to M is an integral curve for the vector field X.

The result is proved using the following theorem on ordinary differential equations depending on parameters:

Let Ω be an open subset of $\mathbb{R}^n$ and $F : \Omega \to \mathbb{R}^n$ a smooth map. Let $a \in \Omega$. Then there exist an interval I containing 0 and an open set V in $\mathbb{R}^n$, $a \in V$, and a (unique) smooth map $\phi : I \times V \to \Omega$ such that

1) $\phi(0,x) = x, \quad x \in V$

2) $\frac{\partial \phi(t,x)}{\partial t} = F(\phi(t,x)) , \ t \in I, \quad x \in V.$

<u>Corollary 6.2.1</u> : Let M be a compact manifold and X a vector field on M. Then there exists a flow (unique) on M whose associated vector field is X.

Proof. Since M is compact we have by the theorem, a local flow $\phi : I \times M \to M$. Let $s \in \mathbb{R}$ and let n be an integer such that $s/n \in I$. Put $\phi_s = (\phi_{s/n})^n = \phi_{s/n} \circ \cdots \circ \phi_{s/n}$, (n times), i.e. $\varphi(s,x) = \phi_s(x)$.

Remark 6.3. Let $A : M \to N$ be a diffeomorphism and X a vector field on M. We define a vector field $A_*(X)$ on N by : $A_*(X)(y) = T_{A^{-1}(y)}(A)\ (X(A^{-1}y))$, for $y \in N$. If φ_t is the local flow generated by X then $A \circ \varphi_t \circ A^{-1}$ is the local flow generated by $A_*(X)$. In particular, when $M = N$, we have $A_*(X) = X$ if and only if $A \circ \varphi_t \circ A^{-1} = \varphi_t$ i.e., A and φ_t commute.

Lie derivatives of vector fields with respect to a vector field.

Let X and Y be vector fields on M. We shall define a vector field, $\theta_X(Y)$, called the Lie derivative of Y with respect to X.

Let $a \in M$. Let φ_t be the local flow generated by X, defined in a neighbourhood of a (We shall assume that I is symmetric around 0). Define $Z_a(t) \in T_a(M)$ by :

$$Z_a(t) = T_{\varphi_t(a)}(\varphi_t^{-1})\ (Y(\varphi_t(a)))$$

(Here $t \in I'$, where $0 \in I' \subset I$ is an interval such that $\varphi_t(a) \in U$, for $t \in I'$)

We define :

$$\theta_X(Y)(a) = \frac{d}{dt} Z_a(t)\Big|_{t=0},$$

where the term on the right is the derivative at 0 of the function

$t \mapsto Z_a(t)$ which has values in the (finite dimensional) vector space $T_a(M)$. Using the chain rule we have

Lemma 6.4. For $s \in I'$, we have

$$T_a(\phi_s)\ \left(\frac{dZ_a(t)}{dt}\Big|_{t=s}\right) = \theta_X(Y)\ (\phi_s(a)).$$

Corollary 6.4.1. If $\theta_X(Y) = 0$, we have $(\phi_t)_* Y = Y$.

Theorem 6.5. We have

$$\theta_X(Y) = [X, Y].$$

Indication of proof. Let f be a C^∞ function on M, with values in $\mathbb{R}$. It is easy to see that there existsa function $g(t,p)$, smooth in (t,p) such that $f \circ \phi_{-t}(p) - f(a) = t.g(t,p)$ and $g(0,p) = -(Xf)(p)$. Then $Z_a(t)f = Y_{\phi_t(a)}(f \circ \phi_{-t})$

$$= Y_{\phi_t(a)} f + t\, Y_{\phi_t(a)}\, g(t, \phi_t(a)).$$

Hence,

$$\lim_{t\to 0} \frac{Z_a(t)f - Y_a f}{t} = \lim_{t \to 0} \frac{Y_{\phi_t(a)} f - Y_a f}{t} + Y_{\phi_t(a)}\, g(t, \phi_t(a))$$

$$= X_a\, Yf + Y_a g(0,a)$$

$$= X_a\, Yf - Y_a\, Xf$$

$$= [X,Y]_a f.$$

Corollary 6.5.1. Let X and Y be vector fields on M such that $[X,Y] = 0$. If ϕ_t and ψ_s are the local flows generated by X and Y then ϕ_t and ψ_s commute.

This follows from the above theorem, Remark 6.3 and the Corollary 6.4.1.

Lie derivative of tensor fields.

Let X be a vector field on M and α a tensor field on M. We can define, in the same way we defined the Lie derivative of a vector field, the Lie derivative $\theta_X(\alpha)$, of the tensor α with respect to X; $\theta_X(\alpha)$ is a tensor of the same covariant and contravariant type as that of α. To define $\theta_X(\alpha)$ we have only to note that $T_{\varphi_t(a)}(\varphi_t^{-1}) : T_{\varphi_t(a)}(M) \to T_a(M)$ induces isomorphisms

$\overset{p}{\otimes} T_{\varphi_t(a)} \otimes \overset{q}{\otimes} T^*_{\varphi_t(a)} \to \overset{p}{\otimes} T_a(M) \otimes \overset{q}{\otimes} T^*_a(M)$. We also have an obvious generalisation of Lemma 6.4. Note that if α is a p-form, $\theta_X(\alpha)$ is defined as a p-form and that $\theta_X(\alpha) = 0$ if and only if $\varphi_t^*(\alpha) = \alpha$.

We list some properties of Lie derivatives:

1) θ_X is a local operator on tensor fields.

2) If f is a real valued function, $\theta_X(f) = Xf$.

3) If Y is a vector field, $\theta_X(Y) = [X,Y]$.

4) If α and α' are tensor fields, we have

$$\theta_X(\alpha \otimes \alpha') = \theta_X(\alpha) \otimes \alpha' + \alpha \otimes \theta_X(\alpha')$$

5) If α is a covariant tensor field of rank p, i.e., a section of $\overset{p}{\otimes} T^*$, we have, if $X_1,\dots,X_p$ are vector fields,

$$\theta_X\alpha(X_1,\dots,X_p) = X\,\alpha(X_1,\dots,X_p) - \sum_i \alpha(X_1,\dots,[X,X_i],\dots,X_p)$$

(Here note that $\alpha(X_1,\dots,X_p)$ is a real valued function and X can be applied on it).

In particular if α is a 1-form and Y a vector field

$$\theta_X < \alpha, Y > = < \theta_X \alpha, Y > + < \alpha, \theta_X Y >$$

or by (3),

(*) $\theta_X < \alpha, Y > = < \theta_X \alpha, Y > + < \alpha, [X,Y] >$

where $< , >$ denotes the scalar product.

To prove 4) and (*) we use the Leibniz formula in the following form. Let A,B,C be finite dimensional vector spaces and $\varphi : A \times B \to C$ a bilinear map. Let f and g be functions from an interval in $\mathbb{R}$ with values in A and B respectively. Define $h(t) = \varphi(f(t), g(t))$. Then we have

$$\frac{dh}{dt}(t_o) = f(t_o)\cdot \frac{dg}{dt}(t_o) + \frac{df}{dt}(t_o) g(t_o)$$

where we have written $\varphi(a,b) = a\cdot b$.

Then 4) follows from (*) by induction.

6) If α and β are differential forms we have

$$\theta_X(\alpha \wedge \beta) = \theta_X(\alpha) \wedge \beta + \alpha \wedge \theta_X(\beta).$$

Interior product and H. Cartan's formula.

Let E be a finite dimension vector space and $v \in E$. Then v defines a linear map $i_v : \overset{p}{\wedge} E^* \to \overset{p-1}{\wedge} E^*$ by :

$$i_v \psi(v_1,\dots,v_{p-1}) = \psi(v, v_1,\dots,v_{p-1}),$$

where $\psi \in \overset{p}{\wedge} E^*$, $v_1,\dots,v_{p-1} \in E$. This map is called the interior product defined by v.

If X is a vector field on M, we have a linear map

$$i_X : \overset{p}{\mathcal{E}}(M) \to \overset{p-1}{\mathcal{E}}(M)$$

with the property

$$i_X\, \alpha(X_1,\dots,X_{p-1}) = \alpha(X, X_1,\dots,X_{p-1})$$

where $\alpha \in \overset{p}{\mathcal{E}}(M)$, and $X_1,\dots,X_{p-1}$ are vector fields.

We then have:

1) i_X is linear over functions: i.e., f is a real valued (smooth) function and α a p-form then

$$i_X(f\cdot\alpha) = f\cdot i_X(\alpha)\ .$$

2) $i_X(\alpha \wedge \beta) = i_X\alpha \wedge \beta + (-1)^p \alpha \wedge i_X\beta$, if $\alpha \in \overset{p}{\mathcal{E}}$ and $\beta \in \overset{q}{\mathcal{E}}$.

3) $i_X^2 = 0$.

4) (H. Cartan's formula) : $\theta_X = i_X d + d i_X$

i.e., if α is a p-form we have $\theta_X\alpha = i_X d\alpha + d i_X\alpha$ where d is the operator of exterior differentiation.

In particular, since $d^2 = 0$, we have

$$d\,\theta_X = \theta_X\, d\ .$$

To prove 4) we write $L_X = i_X d + d i_X$. We have

$L_X(\alpha \wedge \beta) = L_X\alpha \wedge \beta + \alpha \wedge L_X\beta$ and also

$\theta_X(\alpha \wedge \beta) = \theta_X\alpha \wedge \beta + \alpha \wedge \theta_X\beta$, where α is a p-form. Hence it is sufficient to verify the formula when $\alpha = f$ a function and $\alpha = df$,

with f a function. Now $\theta_X f = Xf$ and

$$L_X f = d i_X f + i_X df = i_X df = \langle X, df \rangle = Xf\ .$$

If $\alpha = df$

$$\theta_X\alpha(Y) = X\,\alpha(Y) - \alpha[X,Y] \quad \text{(see (*) in p.28)}$$

$$= X\,Y\,f - [XY - YX]f$$

$$= YXf\ .$$

On the other hand,

$$(d i_X + i_X d)\,(df)\,(Y) = d(i_X df)\,(Y) = d\{\langle X, df\rangle\}(Y)$$

$$= YXf\ .$$

<u>Remark:</u> If X is a vector field and α a p-form such that $\theta_X(\alpha) = 0$, then α is called an 'invariant integral' for the local flow generated by X. In particular when α is a function f with $\theta_X f = Xf = 0$, then f is called a 'first integral' of X.

<u>Formula for the exterior differential.</u>

<u>Proposition 6.6.</u> Let α be a p-form and $X_1,\dots,X_{p-1}$ vector fields. We then have

$$d\alpha(X_1,\dots,X_{p+1}) = \sum_{i=1}^{p+1} (-1)^{i+1} X_i\,\alpha(X_1,\dots,\hat{X}_i,\dots X_{p+1})$$

$$+ \sum_{i<j} (-1)^{i+j}\alpha([X_i,X_j],\dots,\hat{X}_i,\dots,\hat{X}_j,\dots,X_{p+1})\ .$$

where $\hat{}$ over an element means that element is omitted.

In particular for a 1-form α,

$$d\alpha(X,Y) = X\alpha(Y) - Y\alpha(X) - \alpha[X,Y]\ .$$

__Proof.__ We use the formula $\theta_X = di_X + i_X d$ in the proof.

First if α is a 1-form we have

$$\begin{aligned} d\alpha(X,Y) &= i_X d\alpha(Y) \quad \text{(by definition)} \\ &= \{\theta_X\alpha - d(i_X\alpha)\}(Y) \\ &= X\alpha(Y) - \alpha[X,Y] - Y\alpha(X) \quad \text{(By (*) on p. 28 and } di_X\alpha = Y\alpha(X)) \end{aligned}$$

In the general case,

$$\begin{aligned} d\alpha(X_1,\ldots,X_{p+1}) &= i_{X_1} d\alpha(X_2,\ldots,X_{p+1}) \\ &= \theta_{X_1}\alpha(X_2,\ldots,X_{p+1}) - di_{X_1}\alpha(X_2,\ldots,X_{p+1}) . \end{aligned}$$

Using 5), p.27 for the first term and induction for the second term, we obtain the proposition.

__Remarks__

1) If X is a vector field and f a smooth real valued function, $\theta_{f.X} \neq f\theta_X$ in general on tensors. i.e., if α is a tensor field $\theta_{fX}(\alpha) \neq f\theta_X(\alpha)$ (Here $f.X$ is the vector field $a \mapsto f(a)\, X(a)$).

2) If X and Y are vector fields we have

$$\theta_X \circ \theta_Y - \theta_Y \circ \theta_X = \theta_{[X,Y]} \quad \text{and}$$

$$\theta_X \circ i_Y - i_Y \circ \theta_X = i_{[X,Y]} .$$

7. __IMMERSIONS AND SUBMERSIONS__

__Immersions and Submanifolds.__

Let $f : N \to M$ be a smooth map. We say that f is an __immersion__ at a point $b \in N$ if the tangent map $T_b(f) : T_b(N) \to T_{f(b)}(M)$ is

injective. We say that f is an immersion if f is injective at every point of N. We have

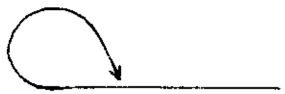
__Theorem 7.1.__ Let $n = \dim M$ and $p = \dim N$. Let f be an immersion at $b \in N$. Then there exists neighbourhoods V of b, U of $f(b)$ with $f(V) \subset U$ and a coordinate system $(x_1,\ldots,x_n)$ in U with $x_i \circ f = 0$ on V for $p < i \leq n$ and such that $\{x_i \circ f\}$, $1 \leq i \leq p$ give a coordinate system for N in V.

Moreover if $T_b(f) : T_b(N) \to T_{f(b)}(M)$ is an isomorphism there exists neighbourhoods V of b and U of $f(b)$ such that $f(V) \subset U$ and $f|V \to U$ is a diffeomorphism (inverse function theorem).

Let M be a manifold and N a subset of M, with the following property: for every $b \in N$ there exists a neighbourhood U of b in M and a coordinate system $(x_1,\ldots,x_n)$ in U (for M) and an integer $p \leq n$ (depending only on N) such that $N \cap U$ is given by $x_i = 0$ for $p < i \leq n$. Then N has a natural structure of a p-dimensional manifold such that the functions $\left\{x_i|_{N \cap U}\right\}_{1 \leq i \leq p}$ form a coordinate system for N in $N \cap U$. (N is provided with the topology induced from M). We then say that N is a submanifold of M. If moreover N is a closed subset of M we say that N is a closed submanifold.

__Remarks:__ 1) If $f: N \to M$ is an immersion such that f is a homeomorphism of N onto $f(N)$, then $f(N)$ is a submanifold and $f : N \to f(N)$ is a diffeomorphism. In this case we say that f is an imbedding.

2) If $f : N \to M$ is an injective immersion, $f(N)$ need not be a submanifold of M.



'The line approaches itself without touching'.

Submersions.

Let $f : M \to N$ be a smooth map. We say that f is a submersion at a point $a \in M$ if $T_a(f) : T_a(M) \to T_{f(a)}(N)$ is surjective. The map f is said to be a submersion if it is a submersion at every point of M.

Theorem (Implicit function theorem) 7.2.

Let $f : M \to N$ be a submersion at $a \in M$. (Let $n = \dim M$ and $q = \dim N$). There exists charts (U, ϕ), (V, ψ) for respectively M and N, with $a \in U$, $f(a) \in V$, $f(U) \subset V$, $\varphi(a) = 0 \in \mathbb{R}^n$, $\psi \circ f(a) = 0 \in \mathbb{R}^q$ with commutativity in the diagram

$$\begin{array}{ccc} U & \xrightarrow{f} & V \\ \downarrow \phi & & \downarrow \psi \\ \varphi(U) & \xrightarrow[F]{} & \psi(V) \end{array}$$

where F is (the restriction of) the map $\mathbb{R}^n \to \mathbb{R}^q$ $(x_1, \dots, x_n) \mapsto (x_1, \dots, x_q)$.

It follows that a submersion is an open map.

8. FROBENIUS THEOREM ON INTEGRABLE SUB-BUNDLES, FOLIATIONS

Sub and quotient bundles of a vector bundle.

Let E be a vector bundle of rank m over M. Suppose that F is a subset of E which satisfies the following conditions:

i) for $a \in M$, the set $F_a = E_a \cap F$ is a vector subspace of dimension p (fixed) of the fibre $E_a = \pi^{-1}(a)$ of E over a.

ii) every point $a \in M$ has a neighbourhood U and a frame $(\sigma_1, \dots, \sigma_m)$ of E over U such that for every $b \in U$, $\{\sigma_1(b), \dots, \sigma_p(b)\}$ form a base for F_b.

Then $\pi/F : F \to B$ has a natural structure of a vector bundle of rank p over M, for which $\{\sigma_1, \dots, \sigma_p\}$ are local frames. We call F a sub-bundle of E.

Let F be a subbundle of E. Let $E/F = \coprod_{b \in M} E_b/F_b$. Let $\eta : E \to E/F$ and $\pi_{E/F} : E/F \to M$ the natural maps. Then $\pi_{E/F} : E/F \to M$ has a structure of a vector bundle for which $(\eta \circ \sigma_{p+1}, \dots, \eta \circ \sigma_m)$ is a frame over U. The bundle E/F is the quotient of E by F.

Integrable sub-bundles of the tangent bundle : Integral Submanifolds.

Let F be a subbundle of rank p of the tangent bundle $T(M)$ of M. A submanifold N of M is said to be an integral manifold for F if the canonical map $T_b i : T_b(N) \to T_b(M)$ maps the tangent space $T_b(N)$ of N at b isomorphically onto the fibre F_b of F at b, for each $b \in N$ (Here $i : N \to M$ is the inclusion map)

Definition: Let F be a subbundle of $T(M)$. We say that F is integrable (or completely integrable) if the following condition is satisfied : if U is an open set of M and X,Y sections of F over U, then $[X,Y]$ is a section of F.

(X and Y are considered as sections of $T(M)$ i.e., as vector fields and $[X,Y]$ is the bracket of vector fields which is a section of $T(M)$ and we require that $[X,Y]_b \in F_b$ for $b \in U$).

Remark: Let F be a subbundle of $T(M)$. We then have a section Ω of $(\Lambda^2 F^* \otimes (T(M)/F))$ i.e., we have for $b \in M$, an alternating bilinear map $F_b \times F_b \to (E/F)_b$. This is defined as follows: Let $v_1, v_2 \in F_b$ and let X,Y be vector fields (in a neighbourhood of b) with $X(a) = v_1$, $X(b) = v_2$. Define $\Omega(v_1, v_2) = \eta_b[X,Y]_b$, where $\eta_b : T_b \to T_b(M)/F_b$ is the natural map. One checks that Ω depends only on v_1, v_2 and not on the extensions X and Y. This section Ω may be called the curvature of the subbundle F. Thus F is integrable if and only if its curvature is identically zero.

Frobenius Theorem.

Theorem 8.1.: The following conditions on a subbundle F of $T(M)$ are equivalent.

1) Through every point $a \in M$, there is an integral submanifold for F.

2) F is integrable.

3) Each point $a \in M$ has a neighbourhood U, a diffeomorphism

$\varphi : U \to V \times W$, where V and W are open subsets of $\mathbb{R}^p$ and $\mathbb{R}^{n-p}$ respectively, such that for each $w \in W$, the set $\varphi^{-1} \circ p_W^{-1}(w)$ ($p_W : V \times W \to W$ the projection) is an integral submanifold of F.

$$\begin{array}{ccc} U & \xrightarrow{\phi} & V \times W \\ & & \big\downarrow p_W \\ & & W \end{array}$$

(In other words, around each point a there exists a system of coordinates $(x_1,\ldots,x_n)$ for M such that the submanifolds $x_{p+1} = c_{p+1},\ldots, x_n = c_n$, where $c_{p+1},\ldots,c_n$ are constants, are integral submanifolds of F.)

Proof. 3) $\Longrightarrow$ 1) is trivial .

To prove 1) $\Longrightarrow$ 2) we use

Lemma 8.2. Let $f : N \to M$ be a smooth map. If X_1 (resp. X), a vector field on N (resp. M) we say that X_1 and X are f-related if for each $b \in N$ we have $T_b(f)(X_1(b)) = X(f(b))$. Suppose that X_1 and Y_1 vector fields on N and X,Y vector fields on M such that X_1 is f-related to X and Y_1 is f-related to Y. Then $[X_1, Y_1]$ is f-related to $[X,Y]$.

To deduce 1) $\Rightarrow$ 2) from the lemma we take f to be the inclusion $i : N \to M$, $X_1 = X|_N$, $Y_1 = Y|_N$.

The essential part of the proof is to show that 2) $\Rightarrow$ 3) . This is done in two steps. First one proves

Lemma 8.3. Suppose that F is integrable. Each point $a \in M$ has a neighbourhood of U and a frame $\{X_1,\ldots,X_p\}$ of F over U such that $[X_i, X_j] = 0$ for $1 \leq i \leq p$, $1 \leq j \leq p$.

The proof of this lemma is essentially algebraic. One then proves

Proposition 8.4. Let $X_1,\ldots, X_p$ be a vector fields in a neighbourhood U of a such that $\{ X_1(b),\ldots,X_p(b)\}$ are linearly independent in $T_b(M)$ for each $b \in U$ and such that $[X_i, X_j] = 0$ in U, for $1 \leq i \leq p$, $1 \leq j \leq p$. Then there exists a coordinate system $(x_1,\ldots,x_n)$ in a neighbourhood V of a such that $X_i = \partial/\partial x_i$, $1 \leq i \leq p$ in V.

Sketch of proof of the proposition.

Choose a coordinate system $(y_1,\ldots,y_n)$ around a with $y_i(a) = 0$ and such that $X_1(a),\ldots, X_p(a)$, $(\partial/\partial y_{p+1})_a,\ldots, (\partial/\partial y_n)_a$ span $T_a(M)$. Let φ_t^i be the local flow associated with X_i, $1 \leq i \leq p$. For $\delta > 0$, let $\Omega = \{(t_1,\ldots,t_p, y_{p+1},\ldots,y_n) \mid |t_i| < \delta,\ |y_j| < \delta\}$. For δ sufficiently small, the map $h : \Omega \to M$

$$h(t_1,\ldots,t_p, y_{p+1},\ldots,y_n) = \varphi^1_{t_1} \circ \ldots \circ \varphi^p_{t_p}(0,\ldots,0, y_{p+1},\ldots,y_n)$$

is well-defined. It is easy to check that

$T_o(h)((\partial/\partial t_i)_o) = X_i(a)$ and $T_o(h)((\partial/\partial y_j)_o) = (\partial/\partial y_j)_a$,

so that h is a diffeomorphism in a neighbourhood V of 0. Next one shows that, for $x \in V$,

$$T_x(h)((\partial/\partial t_i)_x) = X_i h(x), \quad 1 \leq i \leq p.$$

To prove this one uses the fact that φ_t^i and φ_t^j commute for $1\leq i\leq p$, $1\leq j\leq p$

as $[X_i, X_j] = 0$ (see p.26 Corollary 6.5.1).

The implication 2) $\Rightarrow$ 3) is clear from the Proposition.

Foliation.

Let F be an integrable subbundle (of rank p) of $T(M)$. We define a new topology on M by declaring that a subset of M is open if it is a union of integral submanifolds for F. With this topology M becomes, in a natural way, a p-dimensional manifold, which we denote by M_F. A connected component of M_F is called a maximal integral manifold (or a leaf) of F. The leaves define a partition of M, called the foliation of M defined by F. If N is a leaf, note that the inclusion map $i : N \to M$ is an injective immersion; in general N need not be a submanifold of M.

9. LIE GROUPS (Continued)

The Lie algebra of a Lie group.

Let G be a Lie group. A vector field X on G is said to be left invariant if for each $x \in G$ and $g \in G$ we have $X(gx) = T_x(L_g)(X(x))$ where $L_g : G \to G$ is the left translation by g i.e., $L_g(y) = gy$. A left invariant vector field is uniquely determined by its value at the identity element e and given $v \in T_e(g)$ there exists a unique left invariant vector field X with $X(e) = v$. Moreover, if X and Y are left invariant vector fields, so are $\lambda X + \lambda Y$, $(\lambda,\mu \in \mathbb{R})$ and $[X,Y]$. Thus the set of left invariant vector fields on G form a n-dimensional $(n = \dim G)$ Lie algebra over $\mathbb{R}$, called the Lie algebra of G and denoted by $\mathfrak{g}$.

Let G_1 and G_2 be Lie groups and $\varphi : G_1 \to G_2$ a smooth homomorphism. Let $\mathfrak{g}_1$ (resp. $\mathfrak{g}_2$) be the Lie algebra of G_1 (resp. G_2). Then φ induces a Lie algebra homomorphism $\varphi_* : \mathfrak{g}_1 \to \mathfrak{g}_2$, as follows. Let $X \in \mathfrak{g}_1$ and e_i $(i = 1,2)$ be the identity element of G_i. Let $X' \in \mathfrak{g}_2$ with $X'(e_2) = T_{e_1}(\varphi)(X(e_1))$ and define $\phi_*(X) = X'$. Since for $g \in G$ we have $\varphi \circ L_g = L_{\varphi(g)} \circ \varphi$ it is easy to check that X and X' are φ-related. It follows from Lemma 8.2 that φ_* is a Lie algebra homomorphism.

In particular $\varphi : G_1 \to G_2$ is a homomorphism which is an injective immersion, $\mathfrak{g}_1$ can be identified with the Lie subalgebra $\varphi_*(\mathfrak{g}_1)$ of $\mathfrak{g}_2$.

<u>Lie subalgebras and Lie subgroups</u>.

Let G be a Lie group and $H \subset G$. Suppose that H has a structure of Lie group such that the inclusion map $i : H \to G$ is a homomorphism which is an injective immersion. We say that H is a Lie subgroup of G. (A Lie subgroup H need not be a submanifold of G; in fact if it is so, it should be closed in G). The Lie algebra $\mathfrak{h}$ of H is identified by i_* with a subalgebra of $\mathfrak{g}$.

<u>Proposition 9.1</u>. Let G be a Lie group and $\mathfrak{h}$ a Lie subalgebra of $\mathfrak{g}$. Then there exists a Lie subgroup H of G whose Lie algebra is $\mathfrak{h}$.

<u>Idea of Proof</u>. The subalgebra $\mathfrak{h}$ defines an integrable subbundle F of $T(G)$ as follows. The fibre F_x of F at $x \in G$, is the subspace of $T_x(G)$, $\{X(x) \mid X \in \mathfrak{h}\}$. Since for $X, Y \in \mathfrak{h}$, we have $[X,Y] \in \mathfrak{h}$, F is an integrable subbundle of $T(G)$. Let H be the leaf through e of the foliation on G defined by F. One checks that H is a Lie subgroup of G whose Lie algebra is $\mathfrak{h}$, using that if C is a leaf then $g\,C$ is a leaf, for $g \in G$.

<u>Homomorphisms of Lie groups and Lie algebras</u>.

Let G_1 and G_2 be two lie groups with lie algebras $\mathfrak{g}_1$, $\mathfrak{g}_2$. Then the Lie algebra of the Lie group $G_1 \times G_2$ is naturally identified with the direct product $\mathfrak{g}_1 \times \mathfrak{g}_2$. Let $\varphi : G_1 \to G_2$ be a homomorphism of Lie groups and $\varphi_* : \mathfrak{g}_1 \to \mathfrak{g}_2$ the induced homomorphism. The graph $\Gamma = \{(x, \varphi(x));\ x \in G_1\}$, of φ is a (closed) Lie subgroup of $G_1 \times G_2$ whose Lie algebra is the graph $\Gamma' = \{(v, \phi_* v);\ v \in \mathfrak{g}_1\}$ of φ_*. From this we deduce that if φ_1 and φ_2 are two homomorphisms of G_1 into G_2 and $\varphi_{1*} = \varphi_{2*}$ on $\mathfrak{g}_1$, then $\varphi_1 = \varphi_2$ if G_1 is <u>connected</u>.

We now consider the question whether every homomorphism between $\mathfrak{g}_1$ and $\mathfrak{g}_2$ is induced by homomorphism between G_1 and G_2.

<u>Proposition 9.2.</u> Suppose that G_1 is <u>simply connected</u>. If $\psi : \mathfrak{g}_1 \to \mathfrak{g}_2$ is a Lie algebra homomorphism, then there exists a (unique) homomorphism $\varphi : G_1 \to G_2$ such that $\varphi_* = \psi$.

We indicate a proof of this result.

<u>Definition</u>. Let G_1 and G_2 be Lie groups and U a neighbourhood of e in G_1. A smooth map $f : U \to G_1$ is called a local homomorphism if for $x, y \in U$ with $xy \in U$, we have $f(xy) = f(x)\,f(y)$.

Using the monodromy theorem one proves:

Lemma 9.3. Let G_1 be a simply connected lie group and $f : U \to G_2$ be a local homomorphism where U is a connected neighbourhood of e in G_1. Then there exists a (unique) homomorphism (smooth) $\varphi : G_1 \to G_2$ such that $\varphi|_U = f$.

To prove Proposition 9.2., note first that the Lie algebra of $G_1 \times G_2$ is naturally isomorphic to $\mathfrak{g}_1 \times \mathfrak{g}_2$, the direct product of the Lie algebras $\mathfrak{g}_1$ and $\mathfrak{g}_2$. Let $\Gamma' = \{(v, \psi(v)); v \in \mathfrak{g}_1\} \subset (\mathfrak{g}_1 \times \mathfrak{g}_2)$ be the graph of ψ. Then Γ' is a subalgebra of $\mathfrak{g}_1 \times \mathfrak{g}_2$. Let Γ be a Lie subgroup of $G_1 \times G_2$ corresponding to Γ'. The tangent map at identity of the restriction of the projection $pr_{G_1} : G_1 \times G_2 \to G_1$ to Γ is $(v,\psi(v)) \mapsto v$, so that this map is a local diffeomorphism of a neighbourhood of identity in Γ onto a (connected) neighbourhood U of e in G_1. Then the composite f of the maps $(pr_{G_1})^{-1}: U \to \Gamma$ and $pr_{G_2} : \Gamma \to G_2$ is a local homomorphism. Since G_1 is simply connected, by Lemma 9.3, f extends to a homomorphism $\varphi : G_1 \to G_2$ which is the required homomorphism.

The exponential map.

The Lie algebra of the Lie group $GL(n,\mathbb{R})$ can be identified with the Lie algebra $\mathfrak{gl}(n,\mathbb{R})$ of $n \times n$ real matrices. If A and B are $n \times n$ matrices $[A,B]$ is defined to be $AB-BA$). If $A \in \mathfrak{gl}(n,\mathbb{R})$, the series $I + A + A^2 + \cdots + \frac{A^n}{n!} + \cdots$ defines an element in $GL(n,\mathbb{R})$ denoted by $\exp A$, called the exponential of A. Thus $A \mapsto \exp A$ is a map of the Lie algebra of $GL(n,\mathbb{R})$ into $GL(n,\mathbb{R})$. We shall generalise this to any Lie group G.

Considering $\mathbb{R}$ as a Lie group, the vector field $\frac{d}{dt}$ on $\mathbb{R}$ is a left invariant vector field. Let G be a Lie group and X an element of $\mathfrak{g}$. There exists a unique Lie algebra homomorphism of the Lie algebra of $\mathbb{R}$ into $\mathfrak{g}$, sending $\frac{d}{dt}$ into X. Since $\mathbb{R}$ is simply connected, there exists a unique homomorphism $\phi_X : \mathbb{R} \to G$ whose tangent map is the above homomorphism. We define

$$\exp X = \phi_X(1). \qquad (1 \in \mathbb{R})$$

We have

1) $\exp((t_1 + t_2)X) = \exp(t_1X)\exp(t_2X)$, $t_1, t_2 \in \mathbb{R}$
2) $\exp(-tX) = \exp(tX)^{-1}$, $t \in \mathbb{R}$
3) $\exp : \mathfrak{g} \to G$ is a smooth map.
4) $\exp$ is a local diffeomorphism at $0 \in \mathfrak{g}$.

The adjoint representation of a Lie group.

If V is a finite dimensional vector space over $\mathbb{R}$, the group $Aut(V)$ of linear automorphisms is a Lie group (If $\dim V = m$, we can identify $Aut\ V$ with $GL(m,\mathbb{R})$ if we choose a base for V). A smooth homomorphism $\rho : G \to Aut(V)$ is called a representation of the Lie group G on V.

We shall now define a natural representation of a Lie group on its Lie algebra, called the adjoint representation of G. Let $g \in G$ and $Int\ g : G \to G$ be the map $s \mapsto gsg^{-1}$, $s \in G$. Define $Ad\ g : T_e(G) \to T_e(G)$ to be $T_e(Int\ g)$. Identifying $T_e(G)$ with $\mathfrak{g}$, we get a representation $g \mapsto \mathrm{Ad}\, g$ on $\mathfrak{g}$. If $R_g : G \to G$ is the map $s \to sg$, $s \in G$, and $X \in \mathfrak{g}$, it is not hard to prove that $(R_g)_*(X) = \mathrm{Ad}(g^{-1})(X)$.

10. PRINCIPAL BUNDLES AND ASSOCIATED BUNDIES

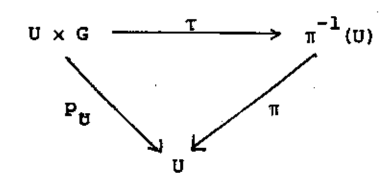
Principal bundles.

Let U be a manifold and G a Lie group. We make G act on $U \times G$ on the right as follows: $((x,s),g) \mapsto (x,sg)$, for $x \in U$, $s, g \in G$. Note that the action of G is free i.e., if $y \in U \times G$ and $yg = g$, $g \in G$, then $g = e$. Moreover given y_1, y_2 with $pr_U(y_1) = pr_U(y_2)$ then there exists a unique element $g \in G$ with $y_2 = y_1 g$ i.e., G acts simply transitively on the fibres of the map $U \times G \to U$.

We now consider the situation where a Lie group G acts on a manifold P, where the situation is 'locally' as above.

Definition. Let $\pi : P \to M$ be a map of smooth manifolds. Suppose that a Lie group G acts on the right on P and that every point $x \in M$ has a neighbourhood U and a diffeomorphism $\tau : U \times G \to \pi^{-1}(U)$ such that

i) the diagram



commutes, and

ii) $\tau(x, sg) = \tau(x,s)g$, for $x \in P$, $s, g \in G$. Then we say that P

is a principal bundle over M with structure group G, or simply that P is a principal G-bundle over M.

Remarks. 1) G acts freely on P.

2) G acts simply transitively on each fibre of $\pi : P \to M$.

The bundle of frames associated to a vector bundle.

As an example of a principal bundle, we shall associate to a vector bundle E of rank m over M, a principal bundle over M with $GL(m, \mathbb{R})$ as structure group, called the bundle of frames of E. Let P be the set of linear isomorphisms of $\mathbb{R}^m$ into the fibres of E (If $\varphi : \mathbb{R}^m \to E_x$ is an isomorphism and $(e_1,\ldots,e_m)$ is the canonical basis in $\mathbb{R}^m$, then $(\varphi(e_1),\ldots,\varphi(e_m))$ is a basis in E_x; conversely given a basis $(f_1,\ldots,f_m)$ of E_x there exists a unique isomorphism of $\mathbb{R}^m$ into E_x sending e_i into f_i. Thus P can be identified with the set of basis ('frames') in the different fibres of E. This explains the terminology). We make $GL(m,\mathbb{R})$ act on P as follows: if $x \in M$, $\varphi : \mathbb{R}^m \to E_x$ an isomorphism and $g \in GL(m,\mathbb{R})$, then $((x,\varphi), g) \mapsto (x, \varphi \circ g)$, g being considered as an isomorphism $\mathbb{R}^m \to \mathbb{R}^m$. P has a natural structure of a manifold (in fact it is an open subset of $E \oplus \cdots \oplus E$, m times) and becomes a principal $GL(m,\mathbb{R})$ bundle under this action.

Note that if there is a smooth section of P(i.e., a smooth map $\sigma : M \to P$, with $\pi \circ \sigma = \mathrm{Id}_M$) the vector bundle E is trivial.

Morphisms of bundles. Gauge transformations.

Let P and P' be principal G-bundles over M and M' respectively. A morphism or a bundle homomorphism from P to P' is a smooth map $h : P \to P'$ such that $h(pg) = h(p).g$, for $p \in P$ and $g \in G$.

It is easily seen that h induces a smooth map $\underline{h} : M \to M$ such that the diagram

$$\begin{array}{ccc} P & \xrightarrow{h} & P' \\ \downarrow \pi & & \downarrow \pi' \\ M & \longrightarrow & M' \end{array}$$

commutes.

If $M = M'$ and if there is a morphism $h : P \to P'$ such that the induced map $\underline{h} : M \to M$ is the identity we say that P and P' are isomorphic. (In this case it is easy to see that $h : P \to P'$ is bijective and $h^{-1} : P' \to P$ is a morphism).

If P is a principal G-bundle over M, a morphism of P into P, which induces identity on the base is called a gauge transformation (such a morphism is an automorphism of P). The gauge transformations of P form a group, called the group of gauge transformations of the principal bundle G.

A bundle isomorphic to the trivial bundle $M \times G$ is called trivial.

<u>Proposition 10.1</u>. Let P be a principal G bundle over M. Then the following conditions are equivalent.

1) P is trivial.

2) There exists a smooth section $\sigma : M \to P$.

<u>Proof.</u> 1) $\Longrightarrow$ 2) is clear, since the trivial bundle admits for instance the section $x \mapsto (x,e)$. To prove 2) $\Longrightarrow$ 1) let $h'(P)$ be the unique element in G such that $\sigma(\pi(p))h'(p) = p$, for $p \in P$. Then $h : P \to M \times G$ defined by $h(p) = (\pi(p), h'(p))$ is an isomorphism of G-bundles.

<u>Associated bundles</u>.

Let P be a principal G-bundle over M and F a manifold with a left action of G. Consider the action of G on $P \times F$ given by :

$$((p,f),g) \mapsto (pg, g^{-1}f).$$

One can show that $P \times F$ is a principal G-bundle over the quotient space $(P \times F)/G$. We denote $(P \times F)/G$ by $P \times_G F$. The map $(p,f) \mapsto \pi(p)$ induces a map $P \times_G F \to M$, which is called the bundle associated to P by the action of G on F. If $p \in P$, p induces a diffeomorphism of F with the fibre over $\pi(p)$ of the map $P \times_G F \to M$.

Any structure on F invariant under the action can be put on the fibres of the associated bundle.

As an example let F be a finite dimensional vector space and $\rho : G \to \mathrm{Aut}(F)$ be a (smooth) homomorphism so that G acts on F by linear transformations. Then $P \times_G F$ is a vector bundle over M, called the vector bundle associated to the representation ρ.

Extension and restriction of the structure group.

Let P be a principal G_1 bundle and $\rho : G_1 \to G_2$ a homomorphism of Lie groups. We can then define a principal G_2-bundle over M, as follows. G_1 operates on G_2 by $(g_1, g_2) \mapsto \rho(g_1)\cdot g_2$, $g_i \in G_i$. The action of G_2 on $P \times G_2$ given by $(p,s)s' = (p, ss')$, $p \in P$, $s, s' \in G_2$ goes over into an action of G_2 on the associated bundle $P \times_{G'} G_2$ and makes of it a principal G_2 bundle. This G_2 is bundle is said to be obtained by extension of the structure group by ρ.

Suppose that H is a Lie subgroup of G and P a G-bundle over M. If there exists an H-principal bundle Q over M such that the G-bundle, obtained from Q by extension of the structure group by the inclusion map $H \to G$, is isomorphic to P(as a G-bundle), we say that the structure group of P can be reduced to H.

The pull-back (or inverse image) of a bundle.

Let P be a principal G-bundle over M and $f: N \to M$ be a smooth map of manifolds. Let $P \times_N M$ be the subset of $P \times N$ consisting of points (p,y) with $\pi(p) = f(y)$ $(p \in P, y \in N)$. G acts on $P \times_N M$ by $(p,y)g = (pg,y)$. With this action $P \times_N M$ becomes a principal G-bundle on N, called the pull-back of P by f (and sometimes denoted by $f^*(P)$). We have a commutative diagram

$$\begin{array}{ccc} f^*(P) & \longrightarrow & P \\ \big\downarrow & & \big\downarrow \pi \\ N & \xrightarrow{\ f\ } & M \end{array}$$

(The maps $f^*(P) \to P$ and $f^*(P) \to N$ are given by the restrictions of the projections of $P \times N$ onto P and N respectively.)

11. EQUIVARIANT FORMS ON A PRINCIPAL BUNDLE

The tangent bundle along the fibres of a principal bundle. Vertical vectors.

Let $\pi : P \to M$ be a principal bundle with structure group G. Let $p \in P$. We denoted by V_p the kernel of the map $T_p(\pi) : T_p(P) \to T_{\pi(p)}(M)$. An element of V_p is called a vertical vector at p. The tangent space at p of the fibre of π through p can be identified with V_p. The vertical vectors $\coprod_{p \in P} V_p$ form a subbundle of $T(P)$, called the tangent bundle along the fibres of P, denoted by T_π or V.

Proposition 11.1. The tangent bundle along the fibres of P is trivial. More precisely there is a canonical isomorphism ψ of the trivial vector bundle $P \times \mathfrak{g}$ with T_π with the following property: for each $g \in G$ the diagram

$$\begin{array}{ccc} P \times \mathfrak{g} & \xrightarrow{\ \psi\ } & T_\pi \\ \big\downarrow R_g \times Ad(g^{-1}) & & \big\downarrow (R_g)_* \\ P \times \mathfrak{g} & \xrightarrow[\ \psi\ ]{} & T_\pi \end{array}$$

is commutative. Here $(R_g)_*$ is the 'differential map' induced by the diffeomorphism R_g, $q \mapsto qg$, $q \in P$ (If $\eta : T_\pi \to P$ is the projection and $v \in T_\pi$, $(R_g)_*(v) = T_{\eta(v)}(R_g)(v)$). If $p \in P$, $\xi \in \mathfrak{g}$, $Id \times Ad(g^{-1})(p, \xi) = (p, Ad(g^{-1})\xi)$.)

Indication of proof. If $p \in P$, consider the orbit map $\sigma_p : G \to P$. Note that $\sigma_p(e) = p$. The tangent map of σ_p at e maps $\mathfrak{g} = T_e(G)$ isomorphically onto V_p. These isomorphisms, as p varies, give ψ. The commutativity of the diagram can be proved using

i) if $X \in \mathfrak{g}$ and $R_g : G \to G$ is the right translation we have $(R_g)_*(X) = \mathrm{Ad}(g^{-1})(X)$ (see p. 43)

ii) commutativity of the diagram

$$\begin{array}{ccc} G & \xrightarrow{p} & P \\ \downarrow R_g & & \downarrow R_g \\ G & \xrightarrow{p} & P \end{array}$$

Remark 11.2. If $X \in \mathfrak{g}$, X defines in natural way a section of the trivial bundle $P \times \mathfrak{g}$, $p \mapsto (p,x)$. Using the isomorphism ψ, we get a section of T_π, and hence a vertical vector field, denoted by $\sigma(X)$ and called the fundamental vector field on P defined by X. If $X,Y \in \mathfrak{g}$, we have $\sigma[X,Y] = [\sigma(X), \sigma(Y)]$. From the commutativity of the diagram in the Proposition, we have: $(R_g)_* \sigma(X)$ is the fundamental vector field corresponding to $\mathrm{Ad}(g^{-1})(X) \in \mathfrak{g}$.

Alternating forms with values in a vector space.

Let E and F be finite dimensional vector spaces over $\mathbb{R}$. It is clear how to define an alternating p-form on E with values in F. For instance a 2-form is a bilinear map $f : E \times E \to F$ with $f(x,x) = 0$ for $x \in E$. Let F_1, F_2, F_3 be (finite dimensional) vector spaces and $\varphi : F_1 \times F_2 \to F_3$ a bilinear map. If α (resp. β) is a p (resp. q) form on E with values in F_1 (resp. F_2) we can define a $(p+q)$ form on E with values F_3, as on page 13, using φ instead of multiplication in $\mathbb{R}$. (We denote this form by $\alpha \wedge_\varphi \beta$). For instance if $p = q = 1$,

$$(\alpha \wedge_\varphi \beta)(X, Y) = \varphi(\alpha(X), \beta(Y)) - \varphi(\alpha(Y), \beta(X))$$

for $X, Y \in E$.

Consider the special case $F_1 = F_2 = F_3 = \mathfrak{g}$, where $\mathfrak{g}$ is a Lie algebra and $\varphi : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ is the map $\varphi(X,Y) = [X,Y]$, $X,Y \in \mathfrak{g}$. In this case we denote $\alpha \wedge_\varphi \beta$ by $[\alpha, \beta]$. Note that if α is a 1-form with values in $\mathfrak{g}$, we have

$$[\alpha, \alpha](X, Y) = [\alpha(X), \alpha(Y)] - [\alpha(Y), \alpha(X)]$$
$$= 2[\alpha(X), \alpha(Y)]$$

We have

i) $[\alpha, \beta] = (-1)^{pq+1} [\beta, \alpha]$

ii) If γ is a r-form with values in $\mathfrak{g}$,

$$(-1)^{pr} [\alpha, [\beta, \gamma]] + (-1)^{qp} [\beta, [\gamma, \alpha]] + (-1)^{rq} [\gamma, [\alpha, \beta]] = 0.$$

It is clear that we can define on a manifold M smooth differential forms with values in a vector space (finite dimensional) F. If α is such a smooth p-form on M, we can define the exterior differential $d\alpha$, which is a $(p+1)$ form with values in F. If we wish, we can define d by the analogue of the formula in Prop. 6.6 (p.30) for d. In particular if α is a 1-form with values in F,

$d\alpha(X, Y) = X\alpha(Y) - Y\alpha(X) - \alpha[X, Y]$, X, Y being vector fields on M.

In the case $F = \mathfrak{g}$, a Lie algebra, for differential forms α, β, γ with values in $\mathfrak{g}$ we have i), ii) above and

$$d[\alpha,\beta] = [d\alpha,\beta] + (-1)^p [\alpha, d\beta] .$$

The Maurer-Cartan form and equation.

We give an illustration of the above notion. Let G be a Lie group with Lie algebra $\mathfrak{g}$. Then there is a canonical 1-form on G with values in $\mathfrak{g}$, called the Maurer-Cartan form on G. This form α is defined as follows. Let v be a tangent vector of G at g. Then there exists a unique left invariant vector field X on G with $X(g) = v$. Define $\alpha(v) = X$. Note that α is essentially the identity map on $T_e(G)$.

Proposition (Maurer-Cartan equation) 11.3.

We have $d\alpha + \frac{1}{2}[\alpha,\alpha] = 0$.

Proof. Let X and Y be left invariant vector fields. Noting that $\frac{1}{2}[\alpha,\alpha](X,Y) = [\alpha(X), \alpha(Y)]$, it suffices to show that $d\alpha[X, Y] + [\alpha(X), \alpha(Y)] = 0$. But

$$d\alpha(X, Y) = X\alpha(Y) - Y\alpha(X) - \alpha[X, Y]$$

$$= -\alpha[X,Y], \text{ as } \alpha(Y) = Y, \ \alpha(X) = X \text{ are constants}$$

$$= -[X,Y] = -[\alpha(X), \alpha(Y)] .$$

Equivariant forms on principal bundles.

Let $\rho : G \to \mathrm{Aut}(F)$ be a finite dimensional representation of G. Let α be a p-form on P with values in F. We say that α is equivariant (with respect to ρ) if $R_g^*\alpha = \rho(g^{-1})\alpha^{(+)}$. (Here $\rho(g^{-1})\alpha$ is the p-form on P defined by $[\rho(g^{-1})\alpha](X_1,\dots,X_p) = \rho(g^{-1})\,\alpha(X_1,\dots,X_p)$ for $X_1,\dots,X_p$ tangent vectors on P. For $p = 0$, the condition means $\alpha(pg) = \rho(g^{-1})\alpha(p)$, $p \in P$, $g \in G$, α being a function from P to V). Since $d R_g^* = R_g^* d$, we see that if α is an equivariant p-form then d is an equivariant (p+1) form.

An equivariant p-form is said to be basic ("coming from the base") or horizontal if $\alpha(X_1,\dots,X_p) = 0$ if at least one of the tangent vectors X_i is vertical. Basic equivariant p-forms can be identified with the sections of the bundle $\Lambda^p T^*(M) \otimes F_\rho$ on M, where F_ρ is the vector bundle on M associated to the representation ρ. Hence such forms are also called p-forms <u>on M</u> with coefficients in the bundle F_ρ. In particular applying this to the case of the trivial 1-dimensional representation of G, we see that a p-form (in the usual sense) on M can be identified with a p-form α on P which is invariant under the action of G and which satisfies $\alpha(X_1,\dots,X_p) = 0$ if one of the X_i is vertical.

Note that if α is basic, the form $d\alpha$, while being equivariant, need not be basic.

+ $R_g^*\alpha$ is the inverse image of α by the map R_g, $p \mapsto pg$.

12. CONNECTION AND CURVATURE

Connections and connection forms.

Definition. Let P be a principal G-bundle. Let $T(P)$ be the tangent bundle of P and T_π the tangent bundle along the fibres. A connection on P is a subbundle $\mathcal{H}$ of $T(P)$ which is supplementary to T_π and which is invariant under the action of G on $T(P)$.

Thus if $\mathcal{H}_p$ is the fibre of $\mathcal{H}$ at p we have

a) $T_p(P) = \mathcal{H}_p \oplus V_p$ b) for $g \in G$, $p \in G$,

$T_p(R_g)(\mathcal{H}_p) = \mathcal{H}_{pg}$. An element of $\mathcal{H}$ is called an horizontal vector and $\mathcal{H}_p$ is called the horizontal space at p.

If $\eta : T(P) \to T_\pi$ is the projection defined by the decomposition $T(P) = \mathcal{H} \oplus T_\pi$ we can consider η as a 1-form on P with values in $\mathfrak{g}$, using the isomorphism of T_π with $P \times \mathfrak{g}$. We denote this 1-form by w and call it the connection form (of the connection).

Thus the form w is defined as follows. Let $v \in T_p(P)$. Write $v = v_1 \oplus h$ with $v_1 \in V_p$, $h \in \mathcal{H}_p$. Under the isomorphism of V_p with $\mathfrak{g}$, v_1 corresponds to an element v_1' in $\mathfrak{g}$. Then we define $w(v)$ to be v_1'.

If $w_p : T_p(P) \to \mathfrak{g}$ is the value of w at p, note that the kernel of w_p is $\mathcal{H}_p$.

The connection form w satisfies the following two conditions.

1) If X_o is a vertical vector at p, $w(X_o)$ is that element X_o' of $\mathfrak{g}$ whose associated fundamental vector field $\sigma(X_o')$ takes the value X_o at p.

2) $(R_g)^* w = (\operatorname{Ad} g^{-1})(w)$.

These properties follow immediately from Proposition 11.1.

Note that the second condition means that w is an equivariant 1-form on P with respect to the adjoint representation of G on its Lie algebra.

Conversely given a 1-form w on P with values in $\mathfrak{g}$ satisfying 1) and 2) above it defines a unique connection whose associated 1-form is w. In fact define $\mathcal{H}_p$ = kernel of $w_p : T_p \to \mathfrak{g}$, where w_p is the value of w at p.

Often we do not distinguish between a connection and the associated connection form.

If P' is a G-bundle over M' and $h : P' \to P$ is a bundle homomorphism and w a connection form on P, it is immediate that $h^*(w)$ is a connection form on P', called the inverse image of w by h.

The curvature form of a connection.

Let P be a G-bundle with a connection. Let $\rho : G \to \operatorname{Aut}(F)$ be a representation of G. If α is an equivariant p-form (with respect to ρ), the form $\alpha \circ H$, where $H : T(P) \to \mathcal{H}$ is the projection onto with respect to the decomposition $T(P) = \mathcal{H} \oplus T_\pi$, is an equivariant form which is clearly <u>basic</u>. (Note that if $X_1, \ldots, X_p$ are tangent vectors at point of P, $\alpha \circ H(X_1, \ldots, X_p) = \alpha(H X_1, \ldots, H X_p)$ where H is the projection on to the horizontal space). We define

$$dw(\alpha) = d\alpha \circ H .$$

i.e., $d_w(\alpha)$ is the basic form of degree $(p+1)$ associated to $d\alpha$.

<u>Definition</u>. The 2-form $d_w(w)$ with values in $\mathfrak{g}$ is defined to be the curvature form of the connection. It is denoted by Ω.

<u>Remark 12.1</u>. 1) $\Omega(X_1, X_2) = dw(HX_1, HX_2)$ (by definition) where X_1, X_2 are tangent vectors at a point of P.

12.2. The curvature form is a <u>basic</u> 2 form with values in $\mathfrak{g}$, equivariant with respect to the adjoint representation. As such it can be identified with a 2-form on M with coefficients in the vector bundle associated to the adjoint representation. (This bundle is called the adjoint bundle of P).

<u>Proposition (Structure equation) 12.3.</u>

We have $\Omega = dw + \frac{1}{2}[w,w]$ where Ω is the curvature form of the connection and w the connection form.

<u>Proof</u>. It suffices to prove that $\Omega(X_o, Y_o) = dw(X_o, Y_o) + \frac{1}{2}[w,w](X_o, Y_o)$ in the following 3 cases 1) X_o, Y_o are both horizontal vectors 2) X_o, Y_o are both vertical 3) X_o vertical and Y_o horizontal.

<u>Case 1</u>. X_o, Y_o both horizontal. Since $w(X_o) = w(Y_o) = 0$, we have $\frac{1}{2}[w,w](X_o, Y_o) = 0$. Hence

$$(X_o, Y_o) = dw(HX_o, HY_o) = dw(X_o, Y_o) = dw(X_o, Y_o) + \frac{1}{2}[w,w](X_o, Y_o) .$$

<u>Case 2.</u> X_o, Y_o both vertical vectors at p. The proof is similar to that of the Maurer-Cartan equation. Let $A, B \in \mathfrak{g}$ be such that, if $X = \sigma(A)$, $Y = \sigma(B)$ be the associated fundamental vector fields on P, we have $X(p) = X_o$, $Y(p) = Y_o$. Then $w(X_o) = A$, $w(Y_o) = B$. Since $\sigma[A,B] = [\sigma(A), \sigma(B)]$, we have $w([X,Y]_p) = [A,B]$. Note that $\frac{1}{2}[w,w](X_o, Y_o) = [w(X_o), w(Y_o)] = [A,B]$.

Now, since $H X_o = H Y_o = 0$, one has $\Omega(X_o, Y_o) = 0$.

On the other hand,

$$\begin{aligned} dw(X_o, Y_o) &= X_o w(Y) - Y_o w(X) - w([X,Y]_p) \\ &= X_o B - Y_o A - w([X,Y]_p) \\ &= -[A,B] \quad \text{(as } B \text{ and } A \text{ are constant)} \\ &= -\frac{1}{2}[w,w](X_o, Y_o) \end{aligned}$$

<u>Case 3</u>. X_o vertical, Y_o horizontal at p. Let A and X be as in case 2. It is easy to see that there exists a neighbourhood U of $\pi(p)$ and a horizontal vector field Y on $\pi^{-1}(U)$ invariant under the action of G and such that $Y(p_o) = Y_o$. We then have $[X,Y] = 0$ on $\pi^{-1}(U)$. In fact the vector field associated to the 1-parameter group $\varphi_t = R_{\exp(tA)}$ is X and $(\varphi_t)_*(Y) = Y$. Hence the Lie derivative $\theta_X(Y)$ is zero which means that $[X,Y] = 0$ (see Theorem 6.5).

Now, since $H(X_o) = 0$, we have $\Omega(X_o, Y_o) = 0$.

On the other hand

$$dw + \frac{1}{2}[w,w] = X_o w(Y) - Y_o w(X) - w([X,Y]_p) + [w(X_o), w(Y_o)]$$
$$= 0$$

since $w(Y) = 0$ (Y being horizontal), $w(X) = A$, $[X,Y]_p = 0$ and $w(Y_o) = 0$. This completes the proof of the structure equation.

Remark 12.4. If X and Y are horizontal vector fields, we have

$$\Omega(X, Y) = (dw + \frac{1}{2}[w,w])(X,Y)$$
$$= Xw(Y) - Yw(X) - w[X,Y] + [w(X), w(Y)]$$
$$= -w[X,Y] \quad \text{since } w(X) = w(Y) = 0.$$

Thus the horizontal bundle $\mathcal{H}$ is integrable (as a subbundle of $T(P)$) if and only the curvature form is zero. A connection with curvature form is zero said to be flat. The curvature form measures the obstruction for the horizontal bundle to be integrable.

Covariant differentiation of forms with values in associated vector bundles.

Let $\rho : G \to Aut(F)$ be a representation . Let α be a p-form on P with values F, which is a basic form of type ρ . We define a basic (p+1) form, $d_w\alpha$, of type ρ by $d_w\alpha = d\alpha \circ H$. Thus $d_w\alpha(X_1,\dots,X_{p+1}) = d\alpha(HX_1,\dots,HX_{p+1})$. The form d_w is the covariant differential of α with respect to the connection.

We shall now give an explicit formula for the covariant differential. For this we observe that $\rho : G \to Aut\ F$ defines a linear

map $\rho' : \mathfrak{g} \to End\ F$, where $End\ F$ is the vector space of endomorphisms of F, as follows. If $A \in \mathfrak{g}$, $v \in F$, we define

$$\rho'(A)v = \lim_{t \to 0} \frac{\rho(\exp tA)v - v}{t} \in V .$$

The map ρ' may be viewed as a bilinear map, still denoted by ρ', $\rho' : \mathfrak{g} \times F \to F$.

Proposition 12.5. If α is a basic p-form of type ρ , we have

$$dw\alpha = d\alpha + w \wedge_{\rho'} \alpha ,$$

where the exterior product is formed using the bilinear map $\rho' : \mathfrak{g} \times F \to F$, noting that w is a form with values in $\mathfrak{g}$ and α a form with values in F.

We shall not prove this proposition here. It may be proved by a method similar to that used in the proof of the structure equation.

Bianchi and Ricci identities.

Proposition (Bianchi identity) 12.6. Let Ω be the curvature form of a connection w . Then $d_w\Omega = 0$ (The covariant differential of the curvature form is zero).

Proof. We use the above formula for the covariant differential. In this case ρ is the adjoint representation and the map $\rho' : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ is given by $(A,B) \to [A,B]$. We have

$$dw\Omega = d\Omega + [w, \Omega]$$
$$= d(dw + \frac{1}{2}[w,w]) + [w,dw + \frac{1}{2}[w,w]]$$
$$= \frac{1}{2}[dw,w] - \frac{1}{2}[w, dw] + [w,dw] + \frac{1}{2}[w,[w,w]]$$
$$= \frac{1}{2}[w,[w,w]]$$
$$= 0.$$

We have used relation i), ii) on p.50 and the relation $d[\alpha,\beta] = [d\alpha,\beta] + (-1)^p [\alpha, d\beta]$, for a p form α.

Proposition (Ricci identity) 12.7. If α is a basic p-form of type ρ, we have

$$d_w^2\alpha = \Omega \wedge_{\rho'} \alpha .$$

We omit the proof.

13. CHERN-WEIL THEORY

Let G be a Lie group with Lie algebra $\mathfrak{g}$. Let Q be a homogeneous polynomial of degree k on , which is invariant under the adjoint action of G on $\mathfrak{g}$. (For example if $G = GL(m, \mathbb{C})$, $\mathfrak{g} = \mathfrak{gl}(n, \mathbb{C})$, we take, for $A \in \mathfrak{gl}(n, \mathbb{C})$, $Q(A)$ = trace A, $Q(A) = \det A$ and more generally $Q(A) = k^{th}$ elementary symmetric function of the eigen-values of A).

Let G be a principal G-bundle with base M. Let w be a connection on P and Ω its curvature. We shall now define a closed form of degree $2k$ on M, by 'substituting' the curvature form in the polynomial Q. The form Ω being a 2-form on P with values in $\mathfrak{g}$, it defines a 2k-form with values in $\overset{k}{\otimes}\mathfrak{g} = \mathfrak{g} \otimes \cdots \otimes \mathfrak{g}$ (k times). Composing this with Q, which can be considered as a linear map $\overset{k}{\otimes}\mathfrak{g} \to \mathbb{R}$, we get a basic 2k-form α on P, which is invariant under G, since Ω is basic and Q is invariant under the adjoint action. Using Bianchi's identity one can show that α is a closed form. Now α can be considered as an (ordinary) form of degree $2k$ on M, still denoted by α; the form α is closed and

hence defines an element of the de Rham Cohomology group $H^{2k}_{DR}(M)$.

Theorem 13.1. The class of α in $H^{2k}_{DR}(M)$ depends only on the polynomial Q and the bundle P and not on the connection w.

Proof. (See Reference [3], page 226, Remark).
Since homotopic maps induce the same map on de Rham cohomology groups (Theorem 4.2), it is enough to show that, given two connection forms w_1 and w_2 on P, they are inverse images of the same connection form γ on a principal G-bundle P' over M', by a bundle homomorphism $P \to P'$ whose projections onto M (maps from M to M') are homotopic. (Note that the curvature form of the inverse image of a connection is the inverse image of the curvature form and that substitution of the curvature form in Q is 'functorial').

We take for P' the principal G-bundle $P' = P \times \mathbb{R}$ on $M \times \mathbb{R}$ and for γ the connection form $\gamma_{(p,t)} = tq^*(w_2) + (1-t)q^*w_1$, $p \in P$, $t \in \mathbb{R}$ and $q : P \times \mathbb{R} \to P$ the projection onto P. It is clear that the inverse images by the inclusions $p \longmapsto (p,0)$ and $p \mapsto (p,1)$ (which are G-morphisms) of P in $P \times \mathbb{R}$ are w_1 and w_2 respectively and the projections of these inclusions onto the base (namely the maps $M \to M \times \mathbb{R}$, $x \to (x,0)$ and $x \to (x,1)$)are clearly homotopic.

Remark 13.2. The element of $H^p_{DR}(M)$ defined by Q is called the **characteristic class** of P corresponding to Q.

Appendix

1. Theorem (smooth partition of unity)

Let M be a smooth (paracompact) manifold. Let $\{U_i\}_{i \in I}$ be an open covering of M. Then there exist smooth functions $\varphi_i : M \to \mathbb{R}$ such that

1) $\varphi_i(x) \geq 0$ for $x \in M$

2) The support of the function φ_i, Supp φ_i, is contained in U.

3) The family of (closed) sets $\{\text{Supp}\ \varphi_i\}$ form a locally finite family (i.e., given a point $x \in M$, there exist a neighbourhood U of x in M and a finite subset J of I such that $U \cap U_i = \emptyset$ for $i \notin J$.

4) $\sum_{i \in I} \varphi_i(x) = 1$ for $x \in M$. (Note that by 3), for $x \in M$, only a finite number of $\varphi_i(x)$ is different from zero).

2. Poincaré Lemma.

Let M be a manifold. We say that M is contractible if the identity map of M is homotopic to a constant map of M into M. (This means that there exists a smooth map $\Phi : \mathbb{R} \times M \to M$ such that $\Phi(1,x) = x$, $x \in M$ and $\Phi(0,x) = x_o$, $x \in M$ where x_o is a fixed point of M.

Proposition. If M is contractible then $H^p_{DR}(M) = 0$ for $p \geq 1$.

Proof. Note that the identity map induces the identity map of $H^p_{DR}(M)$ and that a constant map induces the zero map $\mathcal{E}^p(M) \to \mathcal{E}^p(M)$ for $p \geq 1$ and hence on $H^p_{DR}(M)$ for $p \geq 1$. Now the proposition follows from the

Theorem 4.2.

Remark. Thus every closed p-form $(p \geq 1)$ is a coboundary, if M is contractible.

Corollary. (Poincaré Lemma.) $H^p_{DR}(\mathbb{R}^n) = 0$ for $p \geq 1$.

Proof. $\mathbb{R}^n$ is contractible. Take $\Phi(t,\vec{x}) = t\vec{x}$, for $\vec{x} \in \mathbb{R}^n$.

. A lemma on integral manifolds

Let F be an integral subbundle of $T(M)$. If Y and Z are integral submanifolds of F and if there exists a point $x \in Y \cap Z$, then there exists a neighbourhood V of x in M such that $V \cap Y = V \cap Z$.

This lemma has been implicitly used in the definition of foliation (p. 38).

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