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INFINITE-DIMENSIONAL MANIFOLDS

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Notes on Infinite-dimensional Manifolds.

0. Introduction

The theory of the differential calculus on $\mathbb{R}^n$ is essentially a local theory. Thus it is possible to carry out the procedures of analysis on spaces which "look" like $\mathbb{R}^n$ locally, in some suitable way. These spaces, called <u>manifolds</u>, are locally homeomorphic to some Euclidean space in such a way as to permit a well-defined notion of differentiability of functions on such spaces and indeed of mappings between such spaces. By following these procedures, the methods of analysis can be applied to a large class of curved, non-linear spaces: surfaces, hypersurfaces of Euclidean space, projective spaces and so on.

The differential calculus, however, admits of another generalisation; we have a theory of differentiable functions and mappings on infinite-dimensional Banach spaces, or even Fréchet spaces. Thus we are led to consider infinite-dimensional manifolds, that is spaces locally homeomorphic to Banach spaces.

It is these infinite-dimensional manifolds which seem in many ways the appropriate context in which to place some of the non-linear problems of analysis + differential geometry.

The usefulness of Banach + Hilbert spaces + the methods of linear functional analysis for problems in linear partial differential equations has been apparent for many years. For example, the theory of coercive quadratic forms in Hilbert space can be applied, via Gårding's Inequality, to certain Hilbert spaces of functions to provide existence results for

solutions to elliptic partial differential equations. I refer the reader to Eells, "Elliptic Operators on Manifolds", (Complex Analysis + Its Applications, Vol I, ICTP Trieste 1976, 95–152) for a proof of Hodge's Theorem in this vein.

Again, the theory of unbounded, densely defined operators on Hilbert spaces has been applied by Hormander to solve the $\bar\partial$-problem on pseudo-convex domains. (see L. Hormander, "Introduction to Complex Analysis in Several Variables", North-Holland, Amsterdam 1973.)

However, there are many problems in analysis + geometry in which the objects under study are maps of manifolds, or more generally sections of fibre bundles, rather than functions or maps into linear spaces. Such spaces of maps have, therefore, no global linear structure to exploit. But on the other hand they often inherit a locally linear ie manifold structure, from that of the target manifold which contains the images of the maps.

Thus many interesting spaces of maps admit an infinite-dimensional manifold structure to which the methods of topology, analysis + differential geometry may be applied. Such an approach is often useful as properties of a manifold are frequently reflected by properties of classes of maps into that manifold.

Amongst the most well-known uses of these ideas is the application of Morse Theory on Banach manifolds to the calculus of variations. Recall that Morse Thy relates the topology of a manifold to the number + structure of critical points of functions on that manifold. The calculus of variations can often be put in this context: we regard the action integral as a function on some suitable manifold of maps, where the space of variations to a given map form the tangent space to that map and the Euler-Lagrange operator is seen as some kind of gradient vector field. Then the solutions to the calculus of variations problem are precisely the critical points of the action integral as a function on the manifold.

Palais + Smale (Palais Topology 2 (1963) 299-340, Smale Ann. Math 80 (1964) 382-396, Palais + Smale Bull. Amer. Math. Soc 70 (1964) 165-171) showed that a Morse theory can be given for functions on a Hilbert manifold which satisfy a certain compactness condition on the derivative; the famous condition "C".

Specifically, let $f: M \to \mathbb{R}$ be a C^∞ function on a smooth complete Hilbert manifold M. A critical point $x \in M$ of f is a point where the derivative of f vanishes. A critical point is non-degenerate if the Hessian of f at x is a non-degenerate bilinear form in which case the index of the critical point is the dimension of the negative definite subspace.

f is said to satisfy condition "C" if: for any set $S \subset M$ on which f is bounded and $|df|$ is not bounded away from zero, the closure of S contains a critical point of f.

Then we have the following theorem of Palais + Smale (opera cit.)

Theorem A If f satisfies condition C then

i) If f is bounded below, it attains its minimum on every component of M.

ii) If $a < b$ and $f^{-1}([a,b])$ contains no critical points then $f^{-1}(-\infty, b]$ is diffeomorphic to $f^{-1}(-\infty, a]$.

iii) If $a < b$ are not critical points and $f^{-1}(a,b)$ contains only non-degenerate critical points then $f^{-1}(-\infty, b]$ is diffeomorphic to $f^{-1}(-\infty, a]$ with handles attached corresponding to the critical points in $f^{-1}(a,b)$ where the dimensions of the handles correspond to the indices of the critical points.

[A submanifold N of M with a handle attached is the union of N with the homeomorphic image of the product of two closed discs $D^j \times D^k$ where $S^{j-1} \times D^k$ is mapped diffeomorphically onto $\partial N \cap \mathrm{Im}(D^j \times D^k)$ and $(\mathrm{Int}\, D^j) \times D^k$ is mapped diffeomorphically into $M \setminus N$].

This theorem has been generalised to Banach manifolds by Uhlenbeck (J. Funct. Anal. 10 (1972) 430-445) and Tromba (J. Diff. Geom 12 (1977) 47-86) where of course the notion of non-degeneracy must be redefined to make sense.

Theorem A. can be used to relate the relative homology on M to the number of critical points of a non-degenerate function (see Palais op. cit.) and in the case of functions with degenerate critical points, the Lusternick-Schnirlman category can be used to provide a link between the topology of M and the number of critical points (see Palais Topology 5 (1966) 115-132)

We now give a concrete application of this theory due to Uhlenbeck.

Let M, N be compact C^∞ Riemannian manifolds with metrics g, h respectively. Say a C^2 map $\varphi : M \to N$ is harmonic if it is a critical point of the energy functional $E(\varphi) = \frac{1}{2} \int_M |d\varphi|^2 *1$, here $|d\varphi|^2 = \operatorname{Trace} \varphi^* h$.

The corresponding Euler-Lagrange operator, the tension field, is given by $\tau_\varphi = \operatorname{div}(d\varphi)$.

The harmonic maps include many interesting examples:

i) For $\dim M = 1$ they are just the geodesics of (N, h).
ii) For $N = \mathbb{R}$ they are the harmonic functions.
iii) For $\dim M = 2$ they include the minimal surfaces (with parametric representation). For more information we refer the reader to Eells-Lemaire (Bull. London Math. Soc. 10 (1978) 1-68).

A fundamental problem is to show whether harmonic maps exist in every homotopy class. In the case that N has non-positive sectional curvatures the answer is yes, as was shown by Eells + Sampson (Amer. J. Math. 86 (1964), 109-160), and Uhlenbeck (Trans. Amer. Math. Soc 267 (1981) 569-583) gave a Morse Theory proof as follows:

$E(\varphi)$ does not satisfy condition C on any suitable manifold of maps but a perturbed integral

$$E_\varepsilon(\varphi) = \int_M |d\varphi|^2 + \varepsilon \int_M |d\varphi|^{2m}$$

does on a certain Banach manifold $L_1^{2m}(M,N) \subset C^0(M,N)$ of functions $M \to N$ with $2m$-th integrable derivatives, $m = \dim M$.

Thus Theorem A can be applied to E_ε to get existence of critical points of E_ε. If the sectional curvatures of N are non-positive, it can be shown that the E_ε-critical points are suitably non-degenerate and moreover converge to harmonic

maps as $\varepsilon \to 0$ so that the conclusions of Theorem A apply to the harmonic maps.

Another useful tool for applications is the implicit function theorem. For instance, Eells + Lemaire (in Geometry + Analysis, Springer. Berlin 1982) applied it to the tension field, thought of as a map from the cartesian product of spaces of metrics on M and N and a suitable manifold of maps from M to N, to conclude that under certain non-degeneracy conditions, harmonic maps depend smoothly on the choice of metrics on M and N.

Of course, infinite-dimensional manifolds merit study in their own right. Much of the theory carries over from the finite-dimensional case: the inverse function theorem, Frobenius' theorem, Stokes theorem, differential forms + de Rham cohomology (for most of this, see Lang, "Introduction to Differentiable Manifolds", Interscience, New York 1962). One must, however, exercise a little care when dealing with submersion, immersions, submanifolds + the like as subspaces of Banach spaces do not necessarily have complementary subspaces.

However there are some aspects of the theory which are peculiar to infinite dimensions. Some Banach manifolds do not admit differentiable partitions of unity, $C^0[0,1]$ for example, + thus one cannot guarantee a sufficiently large supply of differentiable functions. Indeed it is an open problem whether every Banach manifold must admit a non-trivial continuously differentiable function. Moreover, in finite-dimensions, it is partitions of unity that one uses to construct the various structures of differential geometry such as connections, metrics + so on. Thus it is a more delicate matter to do differential geometry on infinite-dimensional manifolds.

The topology, also, of Hilbert manifolds has some features peculiar to the infinite-dimensional case.

For example, Eells + Elworthy (Ann. Maths. 91 (1970) 465-485) proved that a smooth separable Hilbert manifold is diffeomorphic to an open set of Hilbert space. This theorem has a remarkable corollary due to Kuiper + Berghelea to the effect that two separable Hilbert manifolds that are homotopy equivalent are in fact diffeomorphic.

The proof is in three parts (see Kuiper + Burghelea Ann. Maths 90 (1969) 379-417)

1) One shows by an application of the Morse theory discussed above that for U an open set of separable Hilbert space H, U is diffeomorphic to $U \times H$. The hard part is showing that a suitable function exists on U with non-degenerate critical points + satisfying condition C.

2) Then a variant of a theorem of Mazur shows that if $f_0 : U \to V$ is a homotopy equivalence of open sets of H, then f_0 induces a diffeomorphism between $U \times H$ and $V \times H$ + by 1) the theorem is true for open sets in Hilbert space.

3) Applying the theorem of Eells + Elworthy gives the general case.

The theorem has also been proved + extended to some Banach manifolds by Eells + Elworthy (see "On Fredholm Manifolds" Actes. Intern. Congrès Math. 1970 Tome 2 215-220).

In this introduction, I have barely scratched the surface of infinite-dimensional manifold theory and its applications. For much more information I can do no better than to refer the reader

to Eells' monograph "A Setting for Global Analysis" (Bull. Amer. Math. Soc 72 (1966) 751-807).

1. Calculus on Banach Spaces

The principle reference in this + the next section is S. Lang's book "Introduction to Differentiable Manifolds" (Interscience, New York, 1962), hereafter referred to as Lang. The material on differentiation in Banach spaces can also be found in most advanced calculus texts eg. H. Cartan "Calcul différentiel", Hermann Paris 1967.

Let E be a locally convex topological vector space over the reals. Recall that if the topology on E can be induced by a norm (and hence by an equivalence class of norms) and if with respect to such a norm, E is a complete metric space, then E is said to be Banachable and E together with such a norm is called a Banach space. Again, if the topology arises from an inner product with respect to which, E is a complete metric space, then E is said to be Hilbertable and E together with such an inner product is said to be a Hilbert space.

Example Let X be a compact topological space and $C^0(X)$, the space of continuous functions on X with the compact-open topology. Then $C^0(X)$ is Banachable with a norm given by $\|f\| = \sup_{x\in X} |f(x)|$, $f \in C^0(X)$.

Any normed vector space may be densely embedded in a Banach space by taking the completion. Indeed many Banach-spaces are defined in this way.

Example Consider $C^0(I)$, the vector space of continuous functions on the unit interval with norm given by $\|f\|_{L_2} = \left(\int_0^1 |f(x)|^2 dx\right)^{1/2}$. This normed vector space is not complete, but we can take a completion to obtain a Banach space

which may be identified with the measureable, square integrable (in the sense of Lebesgue) functions on the interval; $L^2(I)$.

Definition 1.1 Let E, F be Banach spaces, U an open subset of E and $f: U \to F$ a continuous function. f is said to be differentiable at a point $x_0 \in U$ iff there exists a bounded linear map $T: E \to F$ s.t.

$$\frac{\|f(x_0+v) - f(x_0) - Tv\|_F}{\|v\|_E} \to 0 \quad \text{as } v \to 0 \text{ in } E.$$

This map T is called the derivative of f at x_0, denoted $Df(x_0)$, and is easily seen to be unique if it exists.

If f is differentiable at each point of U, we say f is differentiable on U.

(Easy) exercise: The differentiability of f at a point is independant of the choice of equivalent norms on E and F. So is the derivative.

If f is differentiable on U, we have a map $Df: U \to L(E,F)$. $L(E,F)$ - the space of bounded linear maps from E to F - is of course another Banach space and so we may ask whether Df is continuous or differentiable. If it is we obtain a map

$D^2 f = D(Df): U \to L(E, L(E,F))$ which last space we identify with $L^2(E,F)$ - the space of continuous bilinear maps from E to F. If f is sufficiently differentiable we can thus iterate the process to obtain higher derivatives $D^r f: U \to L^r(E;F)$.

Definition 1.2 We say f is C^0 iff f is continuous

C^1 iff df exists + is continuous $: U \to L(E,F)$

C^r iff df is C^{r-1}

and C^∞ if f is C^r $\forall r$.

Definition 1.3 A C^r map $f : U \to F$ is said to be a C^r diffeomorphism iff it is a bijection onto some open set $V \subset F$ and there exists a C^r inverse $V \to U$.

Most of the fundamental theorems of the differential calculus go through in infinite dimensions. We list the most important ones below. Proofs can be found in Lang or Cartan (op.cit).

Theorem 1.4 The composition of C^r maps is a C^r map, $0 \le r \le \infty$ and for $r \ge 1$ and C^r maps $f : U \to V \subset F$, $g : V \to G$

$$D(g \circ f)(x) = Dg(f(x)) \circ Df(x) \quad x \in U.$$

Theorem 1.5 (Taylor's Theorem) Let E, F be Banach spaces and let U be open in E. Let $f : U \to F$ be a C^r map $r \ge 1$ and $x, y \in U$ s.t the segment $\{x+ty : 0 \le t \le 1\} \subset U$.

Then $D^r f(x+ty)(y)^{(r)} = D^r f(x+ty)(\underbrace{y, \dots, y}_{r \text{ times}})$ is continuous in t and we have

$$f(x+y) = f(x) + \frac{Df(x)}{1!}y + \dots + \frac{D^{r-1}f(x)(y)^{(r-1)}}{(r-1)!} + \int_0^1 \frac{(1-t)^{r-1}}{(r-1)!} D^r f(x+ty)(y)^{(r)}\, dt.$$

Here the integral is Banach space valued.

Example Let $f : U \to F$ be a C^1 diffeomorphism onto an open set $V \subset F$ then for $x \in U$ $Df(x)$ is an isomorphism $E \to F$, since

$f^{-1} \circ f = id_E$, $D(id_E) = id_E$ and so $(Df(x))^{-1} = Df^{-1}(f(x))$ by Theorem 1.4.

A partial converse is contained in the following theorem:

Theorem 1.6 (Inverse function theorem). Let E, F be Banach spaces, U an open subset of E and $f : U \to F$ a C^r map, $r \ge 1$. Suppose that for some $x_0 \in U$ $Df(x_0)$ is a linear isomorphism $E \to F$. Then there are neighbourhoods $U_0 \subset U$ and $V_0 \subset F$ of x_0 and $f(x_0)$ respectively s.t $f|U_0$ is a C^r diffeomorphism onto V_0.

We now turn to the question of whether we have a sufficiently large supply of C^r functions. The following exposition relies on Bonic + Frampton "Smooth functions on Banach manifolds" J. Math. Mech 15 (1966) 877–898, to which the reader is referred for more information on the subject.

First we recall some definitions from topology:

Definition 1.7 Let X be a Hausdorff topological space.

X is said to be separable iff X has a dense, countable, subset. A family of subsets of X, $\{A_\alpha\}_{\alpha \in \Delta}$, is said to be locally finite if each point $x \in X$ has a neighbourhood U_x s.t $U_x \cap A_\alpha = \emptyset$ for all but finitely many α.

X is said to be paracompact if every open cover of X admits a locally finite refinement.

Definition 1.8 A Banach space E is said to be C^r-smooth $0 \le r \le \infty$ if there exists a non-trivial real-valued C^r function on E with bounded support. [Recall that the support of a function f is the closure of the set $\{x : f(x) \ne 0\}$. It is denoted $supp(f)$.]

<u>Definition 1.9</u> A topological space, X, is said to admit partitions of unity if for any open cover $\mathcal{U} = \{U_\alpha\}_{\alpha\in\mathcal{Q}}$ of X, there is a family of functions $\{\phi_\beta\}_{\beta\in\mathcal{B}}$ s.t i) each ϕ_β is a continuous function $X \to \mathbb{R}(\geq 0)$

ii) the family $\{\operatorname{supp}(\phi_\beta)\}_{\beta\in\mathcal{B}}$ is locally finite and each $\operatorname{supp}(\phi_\beta) \subset U_{\alpha(\beta)}$ some $\alpha(\beta)$.

iii) $\sum_{\beta\in\mathcal{B}} \phi_\beta(x) = 1 \quad \forall x \in X$. This sum is well-defined since the supports are locally finite. $\{\phi_\beta\}$ is called the partition of unity subordinate to the cover.

If X is an open set of a Banach space (or more generally a Banach manifold, see section 2) then X is said to admit C^r partitions of unity if each ϕ_β is C^r.

The usefulness of partitions of unity is made apparent in the following theorem of Bonic + Frampton (loc.cit)

<u>Theorem 1.10</u> If E is a separable Banach space then the following are equivalent: i) E is C^r smooth

ii) Any open subset of E admits C^r partitions of unity.

iii) for any continuous map $\phi: U \to F$, U an open subset of E and F any Banach space, and any $\varepsilon > 0$, there exists a C^r map $f: U \to F$ s.t $\|f(x) - \phi(x)\| < \varepsilon$, $\forall x \in U$.

Thus we see that once we have C^r partitions of unity, the C^r functions are uniformly dense in the continuous ones + thus a large supply of C^r functions is guaranteed.

<u>Partial proof</u>: We shall prove ii) $\Rightarrow$ iii) as it is typical of many of the proofs involving partitions of unity in manifold theory.

Let $\phi: U \to F$ be continuous then for $x \in U$ by continuity there is a neighbourhood $U_x \subset U$ of x s.t $\|\phi(y) - \phi(x)\|_F < \varepsilon$ for $y \in U_x$. The sets $\{U_x\}_{x\in U}$ cover U and so by (ii) we can take a C^r partition of unity $\{\psi_\alpha\}_{\alpha\in A}$ subordinate to this cover.

Suppose that $\operatorname{supp}\psi_\alpha \subset U_{x_\alpha}$ and define $f_\alpha: U \to F$ by $f_\alpha(y) = \psi_\alpha(y)\,\phi(x_\alpha)$. $\{\operatorname{supp} f_\alpha\}_{\alpha\in A}$ is clearly locally finite and so we can define $f = \sum_\alpha f_\alpha$ which is well-defined + C^r.

Now for $x \in U$ $\quad \|f(x) - \phi(x)\|_F = \|f(x) - \sum_\alpha \psi_\alpha(x)\phi(x)\|_F$

$$= \Big\|\sum_{\alpha:\, x\in U_{x_\alpha}} \psi_\alpha(x)\,(\phi(x_\alpha) - \phi(x))\Big\|_F \leq \sum_\alpha \phi_\alpha(x)\,\varepsilon = \varepsilon .$$

$\blacksquare$

We must now turn our attention to which Banach spaces admit functions of bounded support. The main technique for building such functions is to use the norm of the space:

<u>Lemma 1.11</u> (Bonic + Frampton) Suppose E is a Banach space with a norm α which is C^r on $E \setminus \{0\}$. Then E is C^r smooth.

<u>Proof</u> Choose a C^∞ function $\phi: \mathbb{R} \to \mathbb{R}$ s.t $\phi(x) = 1 \quad |x| \leq 1/2$
$\phi(x) = 0 \quad |x| \geq 1$

Then $\phi \circ \alpha$ is C^r on $E \setminus \{0\}$ by theorem 1.4 and $\operatorname{supp}(\phi\circ\alpha)$ is clearly contained in the unit α-ball. Further $\phi\circ\alpha$ is identically 1 on a neighbourhood of zero + so can be extended to all of E.

<u>Prop^n 1.12</u> If H is a hilbert space with inner product $\langle\, ,\rangle$ then $\|x\| = \langle x, x\rangle^{1/2}$ is C^∞ on $H \setminus \{0\}$. Thus H is C^∞-smooth.

<u>Corollary 1.13</u> All finite-dimensional spaces are C^∞-smooth.

<u>Proof</u> It suffices to show that $x \mapsto \langle x, x\rangle$ is C^∞ on H.

Writing $f(x)$ for $\langle x,x\rangle$, for $x,h \in H$ we have

$$f(x+h) - f(x) = \langle x+h, x+h\rangle - \langle x,x\rangle = 2\langle x,h\rangle + \langle h,h\rangle \quad -①$$

Now by Cauchy-Schwarz inequality we have $|\langle x,h\rangle| \le \|x\|\|h\|$ so that $h \mapsto 2\langle x,h\rangle$ is a bounded linear map $H \to \mathbb{R}$. Thus from ① we have $\dfrac{|f(x+h) - f(x) - 2\langle x,h\rangle|}{\|h\|} = \|h\| \to 0$ as $h \to 0$.

so $Df(x)\,h = 2\langle x,h\rangle$. Now $Df : H \to L(H,\mathbb{R})$ is clearly a bounded linear map and so

$$Df(x+v) - Df(x) = Df(v) \quad \text{ie} \quad D(Df)(x) = Df \in L(H, L(H,\mathbb{R}))$$

so $D^2 f$ is constant + so all subsequent derivatives are easily seen to vanish. Thus f is C^∞. ∎

We may also conclude from lemma 1.11 that all Banach spaces are C^0 smooth since the norm is always continuous (lipschitz even).

However Bonic + Frampton (op. cit) have shown that the space $C^0(I)$ of continuous functions on the interval is not even C^1 smooth.

II. Differentiable Banach Manifolds

Definition 2.1 Let M be a Hausdorff space and r an integer or $r = \infty$.

A C^r atlas for M is a collection $\{(U_\alpha, \phi_\alpha)\}_{\alpha\in A}$ of pairs of open sets + continuous maps so that i) $\{U_\alpha\}_\alpha$ is an open cover of M

ii) ϕ_α is a homeomorphism of U_α onto an open set of some Banach space E_α, each α.

iii) if $U_\alpha \cap U_\beta \neq \emptyset$ then

$$\phi_\alpha \circ \phi_\beta^{-1} : \phi_\beta(U_\alpha \cap U_\beta) \to \phi_\alpha(U_\alpha \cap U_\beta) \quad \text{is a } C^r \text{ diffeomorphism.}$$

Each pair (U_α, ϕ_α) is called a chart of the atlas and the maps $\phi_\alpha \circ \phi_\beta^{-1}$ are called transition maps.

Two C^r atlases $\{(U_\alpha, \phi_\alpha)\}_\alpha$, $\{(V_\beta, \theta_\beta)\}_\beta$ are C^r equivalent if their union is also a C^r atlas: ie for $U_\alpha \cap V_\beta \neq \emptyset$,

$$\phi_\alpha \circ \theta_\beta^{-1} : \theta_\beta(U_\alpha \cap V_\beta) \to \phi_\alpha(U_\alpha \cap V_\beta) \quad \text{is a } C^r \text{ diffeomorphism.}$$

A C^r differentiable structure is then a C^r equivalence class of C^r atlases.

A C^r manifold is a Hausdorff space together with a C^r differentiable structure on it.

Remarks i) It is clear that we need only give a single atlas to determine a differentiable structure.

ii) If $U_\alpha \cap U_\beta \neq \emptyset$, then E_α, E_β are naturally isomorphic by $D(\phi_\alpha \circ \phi_\beta^{-1})(x)$ for $x \in U_\alpha \cap U_\beta$. Thus, by a simple connectedness argument, we see that on components all E_α are mutually isomorphic, to E say, and thus on that component an equivalent atlas $\{V_\beta, \theta_\beta\}_{\beta\in\mathcal{B}}$ may be given s.t. $\theta_\beta : V_\beta \to E$. If such an atlas exists on the whole manifold (i.e. the same E will do for all components) then we say that the manifold is <u>modelled</u> on E.

<u>Examples</u> i) If $U \subset E$ is an open subset of a Banach space then it has a C^∞ structure with atlas provided by the single chart (U, id).

ii) If M, N are C^r manifolds then so is $M \times N$ with atlas $\{(U_\alpha \times V_\beta, (\phi_\alpha, \theta_\beta))\}_{\alpha,\beta}$ where $\{(U_\alpha, \phi_\alpha)\}_\alpha$ and $\{(V_\beta, \theta_\beta)\}$ are atlases for M and N respectively.

<u>Definition 2.2</u> Let M, N be C^r manifolds and $f : M \to N$ a continuous map. f is said to be a <u>C^r map</u> if for each $x \in M$ there are C^r charts $\phi : U \to E$ of M, $\theta : V \to F$ of N with $x \in U$, $f(U) \subset V$ s.t.

$$\theta \circ f \circ \phi^{-1} : \phi(U) \to F \quad \text{is a } C^r \text{ map of Banach spaces.}$$

Again the question arises whether we have a good supply of C^r functions on manifolds + so we briefly return to partitions of unity.

From general topology it is known that if M is a paracompact C^0 manifold, M admits C^0 partitions of unity, see Kelley ("General Topology" Van Nostrand-Reinhold, Princeton New Jersey 1955)

Indeed, if M is at least C^1, M admits locally Lipschitz partitions

of unity (see Palais, Topology $\underline{5}$ (1966) 115-132)

Further Bonic + Frampton have shown that if M is a C^∞ separable manifold modelled on a C^r-smooth Banach space, then M admits C^r partitions of unity (op. cit).

Thus, we see from the results of section 1, that all smooth separable manifolds modelled on Hilbert space admit smooth partitions of unity, which can be used to build Riemannian metrics, connections and so on. It is the abscence of such partitions of unity on certain Banach manifolds which give their study a different flavour.

<u>Tangent spaces</u>.

Let M be a C^r manifold, $r \geq 1$.

To each $x \in M$ we associate a Banach space called the <u>tangent space</u> at x, denoted T_xM, as follows.

Let $\{(U_\alpha, \phi_\alpha)\}_{\alpha\in\mathcal{A}}$ be an atlas for M with $\phi_\alpha : U_\alpha \to E_\alpha$.

Consider 3-tuples (x, U_α, v) with $x \in U_\alpha$, $v \in E_\alpha$. Two such 3-tuples (x, U_α, v) and (x, V_β, w) are said to be equivalent iff

$v = D(\phi_\alpha \circ \phi_\beta^{-1})(\phi_\beta(x))\, w$. It is easy to see, using the chain rule, that we thus obtain an equivalence relationship on these 3-tuples.

Define T_xM to be the set of equivalence classes $\{[(x, U_\alpha, v)]\}$ and put a Banachable topological vector space structure on T_xM by demanding that the map $[(x, U_\alpha, v)] \to v$ of T_xM onto E_α be a bounded linear isomorphism. That this structure is well-defined follows from the fact that the various $D(\phi_\alpha \circ \phi_\beta^{-1})(\phi_\beta(x))$ are bounded linear isomorphisms $E_\beta \to E_\alpha$.

A differentiable map $f: M \to N$ can be seen to induce a map of tangent spaces: $df(x): T_x M \to T_{f(x)} N$ – the <u>derivative at x</u>, as follows: choose charts (U,ϕ), (V,θ) of M and N respectively with $x \in U$, $f(U) \subset V$ and set

$$df(x)([x, U, v]) = [f(x), V, D(\theta \circ f \circ \phi^{-1})(\phi(x))\, v]$$

It is an easy exercise to check that this map is well-defined and continuous for C^1 f.

<u>Example</u> For E, F Banach spaces, with U open in E and $f: U \to F$ a C^1 map, we see that $T_x U = E$ and $df(x) = Df(x): E \to F$.

The disjoint union of all the tangent spaces to M can be given a natural manifold structure, indeed a vector bundle structure. This will be shown in a more general setting in the following section:

<u>Vector-Bundles</u> (see Lang)

<u>Definition 2.3</u> Let E, X be C^r manifolds and $\pi: E \to X$ a C^r surjection. Suppose that i) for $x \in X$, $\pi^{-1}(x) = E_x$ – the <u>fibre at x</u> is a Banachable space and there is a cover $\{U_\alpha\}_{\alpha \in A}$ of X by open sets to each of which is associated a Banach space E_α and a map τ_α

$\tau_\alpha: \pi^{-1}(U_\alpha) \to U_\alpha \times E_\alpha$ such that

ii) τ_α is a C^r diffeomorphism which commutes with π in that the following diagram commutes

$$\begin{tikzcd} \pi^{-1}(U_\alpha) \arrow[r, "\tau_\alpha"] \arrow[dr, "\pi"'] & U_\alpha \times E_\alpha \arrow[d, "\mathrm{proj}_1"] \\ & U_\alpha \end{tikzcd}$$

and further induces a continuous linear isomorphism $\tau_{\alpha,x}: \pi^{-1}(x) \to E_\alpha$ on each fibre $\pi^{-1}(x)$, $x \in U_\alpha$.

iii) For $U_\alpha \cap U_\beta \neq \emptyset$ the map of $U_\alpha \cap U_\beta$ into $GL(E_\alpha, E_\beta)$ given by

$x \mapsto \tau_{\beta,x} \circ \tau_{\alpha,x}$ is C^r – these maps are called transition maps.

Then $\{(U_\alpha, \tau_\alpha)\}_{\alpha \in A}$ is said to be a <u>trivialising cover</u> for π with $\{\tau_\alpha\}$ the <u>trivialising maps</u>. Two such C^r covers are said to be C^r-equivalent if their union is also a C^r trivialising cover. An equivalence class of such covers is said to constitute a <u>vector-bundle structure</u> on π and $\pi: E \to X$ with such a structure is called a <u>vector-bundle</u>.

<u>Remarks</u> i) The third axiom is redundant if the fibres are finite-dimensional.

ii) As with charts above, a connectedness argument shows that on components, all fibres are isomorphic and thus an equivalent trivialising cover can be given where each trivialising map τ_α maps onto $U_\alpha \times F$ for F a fixed Banach space for a given component.

iii) If each fibre is Hilbertable then we say that the vector-bundle is a <u>Hilbert space bundle</u>.

<u>Definition 2.4</u> A C^r <u>vector-bundle morphism</u> between C^r vector bundles $\pi_i : E_i \to X_i$ $(i=1,2)$ is a pair of C^r maps (f, f_0) :

$$f_0 : X_1 \to X_2$$
$$f : E_1 \to E_2$$

s.t. i) The diagram

$$\begin{tikzcd} E_1 \arrow[r, "f"] \arrow[d, "\pi_1"'] & E_2 \arrow[d, "\pi_2"] \\ X_1 \arrow[r, "f_0"] & X_2 \end{tikzcd}$$

commutes and the induced map of fibres is continuous + linear

ii) For each $x \in X$ there are trivialising maps

$$\tau : \pi_1^{-1}(U) \to U \times F$$
$$\tau' : \pi_2^{-1}(V) \to V \times F'$$

for π_1, π_2 respectively with $x \in U$, $f_0(U) \subset V$

s.t. the map of $U \to L(F, F')$ given by $x \mapsto \tau'_{f_0(x)} \circ f_x \circ \tau_x^{-1}$ is C^r.

Here f_x denotes the restriction of f to the fibre at x.

<u>Remark</u> Again the second axiom is redundant if the fibres of E_1, E_2 are finite-dimensional.

Many of the functors defined on the category of Banach spaces induce equivalent functors on the category of vector bundles. A useful technique for building new bundles out of old is provided by the following theorem.

<u>Theorem 2.5</u> Let X be a C^r manifold. Suppose we have an open cover of X, $\{U_\alpha\}_{\alpha \in A}$ together with a family of Banach spaces $\{F_\alpha\}_{\alpha \in A}$ and for each α, β with $U_\alpha \cap U_\beta \neq \emptyset$ a C^r map $g_{\alpha\beta} : U_\alpha \cap U_\beta \to L(F_\beta, F_\alpha)$ s.t.

i) each $g_{\alpha\beta}(x)$ is an isomorphism $F_\beta \to F_\alpha$, $x \in U_\alpha \cap U_\beta$

ii) $g_{\alpha\alpha}(x) = id_{F_\alpha}$, $x \in U_\alpha$

iii) if $U_\alpha \cap U_\beta \cap U_\gamma \neq \emptyset$ then $g_{\alpha\beta}(x)\, g_{\beta\gamma}(x) = g_{\alpha\gamma}(x)$, $x \in U_\alpha \cap U_\beta \cap U_\gamma$

Then there exists a vector bundle $\pi : E \to X$ with trivialising cover $\{(U_\alpha, \Theta_\alpha)\}_{\alpha \in A}$ s.t. $\Theta_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times F_\alpha$ and the $g_{\alpha\beta}$ are the transition maps.

<u>Proof</u> As with tangent spaces, for $x \in X$ consider 3-tuples (x, U_α, v) with $x \in U_\alpha$, $v \in F_\alpha$. Define a relation $\sim$ by $(x, U_\alpha, v) \sim (x, U_\beta, w)$ iff $v = g_{\alpha\beta}(x) w$. $\sim$ is an equivalence relation by virtue of (ii) + (iii)

Let E_x denote the set of equivalence classes.

Define $\Theta_{\alpha,x} : E_x \to F_\alpha$ by $[(x, U_\alpha, v)] \mapsto v$ – this is clearly bijective, and $\Theta_{\alpha x} \Theta_{\beta x}^{-1} = g_{\alpha\beta}(x) : F_\beta \to F_\alpha$.

Put $E = \coprod_{x \in X} E_x$ and define $\pi : E \to X$ so that $\pi^{-1}(x) = E_x$.

Finally define $\Theta_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times F_\alpha$ by $v \mapsto (\pi(v), \Theta_{\alpha, \pi(v)} v)$

Now topologize E so that each Θ_α is a homeomorphism (exercise) and put a C^r-structure on E as follows:

we may assume that each U_α is the domain of a chart φ_α (why?) then writing $\tilde{\Theta}_\alpha$ for $(\varphi_\alpha, id) \circ \Theta_\alpha$ we see that $\{(\pi^{-1}(U_\alpha), \tilde{\Theta}_\alpha)\}_{\alpha \in A}$ form a C^r atlas for E since

$$\tilde{\Theta}_\alpha \circ \tilde{\Theta}_\beta^{-1} = (\varphi_\alpha \circ \varphi_\beta^{-1}, g_{\alpha\beta} \circ \varphi_\beta^{-1})$$ which is C^r since all its ingredients are. It is immediate that π is a C^r map with

respect to this structure and the theorem now follows ∎.

Of course if we start with a manifold M and an atlas $\{(U_\alpha, \varphi_\alpha)\}_{\alpha\in A}$, putting $g_{\alpha\beta}(x) = D(\varphi_\alpha \circ \varphi_\beta^{-1})(\varphi_\beta(x))$ for $x \in U_\alpha \cap U_\beta$ and appealing to theorem 2.5 (the hypotheses of which are valid by the chain rule), we obtain a C^{r-1} vector bundle the fibres of which are precisely the tangent spaces to the manifold. This vector bundle is called the *tangent bundle*, denoted $\tau_M : TM \to M$.

Exercise Let $f : M \to N$ be a C^r map of C^r manifolds. Define $df : TM \to TN$ by $df\,|\,T_xM = df(x)$ - the derivative at x defined above. Show that (df, f) is a C^{r-1} vector bundle morphism.

Example If $U \subset E$ is an open subset of a Banach space E then $TU = U \times E$. If $f : U \to F$ is a C^1 map into another Banach space then $df : TU \to TF$ is simply given by $\underset{U\times E}{\underset{\cap}{(x,v)}} \to \underset{F\times F}{\underset{\cap}{(f(x), Df(x)v)}}$.

Now suppose we are given vector-bundles E, F over X, then using theorem 2.5 we can construct new bundles:
if $\pi : E \to X$ has trivialisation $\{(U_\alpha, \theta_\alpha)\}$ with $\theta_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times F_\alpha$ and transition maps $g_{\alpha\beta}$, consider $\hat{g}_{\alpha\beta} : E_\beta^* \to E_\alpha^*$ given by $\hat{g}_{\alpha\beta}(v) = v \circ g_{\beta\alpha}$. It is easy to see that these new transition maps satisfy the hypotheses of theorem 2.5 and thus we get

a new bundle; *the dual bundle* $E^* \to X$ with fibre $(E_x)^*$ for $x \in X$. Similarly one can construct $E \oplus F \to X$, the *Whitney sum* with fibre $E_x \oplus F_x$ at x and $L(E,F) \to X$ with fibre $L(E_x, F_x)$ at x.

Finally if $\pi : E \to X$ is a C^r vector bundle and $f : B \to X$ a C^r map of C^r manifolds then we have the *pull-back bundle* $f^{-1}E \to B$ with fibre at x given by $E_{f(x)}$, trivialising sets given by $\{f^{-1}(U_\alpha)\}_{\alpha\in A}$ and transition maps by $g_{\alpha\beta} \circ f$.

Submanifolds, immersions + submersions.

Definition 2.6 Let X be a subset of a C^r manifold M. X is said to be a *C^r submanifold* of M if for each $x \in X$ there is a C^r-chart (U, φ) of M with $x \in U$, $\varphi : U \to E_1 \times E_2$ s.t $\varphi^{-1}(E_1 \times \{0\}) = U \cap X$.

Observe that if X is a C^r submanifold, then it has an induced C^r structure with charts of the form $(U \cap X, \mathrm{proj}_1 \circ \varphi | U \cap X)$ where $\mathrm{proj}_1 : E_1 \times E_2 \to E_1$ is projection onto the first factor.

Definition 2.7 A C^r map $f : M \to N$ of C^r manifolds $(r \geq 1)$ is said to be an *immersion at $x \in M$* if there exist charts (U,φ), (V,θ) of M and N with $x \in U$, $f(U) \subset V$, $\varphi : U \to E_1$, $\theta : V \to E_1 \times E_2$ s.t

$$\theta \circ f \circ \varphi^{-1}(v) = (v, 0) \quad \text{for } v \in E_1 .$$

Similarly f is said to be a *submersion at x* if there exist charts (U,ϕ), (V,θ) as before but with $\phi: U \to E_1 \times E_2$ $\theta: V \to E_2$ s.t.

$$\theta \circ f \circ \phi^{-1}(v,w) = w \qquad (v,w) \in \phi(U).$$

A map is said to be an <u>immersion</u> (<u>submersion</u>) if it is an immersion (submersion) at each point.

<u>Examples</u> If $\pi: E \to X$ is a vector bundle then π is a submersion.

If $i: M \hookrightarrow N$ is the inclusion map of a submanifold M of N then i is an immersion.

The inverse function theorem can be used to characterise immersions + submersions in terms of their derivatives.

<u>Definition</u> Let E_0 be a linear subspace of a Banach space E, we say E_0 splits if there exists a complementary linear subspace F s.t. $E_0 \cap F = \{0\}$ and $E_0 \oplus F = E$.

<u>Proposition 2.8</u> Let $f: M \to N$ be a C^r map of C^r manifolds and let $x \in M$. Then i) f is an immersion at x iff $df_x: T_xM \to T_{f(x)}N$ is injective and $\operatorname{Im} df_x$ splits.

ii) f is a submersion iff $df_x: T_xM \to T_{f(x)}N$ is surjective and $\operatorname{Ker} df_x$ splits.

<u>Proof</u> Since the result is local it suffices to work in a chart and thus to prove the result when $f: U \to F$ is a map of an open set of a Banach space E into a Banach space F.

Suppose then that for some $x_0 \in U$. $Df(x_0)$ is injective and its image splits. Then $F = \operatorname{Im} Df(x_0) \oplus E_2$ say.

Define $\phi: U \times E_2 \to F$ by $\phi(x,v) = f(x) + v$.

Now $D\phi(x_0, 0)$ is clearly an isomorphism $E \times E_2 \to \operatorname{Im} Df(x_0) \oplus E_2$ so by the inverse function theorem there is a neighbourhood U_0 of $(x_0, 0)$ in $E \times E_2$ on which ϕ is a diffeomorphism onto a neighbourhood V_0 of $f(x_0)$ in F. (V_0, ϕ^{-1}) is a chart for F and on $f^{-1}(V_0)$ we have $\phi^{-1} \circ f(v) = (v, 0)$ since $f(v) = \phi((v,0))$. Thus f is an immersion at x_0. The converse is trivial. Thus part (i) is proved. Part (ii) is proved in a similiar manner.

∎

<u>Exercise</u> If $f: M \to N$ be a C^r submersion. Then i) f is open.

ii) if $x_0 \in \operatorname{Im} f$, then $f^{-1}(x_0)$ is a C^r submanifold of M.

All the results + definitions in this section have natural analogues for vector-bundles and vector bundle morphisms (see Lang). However we will only recall the definition of a sub-bundle which we need later.

<u>Definition 2.9</u> Let E_0 be a subset of a C^r vector bundle $\pi: E \to X$. Then E_0 is a C^r vector subbundle of E if there exists a trivialising cover $\{(U_\alpha, \theta_\alpha)\}$ for E with

$$\theta_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times E^1_\alpha \times E^2_\alpha \qquad \text{s.t.}$$

$$E_0 \cap \pi^{-1}(U_\alpha) = \Theta_\alpha^{-1}(U_\alpha \times E_\alpha^1 \times \{0\}).$$

It is easy to see that $\{(U_\alpha, \mathrm{proj}_1 \circ \Theta_\alpha)\}$ then provide a trivialising cover for E_0 making it a C^r vector-bundle. Here, of course, proj_1 is the obvious projection onto the first factor.

Riemannian + Finsler structures.

Definition 2.10 Let $\pi: E \to M$ be a C^r Hilbert bundle. A C^r Riemannian metric on π is a choice of inner product $\langle\ ,\ \rangle_x$ on each fibre E_x, $x \in M$, s.t. for a trivialising cover $\{(U_\alpha, \Theta_\alpha)\}_{\alpha\in\mathcal{A}}$ there exist C^r maps

$A_\alpha : U_\alpha \to \mathrm{Pos}\, H_\alpha$ s.t. for $u, v \in E_x$ and $x \in U_\alpha$ we have

$$\langle u, v\rangle_x = \langle A_\alpha(x)\Theta_{\alpha,x} u\ ,\ \Theta_{\alpha,x} v\rangle.$$

Here $\mathrm{Pos}\, H_\alpha = \{T \in GL(H_\alpha) : T^* = T \text{ and } \langle Tu, u\rangle > 0 \text{ for } u \in H_\alpha \setminus \{0\}\}$ for some fixed inner product $\langle\ ,\ \rangle$ on H_α.

A manifold with a Riemannian metric on its tangent bundle is called a Riemannian manifold.

If M admits C^r partitions of unity then any Hilbert bundle over M admits a C^r Riemannian metric built by "gluing" together the local Riemannian metrics given by a trivialisation. Thus any separable manifold modelled on Hilbert space is a Riemannian manifold.

However there are many cases where we might wish to consider

vector-bundles with non-Hilbertable fibres or manifolds without sufficiently good partitions of unity. Here, even though we might not have an inner product on each fibre, we may nevertheless have a suitably continuous choice of norm.

Definition 2.11 Let $\pi: E \to B$ be a (possibly C^0) vector bundle and $\|.\|: E \to \mathbb{R}$ a continuous function. $\|.\|$ is said to be a Finsler structure for E if there is a trivialising cover $\{(U_\alpha, \Theta_\alpha)\}_{\alpha\in\mathcal{A}}$ for π s.t.

i) $v \mapsto \|\Theta_{\alpha,x}^{-1}(v)\|$ is an admissible norm for E_α, $x \in U_\alpha$.

ii) for each $x_0 \in U_\alpha$ and $K > 1$ there is a neighbourhood $U_0 \subset U_\alpha$ of x_0 s.t. for $y \in U_0$

$$\tfrac{1}{K}\|\Theta_{\alpha,y}^{-1}(v)\| \le \|\Theta_{\alpha,x}^{-1}(v)\| \le K\|\Theta_{\alpha,y}^{-1}(v)\|\ ,\quad v \in E_\alpha.$$

A manifold with a Finsler structure on its tangent bundle is said to be a Finsler manifold.

Example Let E be a Banach space with admissible norm $N(.)$. Then the trivial bundle $E \times E$ clearly has a Finsler structure given by $\|(x, v)\| = N(v)$. This is called the flat Finsler structure.

Using a C^0 partition of unity it is easy to show that any vector-bundle over a paracompact base admits a Finsler structure. In particular every paracompact C^1 manifold is a Finsler manifold.

The following proposition lists some straightforward consequences of Definition 2.11. The proofs are trivial.

Proposition 2.19 If V is a Banach space, V is a complete Finsler manifold with the flat Finsler structure.

Proof In fact $d(p,q) = \|p-q\|$ (Exercise).

Theorem 2.20 Let M be a submanifold of a Finsler manifold N. Then M has a natural Finsler structure induced by restriction of that of N. Further if M is closed and N complete, then M is complete.

Corollary 2.21 If M is a closed submanifold of a Banach space, then M has a natural complete Finsler structure.

Proof If $i: M \to N$ is the inclusion map then TM is a subbundle of i^*TN (exercise) and thus carries an induced Finsler structure by Proposition 2.12 parts (i) + (iv).

Now if d^M, d^N are the metrics induced on M and N respectively, since a path in M is a path in N we see that

$$d^N(p,q) \le l(\sigma) \quad \text{for any } \sigma \text{ a path in } M \text{ joining } p,q \in M.$$

Thus $d^N(p,q) \le d^M(p,q)$ and so if N is complete, an M-cauchy sequence is N-cauchy + thus convergent in N and thus in M if M is closed. This completes the proof. The corollary follows immediately from the theorem + proposition 2.19.

Note The principal reference for the material on Finsler structures is Palais, Topology $\underline{5}$ (1966) 115-132. to which the reader is referred for more information.

III: Some Banach Spaces of Sections.

Let Ω be a bounded open subset of $\mathbb{R}^n$, denote by $C_c^\infty(\Omega, \mathbb{R}^m)$ the space of C^∞ functions $\Omega \to \mathbb{R}^m$ with support contained in Ω.

For any multi-index $\alpha = (\alpha_1, \dots \alpha_n) \in (\mathbb{Z}^+)^n$ put $|\alpha| = \alpha_1 + \dots + \alpha_n$ and denote by D^α the differential operator $\dfrac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \dots \partial x_n^{\alpha_n}}$

We introduce the following norms on $C_c^\infty(\Omega, \mathbb{R}^m)$:

$$\|f\|_{C^0(\Omega,\mathbb{R}^m)} = \sup_{x\in\Omega} |f(x)|$$

$$\|f\|_{C^k(\Omega,\mathbb{R}^m)} = \sum_{|\alpha|\le k} \|D^\alpha f\|_{C^0} \qquad , k \in \mathbb{Z}^+$$

$$\|f\|_{L_k^p(\Omega,\mathbb{R}^m)} = \Big(\sum_{|\alpha|\le k} \int_\Omega |D^\alpha f|^p \, dx \Big)^{1/p} \qquad k\in\mathbb{Z}^+, \ p \ge 1$$

$$\|f\|_{\mathcal{C}_k^p(\Omega,\mathbb{R}^m)} = \|f\|_{C^{k-1}} + \sum_{|\alpha|= k} \Big(\int_\Omega |D^\alpha f|^p \, dx \Big)^{1/p} \qquad k\in\mathbb{Z}^+, k\ge 1, p \ge 1$$

We complete $C_c^\infty(\Omega, \mathbb{R}^m)$ with respect to these norms to obtain Banach space $C_0^k(\Omega, \mathbb{R}^m)$, $L_{0,k}^p(\Omega, \mathbb{R}^m)$, $\mathcal{C}_{0,k}^p(\Omega, \mathbb{R}^m)$

Remarks i) It is clear that the spaces C_0^k are just the C^k functions with support contained in Ω.

ii) The spaces L_k^p are called Sobolev spaces and can be shown to consist of L^p functions with L^p derivatives in the sense of distributions up to order k.

Observe that the spaces L_k^2 are Hilbertable with inner product $\langle f, g\rangle_{L_k^2} = \sum_{|\alpha|\le k} \int \langle D^\alpha f, D^\alpha g\rangle$. These spaces are very important in applications of functional analysis to partial differential equations – both linear + non-linear.

We now extend these concepts to spaces of sections of a vector bundle.

Let $\pi : E \to M$ be a C^∞ vector bundle with finite dimensional fibres over a C^∞ connected compact manifold.

Recall that a map $\sigma : M \to E$ is a <u>section</u> of E if the following diagram commutes:

$$\begin{tikzcd} M \arrow[r, "\sigma"] \arrow[dr, "id"'] & E \arrow[d, "\pi"] \\ & M \end{tikzcd} \qquad \text{ie} \quad \pi \circ \sigma = id_M .$$

Let $C^\infty(E)$ denote the space of smooth sections of E – it is clearly a vector-space with addition + scalar multiplication defined pointwise.

If $M \times \mathbb{R}^m \to M$ is a trivial bundle we can clearly identify $C^\infty(M \times \mathbb{R}^m)$ with $C^\infty(M, \mathbb{R}^m)$, the space of smooth functions into $\mathbb{R}^m$ by identifying a function with its graph. Denote $C^\infty(M \times \mathbb{R})$ by $C^\infty(M)$

<u>Definition 3.1</u> A <u>connection</u> ∇ on $\pi : E \to M$ is an operator

$$\nabla : C^\infty(TM) \times C^\infty(E) \to C^\infty(E)$$

, written $(X, \sigma) \mapsto \nabla_X \sigma$ – the <u>covariant derivative</u> of σ by X, $X \in C^\infty(TM)$, $\sigma \in C^\infty(E)$ s.t.

i) $X \to \nabla_X \sigma$ is $C^\infty(M)$-linear, $\sigma \in C^\infty(E)$

ii) $\sigma \to \nabla_X \sigma$ is $\mathbb{R}$-linear, $X \in C^\infty(TM)$

iii) $\nabla_X(f\sigma) = df(X)\cdot\sigma + f\nabla_X\sigma$, $f \in C^\infty(M)$, $\sigma \in C^\infty(E)$.

A <u>Riemannian structure</u> on E is a connection ∇ together with

a Riemannian metric g s.t.

$$d(g(\sigma,\tau))(X) = g(\nabla_X\sigma, \tau) + g(\sigma, \nabla_X\tau) \quad, X \in C^\infty(TM) \quad \sigma, \tau \in C^\infty(E)$$

and a bundle with a given Riemannian structure is called a <u>Riemannian bundle</u>. A partition of unity argument shows that any finite dimensional vector bundle over a finite-dimensional paracompact base admits a Riemannian structure.

If E and F are Riemannian bundles over M then we induce a Riemannian structure on $L(E,F)$ by setting

$$\langle A, B \rangle_x = \sum_i \langle Ae_i, Be_i \rangle_{F_x} \quad, x \in M,$$

where $\{e_i\}$ is an orthonormal basis for E_x and $A, B \in L(E_x, F_x)$ – this is the <u>Hilbert-Schmidt</u> inner product. We then define the connection on $L(E,F)$ by

$$(\nabla_X A)\cdot\sigma = \nabla^F_X(A\cdot\sigma) - A\cdot\nabla^E_X\sigma$$

– the "Leibnitz rule" for $X \in C^\infty(TM)$ $A \in C^\infty(L(E,F))$ $\sigma \in C^\infty(E)$.

In particular if M is a Riemannian manifold, the Levi-Civita connection on TM provides a Riemannian structure on $TM \to M$, thus if $E \to M$ is a Riemannian bundle we have an induced Riemannian structure on $L(TM, E)$.

Then, if $\xi \in C^\infty(E)$, $\nabla\xi \in C^\infty(L(TM,E))$ which we can then covariantly differentiate to get $\nabla^2\xi \in C^\infty(L(TM, L(TM,E)))$ which we identify with $C^\infty(L^2(TM;E))$ in the obvious way. Iterating, we define the j^{th} covariant derivative of ξ, $\nabla^j\xi \in C^\infty(L^j(TM,E))$.

We can now define the norms on $C^\infty(E)$ for E a Riemannian bundle over a compact Riemannian manifold M.

Letting $|\,|$ denote indifferently the Finsler structure on $L^j(TM,E)$ $j\geq 0$ induced by the Riemannian structures, put

$$\|\xi\|_{C^0(E)} = \sup_{x\in M} |\xi(x)|_x$$

$$\|\xi\|_{C^k(E)} = \sum_{j=0}^{k} \|\nabla^j\xi\|_{C^0}$$

$$\|\xi\|_{L^p_k(E)} = \sum_{j=0}^{k} \left(\int_M |\nabla^j\xi(x)|_x^p * 1_M\right)^{1/p}$$

$p>1$ where $*1_M$ is the volume element on M.

$$\|\xi\|_{\mathcal{C}^p_k(E)} = \|\xi\|_{C^{k-1}(E)} + \left(\int_M |\nabla^k\xi(x)|_x^p \; *1_M\right)^{1/p}, \quad k\geq 1 \quad p\geq 1.$$

The corresponding Banach spaces obtained by completion are denoted $C^k(E)$, $L^p_k(E)$, $\mathcal{C}^p_k(E)$

Lemma 3.2 The map $\xi \mapsto \nabla\xi$ of $C^\infty(E)$ into $C^\infty(L(TM,E))$ has a bounded linear extension

$$C^{k+1}(E) \to C^k(L(TM,E))$$
$$L^p_{k+1}(E) \to L^p_k\,(L(TM,E))$$
$$\mathcal{C}^p_{k+1}(E) \to \mathcal{C}^p_k(L(TM,E))$$

Proof: Trivial.

Lemma 3.3 If $\phi\in C^\infty(M)$ the map $\xi\mapsto \phi.\xi$ of $C^\infty(E)$ into itself has a bounded linear extension to $C^k(E)$, $\mathcal{C}^p_k(E)$ and $L^p_k(E)$

Proof In fact $\|\phi.\xi\|_{C^k(E)} \leq \|\phi\|_{C^k(M\times\mathbb{R})}\|\xi\|_{C^k(E)}$

$\|\phi.\xi\|_{L^p_k(E)} \leq \|\phi\|_{C^k(M\times\mathbb{R})}\|\xi\|_{L^p_k(E)}$ etc. Use the Leibnitz rule

It is possible to give another, perhaps less abstract, description of these spaces of sections by seeing what they look like locally. To this end, let $\{(U_i,\phi_i)\}_{1\leq i\leq N}$ be a finite atlas for M with $\{(U_i,\Theta_i)\}_{1\leq i\leq N}$ a trivialising cover for E. Let $\{\psi_i\}_{1\leq i\leq N}$ be a partition of unity subordinate to $\{U_i\}$.

Letting $\mathcal{B}$ denote indifferently C^k, L^p_k or $\mathcal{C}^p_k$ we have

Theorem 3.4 Let σ be a section of E, write $\sigma_i = \psi_i\sigma$ so that $\sigma = \sum_{i=1}^{N}\sigma_i$. Then $\sigma\in\mathcal{B}(E)$ iff $\Theta_i\circ\sigma_i\circ\phi_i^{-1}\in\mathcal{B}_0(\phi_i(U_i),\mathbb{R}^m)$ where we have as usual identified functions + sections of trivial bundles. Further $\sigma\mapsto\sum_i\|\Theta_i\circ\sigma_i\circ\phi_i^{-1}\|_{\mathcal{B}(\phi_i(U_i),\mathbb{R}^m)}$ is an equivalent norm on $\mathcal{B}(E)$.

Corollary 3.5. The spaces $\mathcal{B}(E)$ are defined independently of the choice of Riemannian structures on E and M.

Pf (Sketch) The first part is straightforward.
For the equivalence of norms: $\sigma = \sum\sigma_i$ so $\|\sigma\|\leq\sum_i\|\sigma_i\|$.
Further $\|\sigma_i\| = \|\psi_i\sigma\| \leq K_i\|\sigma\|$ by Lemma 3.3 and so

$\frac{1}{(\sum_i k_i)} \sum_i \|\sigma_i\| \leq \|\sigma\| \leq \sum_i \|\sigma_i\|$. Thus we need only show that

$\|\sigma_i\|_{B(E)}$ and $\|\Theta_i \circ \sigma_i \circ \phi_i^{-1}\|_{B(\phi_i(U_i), \mathbb{R}^m)}$ are equivalent.

The crucial step here is the following: writing f_i for $\Theta_i \circ \sigma_i \circ \phi_i^{-1}$ and calculating in local co-ordinates using the compactness of $\operatorname{supp} \psi_i$, we can show that there exist positive constants c_1^i, c_2^i s.t

$$c_1^i \Big(\sum_{|\alpha|=1} |D^\alpha f_i(x)| - |f_i(x)| \Big) \leq \|\nabla \sigma_i(x)\| \leq c_2^i \Big(\sum_{|\alpha|=1} |D^\alpha f_i(x)| + |f_i(x)| \Big).$$

Induction provides similiar inequalities for higher orders of differentiability and the rest of the proof follows by taking supremums or integrating over M as appropriate.

<u>Exercise</u> (tedious): Fill in the details of the above proof.

Much of the usefulness of Sobolev spaces is a consequence of the following Sobolev - Rellich - Kondrachov embedding theorems.

<u>Theorem 3.6</u> For $\pi: E \to M$ a C^∞ vector bundle of finite rank over a smooth compact n-dimensional manifold:

i) we have a compact linear inclusion $L^p_k(E) \hookrightarrow C^s(E)$ if $k - n/p > s$

ii) we have a continuous linear inclusion $L^p_r(E) \hookrightarrow L^q_s(E)$ if $r \geq s$ $\quad r - n/p \geq s - n/q$

and the inclusion is compact if the inequalities are strict.

[Recall that a linear map of Banach spaces is compact if it maps bounded sets into precompact ones]

Proofs of theorem 3.6 may be found in

i) A.P. Calderón Proc. Symp. A.M.S vol 4 (1961) 33-50.

ii) R.T. Seeley. Trans. Amer. Math. Soc. 117 (1965) 167-204 and in most books on partial differential equations.

A. Some Differential Geometry

We recall some facts about the exponential map. Let M be a C^∞ Riemannian manifold. As we saw in section II, we can use the metric on M to define the length of a path joining two points.

Fix a point $m \in M$, then for $p \in M$ near m there exists a unique C^1 path $\gamma_p : [0,1] \to M$ with $\gamma(0) = m$, $\gamma(1) = p$ that minimises length amongst all such paths. γ_p is called the minimising geodesic joining m and p. Let $v = \left.\frac{d\gamma_p}{dt}\right|_{t=0} \in T_m M$ and define the exponential map by $\exp v = p$. From the theory of existence and uniqueness of ordinary differential equations, it can be shown that exp is defined + C^∞ on a neighbourhood of the zero section in TM. Indeed, if we let $\text{Exp} = (\tau, \exp)$: where τ is the bundle projection $TM \to M$, then an application of the inverse function theorem shows that Exp maps a neighbourhood $\mathcal{U}$ of the zero section in TM C^∞ diffeomorphically onto a neighbourhood $\mathcal{W}$ of the diagonal in $M \times M$.

For more information, see Kobayashi + Nomizu, "Foundations of Differential Geometry" Vol I, (Interscience, New York, 1963, 1969).

B. The Canonical Example.

We now construct a C^∞ differentiable structure on the space of continuous maps of a compact space into a finite-dimensional manifold. The method of proof is a model for all such constructions.

Theorem 4.1 (Eells, Symp. Inter de Topologia Alg. Mexico (1956); 1958 303-308) Let S be a compact Hausdorff space and M a smooth finite-dimensional manifold. Let $C^0(S,M)$ denote the space of continuous maps from S into M with the compact-open topology. Then $C^0(S,M)$ has a natural C^∞ differentiable structure and thus is an (infinite-dimensional) C^∞ manifold.

Proof We construct an atlas for $C^0(S,M)$.

First, we equip M with a Riemannian metric using partitions of unity. Now, for $f \in C^0(S,M)$ let $f^{-1}TM \to S$ denote the pull-back of TM by f. $f^{-1}TM$ is a C^0 vector bundle with a Riemannian metric induced by that on TM. Thus, just as in section III, the space of continuous sections of $f^{-1}TM$, $C^0(f^{-1}TM)$, becomes a Banach space when given the sup norm, denoted $\|\cdot\|_f$. It is easily identified with the continuous maps $\sigma : S \to TM$ s.t. $\tau_M \circ \sigma = f$ — the "continuous variations of f".

We will construct a chart that maps an open neighbourhood of f into $C^0(f^{-1}TM)$, thus we may identify $C^0(f^{-1}TM)$ with the tangent space of $C^0(S,M)$ at f.

Consider $\text{Exp} = (\tau_M, \exp)$ the exponential map induced by the Riemannian structure on M. From section A we know that there is a neighbourhood $\mathcal{U}$ of the zero section of TM that is mapped diffeomorphically by Exp onto a neighbourhood $\mathcal{W}$ of the diagonal of $M \times M$

Put $\mathcal{U}_f = f^{-1}\mathcal{U}$ - an open neighbourhood of the zero section of $f^{-1}TM$, then $\mathcal{O}_f = C^0(\mathcal{U}_f)$ is an open subset of $C^0(f^{-1}TM)$ since S is compact.

Now let $\mathcal{W}_f$ denote $\{g \in C^0(S,M) : (f,g)(S) \subset \mathcal{W}\}$ which is open in $C^0(S,M)$.

Define $\phi_f : \mathcal{O}_f \to \mathcal{W}_f$ by $\phi_f(y) = \exp \circ y$, $y \in \mathcal{O}_f$.

ϕ_f is clearly bijective and is continuous since $\exp$ is. Moreover it has a continuous inverse given by $\phi_f^{-1}(g)(s) = \exp_{f(s)}^{-1}(g(s))$ $s \in S$, $g \in \mathcal{W}_f$.

Thus ϕ_f is a homeomorphism.

We claim that the collection $\{(\mathcal{W}_f, \phi_f^{-1})\}_{f \in C^0(S,M)}$ is a smooth atlas, to show which we must prove the smoothness of the transition maps

To this end, let $f, f' \in C^0(S,M)$ with $\mathcal{W}_f \cap \mathcal{W}_{f'} \neq \emptyset$ and denote by ϕ. $\phi_{f'}^{-1} \circ \phi_f : \phi_f^{-1}(\mathcal{W}_f \cap \mathcal{W}_{f'}) \to \phi_{f'}^{-1}(\mathcal{W}_f \cap \mathcal{W}_{f'})$.

Let $y \in \phi_f^{-1}(\mathcal{W}_f \cap \mathcal{W}_{f'})$ then $\phi(y)(s) = \exp_{f'(s)}^{-1} \circ \exp_{f(s)} y(s)$

$= (g \circ y)(s)$ where we have (*)

defined g on an open neighbourhood $\mathcal{O}$ of $y(S)$ in $f^{-1}TM$ by $g|\mathcal{O}_s = \exp_{f'(s)}^{-1} \circ \exp_{f(s)}$. Thus g is a continuous fibre map $\mathcal{O} \to f'^{-1}TM$.

Let $D_2^i g$ be the fibre map $\mathcal{O} \to L^i(f^{-1}TM, f'^{-1}TM)$ given by

$D_2^i g|\mathcal{O}_s = D^i(g|\mathcal{O}_s)$ $s \in S$ - the "ith fibre derivative". These maps exist + are continuous for each i since $\exp$ is C^∞.

Now let $h \in C^0(f^{-1}TM)$ be sufficiently small that $(y+h)(S) \subset \mathcal{O}$. Then

$$g(y(s)+h(s)) - g(y(s)) - D_2 g(y(s)) \cdot h(s) = \Theta_y(h(s)) \cdot h(s) \quad \text{—}①$$

where $\Theta_y : \mathcal{O} \to L(f^{-1}TM, f'^{-1}TM)$ a continuous fibre map given by

$$\Theta_y(v) = \int_0^1 D_2 g(y(s)+tv) - D_2 g(y(s))\, dt \quad \text{on } \mathcal{O}_s$$

Taking norms + suprema over ① we have

$$\| g \circ (y+h) - g \circ y - (D_2 g) \circ y \cdot h \|_{f'} \leq \| \Theta_y \circ h \| \, \| h \|_f$$

Since Θ_y is continuous, composition with Θ_y is a continuous map of h and since $\Theta_y = 0$ it follows that for $\varepsilon > 0$ $\exists \delta$ s.t. $\|h\|_f < \delta \Rightarrow \|\Theta_y \circ h\| < \varepsilon$.

$h \mapsto D_2 g \circ y \cdot h$ is easily seen to be a bounded linear map $f^{-1}TM \to f'^{-1}TM$ and so we conclude that ϕ is differentiable + $D\phi(y) = D_2 g \circ y$. Repeating the whole argument from (*) replacing g with $D_2 g$ we inductively prove that ϕ is C^∞ which completes the proof. ■

<u>Remarks</u> i) The same method of proof shows that a different choice of metric + thus exponential map gives rise to compatible charts so that the differentiable structure on $C^0(S, M)$ depends only on that of M.

ii) The restriction of M to finite dimensions is inessential. In fact the main requirement is that M should admit a connection + thus an exponential map, in which case the proof goes through as above. For details see Eliasson. J. Diff. Geom. 1 (1967) 169-194.

iii) An examination of the above proof shows that the crucial step in the argument, that of showing smoothness of the transition maps, since composition with a suitably differentiable fibre map is a smooth map of Banach spaces of sections. It is this composition property that is the main requirement for a class of maps to admit a manifold structure, as we shall see in the next section.

C. Section Functors + Manifold Models

The main reference for this section is

Palais. "Foundations of Global Non-linear Analysis" Benjamin New York (1968), hereafter refered to as <u>Foundations</u>

See also Eliasson op. cit.

Let M be a C^∞ compact manifold and $VB(M)$ the category of C^∞ vector bundles of finite rank over M.

<u>Definition 4.2</u> Γ is called a <u>section functor</u> if for each $E \in VB(M)$ there is a Banachable space of sections of E, $\Gamma(E)$ s.t

i) $C^\infty(E)$ is a dense linear subspace of E

ii) for $E, F \in VB(M)$ and each $A \in C^\infty(L(E,F))$, the map

$\xi \mapsto A\cdot\xi$ is in $L(\Gamma(E), \Gamma(F))$ ie

$$\|A\cdot\xi\|_F \le \text{const} \;\|\xi\|_E \quad , \xi \in C^\infty(E) \quad \text{where} \quad A\cdot\xi(s) = A(s)\,\xi(s)$$

Thus Γ is a covariant functor from $VB(M)$ into the category of Banach-spaces.

<u>Exercise</u> The functors L^p_k, C^k, $\mathcal{Q}^p_k$ are section functors. (Use Leibnitz rule)

<u>Definition 4.3</u> A section functor Γ on $VB(M)$ is called a manifold model if i) there is a continuous inclusion $\Gamma(E) \hookrightarrow C^0(E)$. $E \in VB(M)$

ie
$$\|\xi\|_{C^0} \le \text{const}\;\|\xi\|_\Gamma$$

ii) there is a continuous inclusion $\Gamma(L(E,F)) \hookrightarrow L(\Gamma(E), \Gamma(F))$ $\forall E, F \in VB(M)$

ie.
$$\|A\cdot\xi\|_\Gamma \le \text{const}\;\|A\|_\Gamma\,\|\xi\|_\Gamma \quad \forall A \in C^\infty(E,F) \quad \xi \in C^\infty(E).$$

iii) let $E, F \in VB(M)$ and $U \subset E$ open with each fibre non-empty, $\bar{\Phi} : U \to F$ be a C^∞ fibre map. Then we have an induced continuous map $\Gamma(\bar\Phi) : \Gamma(U) \to \Gamma(F)$, $\Gamma(\bar\Phi)(\xi) = \bar\Phi \circ \xi$.

In fact as in theorem 4.1 such a $\bar\Phi$ induces a smooth map $\Gamma(U) \to \Gamma(F)$.

<u>Lemma 4.4</u> Let $\bar\Phi : U \to F$ be a smooth fibre map as above. Then $\Gamma(\bar\Phi) : \Gamma(U) \to \Gamma(F)$ is a C^∞ map with $D^i\Gamma(\bar\Phi) = \Gamma(D_2^i\bar\Phi)$ where $D_2^i\bar\Phi : U \to L^i(E,F)$ is the i^{th} fibre derivative.

<u>Proof</u> Since $\bar\Phi$ is C^∞ so is each $D_2^i\bar\Phi$ + so it suffices to prove that $\Gamma(\bar\Phi)$ is C^1 and $D\Gamma(\bar\Phi) = \Gamma(D_2\bar\Phi)$ and then

iterate the argument replacing $\overline{\Phi}$ by $D_2\overline{\Phi}$.

Now, for $\xi \in \Gamma(U)$ and $h \in \Gamma(E)$ sufficiently small that $(\xi + h)S \subset U$ we have

$$\overline{\Phi} \circ (\xi + h) - \overline{\Phi} \circ \xi - D_2\overline{\Phi} \circ \xi \cdot h = \Theta(\xi, h) \cdot h \qquad — ①$$

where $\Theta(u,v) = \int_0^1 D_2\overline{\Phi}(u + tv) - D_2\overline{\Phi}(u)\, dt$ is a C^∞ fibre map defined on an open neighbourhood $\mathcal{O}$ of $U \times \{0\}$ in $E \oplus F$ into $L(E, F)$.

Taking norms in ① we have

$$\| \Gamma(\Phi)(\xi + h) - \Gamma(\Phi)(\xi) - \Gamma(D_2\Phi)(\xi) \cdot h \| = \| \Gamma(\Theta)(\xi, h) \cdot h \|$$

By axioms (ii) and (iii) $\xi \mapsto \Gamma(D_2\Phi)(\xi)$ is a continuous map into $L(\Gamma(E), \Gamma(F))$ and $\| \Gamma(\Theta)(\xi, h) \| \leq \text{const} \| \Gamma(\Theta)(\xi, h) \| \, \|h\|$. Since $\Gamma(\Theta)(\xi, 0) = 0$ and $\Gamma(\Theta)$ is continuous, for $\varepsilon > 0$ there is a $\delta > 0$ s.t. $\|h\| < \delta \Rightarrow \text{const} \| \Gamma(\Theta)(\xi, h) \| < \varepsilon$, whence $\Gamma(\Phi)$ is C^1 with $D\Gamma(\Phi) = \Gamma(D_2\Phi)$. Then induction completes the proof. ∎

It should now be clear that manifold models Γ induce manifold structures just as C^0 did.

<u>Theorem 4.5</u> Let Γ be a manifold model on $VB(M)$ and N a smooth finite dimensional manifold. Then there is a well-defined set of maps $\Gamma(M,N) \subset C^0(M,N)$ with a C^∞ differentiable structure modelled on the Banach spaces $h^{-1}TN$, $h \in C^\infty(M,N)$.

<u>Proof</u> Let $h \in C^\infty(M,N)$ and let ψ_h be the chart for $C^0(M,N)$ about h constructed in Theorem 4.1

$\Gamma(h^{-1}U) \subset C^0(h^{-1}U)$ and so we define $\Gamma(S,M)$ to be the union of the images of $\Gamma(h^{-1}U)$ under ψ_h^{-1} for $h \in C^\infty(M,N)$.

For $h, h' \in C^\infty(M,N)$ $\psi_{h'} \circ \psi_h^{-1}$ is induced by a smooth fibre map and so by lemma 4.4 the restriction of ψ_h to $\Gamma(h^{-1}(U))$ provides an C^∞ atlas for $\Gamma(S,M)$. Again, a different choice of exponential map will provide compatible charts. ∎

We can exhibit the manifold structure in another way, which has the advantage of identifying the maps $\Gamma(S,M)$ explicitly and embeds $\Gamma(S,M)$ as a submanifold of a Banach space thus providing a Finsler structure on $\Gamma(S,M)$ by theorem 2.20.

Let N be a smooth paracompact finite dimensional manifold embedded as a closed submanifold of some $\mathbb{R}^p$; such an embedding exists for any N if p is big enough by a theorem of Whitney.

Writing $\Gamma(M \times \mathbb{R}^p)$ as $\Gamma(M, \mathbb{R}^p)$ making the usual identification with functions + sections we define $\Gamma(M,N)$ to be $\{ f \in \Gamma(M, \mathbb{R}^p) : f(x) \in N \ \forall x \in M \}$.

<u>Theorem 4.6</u> $\Gamma(M,N)$ is a closed C^∞ submanifold of $\Gamma(M, \mathbb{R}^p)$ and thus has a complete Finsler structure induced by the flat one on $\Gamma(M, \mathbb{R}^p)$

<u>Proof</u> We put a Riemannian metric g on $\mathbb{R}^p$ so that N is a totally geodesic submanifold (this can always be done).

Let $h \in C^\infty(M,N)$ and let (U_h, ψ_h) be the chart for $\Gamma(M, \mathbb{R}^p)$ at h constructed in theorem 4.1.

$h^{\cdot}T\mathbb{R}^p = h^{\cdot}TN \oplus h^{\cdot}V(N)$ where $V(N) \to N$ is the normal bundle of N in $\mathbb{R}^p$ thus $\Gamma(h^{\cdot}T\mathbb{R}^p) = \Gamma(h^{\cdot}TN) \oplus \Gamma(h^{\cdot}V(N))$.

Now $\psi_h^{-1}(y) = \exp \circ y$ for $y \in \Gamma(h^{\cdot}T\mathbb{R}^p)$ where $\exp$ is the exponential map corresponding to g. Thus since N is totally geodesic in $\mathbb{R}^p$ $\exp y(x) \in N$ iff $y(x) \in TN$ thus

$\psi_h^{-1}(y) \in \Gamma(M,N)$ iff $y \in \Gamma(h^{-1}TN)$. So (U_h, ψ_h) provides a submanifold chart at h. The theorem now follows since

$C^\infty(M,N)$ is dense in $\Gamma(M,N)$.

The following is easy to prove using lemma 4.4.

<u>Theorem 4.7</u> Let N, N' be smooth finite dimensional manifolds and $\theta : N \to N'$ a C^∞ map. Then $\Gamma(\theta) : \Gamma(S,N) \to \Gamma(S,N')$ given by $f \mapsto \theta \circ f$ is a C^∞ map.

<u>D. Some Manifold Models.</u>

We now identify some functors which are manifold models + to which the above theory can be applied.

i) The functors C^k $0 \le k < \infty$ are easily seen to be manifold models using the Leibnitz rule and the chain rule.

ii) Sobolev-type functors are slightly more difficult, since it is a bit tricky to show they behave well under composition. However a partition of unity argument shows that it is sufficient to check the axioms on compactly supported sections of a trivial bundle over a disc (cf Thm 3.4)

<u>Theorem 4.8</u> L^p_k is a manifold model for $pk > n$

<u>Proof</u>: $L^p_k \hookrightarrow C^0$ is just Sobolevs theorem (Theorem 3.6)

The technical tool required to show that L^p_k satisfies axioms (ii) + (iii) is the following multiplication lemma of Palais which follows from a straightforward application of Sobolevs theorem + Holder's Inequality.

<u>Lemma 4.9</u> (Palais - Foundations) Let $1 \le p < \infty$ $k > n/p$ and let $\beta_1 \dots \beta_r$ be n-multi-indices with $|\beta_i| > 0$ $\Sigma |\beta_i| \le k$.

Then $(f_1 \dots f_r) \mapsto D^{\beta_1} f_1 \dots D^{\beta_r} f_r$ is a continuous r-linear map of $\oplus^r L^p_{0,k}(D^n, \mathbb{R})$ into $L^p_0(D^n, \mathbb{R})$.

Now, if $f : D^n \times \mathbb{R}^p \to D^n \times \mathbb{R}^q$ is a smooth fibre map, to prove axiom (iii) it suffices to show that

$$s = (s_1 \dots s_p) \longmapsto D^\alpha(f(\cdot, s_1, \dots, s_p))$$

is a continuous map from $L^p_k \to L^p$. $0 \le |\alpha| \le k$.

The case $|\alpha| = 0$ follows since $L^p_{0,k}$ maps are continuous $pk > n$

Now for $|\alpha| \ge 1$ $D^\alpha f(\cdot, s_1 \dots s_p)$ is a sum of terms of the form

$$(D^\beta f)(\cdot, s_1 \dots s_p)\, D^{\gamma_1} s_{i_1} \dots D^{\gamma_r} s_{i_r} \quad \text{with } \Sigma|\gamma_i| \le k \quad |\gamma_i| > 0$$

Since $D^\beta f$ is continuous, $(s_1 \dots s_p) \mapsto D^\beta f(\cdot, s_1 \dots s_p)$ is continuous $L^p_k \to C^0$

By lemma 4.9 $(s_1 \dots s_p) \mapsto D^{\gamma_1} s_{i_1} \dots D^{\gamma_r} s_{i_r}$ is continuous $L^p_k \to L^p$ and now axiom (iii) follows since multiplication is clearly continuous $C^0 \times L^p \to L^p$.

Axiom (ii) follows similiarly. ∎

<u>Remark</u> A similiar proof shows that $C^0 \cap L^p_{n/p}$ with norm given by $\|\cdot\| = \|\cdot\|_{C^0} + \|\cdot\|_{L^p_{n/p}}$ is a manifold model.

In the same order of ideas:

<u>Theorem 4.10</u> $\mathcal{C}\ell^p_k$ is a manifold model $p, k \ge 1$

<u>Proof</u> As before we shall prove axiom (iii) for trivial bundles over discs.

For $f : D^n \times \mathbb{R}^p \to D^n \times \mathbb{R}^q$ a smooth fibre map we must show that

$$s = (s_1 \dots s_p) \mapsto D^\alpha(f(\cdot, s_1 \dots s_p)) \text{ is continuous } \mathcal{C}\ell^p_k \to C^0 \quad 0 \le |\alpha| < k$$
$$\mathcal{C}\ell^p_k \to L^p \quad |\alpha| = k.$$

For $|\alpha| < k$ this follows by the continuity of all the derivatives involved have order less than k + are thus continuous.

For $|\alpha| = k$ the only case where non-continuous derivatives can appear is in terms of the form

$$(D^\alpha f)(\cdot, s_1 \dots s_p)\, D^\beta s_i \qquad |\beta| = k$$

and so continuity is assured since multiplication $C^0 \times L^p \to L^p$ is continuous ∎

In particular, $C^0 \cap L^2_1 = \mathcal{C}\ell^2_1$ is a manifold model. The resulting manifold $\mathcal{C}\ell^2_1(M,N)$ is of particular interest in the theory of harmonic maps, since a theorem of Hildebrandt et. al. shows that L^2_1-continuous harmonic maps are smooth. Thus $\mathcal{C}\ell^2_1(M,N)$ might be the appropriate manifold on which to do critical point theory for the energy functional.

<u>Remark</u> Maps $M \to N$ may be considered as sections of a trivial bundle $M \times N \to M$. The whole theory of manifold models discussed here can be carried through to show that spaces of sections of fibre bundles have a natural manifold structure. I refer the interested reader to Palais – Foundations.