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SUMMER WORKSHOP ON FIBRE BUNDLES AND GEOMETRY

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THE TOPOLOGY OF MANIFOLDS

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The topology of manifolds

J. Eells (Warwick. Winter 1982)

1. Manifolds.

A space X is an n-manifold if every point has a nbd. homeomorphic to $\mathbb{R}^n$. Assume further one of the following equivalent properties:

1) X is metrizable;
2) X is paracompact;
3) X is separable;
4) X is expressible as a countable union of compact sets.

In any case, X is locally compact and locally contractible. (Even more, X is an absolute nbd retract.) We shall also suppose X connected, unless otherwise specified.

Properties 1. X has the homotopy type of a locally finite simplicial complex. If X is compact, then it has the homotopy type of a finite simplicial complex.

2. X may not be homeomorphic to a simplicial complex (be triangulable — but we require the Brouwer property of having the star of each simplex admit a flat Euclidean embedding).

3. X may have inequivalent triangulations.

4. X may not admit a differential structure. It may admit several inequivalent such structures. These statements are true even if X admits a simplicial structure.

5. Every differentiable manifold has a unique simplicial structure with compatibility.

Even for compact manifolds there are examples
1) of manifolds of the same homotopy type which are not homeomorphic;
2) of manifolds which are homeomorphic

and simplicial, and not simplicially isomorphic;
3) of simplicially isomorphic manifolds which are differentiable, and yet not differentiably isomorphic.

2. Homology / cohomology.

(A) Fix a space X and coefficient group G.
<u>Singular chain group</u> $(C(X;G), \partial)$

If R is commutative ring with unit, $C^*(X,R)$ with ring structure: If $u \in C^p$, $v \in C^q$, have $u \smile v \in C^{p+q}$

$$u \smile v\,(s_{p+q}) = u(s'_p)\, v\, s''_q, \text{ where } s_{p+q} = s'_p s''_q$$

$$\delta(u \smile v) = \delta u \smile v + (-1)^p u \smile \delta v$$

Thus $(C^*(X;R), \delta)$ determines <u>cohomology ring</u> $H^*(X;R)$. It is commutative, in sense $v \smile u = (-1)^{pq} u \smile v$.

A (cont) map $f: X \to Y$ induces group homomorphism $f_*: H(X;G) \to H(Y;G)$ and ring homomorphism

$$f^*: H^*(Y;R) \to H^*(X;R)$$

$$f^*(u \smile v) = f^*u \smile f^*v.$$

(B) From the pairing $\langle\ \rangle$

$$C^*(X,R) \times C(X,R) \to R$$

we have pairing <u>cap product</u>

$$C^p \times C_{p+q} \to C_q$$

given by

$$u \cap s_{p+q} = u(s'_p)\, s''_q, \text{ with}$$

$$\langle v, u \cap s_{p+q} \rangle = \langle v \cup u, s_{p+q} \rangle.$$

Then $\partial(u \cap s) = u \cap \partial s + (-1)^p \delta u \cap s$, so that we have induced pairing.

$$H^p \times H_{p+q} \xrightarrow{\cap} H_q.$$

Otherwise said, $\boxed{H(X,R) \text{ is an } H^*(X;R)\text{-module}}$.

We have change of ring formula for a map $f: X \to Y$:

$$\boxed{f_*(f^*v \cap c) = v \cap f_* c}$$

(C) As a first crack at duality:

<u>Theorem</u> <u>If X is compact n-manifold then $H(X;\mathbb{Z}_2)$ is a free rank 1 $H^*(X;\mathbb{Z}_2)$-module. There is a unique homology class $X \in H_n(X;\mathbb{Z}_2)$ such that $\mathcal{D}: H^p(X;\mathbb{Z}_2) \to H_{n-p}(X;\mathbb{Z}_2)$ is an isomorphism $\forall p$, where</u>

$$\mathcal{D}u = u \cap X.$$

<u>Df</u> If $f: X \to Y$ is a map between compact n-manifolds, its <u>$\mathbb{Z}_2$-degree</u> $d_f \in \mathbb{Z}_2$ is given by $f_*(X) = d_f Y \in H_n(Y;\mathbb{Z}_2)$.

<u>Prop</u> <u>If $d_f \neq 0$ then f^* embeds $H^*(Y;\mathbb{Z}_2)$ into $H^*(X;\mathbb{Z}_2)$.</u>

Pf. $f_*(\mathcal{D}f^*u) = u \cap f_*X = d_f \mathcal{D}u$. //

<u>Example</u>. The Euler characteristic

$$\chi(X) = \sum_p (-1)^p \beta_p(X;\mathbb{Z}_2) = 0 \text{ if } n \text{ is odd}$$
$$\equiv \beta_k(X;\mathbb{Z}_2) \bmod 2 \text{ if } n = 2k.$$

(D) Define the <u>intersection pairing</u>

$$\circ : H_{n-p}(X;\mathbb{Z}_2) \times H_{n-q}(X;\mathbb{Z}_2) \to H_{n-(p+q)}(X;\mathbb{Z}_2)$$

by $a \circ b = \mathcal{D}(\mathcal{D}^{-1}a \cup \mathcal{D}^{-1}b)$.

Given a map $f : X \to Y$ between compact manifolds of dimensions m, n, the <u>umkehrungshomomorphismus</u>

$$\begin{array}{ccc} f_! : H_{m-p}(Y;\mathbb{Z}_2) & \longrightarrow & H_{n-p}(X;\mathbb{Z}_2) \\ \mathcal{D}_Y \uparrow & & \uparrow \mathcal{D}_X \\ H^p(Y;\mathbb{Z}_2) & \xrightarrow{f^*} & H^p(X;\mathbb{Z}_2) \end{array}$$

is defined by $f_! = \mathcal{D}_X \circ f^* \circ \mathcal{D}_Y^{-1}$.

<u>Alternative definition of cup product</u>:

$$\sum_{p+q=n} H_p(X,R) \otimes H_q(X,R) \xrightarrow{\alpha} H_n(X \times X, R)$$

embeds as a direct summand. Ditto for

$$\sum_{p+q=n} H^p(X,R) \otimes H^q(X,R) \xrightarrow{\alpha} H^n(X \times X, R).$$

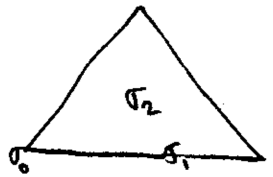
The diagonal map $\Delta : X \to X \times X$ induces $H^n(X \times X, R) \xrightarrow{\Delta^*} H^n(X, R)$, and

$$\Delta^* \alpha(u \otimes v) = u \cup v.$$

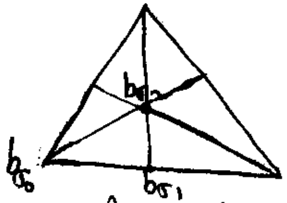
By way of contrast, we observe that there is not a corresponding composition in homology.

3. Simplicial manifolds.

(A) Let X be a simplicial n-manifold and K a simplicial realisation. We denote by K′ its first barycentric subdivision; thus the vertices of K′ are identified with the simplices of K (correspondence written $b_\sigma \longleftrightarrow \sigma$), and the simplices of K are the finite increasing sequences $b_{\sigma_0} \dots b_{\sigma_p}$ with σ_{i-1} a face of σ_i in K.



In K



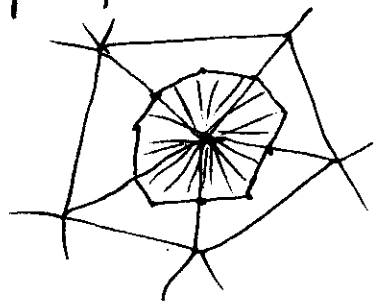
In K′

Say that a p-simplex σ' of K′ is dual to a simplex σ of K if $\dim \sigma = n-p$ and $\sigma' = (b_{\sigma_p} \dots b_{\sigma_n})$ with $\sigma_p = \sigma$. Let $\mathcal{D}\sigma$ be the subpolyhedron of K′ consisting of all dual simplices, together with their boundaries.

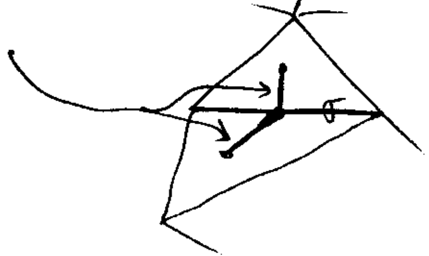
Thus

1) If $p = n$, then $\mathcal{D}\sigma = b_\sigma$;

2) if $p = 0$, then $\mathcal{D}\sigma$ = the n-cell



3) If $n = 2$ and $p = 1$, then $\mathcal{D}\sigma$ is the 1-cell



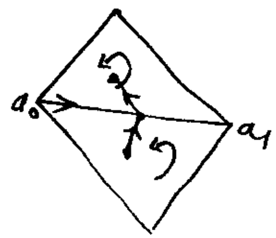
4) If $\sigma = (a_0 \dots a_p)$ then

$$\mathcal{D}\sigma = \bigcap_{i=0}^{p} \mathcal{D}a_i$$

5) Each $\mathcal{D}\sigma_p$ is a cone at b_{σ_p} over $\bigcup\{\mathcal{D}\sigma_{p+1} : \sigma_{p+1} \text{ is a coface of } \sigma_p\}$. In particular, each $\mathcal{D}\sigma$ is a cell.

The cells $\mathcal{D}_\sigma$ form a complex $K_\perp$, called the <u>dual complex of</u> K. Its cells are in bijective correspondence ($\sigma \longleftrightarrow \mathcal{D}_\sigma$) with the simplices of K.

6) If we fix an orientation of the simplices of K, that determines an orientation of the dual simplices of K'. If $\sigma = (a_0 \dots a_p)$ then we can write $\mathcal{D}\sigma = \sum \sigma'$ as an (n-p)-chain of K'.



7) We can define the coboundary operator δ directly on the simplices of K. With it we have the

<u>Proposition</u>. <u>Let</u> X <u>be a compact n-manifold</u>. <u>Then $\mathcal{D}$ induces an isomorphism</u>

$$\mathcal{D} : C^p(K;\mathbb{Z}_2) \longrightarrow C_{n-p}(K_\perp;\mathbb{Z}_2),$$

<u>with</u> $\mathcal{D}\delta u = \partial\mathcal{D} u$.

It follows that $\mathcal{D}$ induces an isomorphism at cohomology/homology level — and we know that these groups are topological invariants. In this fashion we can obtain a proof of the version of Poincaré duality stated in § 2.

(B) We now introduce the basic notion of <u>orientability</u>; this can be defined for those simplicial complexes K which are <u>n-circuits</u>; i.e., such that

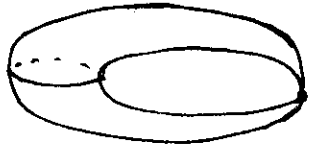
a) every simplex of K is the face of some n-simplex;

b) every (n-1)-simplex is the common face of exactly two n-simplices;

c) any two n-simplices can be joined by a sequence of n- and (n-1)-simplices.

Say that an n-circuit is <u>orientable</u> if we can orient each n-simplex so that each oriented $(n-1)$-simplex (induced orientation) is the oriented face of exactly 1 positively oriented n-simplex (and ∴ of exactly 1 negatively oriented n-simplex.

Every simplicial n-manifold is an n-circuit. A pinched torus is an orientable n-circuit which is not a manifold:



If K is a finite oriented n-manifold then its oriented n-simplices form an n-cycle; so the n-cells of $K_\perp$. (If K is nonorientable, then K and $K_\perp$ have n-cycles mod 2, a key feature of the preceding proposition.) These various n-cycles are nonhomologous to zero, and are called the fundamental n-cycles of K.

(C) Let K be an oriented finite simplicial manifold. Choose an order of its vertices, coherent with the orientation. Let $\rho : K' \to K$ be the simplicial map defined by letting $\rho(b_\sigma) =$ first vertex of σ. (Then at homology level, ρ_* is the inverse isomorphism of π_*, where $\pi(\sigma) = \sum \sigma'$ is the standard barycentric subdivision map). Let $\rho^\# : C^p(K) \to C^p(K')$ be the induced cochain map. We define characteristic cochains of K: For each oriented simplex σ of K let

$$\langle u_\sigma, \tau \rangle = \pm 1 \text{ if } \tau = \pm\sigma$$
$$= 0 \text{ otherwise.}$$

If $z_n' = \sum \sigma_n'$ is the fundamental n-cycle $/\mathbb{Z}$ of the oriented K', we define now

$$\mathcal{D} u_{\sigma_p} = \rho^\# u_{\sigma_p} \cap z_n',$$

using the same formula for $\cap$ as in the singular case

Proposition. Extending $\mathcal{D}$ by linearity gives a homomorphism $\mathcal{D} : C^p(K) \to C_{n-p}(K')$ satisfying $\partial \mathcal{D}(u) = (-1)^{n-p} \mathcal{D} \delta u$.

Proof $\partial \mathcal{D} u = \partial(\rho^\# u \cap z_n') = \cancelto{0}{\rho^\# u \cap \partial z_n'} + (-1)^{n-p} \delta(\rho^\# u) \cap z_n' = (-1)^{n-p} \mathcal{D} \delta u$. //

Consolidation of the dual simplices of K' into dual cells of $K^\perp$ leads to the isomorphism

$$\mathcal{D} : H^p(K) \to H_{n-p}(K_\perp):$$

Theorem Let X be a compact oriented simplicial n-manifold. Then the fundamental n-cycle of any simplicial realisation determines a class $X \in H_n(K_\perp)$ such that $\mathcal{D} u = u \cap X$ defines an isomorphism

$$\mathcal{D} : H^p(X; \mathbb{Z}) \to H_{n-p}(X; \mathbb{Z})$$

for all p.

Remark. If X is compact and oriented, then $H_p(X) = F_p \oplus T_p$, where $F_p = \mathbb{Z}^{\beta_p}$ and $T_p = \mathbb{Z}_{\tau_p^1} \oplus \cdots \oplus \mathbb{Z}_{\tau_p^{k_p}}$ with $\tau_p^i \mid \tau_p^{i+1}$ the torsion coefficients of T_p. Similarly, $H^p(X) = F_p \oplus T_{p-1}$. We have

$$\beta_p = \beta_{n-p} \quad \text{and} \quad \tau_{p-1}^i = \tau_{n-p}^i \quad (1 \le i \le k_p).$$

Note in particular that $F_n = \mathbb{Z}$, $T_{n-1} = 0$, which can be proved directly for compact oriented n-circuits; along with $F_n = 0$, $T_{n-1} = \mathbb{Z}_2$ if X is a nonorientable n-circuit.

(D) Example. If V is an n-dimensional vector space $/\mathbb{R}$, an orientation of V is a choice of ray in the 1-dimensional space $\Lambda^n V$. Given a Euclidean structure on V, let α_0 be the unit vector in the orientation ray. Then we have the duality isomorphism

$$\mathcal{D}: \Lambda^p V^* \longrightarrow \Lambda^{n-p} V$$

given by $\mathcal{D}\omega = \omega \lrcorner \alpha_0$ (interior product; i.e., for all $\theta \in \Lambda^{n-p} V^*$,

$$\langle \theta, \mathcal{D}\omega \rangle = \langle \theta \wedge \omega, \alpha_0 \rangle.$$

Thus $\mathcal{D}$ is the duality of projective geometry. Combining $\mathcal{D}$ with the isomorphism $\Phi: \Lambda^{n-p} V \longrightarrow \Lambda^{n-p} V^*$ given by the induced Euclidean structure, we have the star isomorphism

$$\begin{tikzcd} \Lambda^p V^* \arrow[rr, "*"] \arrow[dr, "\mathcal{D}"'] & & \Lambda^{n-p} V^* \\ & \Lambda^{n-p} V \arrow[ur, "\Phi"'] & \end{tikzcd}$$

If X is a compact oriented smooth Riemannian n-manifold, then $H^p(X;\mathbb{R})$ can be represented by harmonic differential forms; these are unique in each real cohomology class. If ω is a harmonic p-form, then $*\omega$ is a harmonic $(n-p)$form. I.e., $*$ gives Poincaré duality$/\mathbb{R}$ in the Riemannian case.

(D) Example. Let X be a compact oriented $2k$ manifold. Consider $H^k(X;\mathbb{Q})$; then the cup product defines a bilinear form $H^k(X;\mathbb{Q}) \times H^k(X;\mathbb{Q}) \longrightarrow \mathbb{Q}$, which is

$$\left.\begin{array}{c}\text{skew symmetric}\\ \text{symmetric}\end{array}\right\} \text{if } k \text{ is } \left.\begin{array}{c}\text{odd}\\ \text{even}\end{array}\right\}$$

The index (or signature) of a $4r$-dimensional compact oriented manifold is the signature of that nondegenerate quadratic form; i.e.,

$$\text{sign}(X) = \text{number}(+\text{ eigenvalues}) - \text{number}(-\text{ eigenvalues}).$$

Calculate the index of some easy examples.

In the case of $\dim X = 4r+2$, that index form is skew symmetric and nondegenerate. Then $\beta_k(X) \equiv 0 \bmod 2$, so $\chi(X)$ is even.

(E) Example. Let X be a compact oriented simplicial n-manifold, and $f: X \to X$ a map; let $\Gamma_f = \{(x, f(x)) \in X \times X\}$ denote its graph. Then Γ_f and $\Delta = \Gamma_{id}$ represent classes in $H^n(X \times X)$. Their

intersection class $\Gamma_f \circ \Delta$ defines an integer Λ_f called the <u>Lefschetz number</u> of f, giving the algebraic number of fixed points of f. It can also be represented by
$$\Lambda_f = \sum (-1)^p \operatorname{Trace} f_{p*} = \sum (-1)^p \operatorname{Trace} f_{p\#},$$
where $f_{p*}: H_p(X;\mathbb{Q}) \to H_p(X;\mathbb{Q})$ is the homomorphism induced from $f_{p\#}: C_p(X;\mathbb{Q}) \to C_p(X;\mathbb{Q})$ induced from f.

In particular, $\Lambda_{id} = \Gamma_{id} \circ \Gamma_{id} = \chi(X)$.

More generally the <u>coincidence index</u>
$$\Lambda_{f,g} = \Gamma_f \circ \Gamma_g$$
of two maps $f, g: X \to Y$ between two compact oriented simplicial n-manifolds has the representation
$$\Lambda_{f,g} = \sum (-1)^p \operatorname{Trace} \mathcal{D}_X\, g^{n-p*}\, \mathcal{D}_Y^{-1}\, f_p^*.$$

$\clubsuit$ In the smooth case, if the fixed points (a_i) of $f: X \to X$ are isolated then
$$\Lambda_f = \sum_i \operatorname{sign} \det (I - f_*(a_i)).$$

4. Differential manifolds.

(A) Suppose X is a smooth n-manifold. We denote its tangent vector bundle by TX, its dual bundle by T^*X. The sections of its exterior p^{th} power $\Lambda^p T^*X$ are called <u>(exterior) differential p-forms on X</u>. Let $\mathcal{A}^p$ denote the vector space $/\mathbb{R}$ of the smooth exterior p-forms on X. The direct sum $\mathcal{A}^* = \sum_{p=0}^{n} \mathcal{A}^p$ is a commutative algebra $/\mathbb{R}$ with unit 1 (viewed as the constant function $X \to \mathbb{R}$), and multiplication
$$(\theta \wedge \omega)(v_1, \dots, v_{p+q}) =$$
$$\sum \varepsilon_\sigma\, \theta(v_{\sigma 1}, \dots, v_{\sigma p})\, \omega(v_{\sigma(p+1)}, \dots, v_{\sigma(p+q)})$$
summed over all permutations σ of $(1, \dots, p+q)$ and ε_σ the sign of σ, with $\sigma(1) < \dots < \sigma(p)$ and $\sigma(p+1) < \dots < \sigma(p+q)$.

Exterior differentiation gives an antiderivation $d: \mathcal{A}^* \to \mathcal{A}^*$:

1) $d: \mathcal{A}^p \to \mathcal{A}^p$ is linear;
2) $d(\theta \wedge \omega) = d\theta \wedge \omega + (-1)^p \theta \wedge d\omega$ for

$\theta \in \mathcal{A}^p$, $\omega \in \mathcal{A}^*$;

3) $df = \frac{\partial f}{\partial x^i} dx^i$ for $f \in \mathcal{A}^0$ in any chart;

4) $dd\omega = 0$ for all $\omega \in \mathcal{A}^*$.

5) If $f : X \longrightarrow Y$ is a smooth map between manifolds, then f induces an algebra homomorphism $f^* : \mathcal{A}^*(Y) \longrightarrow \mathcal{A}^*(X)$, with $f^* d = d f^*$.

(B) Let $\mathcal{C} = \mathcal{C}(X, \mathbb{R})$ denote the vector space$/\mathbb{R}$ of smooth p-chains on X. We have the bilinear pairing

$$\int : \mathcal{A}^p \times \mathcal{C}_p \longrightarrow \mathbb{R}$$

given by $(\omega, s) \rightarrow \int_s \omega$.

Stokes' theorem

$$\int_s d\omega = \int_{\partial s} \omega$$

shows that $\int$ induces a bilinear pairing

$$\int : H^p(\mathcal{A}^*, d) \times H_p(\mathcal{C}, \partial) \longrightarrow \mathbb{R}$$

Thus $\mathrm{Hom}(H(\mathcal{C}, \partial), \mathbb{R}) = H^p(\mathcal{A}^*, d)$

__Lemma.__ __The inclusion map induces an identification__ $H(\mathcal{C}, \partial) = H(X; \mathbb{R})$, i.e., the homology of smooth chains is the same as that based on continuous chains; in particular, is a topological invariant of X.

__deRham's Theorem.__ __Integration induces an algebra isomorphism__

$$\int : H^*(\mathcal{A}^*, d) \longrightarrow H^*(X; \mathbb{R}).$$

(C) We can define orientability of a smooth manifold X as follows: Say X __is orientable__ if there is a smooth atlas $\mathcal{U} = (U_i, \theta_i)_{i \in I}$ of X such that the Jacobians of $\theta_i \cdot \theta_j^{-1} > 0$ whenever $U_i \cap U_j \neq \emptyset$. To orient X is to choose such an atlas.

__Example.__ P^n is orientable for $n \equiv 1 (2)$;
P^n is nonorientable for $n \equiv 0 (2)$.

(D) Suppose now that X is compact and oriented. Then de Rham's theorem can be formulated as follows:

let $z_1, \dots, z_{\beta_p}$ be a base for $H_p(X;\mathbb{R})$, and let $z_1, \dots, z_{\beta_p}$ also denote cycles representing those classes. Then

1) If ω is a closed p-form ($d\omega = 0$) such that

$$\int_{z_j} \omega = 0 \text{ for } 1 \leq j \leq \beta_p,$$

then $\exists\ \eta \in \mathcal{A}^{p-1}$ such that $d\eta = \omega$.

2) Given numbers $(\pi_j)_{1 \leq j \leq \beta_p}$, $\exists$ closed p-form ω:

$$\int_{z_j} \omega = \pi_j \qquad (1 \leq j \leq \beta_p).$$

3) If $\omega \in \mathcal{A}^p$, $\theta \in \mathcal{A}^{n-p}$ are closed forms then

$$\int_X \omega \wedge \theta = \int_{\omega \cap X} \theta = \int_{\mathcal{D}\omega} \theta. \text{ And}$$

$$\int_X : H^p(\mathcal{A}^*, d) \times H^{n-p}(\mathcal{A}^*, d) \longrightarrow \mathbb{R}$$

is a nonsingular bilinear pairing. Otherwise said, de Rham's theorem (part 3)) determines an isomorphism

$$H^p(X;\mathbb{R}) \longrightarrow H_{n-p}(X;\mathbb{R}).$$

(E) The degree of a map $f : X \to Y$ between compact oriented n-manifolds can be defined

1) in the simplicial case, taking a simplicial approximation and making an algebraic count of the number of n-simplexes (oriented) in $f^{-1}(\tau)$ for an n-simplex τ of Y.

2) in the differentiable case, as follows:

a) for any n-form $\omega \in \mathcal{A}^n(Y)$ we have

$$\int_X f^*\omega = d_f \int_Y \omega.$$

b) for any regular value $b \in Y$ (i.e., point such that for any $x \in f^{-1}(y)$ the differential $f_*(x)$ is an isomorphism),

$$d_f = \sum_{x \in f^{-1}(b)} \operatorname{sign} f_*(x).$$

5. Alexander – Pontrjagin duality

(A) **Theorem.** Let X be a connected oriented n-manifold and Y a finite polyhedron in X. Then for any coefficient group G, there is a canonical isomorphism

$$a : H_{n-p}(X, X-Y; G) \xrightarrow{\approx} H^p(Y; G)$$

for all p.

From the corollaries below, interpret the isomorphism a.

Corollary. If $\dim Y = r$, then $H_{n-p}(X, X-Y) = 0$ for $p > r$.

Corollary. If Y is a closed oriented submanifold with $\operatorname{codim}(X,Y) = m$, then we have a canonical isomorphism

$$\begin{tikzcd} H_q(X, X-Y) \arrow[rr, "\alpha"] \arrow[dr, "a"'] & & H_{q-m}(Y) \\ & H^{n-q}(Y) \arrow[ur, "\mathcal{D}_Y"'] & \end{tikzcd}$$

There are analogous isomorphisms in cohomology:

Example. Letting $\Delta : X \to X \times X$ be the diagonal embedding, we have the isomorphism

$$\alpha : H^{q-n}(X) \xrightarrow{\approx} H^q(X\times X, X\times X - \Delta).$$

In particular, taking $q = n$, consider the class $\alpha(1) \in H^n(X \times X, X\times X - \Delta)$.

(B) **Corollary.** If for some r we have

$$H_{r-1}(X; G) = 0 = H_r(X; G),$$

then for any finite polyhedron $Y \subset X$ we have the linking isomorphism

$$\mathcal{L} : H^{n-r}(Y; G) \xrightarrow{\approx} H_{r-1}(X-Y; G).$$

Proof. From the homology sequence of $(X, X-Y)$:

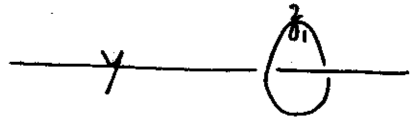
$$\cdots \to \overset{0}{\cancel{H_r(X;G)}} \to H_r(X, X-Y; G) \xrightarrow{\partial} H_{r-1}(X-Y; G) \to \overset{0}{\cancel{H_{r-1}(X;G)}}$$

combined with Alexander – Pontrjagin duality, we see that $\mathcal{L} = \partial \circ a^{-1}$ is an isomorphism in the appropriate dimension.

Example. For $1<r<n$ $(n \geq 3)$ we have for $Y \subset S^n$,

$$\mathcal{L}: H^{n-r}(Y) \xrightarrow{\approx} H_{r-1}(S^n - Y).$$

For instance, $H^1(Y) \approx H_1(S^3 - Y)$. If we think of S^3 as $\mathbb{R}^3 \cup \infty$ and take $Y = x_1$ axis, then



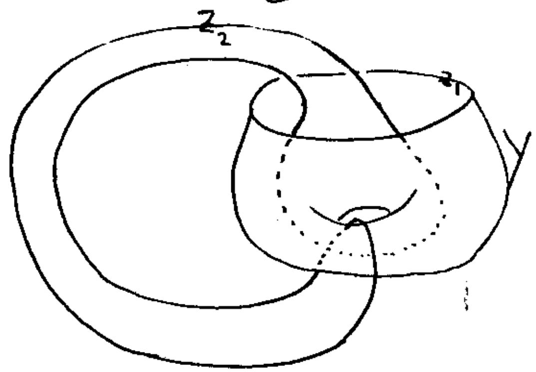
More generally, if we write $\mathbb{R}^{n+1} = \mathbb{R}_p \times \mathbb{R}^{n-p+1}$ where $\mathbb{R}_p = \{(x_1, \dots, x_p, 0, \dots, 0)\}$

$\mathbb{R}^{n-p+1} = \{(0, \dots, 0, x_{p+1} \cdots x_{n+1})\}$,

then their unit spheres S_{p-1} and S^{n-p} are linked.

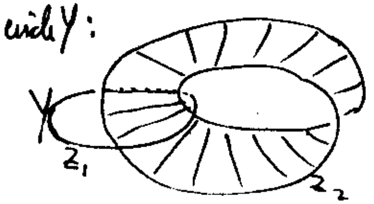
Example. Take $X = \mathbb{R}^2$, $Y = 0$, $r = 2$: $\mathbb{Z} = H^0(0) = H_1(\mathbb{R}^2 - 0)$. The circle S^1 links 0.

Example. To illustrate the essential homological character of linking, Alexandroff-Hopf draw this picture of a jug Y with handle:



$z_1 \sim 0$ in Y; z_1, z_2 are not linked in Y, $S^3 - Y$.

Example. To illustrate the importance of coefficients, consider circle Y:



in S^3

z_1 and z_2 are linked generators in $H^1(Y) \approx H_1(\mathbb{R}^3 - Y)$. But $z_2 \sim 0$ mod 2 so that z_1 and z_2 are not linked $/\mathbb{Z}_2$.

(C) Corollary (Jordan-Brouwer). Let X be an oriented n-manifold with $\beta_1(X) = 0$. Let Y be a compact oriented $(n-1)$-submanifold with k components. Then $X - Y$ has $k+1$ components.

Proof.

$$\begin{tikzcd} H_1(X;\mathbb{Q}) \arrow[r] \arrow[d, phantom, "\shortparallel"] & H_1(X,X-Y;\mathbb{Q}) \arrow[r, "\partial"] \arrow[d, "a", "\approx"'] & H_0(X-Y;\mathbb{Q}) \arrow[r] & H_0(X;\mathbb{Q}) \arrow[r] \arrow[d, phantom, "\wr\shortparallel"] & 0 \\ 0 & H^{n-1}(Y;\mathbb{Q}), & & \mathbb{Q} & \end{tikzcd}$$

so

$$H_0(X-Y;\mathbb{Q}) \approx \mathbb{Q} \oplus H^{n-1}(Y;\mathbb{Q});$$

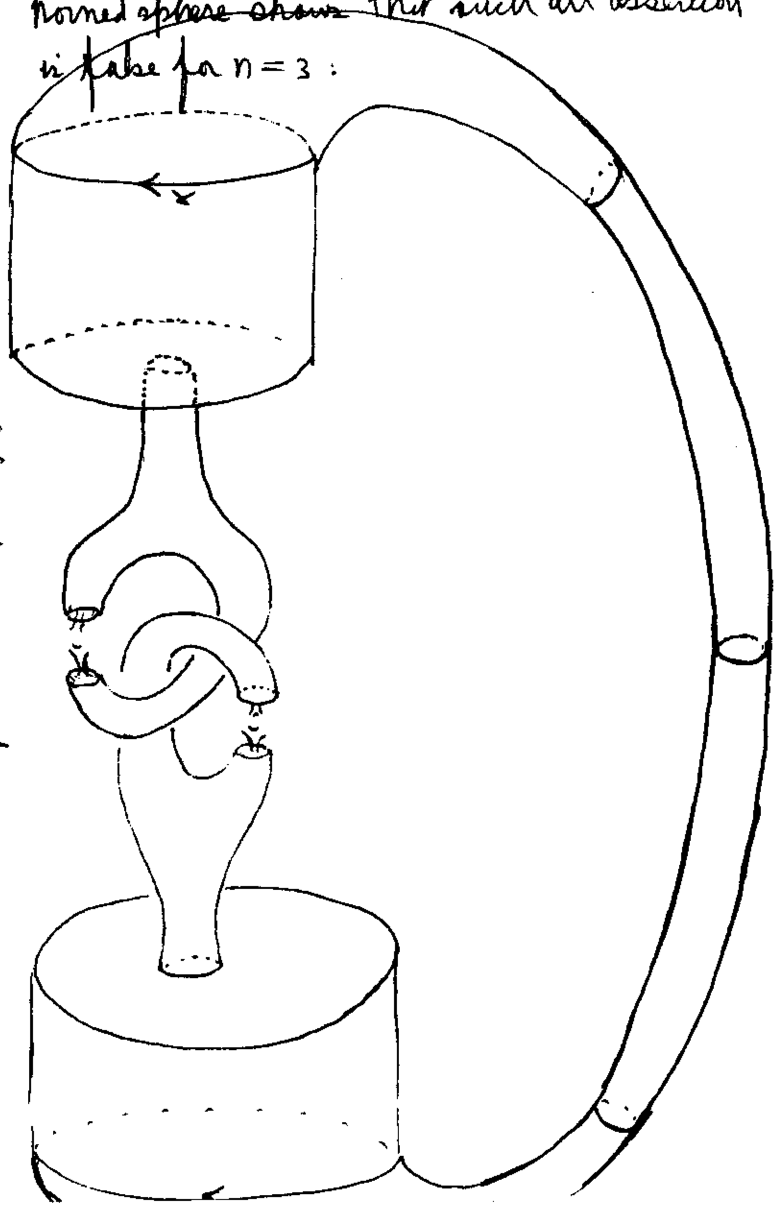
$$\beta_0(X-Y) = 1 + k \quad //$$

Examples. The real projective spaces $P^{2k} \hookrightarrow P^{2k+1}$ show the need of orientability of Y: $\beta_1(P^{2k+1}) = 0$, $k=1$; $\beta_0(P^{2k+1} - P^{2k}) = 1$.

The circle $S^1 \subset T^2$ embedded as a meridian shows the need of the condition $\beta_1(X) = 0$.

Example. Let $X = S^n$ $(n \geqslant 2)$ and Y a homeomorphic image of S^{n-1} (we shall obtain the isomorphism a for arbitrary closed sets in due course). Then Y separates S^n into 2 subsets having Y as common boundary. When $n=2$ that is the

Jordan curve theorem; for that dimension there is the complementary property that both components are open 2-cells. Alexander's horned sphere shows that such an assertion is false for $n=3$:



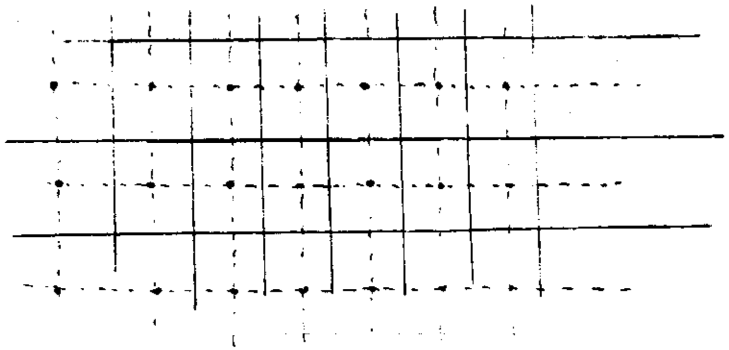
This surface is homeomorphic to S^2.
The bounded component of its complement in $\mathbb{R}^3$ is a 3 cell.
The unbounded component has infinitely generated π_1.

(D) Remark. Simplicially speaking, if z_{p-1} is a $(p-1)$cycle of a simplicial subdivision K of S^n (for simplicity) and $z^{\perp}_{n-p}$ an $(n-p)$-cycle of $K_{\perp}$, one of which bounds in S^n (say $z_{p-1} = \partial c_p$), then the linking $\mathcal{L}(z^{\perp}_{n-p}, z_{p-1}) = z_{n-p} \circ c_p \in \mathbb{Z}$.

That gives the homomorphism of Alexander duality: If Y is a closed polyhedron in K then any $(n-p)$cycle of $K_{\perp}$ bounds in S^n and is in general position with respect to Y.
Linking defines a bilinear pairing of cycles

$$Z_{n-p}(S^n - Y) \times Z_{p-1}(Y) \to \mathbb{Z}.$$

Alexandroff-Hopf presented Alexander duality in case $X = \mathbb{R}^n$ by considering dual subdivisions by cubes



(E) Linking can be described in terms of differential forms in the smooth case!

Example. In $\mathbb{R}^n - 0$ $(n \geq 0)$ the $(n-1)$ form

$$\omega(x) = k(n) \sum_{i=1}^{n} (-1)^{i+1} x_i (x_1^2 + \dots + x_n^2)^{-n/2} dx_1 \wedge \dots \wedge \widehat{dx_i} \wedge \dots \wedge dx_n$$

is closed. For any oriented n-disc D whose boundary ∂D does not contain 0 we have

$$\int_{\partial D} \omega = \left. \begin{array}{ll} \pm 1 & \text{if } 0 \in D \\ 0 & \text{otherwise} \end{array} \right\} \quad k(n)^{-1} = \operatorname{vol} S^{n-1}.$$

Example. Let $f : S^{2n-1} \to S^n$ $(n \geq 2)$ be a smooth map transversal over $a, c \in S^n$. Then $f^{-1}(a)$ and $f^{-1}(c)$ are smooth oriented $(n-1)$-manifolds. Their linking number $\gamma_f \in \mathbb{Z}$ is a homotopy invariant of f, and can be represented as follows: Let $\omega \in \mathcal{A}^n(S^n)$ be dual to $1 \in \mathcal{A}^0(S^n)$, and choose $\xi \in \mathcal{A}^{n-1}(S^{2n-1})$ such that $d\xi = f^*\omega$. Then

$$\gamma_f = \int_{S^{2n-1}} \xi \wedge f^*\omega = \int_{f^{-1}(a)} \xi .$$

γ_f is the Hopf invariant of f.

Chapter II. Sheaf machinery

1. Sheaves

(A) A continuous surjective map $\pi : \mathcal{S} \to X$ between topological spaces defines a sheaf if every point $s \in \mathcal{S}$ has an open neighborhood O such that $\pi | O$ is a homeomorphism of O onto an open neighborhood of $\pi(s)$ in X. Thus π is a local homeomorphism.

A sheaf of abelian groups (of rings, of modules ..) is a sheaf such that each stalk $\mathcal{S}_x = \pi^{-1}(x)$ is an abelian group, and the group operation $+$ is continuous: If $\mathcal{S} +_\pi \mathcal{S} = \{ (s,t) \in \mathcal{S} \times \mathcal{S} : \pi(s) = \pi(t) \}$, then $(s,t) \to s-t$ is a continuous map $\mathcal{S} +_\pi \mathcal{S} \to \mathcal{S}$.

A homomorphism of sheaves over the same space X

$$\begin{tikzcd} \mathcal{S} \arrow[rr, "h"] \arrow[dr, "\pi"'] & & \tilde{\mathcal{S}} \arrow[dl, "\tilde{\pi}"] \\ & X & \end{tikzcd}$$

is a continuous map $h : \mathcal{S} \to \tilde{\mathcal{S}}$ preserving stalks (i.e., $\pi = \tilde{\pi} \circ h$). If these are sheaves

of abelian groups, then we require that each $h|_{\mathcal{S}_x} : \mathcal{S}_x \to \tilde{\mathcal{S}}_x$ is a group homomorphism. Note that h is a local homeomorphism.

(B) Sheaves are usually constructed via presheaves (S_U, r^U_V):
Suppose we assign a set S_U to each open $U \subset X$ and a map

$$r^U_V : S_U \to S_V$$

to each pair of open subsets with $V \subset U$:

a) $r^U_U = id$

b) $W \subset V \subset U \Rightarrow r^U_W = r^V_W r^U_V$.

If each S_U is an abelian group, of course we require r^U_V to be a homomorphism.

For each $x \in X$ let $\mathcal{S}_x = \lim_{U \to x} S_U$.
direct limit through the neighborhoods of x.
If $f \in S_U$ then f determines an $f_x \in \mathcal{S}_x$ called the germ of f.

Let $\mathcal{S} = \bigcup \{ \mathcal{S}_x : x \in X \}$, and $\pi : \mathcal{S} \to X$

is obviously defined. A base for the topology of $\mathcal{S}$ is given by the subsets $\{r^{U}_{x}(f) : x \in U\}$ as U varies over the open subsets of X and f varies over U. It is a straight-forward matter to verify that we have defined a sheaf from the data of a presheaf.

A *section* of a sheaf $\pi : \mathcal{S} \to X$ over an open $U \subset X$ is a continuous map $s : U \to \mathcal{S}$ such that $\pi \circ s = id_U$. The sections over U form a set S_U; and if $V \subset U$ then restriction $r^U_V : S_U \to S_V$ is defined; (S_U, r^U_V) is a presheaf, the *canonical presheaf* of the sheaf $\pi : \mathcal{S} \to X$.

(C) A *subsheaf* $\pi' : \mathcal{S}' \to X$ of a sheaf $\pi : \mathcal{S} \to X$ requires

a) $\mathcal{S}'$ open in $\mathcal{S}$

b) $\pi | \mathcal{S}' = \pi'$, and maps onto X.

If $\pi : \mathcal{S} \to X$ is a sheaf of abelian groups, then $\pi'^{-1}(x) = \mathcal{S}' \cap \pi^{-1}(x)$ is a subgroup of $\mathcal{S}_x$ for all $x \in X$.

(D) *Example 1.* A very important sheaf of abelian groups is given by the product $\mathcal{S} = X \times G$ of a topological space X and an abelian group G with its discrete topology. $\pi : X \times G \to X$ is the projection map. Such sheaves are said to be *simple*.

Example 2. For any space X let $S_U = C^0(U, \mathbb{R})$ be the vector space $/\mathbb{R}$ of continuous functions $U \to \mathbb{R}$ (or of all functions $U \to \mathbb{R}$; or of all continuous functions on U with values in a vector bundle $/X$, etc.) With restrictions r^U_V we have a presheaf of vector spaces, determining the sheaf of germs of continuous functions $\mathcal{C}_X(\mathbb{R})$ on X (The space $\mathcal{S}$ of these sheaves are not Hausdorff.)

Similarly, if X is a smooth manifold, we can define the sheaf of germs of smooth functions (of germs of smooth sections of a smooth vector bundle over X). Similarly if X is a domain in $\mathbb{C}^n$, we can define the sheaf of germs of holomorphic functions. The first sheaf construction was made by Riemann — his covering Riemann surfaces.

Example 3. A locally simple sheaf has components which are *covering spaces*.

Example 4. Given sheaves $\pi : \mathcal{S} \to X$, $\rho : \mathcal{T} \to Y$ and a map $f : X \to Y$, an f-*homomorphism* $\bar{f} : \mathcal{S} \to \mathcal{T}$ is a continuous map wrt which the diagram

$$\begin{tikzcd} \mathcal{S} \arrow[r, "\bar{f}"] \arrow[d, "\pi"'] & \mathcal{T} \arrow[d, "\rho"] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

is commutative, and the restrictions $\bar{f} | \mathcal{S}_x \to \mathcal{T}_{f(x)}$ are homomorphisms. Say that $\pi : \mathcal{S} \to X$ is *modeled on* $\rho : \mathcal{T} \to Y$ if every $x \in X$ has a neighbourhood U and homeomorphism $f : U \to Y$ and $\bar{f} : \mathcal{S}|U \to \mathcal{T}$ such that $\bar{f}$ is an f-isomorphism. Thus a *differential structure* on a topological n-manifold is a subsheaf $\mathcal{D}_{X,\mathbb{R}}$ of $\mathcal{C}_{X,\mathbb{R}}$ which is modeled on $\mathcal{D}_{\mathbb{R}^n,\mathbb{R}}$.

(E) *Lemma.* *If* $\to S_n \to S_{n+1} \to \cdots$ *is an exact sequence of presheaves of abelian groups* $/X$, *then the induced sequence*

$$\to \mathcal{S}_n \to \mathcal{S}_{n+1} \to \cdots$$

of sheaves is exact.

Lemma. *If* $\mathcal{S}'$ *is a subsheaf of a sheaf* $\mathcal{S}$ *of abelian groups, there is a sheaf* $\mathcal{S}''$, *unique up to isomorphism, and an epimorphism* $\mathcal{S} \to \mathcal{S}''$ *such that*

$$0 \to \mathcal{S}' \to \mathcal{S} \to \mathcal{S}'' \to 0$$

is exact.

2. The orientation sheaf.

(A) The local singular homology groups /$\mathbb{Z}$ of a space X at a point $x \in X$ are

$$LH_p(x) = H_p(X, X-x).$$

If U is any open or closed neighborhood of x, then $LH_p(x) = H_p(U, U-x)$; for $X = \mathrm{Int}(X-x) \cup \mathrm{Int}\, U$, to which we apply excision.

Proposition. If $X = \mathbb{R}^n$, then

$$LH_p(x) = \left.\begin{matrix} \mathbb{Z} & \text{for } p = n \\ 0 & \text{for } p \neq n \end{matrix}\right\}$$

Proof. Let D^n be an x-centred disc with boundary S^{n-1}, which is a deformation retract of $D^n - x$. The inclusion $\eta : (D^n, S^{n-1}) \to (D^n, D^n - x)$ induces an exact ladder

$$\begin{tikzcd}
\cdots \arrow[r] & H_p(S^{n-1}) \arrow[r] \arrow[d] & H_p(D^n) \arrow[r] \arrow[d] & H_p(D^n, S^{n-1}) \arrow[r, "\partial"] \arrow[d, "\eta_*"] & H_{p-1}(S^{n-1}) \arrow[r] \arrow[d] & \cdots \\
& H_p(D^n - x) \arrow[r] & H_p(D^n) \arrow[r] & H_p(D^n, D^n - x) \arrow[r] & H_{p-1}(D^n - x) \arrow[r] & \cdots
\end{tikzcd}$$

The 5-lemma insures that the indicated η_*-arrow is an isomorphism, as is ∂. //

A locally compact paracompact space X is a homology n-manifold /$\mathbb{Z}$ if the local homology groups at each of its points are those of $\mathbb{R}^n$.

(B) The support $|s|$ of a singular simplex $s : \Delta \to X$ is its image $s(\Delta)$. The support $|c|$ of a singular chain $c = \sum a_i s_i$ (reduced, so $a_i \neq 0$) is $\bigcup |s_i|$. We broaden our concept of singular chains to admit infinite sums $\sum a_i s_i$, provided that in its reduced form $(|s_i|)$ form a locally finite family in X; i.e., every $x \in X$ has a neighborhood meeting only finitely many $|s_i|$. We let $(\tilde{S}(X), \partial)$ denote the resulting complex of locally finite singular chains of X.

For any coefficient group G we define the sheaf $\tilde{\mathcal{S}} \to X$ of locally finite singular chains of X with coefficients in G through the presheaf

$$\tilde S_U = \tilde S(X;G)/\tilde S(X-U;G),$$

$$\tilde r^U_V : \tilde S_U \to \tilde S_V \quad \text{if} \quad U \supset V$$

the homomorphism induced by the inclusion

$$(\tilde S(X), \tilde S(X-U)) \to (\tilde S(X), \tilde S(X-V))$$

Lemma. Let X be a homology n-manifold. Then the inclusion map $\eta: S_U \to \tilde S_U$ induces an isomorphism of sheaves $\mathcal{S} \to \tilde{\mathcal{S}}$.

Here $(S_U = S(X)/S(X-U), r^U_V)$ is the presheaf of finite singular chains, defining the sheaf $\mathcal{S}$.

Proof. η is indeed a monomorphism, for if $c \in S(X)$ and $\eta(c) \in \tilde S(X-U)$, then $c \in S(X-U)$ for any open U with compact closure. For such a U, if $c = \sum a_i s_i \in \tilde S_U$, then only finitely many $|s_i| \cap \bar U \neq \emptyset$. Thus if we

$c = c' + c''$ with $|c''| \subset X-U$, we have $\eta(c') = c$; therefore η is surjective. //

(C) Let X be a homology manifold and consider the derived sheaf $H(\mathcal{S}) \approx H(\tilde{\mathcal{S}})$ determined by the derived presheaf

$$\{H_U = H(X, X-U), r^U_V\}.$$

The stalk H_x over each point is $LH(x)$. We shall call the sheaf $H(\mathcal{S}) \to X$ the orientation sheaf of X, and denote it by $\mathcal{T} \to X$. It is also called the sheaf of twisted integers $/X$. Of course, if H_U had been based on a coefficient group G, then we would have produced $\mathcal{T}_G \to X$, the sheaf of twisted groups G over X.

An n-manifold X has $\mathcal{T} \to X$ as covering space (not necessarily connected), because the orientation sheaf is locally simple. And X is

orientable if $\mathcal{T} \to X$ is globally simple. (More generally, for any coefficient group G we have the notion of orientability of X with respect to G.) To orient X at x is to choose one of the two isomorphisms
$$H_n(X, X-x; \mathbb{Z}) \to \mathbb{Z};$$
i.e, one of the two generators of $H_n(X, X-x)$; and X is orientable iff we can choose that isomorphism coherently over all X. We note that orientability is a topological invariant of X.

Exercise

(1) If X is a simplicial and G a coefficient group, let $p: \mathcal{T}_G \to X$ be the associated G-orientation sheaf. Then $\mathcal{T}_G$ is simple over the open star of each simplex; let $\mathcal{T}_\sigma = p^{-1}(b_\sigma)$. For each pair of simplices σ, τ such that σ is a face of τ we choose a path $w_{\sigma\tau}$ from b_τ to b_σ. Then $w_{\sigma\tau}$ induces an isomorphism $\mathcal{T}_\sigma \xrightarrow{\cong} \mathcal{T}_\tau$.

And $\sigma < \tau < \varsigma \Rightarrow w_{\tau\varsigma} w_{\sigma\tau} = w_{\sigma\varsigma}$. Use this to show that our two definitions of orientability of a simplicial manifold coincide.

Exercise. Show that our two definitions of orientability of a smooth manifold coincide.

3. Čech cohomology.

(A) Let $\mathcal{U}=(U_i)_{i\in I}$ be an indexed locally finite open cover of a space X. Its *nerve* $N(\mathcal{U})$ is the locally finite simplicial complex whose vertices are the U_i, and whose simplices are the nonvoid intersections $U_{i_0}\cap\cdots\cap U_{i_p}=U_{i_0\cdots i_p}$.

If $\mathcal{W}=(V_i)_{i\in I}$ is a shrunk refinement of $\mathcal{U}$ ($V_i \Subset U_i$ means $V_i\subset \overline{V_i}\subset U_i$ all $i\in I$), then we may take a partition of unity $(\lambda_i)_{i\in I}$ subordinate to $\mathcal{U}$ (i.e., the supports $|\lambda_i|\subset U_i$ for all $i\in I$ and $\lambda_i|_{V_i}\equiv 1$). Define the map $f: X\to N(\mathcal{U})$ by letting $f(x)=$ the point of $N(\mathcal{U})$ whose barycentric coordinates are $\lambda_i(x)$; we note that if $i_0,\dots,i_k$ are those indices for which $\lambda_{i_j}(x)>0$, then $x\in U_{i_0\cdots i_k}$. If a_i is the vertex of $N(\mathcal{U})$ corresponding to U_i, then $f^{-1}(\mathrm{St}\,a_i)\subset U_i$.

We remark that that construction yields *Alexandroff's approximation theorem*. *If X is a paracompact space and $\mathcal{U}=(U_i)_{i\in I}$ a locally finite open cover, then there is a map $f: X\to N(\mathcal{U})$ such that $f^{-1}(y)\in U_i$ (some $i\in I$) for every $y\in f(X)\subset N(\mathcal{U})$.*

Corollary: If X is compact and metric then for every $\varepsilon>0$ there is an ε-map of X onto a finite polyhedron N_ε (i.e., for all $y\in N_\varepsilon$, $\operatorname{diam} f^{-1}(y)<\varepsilon$).

Remark: If K is a locally finite complex and $\mathcal{U}=\{\mathrm{St}\,a : a \text{ vertex of } K\}$ then $K\cong N(\mathcal{U})$.

(B) If $\mathcal{W}=(V_j)_{j\in J}$ is any refinement of $\mathcal{U}=(U_i)_{i\in I}$ and we choose any map $\tau: J\to I$ between indexing sets such that $V_j\subset U_{\tau_j}$ for all $j\in J$, then τ defines a simplicial map $\tau: N(\mathcal{W})\to N(\mathcal{U})$. The induced homomorphism $\tau_*: H(N(\mathcal{W}))\to H(N(\mathcal{U}))$ is independent of the choices made in the definition of τ; see (C) below.

The open covers of X form a directed set under inclusion, relative to which $H(N(\mathcal{U}))$ form an inverse system with respect to the homomorphisms τ_*. The *Čech homology group* $\check{H}(X)=\varprojlim H(N\mathcal{U})$.

Similarly for Čech cohomology group

$$\check{H}^*(X) = \varinjlim H^*(N(\mathcal{U}))$$

<u>Remark</u>. Čech homology is useful for compact spaces — and less so otherwise. Čech cohomology is better (e.g., the direct limit of exact cohomology sequences based on coverings is again exact.)

(C) We next generalise that construction by considering the cohomology of X with coefficients in a sheaf.

Let $\mathcal{U}$ be an open cover of X and $S_0 = \{S_U, r^U_V\}$ a presheaf of (abelian groups) which is small of order $\mathcal{U}$; i.e., all the open sets U in indexing belong to some member of $\mathcal{U}$.

A <u>p-cochain of $\mathcal{U}$ with coefficients in S_0</u> is a function f assigning to each simplex $i_0 \dots i_p$ of $N(\mathcal{U})$ an element $f(i_0 \dots i_p) \in S_{i_0 \dots i_p} = S_{U_{i_0} \cap \dots U_{i_p}}$. We let $C^p(\mathcal{U}, S_0)$ denote the group of these p-cochains.

<u>The coboundary operator</u>

$$d : C^p(\mathcal{U}, S_0) \longrightarrow C^{p+1}(\mathcal{U}, S_0)$$

is the homomorphism defined by

$$(df)(i_0 \dots i_{p+1}) = \sum_{j=0}^{p+1} (-1)^j (r_j f)(i_0 \dots \hat{i}_j \dots i_{p+1}),$$

where r_j is the restriction map of the presheaf S_0 mapping $S_{i_0 \dots \hat{i}_j \dots i_{p+1}} \to S_{i_0 \dots i_{p+1}}$.

Then $dd = 0$.

If $\mathcal{W} = (V_j)_{j \in J}$ is a refinement of $\mathcal{U}$, we choose a map $\tau : J \to I$ with $V_j \subset U_{\tau j}$; that induces a homomorphism

$$\tau : C^p(\mathcal{U}, S_0) \to C^p(\mathcal{W}, S_0)$$

defined by

$$(\tau f)(j_0 \dots j_p) = r^\tau f(\tau j_0 \dots \tau j_p),$$

where r^τ is the restriction map

$$S_{\tau j_0 \dots \tau j_p} \to S_{j_0 \dots j_p}.$$

We have $d\tau = \tau d$, so τ induces the homomorphism

$$\tau^* = \tau^*(\mathcal{U}, \mathcal{W}) : H^p(\mathcal{U}, S_0) \to H^p(\mathcal{W}, S_0).$$

Lemma. *τ^* is independent of the choices made in its definition.*

Proof. If $\bar\tau$ is another such map, then $V_j \subset U_{\tau_j} \cap U_{\bar\tau_j}$. Define the homomorphism $k : C^p(\mathcal{V}, S_0) \to C^{p-1}(\mathcal{W}, S_0)$ by

$$(kf)(j_0 \ldots j_{p-1}) = \sum_{i=0}^{p-1} (-1)^i (r_i f)(\tau j_0 \ldots \tau j_i \bar\tau j_i \ldots \bar\tau j_{p-1}),$$

where r_i is the restriction map

$$S_{\tau j_0 \ldots \tau j_i \bar\tau j_i \ldots \bar\tau j_{p-1}} \to S_{j_0 \ldots j_{p-1}}.$$

Then
$$\left.\begin{aligned} \bar\tau - \tau &= dk + kd && \text{if } p \geq 1 \\ &= kd && \text{if } p = 0 \end{aligned}\right\}$$

Thus $\bar\tau^* = \tau^*$. //

Again, the $H^p(\mathcal{V}, S_0)$ form a direct system of abelian groups, whose limit we denote by

$$H^p(X, S_0) = \varinjlim H^p(\mathcal{V}, S_0).$$

If $\mathcal{S}$ is a sheaf of abelian groups $/X$,

we let S_0 denote its presheaf of sections and define

$$H^p(X, \mathcal{S}) = H^p(X, S_0).$$

Proposition. *If $\mathcal{V}$ is an open cover of X and S_0 is a presheaf based on all open subsets of X then*

$$H^0(\mathcal{V}, S_0) = S_X;$$

in particular, it is independent of $\mathcal{V}$. If $\mathcal{S} \to X$ is a sheaf then

$$H^0(X, \mathcal{S}) = \Gamma(X, \mathcal{S}),$$

the group of all sections of $\mathcal{S}$.

Proof. By definition $H^0(\mathcal{V}, S_0) = \operatorname{Ker} d$ on $C^0(\mathcal{V}, S_0)$. If $f \in Z^0(\mathcal{V}, S_0)$ then $df(i,j) = 0$, so $f(i) = f(j)$ in U_{ij}. It follows that f determines an elt of S_X. Conversely, each $f \in S_X$ defines by restriction an element in each S_U, and $f(i) = f(j)$ in U_{ij}. The second statement follows since $H^0(\mathcal{V}, S_0) \simeq \Gamma(X, \mathcal{S})$. //

(D) If $0 \to \mathcal{A}_0 \xrightarrow{\varphi} \mathcal{B}_0 \xrightarrow{\psi} \mathcal{C}_0 \to 0$ is an exact sequence of presheaves over a space X, then for any open cover $\mathfrak{U}$ we have the exact sequence

$$0 \to C^p(\mathfrak{U}, \mathcal{A}_0) \to C^p(\mathfrak{U}, \mathcal{B}_0) \to C^p(\mathfrak{U}, \mathcal{C}_0) \to 0,$$

from which we conclude that

$$0 \to H^0(X, \mathcal{A}_0) \to H^0(X, \mathcal{B}_0) \to \cdots$$

$$H^p(X, \mathcal{A}_0) \to H^p(X, \mathcal{B}_0) \to \cdots$$

is exact.

Now suppose $0 \to \mathcal{A} \to \mathcal{B} \to \mathcal{C} \to 0$ is an exact sequence of sheaves. Then

$$0 \to \mathcal{A}_0 \xrightarrow{\varphi} \mathcal{B}_0 \xrightarrow{\psi} \mathcal{C}_0$$

is exact, where $\mathcal{A}_0$ is the canonical presheaf of $\mathcal{A}$, etc. But the right hand arrow is generally not surjective. However,

Proposition. If X is paracompact and $0 \to \mathcal{A} \xrightarrow{\varphi} \mathcal{B} \xrightarrow{\psi} \mathcal{C} \to 0$ is exact, then so is the induced sequence

$$0 \to H^0(X, \mathcal{A}) \to H^0(X, \mathcal{B}) \to \cdots$$

$$H^p(X, \mathcal{A}) \to \cdots$$

First we prove the

Lemma. Let $f \in C^p(\mathfrak{U}, \mathcal{C}_0)$. Then there is a refinement $\mathfrak{W} = (V_x)_{x \in X}$ (X is taken as the indexing set) of $\mathfrak{U} = (U_i)_{i \in I}$ and a map $\tau : X \to I$ such that τf is in the image $h(C^p(\mathfrak{W}, \overline{\mathcal{C}}_0))$, where $\overline{\mathcal{C}}_0 = \psi(\mathcal{B}_0)$ and $h : \overline{\mathcal{C}}_0 \to \mathcal{C}_0$ is the inclusion.

$$\begin{array}{ccccc} & & C^p(\mathfrak{U}, \overline{\mathcal{C}}_0) & \xrightarrow{h} & C^p(\mathfrak{U}, \mathcal{C}_0) \\ & & \downarrow \tau & & \downarrow \tau \\ C^p(\mathfrak{W}, \mathcal{B}_0) & \xrightarrow{\psi} & C^p(\mathfrak{W}, \overline{\mathcal{C}}_0) & \xrightarrow{h} & C^p(\mathfrak{W}, \mathcal{C}_0) \end{array}$$

Proof. $\mathfrak{U}$ being locally finite, take any covering $(W_i)_{i \in I}$ with $\overline{W}_i \subset U_i$ for all $i \in I$. For each $x \in X$ we will construct a neighborhood V_x:

1) $x \in U_i \Rightarrow V_x \subset U_i$;
   $x \in W_i \Rightarrow V_x \subset W_i$;

2) $V_x \cap W_i \neq \emptyset \Rightarrow V_x \subset U_i$;

3) $x \in U_{i_0 \cdots i_p} \Rightarrow$ there is a section

$g_x \in \Gamma(V_x, \mathcal{B})$: the induced map $\psi : \Gamma(V_x, \mathcal{B}) \longrightarrow \Gamma(V_x, \mathcal{C})$ maps g_x onto $f(i_0 \dots i_p)|V_x$; note that $V_x \subset U_{i_0 \dots i_p}$ by (1).

Namely, by definition $\mathcal{C}$ is the quotient sheaf $\mathcal{B}/\mathcal{A}$. We take any p-simplex $(i_0 \dots i_p)$ of $N(\mathcal{U})$, as $f(i_0 \dots i_p) \in \mathcal{C}_{i_0 \dots i_p}$ and $\psi : \mathcal{B}_{i_0 \dots i_p} \to \mathcal{C}_{i_0 \dots i_p}$. Thus we satisfy 3) if we take a neighbourhood V'_x of x meeting only finitely many elements of $\mathcal{U}$ and with $V'_x \subset U_{i_0 \dots i_p}$. Once 3) is satisfied we construct V_x to satisfy 1), 2) by taking suitable intersections of V'_x with U_i, W_i.

We let $\mathcal{W}' = (V_x)_{x \in X}$ and define $\tau : X \to I$ by τx an index such that $x \in W_{\tau x}$; thus $\tau f \in C^p(\mathcal{W}', \mathcal{C}_0)$. We will next show that there is a $g \in C^p(\mathcal{W}', \mathcal{B}_0)$ such that $h\psi g(x_0 \dots x_p) = f(\tau x_0 \dots \tau x_p)$ for any $(x_0 \dots x_p) \in N(\mathcal{W}')$. We note that

$x \in U_{\tau x_0 \dots \tau x_p}$; namely for all $0 \leq k \leq p$, $V_{x_k} \subset W_{\tau x_k}$ and $V_{x_0} \cap V_{x_k} \neq \emptyset$, whence $V_{x_0} \cap W_{\tau x_k} \neq \emptyset \Rightarrow V_{x_0} \subset U_{\tau x_k} \Rightarrow$ $x_0 \in V_{x_0} \subset U_{\tau x_0 \dots \tau x_p}$, by 1) and 2).
By 3), $\exists\, g_0 \in \Gamma(V_{x_0}, \mathcal{B}) : \psi g_0 = f(\tau x_0 \dots \tau x_p)|V_{x_0}$; and therefore over $V_{x_0 \dots x_p}$.
Thus we construct the cochain g from the sections g_0. //

To prove the proposition it suffices to show that h^* is an isomorphism. That is an immediate consequence of the lemma and the fact that both τ and h induce chain homomorphisms. //

(E) A sheaf $\mathcal{S} \to X$ of abelian groups is <u>fine</u> if for every locally finite open cover $\mathfrak{U} = (U_i)_{i \in I}$ there is a system $(\lambda_i)_{i \in I}$ of endomorphisms of $\mathcal{S}$:

1) for each $i \in I$, if
$$|\lambda_i| = X - \{x : \text{for every } s \in \mathcal{S}_x,\ \lambda_i(s) = 0\}$$
then $|\lambda_i| \subset U_i$;

2) $\sum_{i \in I} \lambda_i(s) = s$ for every $s \in \mathcal{S}$.

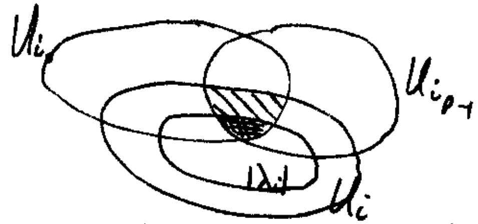
<u>Example 1</u>. Let X be paracompact and $\mathcal{S}$ the sheaf of maps into a commutative ring A with discrete topology. Then $\mathcal{S} \to X$ is fine. For if $\mathfrak{U} = (U_i)_{i \in I}$ is a lfoc of X & $\mathfrak{V} = (V_i)_{i \in I}$ is a shrunk refinement, we well order I and set
$$\lambda_i(x) = \begin{cases} 1 & \text{if } x \in V_i - \bigcup_{j<i} V_j \cap V_i \\ 0 & \text{elsewhere.} \end{cases}$$
Then each λ_i defines an endomorphism of $\mathcal{S}$; furthermore $\sum \lambda_i = \mathrm{Id}$, for $x \in X$ there is but a single index i for which $\lambda_i(x) \neq 0$, and $|\lambda_i| \subset V_i \subset U_i$.

<u>Example 2</u>. Let X be paracompact and $\mathcal{S} \to X$ the sheaf of continuous functions on X. Then $\mathcal{S} \to X$ is fine, using a continuous partition of unity $(\lambda_i)_{i \in I}$ subordinate to a lfoc $\mathfrak{U} = (U_i)_{i \in I}$. Ditto for the sheaf of smooth functions on a smooth manifold. On the other hand, the sheaf of analytic functions on an analytic manifold is not fine.

<u>Theorem</u>. If X is paracompact and $\mathcal{S} \to X$ is fine, then $H^p(X, \mathcal{S}) = 0$ for $p > 0$.

<u>Proof</u>. We prove that $H^p(\mathfrak{U}, \mathcal{S}) = 0$ for every lfoc $\mathfrak{U} = (U_i)_{i \in I}$. Let $(\lambda_i)_{i \in I}$ be as above, and define the homomorphism $k^i : C^p(\mathfrak{U}, \mathcal{S}) \to C^{p-1}(\mathfrak{U}, \mathcal{S})$ by

$$(k^i f)(i_0\cdots i_{p-1}) = \begin{cases} (\lambda_i f)(i\, i_0\cdots i_{p-1}) & \text{on } U_{i i_0\cdots i_{p-1}} \\ 0 & \text{elsewhere on } U_{i_0\cdots i_{p-1}} \end{cases}$$



Then set $kf = \sum_{i\in I} k^i f$ to obtain an element of $C^{p-1}(\mathcal{U},\mathcal{S})$ which satisfies

$$d(kf) + k(df) = \sum \lambda_i f = f .$$

Thus $df = 0 \Longrightarrow f = d(kf)$. //

Say that $\mathcal{S}\to X$ is *homologically fine* if for every lfoc $\mathcal{U}$ we have $H^p(\mathcal{U},\mathcal{S}) = 0$. For instance, if there is a system $(\lambda_i)_{i\in I}$:

$$\sum \lambda_i f = f - (dhf + hdf)$$

for some homomorphism $h : C^p(\mathcal{U},\mathcal{S}) \to C^{p-1}(\mathcal{U},\mathcal{S})$.

4. Applications of sheaf machinery.

(A) A space is a *hlc^0-space* if for every $x\in X$ and neighbourhood U of x there is a neighbourhood V of x such that for every $y\in V$ there is a singular 1-simplex $s : [0,1]\to U$ such that $s(0)=x$, $s(1)=y$. Thus X is hlc^0 iff X is locally arc-connected.

An *hlc^p-space* is one such that every singular p-cycle in V bounds in U. Say that X is a *hlc-space* if it is hlc^p for all p.

~~Example: every locally contractible space is hlc.~~

Theorem. If X is a *paracompact hlc space*, then $\check{H}^*(X) \approx H^*(X)$ canonical isomorphism.

Proof 1) Let $\mathcal{S}^p \to X$ be the sheaf of singular p-cochains. If d denotes the cochain differential then $0\to\mathcal{Z}^p\to\mathcal{S}^p\to\mathcal{Z}^{p+1}\to 0$ is an exact sequence of sheaves, surjectivity of $\mathcal{S}^p\to\mathcal{Z}^{p+1}$ being a consequence of hlc: Take $(f)\in\mathcal{Z}^{p+1}_x$, and a neighbourhood U of x & $f\in(f)$ such that $df=0$ there. Let V be a neighbourhood of x such that every $(p+1)$-cycle in V bounds in U.

We define $g: B_p(V) \to \mathbb{Z}$ by $g(\partial c) = f(c)$. Then g is well defined, for if $\partial c = \partial c'$ then $f(c) = f(c')$. Using hlc hypothesis again, we find a neighborhood W of x: every p-cycle bounds in V; thus we define g on $Z_p(W)$. Now $Z_p(W)$ is a direct summand of $C_p(W)$ (since the quotient is free), whence we can extend g over $C_p(W)$ by defining it to be 0 on the other summand. It follows that g determines an element of $\mathcal{S}^p_x$ which maps onto (f) by d.

<u>Step 2)</u> $hlc^0 \Rightarrow \mathcal{Z}^0 = \mathbb{Z}$ as a simple sheaf.

<u>Step 3</u> $\mathcal{S}^p$ is fine, by Example 1 above.

From the long exact sequence

$$0 \to H^0(X,\mathcal{Z}^p) \to H^0(X,\mathcal{S}^p) \to H^0(X,\mathcal{Z}^{p+1}) \xrightarrow{\delta} H^1(X,\mathcal{Z}^p) \to \cdots$$

$$H^{q-1}(X,\mathcal{S}^p) \to H^{q-1}(X,\mathcal{Z}^{p+1}) \xrightarrow{\delta} H^q(X,\mathcal{Z}^p) \to \underset{\substack{\| \\ 0}}{H^q(X,\mathcal{S}^p)}$$

Thus for $q \geq 2$, $H^q(X,\mathcal{Z}^p) \underset{\approx}{\xleftarrow{\delta}} H^{q-1}(X,\mathcal{Z}^{p+1})$

$$\underset{\approx}{\xleftarrow{\delta}} H^{q-2}(X,\mathcal{Z}^{p+2}) \underset{\approx}{\xleftarrow{\delta}} \cdots \underset{\approx}{\xleftarrow{\delta}} H^1(X,\mathcal{Z}^{p+q-1})$$

But from

$$0 \to H^0(X,\mathcal{Z}^{p+q-1}) \to H^0(X,\mathcal{S}^{p+q-1}) \to H^0(X,\mathcal{Z}^{p+q})$$
$$\xrightarrow{\delta} H^1(X,\mathcal{Z}^{p+q-1}) \to 0$$

we have

$$H^1(X,\mathcal{Z}^{p+q-1}) \underset{\approx}{\xleftarrow{\delta}} \Gamma(X,\mathcal{Z}^{p+q}) / d\Gamma(X,\mathcal{S}^{p+q-1}).$$

Combining these isomorphisms and setting $p=0$ gives

$$H^q(X,\mathcal{Z}^0) \xleftarrow{\approx} \Gamma(X,\mathcal{Z}^q) / d\Gamma(X,\mathcal{S}^{q-1}).$$

But Step 2 $\Rightarrow H^q(X,\mathcal{Z}^0) = \check{H}^q(X)$.

The theorem follows, because (the lemma below)

$$\Gamma(X,\mathcal{Z}^q) / d\Gamma(X,\mathcal{S}^{q-1}) = H^q(X). \quad //$$

<u>Lemma. The homomorphism</u>

$$h: S^p(X) \to \Gamma(X,\mathcal{S}^p)$$

<u>defined by</u> $h(u)(x) = r^X_x(u)$ <u>induces an</u>

isomorphism $h^*: H^p(X) \to \Gamma(X, \mathcal{Z}^p)/d\Gamma(X, \mathcal{S}^{p-1})$.

Proof. To see surjectivity, take any $\varphi \in \Gamma(X, \mathcal{S}^p)$. For any $x \in X$ there is a neighbourhood U_x and $\varphi_x \in S^p(U_x)$ representing φ_x. Then $\mathcal{U} = (U_x)_{x \in X}$ is a cover of X, which we well order. Then for any singular p-simplex s in X we set

$u(s) = 0$ if $|s|$ is not in some element of $\mathcal{U}$;
$= \varphi_x(s)$ where x is the index of the first element of $\mathcal{U}$ containing s. Then $u \in S^p(X)$, and $h(u) = \varphi$.

Injectivity of h^* is clear. //

(B) That proof is begging for abstraction, and we provide that in the next section. Before doing so, however, let us note first several situations in which we can imitate our procedure:

Theorem (de Rham) If X is a smooth manifold,

then) $\check{H}^*(X)$ is canonically isomorphic to the cohomology based on smooth cochains.

2) $\check{H}^*(X, \mathbb{R})$ is canonically isomorphic to the cohomology based on smooth differential forms.

For these we need to know that there are smooth partitions of unity subordinate to any any lfoc X. And that if $\mathcal{S}^p \to X$ denotes the sheaf of smooth p-forms on X, then for all $p \geq 0$

$$0 \to \mathcal{Z}^p \to \mathcal{S}^p \xrightarrow{d} \mathcal{Z}^{p+1} \to 0$$

is an exact sheaf sequence, with $\mathcal{Z}^0$ the simple sheaf $X \times \mathbb{R}$. That is simply a sheaf formulation on Poincaré's lemma: If ω is a closed $p+1$ form in a contractible open set U in $\mathbb{R}^n$ Then $\omega = d\eta$ for some p-form on U. Indeed, we can take $\eta = k\omega$, where k is a cone operator on U.

(C) <u>Theorem (Poincaré duality). Let X be an n-manifold. Then there is a canonical isomorphism</u> $\mathcal{D}: H^p(X, \mathcal{T}_X) \to H_{n-p}(\tilde{S}(X))$ <u>for all</u> $0 \leq p \leq n$.

<u>Proof</u>. Let $\mathcal{S}$ denote the sheaf of locally finite singular chains on X; i.e., that determined by the presheaf $\{\tilde{S}_U = \tilde{S}(X, X-U), r^U_V\}$. Then for $p \neq n$

$$0 \to \mathcal{Z}_p \to \mathcal{S}_p \xrightarrow{\partial} \mathcal{Z}_{p-1} \to 0$$

is an exact sequence of sheaves by

Consider the sheaf homomorphism

$$\partial : \tilde{S}_n(X)/\tilde{S}_n(X-U) \to \tilde{S}_{n-1}(X)/\tilde{S}_{n-1}(X-U).$$

Its kernel is induced from that of $\tilde{S}_n(X) \xrightarrow{\partial} \tilde{S}_{n-1}(X)/\tilde{S}_{n-1}(X-U)$, which is $\{c \in \tilde{S}_n(X) : |\partial c| \subset X-U\} = Z_n(\tilde{S}(X)/\tilde{S}(X-U))$. Thus that kernel is $H_n(\tilde{S}(X, X-U))$. We conclude that

$0 \to \mathcal{T}_X \to \mathcal{S}_n \xrightarrow{\partial} \mathcal{Z}_{n-1} \to 0$ is an exact sequence of sheaves.

Next, the technique of barycentric subdivision of chains shows that $\mathcal{S}$ is homologically fine: There is a family $(\lambda_i)_{i \in I}$ of endomorphisms of $\mathcal{S}$ subordinate to any lfoc $\mathcal{U} = (U_i)_{i \in I}$ and an endomorphism k such that

$$\sum_{i \in I} \lambda_i + kd + dk = id.$$

Finally, we set $\mathcal{S}^p = \mathcal{S}_{n-p}$; then

$$0 \to \mathcal{Z}^{n-p} \to \mathcal{S}^{n-p} \xrightarrow{\partial} \mathcal{Z}^{n-p+1} \to 0$$

$$0 \to \mathcal{Z}^0 = \mathcal{T}_X \to \mathcal{S}^0 \to \mathcal{Z}^1 \to 0, \text{ so}$$

$$H^q(X, \mathcal{Z}^{n-p}) \cong \dots \cong \Gamma(X, \mathcal{Z}^{n-p+q})/\partial\Gamma(X, \mathcal{S}^{n-p+q-1}).$$

Taking $n = p$ gives

$$\begin{array}{ccc} H^q(X, \mathcal{Z}^0) & \xleftarrow{\approx} & \Gamma(X, \mathcal{Z}_{n-q})/\partial\Gamma(X, \mathcal{S}_{n-q+1}) \\ \| & & \\ \check{H}^q(X, \mathcal{T}_X) & \xleftarrow{\approx} & H_{n-p}(\tilde{S}(X)) \ // \end{array}$$

Corollary. X is nonorientable iff $H_n(\tilde{S}(X)) = 0$.
X is orientable iff $H_n(\tilde{S}(X)) = \mathbb{Z}$.

For X is orientable iff $\mathcal{J}_X$ is the simple sheaf $X \times \mathbb{Z}$.

(D) Čech cohomology can be further generalised to cohomology based on cochains whose supports have special properties: For an arbitrary topological space X a family (Φ) is a collection of closed paracompact subsets of X such that

1) $A, B \in \Phi \Rightarrow A \cup B \in \Phi$;

2) $A \in \Phi$ and B closed in $A \Rightarrow B \in \Phi$;

3) if $A \in \Phi$, then there is a closed neighbourhood V of A such that $V \in \Phi$.

Example. If X is paracompact, then we could take Φ to be the totality of closed subsets of X; the resulting cohomology

is that already described.

Example. If X is locally compact, we could take for Φ the family $(\mathcal{K})$ of all compact subsets of X.

If $\mathcal{S} \to X$ is a sheaf, let $\Gamma_\Phi(X, \mathcal{S})$ denote the subgroup of all sections $\varphi \in \Gamma(X, \mathcal{S})$ such that its support $|\varphi| \in \Phi$. $H^p_\Phi(X, \mathcal{S})$ is defined accordingly.

We have the following version of the Poincaré duality theorem:
Let X be an n-manifold. Then there is a canonical isomorphism

$$\mathcal{D} : H^p_{\mathcal{K}}(X, \mathcal{J}_X) \to H_{n-p}(X),$$

where the left member is cohomology with compact supports (and twisted coefficients).

The proof is the same.

Corollary. X noncompact $\Rightarrow H_n(X) = 0$.

(E) The same proof produces Poincaré duality in the forms

$\mathscr{D} : H^p(X) \xrightarrow{\approx} H_{n-p}(\tilde{S}(X); \mathcal{T}_X)$

$\mathscr{D} : H^p_k(X) \xrightarrow{\approx} H_{n-p}(X; \mathcal{T}_X)$.

And also with both cochains and chains with (Φ)-supports.

<u>Corollary</u>. X is <u>nonorientable iff</u> $H^n_k(X) \approx \mathbb{Z}_2$.
X is <u>orientable iff</u> $H^n_k(X) = \mathbb{Z}$.

The isomorphisms $\mathscr{D}$ are the inverses of the isomorphisms $\tilde{\mathscr{D}}$ built out of connecting homomorphisms in the proof. We now identify $\mathscr{D}$:

Let X be that class in $H_n(\tilde{S}(X), \mathcal{T}_X)$ such that $\tilde{\mathscr{D}}(X) = 1$ in $H^0(X)$. Call X the <u>fundamental class of</u> X. We have

$$\begin{array}{ccc}
H^q(X) \times H_{p+q}(\tilde{S}(X), \mathcal{T}_X) & \xrightarrow{\frown} & H_p(\tilde{S}(X), \mathcal{T}_X) \\
\downarrow{\scriptstyle Id \times \tilde{\mathscr{D}}} & & \downarrow{\scriptstyle \tilde{\mathscr{D}}} \\
H^q(X) \times H^{n-p-q}(X) & \xrightarrow{\smile} & H^{n-p}(X)
\end{array}$$

$\tilde{\mathscr{D}}(v \frown z) = v \smile \tilde{\mathscr{D}} z$.

Defining $\mathscr{D} : H^p(X, \mathcal{T}_X) \to H_{n-p}(\tilde{S}(X))$ by $\mathscr{D}(v) = v \frown X$, we note that

$$\mathscr{D} = \tilde{\mathscr{D}}^{-1}.$$

(For $\tilde{\mathscr{D}}$ is an isomorphism, and $\mathscr{D}$ a homomorphism with $\tilde{\mathscr{D}}\mathscr{D}(v) = \tilde{\mathscr{D}}(v \frown X) = v \smile \tilde{\mathscr{D}} X = v$.)

(F) Another consequence is a general form of Alexander–Pontrjagin duality, toward which we proceed as follows:

<u>Lemma</u>. <u>Let</u> Y <u>be a closed subspace of</u> X <u>and</u> $\mathcal{S} \to X$ <u>a sheaf. There is one and only one sheaf</u> $\bar{\mathcal{S}} \to X$ <u>such that</u> $\bar{\mathcal{S}}|_Y = \mathcal{S}$ <u>and</u> $\bar{\mathcal{S}}|_{X-Y} = 0$.

<u>Proof</u>. For U open in X let $\overline{S}_U$ be the module of all $\overline{\varphi}$ determined by $\varphi \in \Gamma(U \cap Y, \mathcal{S})$, being 0 in $U - U \cap Y$. That determines a presheaf defining $\overline{\mathcal{S}} \to X$. Uniqueness follows from the next result. //

For any family (Φ) of X, the <u>induced family</u> $\Phi(Y)$ of Y is the collection of those sets of Φ which are contained in Y. Thus if Y is closed, then $\Phi(Y) = \Phi \cap Y$. We note that $\Phi(Y)$ satisfies 1), 2), 3) of (D).

In the preceding lemma we associated a map $\overline{\varphi} : X \to \overline{\mathcal{S}}$ with each $\varphi \in \Gamma(Y, \mathcal{S})$. Clearly $\overline{\varphi} \in \Gamma(X, \overline{\mathcal{S}})$ if either

1) Y is closed; or

2) Y is open and $\varphi \in \Gamma_{\Phi(Y)}(Y, \mathcal{S})$.

Thus in either case we have a canonical isomorphism

$$\Gamma_{\Phi(Y)}(Y, \mathcal{S}) \xrightarrow{\approx} \Gamma_{\Phi}(X, \overline{\mathcal{S}}).$$

In particular, we have the

<u>Proposition</u>. <u>Given any sheaf $\mathcal{S}$ on the open or closed subspace Y of X and any family Φ on X, we have the induced isomorphism</u>

$$H^p_{\Phi(Y)}(Y, \mathcal{S}) \xrightarrow{\approx} H^p_{\Phi}(X, \overline{\mathcal{S}}).$$

Now if Φ is a family of X, Y closed in X, $\mathcal{B} \to X$ a sheaf and $\mathcal{A} = \mathcal{B}|_Y \to Y$, let $\overline{\mathcal{A}}$ denote the extension of $\mathcal{A}$ to X as described in the lemma. If $\mathcal{C} = \mathcal{B}/\overline{\mathcal{A}}$, we have the exact sequence

$$0 \to \overline{\mathcal{A}} \to \mathcal{B} \to \mathcal{C} \to 0.$$

<u>Theorem</u>. <u>We have the exact sequence</u>

$$0 \to H^0_{\Phi(Y)}(Y, \mathcal{A}) \to H^0_{\Phi}(X, \mathcal{B}) \to H^0_{\Phi(X-Y)}(X, \mathcal{C}) \to \cdots$$

Proof. This is the cohomology sequence

$$\cdots \to H^{p}_{\Phi}(X, \overline{\mathcal{a}}) \to H^{p}_{\Phi}(X, \mathcal{B}) \to H^{p}_{\Phi}(X, \mathcal{C}) \to H^{p+1}_{\Phi}(X, \overline{\mathcal{a}}) \to \cdots$$

combined with the Proposition. //

Remark. We must not confuse the cohomology of $X-Y$ with the relative cohomology $H^p_{\mathcal{K}}(X, Y)$. However, if X is compact and Y closed in X, then their Čech cohomologies are canonically isomorphic

$$H^p_{\mathcal{K}}(X-Y) \xrightarrow{\approx} \check{H}^p(X, Y).$$

Such a statement is easily seen to be false if X is not compact.

Corollary (Alexander - Pontrjagin duality). Let X be an n-manifold and Y a closed nonvoid subset. There is a canonical isomorphism

$$a : H^{\Phi}_{n-p}(X, X-Y) \xrightarrow{\approx} H^{p}_{\Phi|Y}(Y, \mathcal{J}_X)$$

for all $0 \leq p \leq n$.

Proof. We have the ladder

$$\begin{array}{ccccccc}
\cdots \to H^{\Phi|X-Y}_{n-p}(\tilde{S}(X-Y)) & \to & H^{\Phi}_{n-p}(\tilde{S}(X)) & \to & H^{\Phi}_{n-p}(\tilde{S}(X, X-Y)) & \to & \cdots \\
\approx\downarrow \mathcal{D}^{-1} & & \downarrow a & & \approx\downarrow \mathcal{D}^{-1} & & \\
H^{p}_{\Phi|X-Y}(X-Y, \mathcal{J}_X) & \to & H^{p}_{\Phi}(X, \mathcal{J}_X) & \to & H^{p}_{\Phi}(Y, \mathcal{J}_Y) & \to & \cdots,
\end{array}$$

and the conclusion follows from the 5-lemma.

Corollary $a : H_{n-p}(X, X-Y) \xrightarrow{\approx} H^p_{\mathcal{K}}(Y, \mathcal{J}_X)$.

(G) *Exercise.* If (Y, Z) is a closed pair in the n-manifold X, then we have an isomorphism

$$H^{\Phi}_{n-p}(Y, Z) \xrightarrow{\approx} H^{p}_{\Phi|X-Z}(X-Z, X-Y, \mathcal{J}_X).$$

Remark. For a closed subset Y of an arbitrary space X we say that (X, Y) is a relative n-manifold if $X-Y$ is an n-manifold. If Y is itself an $(n-1)$-manifold then (X, Y) is called a bounded manifold, with boundary Y.

If X is compact, then we have the Čech isomorphism $\check{H}^p(X,Y) \xrightarrow{\approx} H^p_{\mathcal{R}}(X-Y)$.

Duality follows:

$$\mathcal{D} : \check{H}^p(X,Y) \longrightarrow H_{n-p}(X-Y, \mathcal{T}_{X-Y}).$$

Also, if X is a compact bounded n-manifold there is

$$\mathcal{D} : H^p(X, \partial X) \xrightarrow{\approx} H_{n-p}(X, \mathcal{T}_X).$$

<u>Exercise 2</u>. If Y_1 closed in Y, which is closed in X, then Y_1 is closed in X. Set $Y_2 = Y - Y_1$. Show

$$\check{H}^p(X - Y_2, Y) \approx \check{H}^p(X-Y).$$

If (X,Y) is a relative n-manifold, then $(X-Y_1, Y_2)$ is a relative n-manifold, with

$$\mathcal{D} : \check{H}^p(X-Y_2, Y_1) \xrightarrow{\approx} H_{n-p}(X-Y_1, Y_2; \mathcal{T}).$$

<u>Exercise 3.</u> Formulate precisely a version of Alexander-Pontryagin duality of the form:

Let (X, A) be a relative n-manifold, (Y, B) a closed pair of (X, A). Then

$$\check{H}^p(Y, B) \xrightarrow{\approx} H_{n-p}(X-A, X-A-Y; \mathcal{T})$$

5. Gratings and Leray gratings.

(A) Definitions. Let A be a principal ideal domain and K a unitary A-module (i.e., K is an abelian group with A as a ring of operators such that $1 \in A$ acts as the identity operator). K is a differential A-module if there is an endomorphism d of K such that $d \circ d = 0$. If K is graded (i.e., K is represented as a direct sum of submodules K^p), then we require that $d: K^p \longrightarrow K^{p+1}$. If K is an A-algebra (i.e., K has (associative) ring structure, with product denoted by $\cup$, and satisfies $K^p \cup K^q \longrightarrow K^{p+q}$ if K is graded), then we require further that

$$d(\alpha \cup \beta) = d\alpha \cup \beta + (-1)^p \alpha \cup d\beta, \text{ where } \alpha \in K^p,\ \beta \in K.$$

If K is a differential graded A-algebra, we let $Z(K) = \Sigma Z^p(K)$ denote the kernel of d, B(K) – the image d(K). Then B(K) is a two-sided ideal in Z(K), and the quotient $H(K)=Z(K)/B(K)$ is called the cohomology algebra of K, and $H(K) = \Sigma H^p(K)$, $H^p(K) = Z^p(K)/B^p(K)$.

Remark. If K is a graded A-algebra with unit u, then $u \in K^o$ and $du = 0$.

Definitions. Let X be a topological space and K an A-module. We say that K is an A-module with supports in X if for each $\alpha \in K$ there is associated a closed set $|\alpha| \subset X$ such that

1) $|\alpha + \beta| \subset |\alpha| \cup |\beta|$ for all $\alpha, \beta \in K$;

2) $|a\alpha| \subset |\alpha|$ for all $a \in A$, $\alpha \in K$;

3) $|0| = \emptyset$.

If K is a differential A-module, we require

4) $|d\alpha| \subset |\alpha|$ for all $\alpha \in K$.

If K is graded, we require

5) $|\alpha + \beta| = |\alpha| \cup |\beta|$ if $\alpha \in K^p$, $\beta \in K^q$ and $p \neq q$.

If K is an A-algebra, we require

6) $|\alpha \cup \beta| \subset |\alpha| \cap |\beta|$ for all $\alpha, \beta \in K$.

If K is an A-algebra with unit u, we require

7) $|u| = X$.

We will say that K is torsion free if

2') $|a\alpha| = |\alpha|$ if $a \neq 0$

We will say that K is separated if

3') $\alpha = 0$ if and only if $|\alpha| = \emptyset$

Remark. If K_o is an A-module with supports in X and N is the submodule consisting of those $\alpha \in K$ with $|\alpha| = \emptyset$, then the quotient $K = K_o/N$ is a separated A-module with supports in X, where $|\tilde{\alpha}| = |\alpha|$ for any $\alpha \in \tilde{\alpha} \in K$.

Remark. K is always torsion free if A is a field.

Definition. An A-grating on X is a separated differential A-module K with supports in X, which is graded by non-negative degrees (i.e., $K = \Sigma_{p \geq o} K^p$). An A-algebra grating on X is an A-grating which is also an A-algebra.

Definition. Let X and Y be topological spaces and $f: X \longrightarrow Y$ a continuous map. If K and L are A-modules with supports in X and Y, resp, and if $h: L \longrightarrow K$ is an A-homomorphsim, then we say that f and h are compatible if $|h\beta| \subset f^{-1}(|\beta|)$ for all $\beta \in L$.

$$\begin{array}{ccc} K & \xleftarrow{h} & L \\ | & f & | \\ X & \longrightarrow & Y \end{array}$$

If K and L are A-gratings on X we say that $h: L \longrightarrow K$ is a <u>grating homomorphism</u> if h is a cochain map (i.e., $dh = hd$) and is compatible with the identity $i: X \longrightarrow X$; thus $|h\beta| \subset |\beta|$ for all $\beta \in L$.

<u>Remark</u>. Other algebraic notions (e.g., submodules, direct sums) can easily be formulated for modules with supports and gratings.

(B) <u>Definition</u>. If L is an A-module with supports in Y, and $f: X \longrightarrow Y$ is a continuous map, then we construct an A-module $K(f,L)$ with supports in X as follows: As an A-module, we take $K(f,L) = L$; to each $\beta \in K(f,L)$ we associate the closed subset (of X)

$$|\beta|_X = f^{-1}(|\beta|).$$

If L is an A-grating on Y, then the <u>A-grating</u> $f^{-1}L$ <u>on</u> X <u>induced by</u> f <u>and</u> L is the quotient of $K(f,L)$ by the submodule of those elements with void X-support.

<u>Definition</u>. If K is an A-grating on X and Y is a subspace of X, the <u>Y-section of</u> K is the A-grating on Y induced by the inclusion map $i: Y \longrightarrow X$. We will denote the Y-section of K by $Y \circ K$; let $Y \circ \alpha$ denote the element of $Y \circ K$ determined by $\alpha \in K$.

<u>Remarks</u>. 1. The notions of Y-section and induced grating of course extend to (separated) modules with supports. In particular, $X \circ K$ is the separated module associated with K.

2. If Z is a subspace of Y, we have the transitivity relation $Z \circ (Y \circ \alpha) = Z \circ \alpha$.

3. If K is torsion free, then so is $Y \circ K$.

<u>Remark</u>. An equivalent construction of $Y \circ K$ is the following: Let $K_{X-Y} = \{\alpha \in K : |\alpha| \cap Y = \emptyset\}$, and set $Y \circ K = K/K_{X-Y}$. We define supports in $Y \circ K$ by setting $|Y \circ \alpha| = Y \cap |\alpha|$ for all $Y \circ \alpha \in Y \circ K$; note that if $Y \circ \alpha = Y \circ \alpha'$, then $Y \cap |\alpha| = Y \cap |\alpha'|$. The differential operator $d(Y \circ \alpha)$ is defined as $Y \circ (d\alpha)$.

(C) <u>Remark</u>. If K is an A-module with supports in X, then K <u>determines a sheaf</u> $\mathcal{S}(K)$ <u>of</u> A-<u>modules</u> <u>over</u> X as follows. For each open set $U \subset X$ let $S_U = U \circ K$; then if V is open in U we have a natural map

$$r^U_V : S_U \longrightarrow S_V$$

which is clearly transitive $S_U \longrightarrow S_V \longrightarrow S_W$ and is the identity on $S_U \to S_U$. Thus $\{S_U, r^U_V\}$ is a presheaf and therefore determines a definite sheaf $\mathcal{S}(K)$ of A-modules on X.

On the other hand, if $\mathcal{S}$ is a sheaf of A-modules over X, then $\mathcal{S}$ <u>determines</u> <u>an</u> A-<u>module</u> $K(\mathcal{S})$ <u>with</u> <u>supports</u> <u>in</u> X as follows: Set $K(\mathcal{S}) = \Gamma(X, \mathcal{S})$; for each $\alpha \in K(\mathcal{S})$ let $|\alpha| =$

$\{x \in X : \alpha(x) \neq 0\}$. Then $|\alpha|$ is a closed set, and 1), 2), 3) of (A) are satisfied; furthermore $K(\mathcal{S})$ is clearly separated. Thus we have an explicit identification of the sheaves of A-modules over X and the separated A-modules with supports in X. Furthermore, the correspondence identifies differential graded sheaves with A-gratings (and preserves ring structure). If $h : L \to K$ is a grating homomorphism, then h induces a sheaf homomorphsim $\mathcal{S}(L) \to \mathcal{S}(K)$, and conversely.

If K is an A-module with supports in X, we have a natural homomorphism

$$h : K \longrightarrow \Gamma(X, \mathcal{S}(K))$$

defined by $(h\alpha)x = r^X_x(\alpha)$. If K is separated, then h is one-one; in general, however, h is not onto.

Definition. If K is a separated A-module with supports in X, the module $\Gamma(X, \mathcal{S}(K))$ is called the completion of K. If h is onto, we say that K is complete.

Remark. If $\mathcal{S}$ is a sheaf, then the associated grating is complete, as we have seen in the preceding section. In particular the completion of a grating is complete.

(D) Definition. Let K be an A-grating on X. Then K is locally acyclic if $H(x \circ K) \approx A$ for all $x \in X$; i.e., $H^p(x \circ K) \approx A$ if $p = 0$, $= 0$ if $p > 0$.

Remark. If K is locally acyclic and $\mathcal{S}(K)$ is the associated differential graded sheaf, then we have the exact sheaf sequence

$$0 \longrightarrow A \longrightarrow \mathcal{S}^0(K) \xrightarrow{d} \mathcal{S}^1(K) \xrightarrow{d} \mathcal{S}^2(K) \longrightarrow \cdots .$$

(E) Definition. Let K be a separated A-module with supports in X. We say that K is fine if for each locally finite open cover $\mathcal{U} = (U_i)_{i\in I}$ of X there are endomorphisms $(\phi_i)_{i\in I}$ of K such that for all $\alpha \in K$ we have

1) $|\phi_i\alpha| \subset U_i \cap |\alpha|$,
2) $\sum_{i\in I} \phi_i\alpha = \alpha$.

If K is graded, we require that $\phi_i : K^p \longrightarrow K^p$ for all p.

Remark. The elements $\phi_i\alpha$ form a locally finite family in K; i.e., every $x \in X$ has a neighborhood which intersects only finitely many of the sets $|\phi_i\alpha|$. Then the sum 2) always represents an element of $\Gamma(X, \mathcal{S}(K))$; we require that the sum represent an element of K (using the fact that K is separated, and hence identified to a submodule of $\Gamma(X, \mathcal{S}(K))$.

Remark. A separated module K with supports is fine if and only if the associated sheaf $\mathcal{S}(K)$ is fine. In particular, if K is fine then so is its completion.

Proposition. Let K be a fine grating on the paracompact space X. If every locally finite sum $\Sigma\alpha_i$ converges in K, then K is complete.

Proof. We must show that $h : K \longrightarrow \Gamma(X, \mathcal{S}(K))$ is onto; take $\phi \in \Gamma(X, \mathcal{S}(K))$. Then every point $x \in X$ has a neighborhood U_x such that $\phi(x) \in S_x(K)$ can be represented by an element in $U_x \circ K$. Let $\mathcal{U} = (U_i)_{i\in I}$ be a locally finite refinement of the covering $(U_x)_{x\in X}$, and let $(\psi_i)_{i\in I}$ be a partition of unity subordinate to $\mathcal{U}$. Then each $\psi_i\phi \in \Gamma(X, \mathfrak{S}(K))$ and furthermore is represented by an element $\alpha_i \in K$. The sum $\alpha = \sum \alpha_i$ is therefore an element of K such that $h(\alpha) = \phi$.

Corollary. If X is compact and K is fine, then K is complete.

(F) Definition. Let X be a topological space. A Leray A-grating K on X is an A-algebra grating on X such that

1) K is complete;
2) K is locally acyclic;
3) K is fine.

If X is paracompact, we have seen that the Čech cochains on X with coefficients in A form a Leray A-grating K; furthermore, K has a unit and is torsion free.

Fundamental Theorem (Leray). If X is a paracompact space, then the cohomology rings of any two Leray A-gratings are canonically isomorphic.

Proof. Let K be a Leray A-grating and $\mathfrak{S}(K)$ the associated differential graded sheaf. Then since K is locally acyclic we have the exact sheaf sequence

$$0 \longrightarrow A \longrightarrow \mathfrak{S}^0(K) \xrightarrow{d} \mathfrak{S}^1(K) \xrightarrow{d} \dots .$$

This is in fact a resolution of A, because K is fine (i.e., $H^p(X, \mathfrak{S}(K)) = 0$ if $p > 0$). It follows from the previous section that the Čech cohomology group $H^p(X,A)$ is canonically isomorphic to $H^p(\Gamma(\mathfrak{S}(K)))$, the $p^{\underline{th}}$ cohomology group of the

complex of the resolution. Since $h : K \longrightarrow \Gamma(\mathfrak{S}(K))$ is a cochain isomorphism (K being complete and separated), it follows that h induces an isomorphism of their associated cohomology groups (and in fact, h is a ring isomorphism). Thus we have

$$H^p(K) \approx H^p(\Gamma(\mathfrak{S}(K))) \approx H^p(X,A)$$

for all $p \geq 0$. The proof that this isomorphism preserves ring structure is exactly the same as that given in the previous section.

Remark. It follows the cohomology ring h(K) of any Leray A-grating is anti-commutative, for the multiplication in A is commutative and the Čech cohomology ring is anti-commutative.

Remark. The above proof shows that the conclusion follows for an A-algebra grating K wich satisfies 1), 2) and 3') K is homologically fine; i.e., $H^p(\mathcal{U}, \mathfrak{S}(K)) = 0$ if $p > 0$ for all locally finite open covers $\mathcal{U}$ of X.