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SUMMER WORKSHOP ON FIBRE BUNDLES AND GEOMETRY

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LIMIT BEHAVIOURS OF SOLUTIONS FOR SOME PROBLEMS OF PARTIAL DIFFERENTIAL EQUATIONS AND THEIR APPLICATIONS

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I. Statement of problems and principal results

Let Ω be a bounded open set in $\mathbb{R}^n$ containing the origin with smooth boundary Γ.

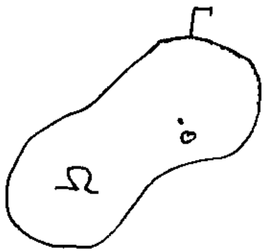

Fig. 1

Here we restrict ourselves to the case $n=2$ or 3 which is more important in applications. For any $v \in L^2(0,T)$ we consider the following problem of parabolic equations

$$
(\mathrm{I}) \quad \begin{cases} \dfrac{\partial y}{\partial t} + A y = v(t)\,\delta(x) & \text{in } Q = \Omega \times (0,T), \\ y = 0 & \text{on } \Sigma = \Gamma \times (0,T), \\ y(x,0) = 0 & \text{in } \Omega, \end{cases}
$$

Where A is a second order elliptic operator of divergence form with variable coefficients:

$$
A\phi = -\sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial \phi}{\partial x_j}\right)
$$

with $a_{ij}(x)$ suitably smooth and

$$
\sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j \geqslant \alpha|\xi|^2, \quad \alpha>0 \text{ constant} \quad \forall\, \xi_i \in \mathbb{R}
$$

for a.e. $x \in \Omega$.

and $\delta(x)$ is the Dirac mass at the origin.

This problem admits a unique weak solution $y = y(t;v)$ which is only in $L^2(Q)$ (cf. [1], [2]). By transposition this solution is defined by the following Green's formula:

$$
\int_Q y(t;v)\,\psi\, dx\, dt = \int_0^T \varphi(0,t;\psi)\, v(t)\, dt \quad \forall\, \psi \in L^2(Q),
$$

Where $\varphi = \varphi(t;\psi)$ is the solution of the adjoint problem

$$
(\mathrm{II}) \quad \begin{cases} -\dfrac{\partial \varphi}{\partial t} + A^*\varphi = \psi & \text{in } Q, \\ \varphi = 0 & \text{on } \Sigma, \\ \varphi(x,T) = 0 & \text{in } \Omega. \end{cases}
$$

in which A^* is the formal adjoint operator of A:

$$A^*\varphi = -\sum_{i,j=1}^{n} \frac{\partial}{\partial x_j}\left(a_{ij}(x)\frac{\partial \varphi}{\partial x_i}\right).$$

The goal of this work is to give an approximation of problem (I) and to prove the convergence of the approximate solution y_ε to the solution y of (I) as $\varepsilon \to 0$. Here we construct the approximate problem by excision of a small domain converging to the origin, on the boundary of which some flux conditions are imposed. Precisely speaking, for $\varepsilon > 0$ small enough, let Γ_ε be the surface of the spheroid

$$B_\varepsilon = \{ x \mid |x| < \varepsilon \},$$

setting

$$\Omega_\varepsilon = \Omega \setminus \overline{B_\varepsilon},$$
$$Q_\varepsilon = \Omega_\varepsilon \times (0,T),$$
$$\Sigma_\varepsilon = \Gamma_\varepsilon \times (0,T),$$

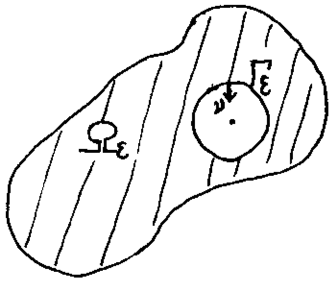

Fig. 2

We consider the following approximation of problem (I):

$$(\mathrm{I})_\varepsilon \begin{cases} \dfrac{\partial y_\varepsilon}{\partial t} + A y_\varepsilon = 0 & \text{in } Q_\varepsilon, \\ y_\varepsilon|_{\Gamma_\varepsilon} = C_\varepsilon(t) \text{ (unknown function of } t\text{) and} & \\ \displaystyle\int_{\Gamma_\varepsilon} \frac{\partial y_\varepsilon}{\partial \nu_A}\, dS = v_\varepsilon(t) \text{ for a.e. } t \in (0,T), & \text{on } \Sigma_\varepsilon, \\ y_\varepsilon = 0 & \text{on } \Sigma, \\ y_\varepsilon(x,0) = 0 & \text{in } \Omega_\varepsilon, \end{cases}$$

where $\dfrac{\partial y_\varepsilon}{\partial \nu_A}$ is the conormal derivative associated to A:

$$\frac{\partial y_\varepsilon}{\partial \nu_A} = \sum_{i,j=1}^{n} a_{ij}\nu_i \frac{\partial y_\varepsilon}{\partial x_j}$$

and $\vec{\nu} = (\nu_1, \cdots, \nu_n)$ is the unit normal on Γ_ε pointing towards the interior of B_ε.

It is easy to verify that for any $v_\varepsilon \in L^2(0,T)$ problem $(\mathrm{I})_\varepsilon$ admits a unique solution $y_\varepsilon(t; v_\varepsilon)$. For the convergence of the approximate solution y_ε to the original solution y, we have the following (cf. [3])

Theorem 1 (resp. Theorem 1 bis): As $\varepsilon \to 0$, if

$$v_\varepsilon(t) \longrightarrow v(t) \text{ in } L^2(0,T) \text{ weakly (resp. strongly)},$$

then

$$y_\varepsilon(t; v_\varepsilon) \text{ (extended by 0 or by } C_\varepsilon(t)) \longrightarrow y(t; v)$$

$$\text{in } L^2(Q) \text{ weakly (resp. strongly)}.$$

II. Proof of theorem 1 (for $n=3$).

We introduce the adjoint problem of (I_ε), which is also a corresponding approximation of problem (II):

$$(II_\varepsilon)\begin{cases} -\dfrac{\partial \varphi_\varepsilon}{\partial t} + A^* \varphi_\varepsilon = \psi_\varepsilon & \text{in } Q_\varepsilon, \\ \varphi_\varepsilon = d_\varepsilon(t) \text{ (unknown function of } t\text{) and} & \\ \displaystyle\int_{\Gamma_\varepsilon} \frac{\partial \varphi_\varepsilon}{\partial \nu_{A^*}} dS = 0 \quad \text{for a.e. } t \in (0,T), & \text{on } \Sigma_\varepsilon, \\ \varphi_\varepsilon = 0 & \text{on } \Sigma, \\ \varphi_\varepsilon(x,T) = 0 & \text{in } \Omega_\varepsilon. \end{cases}$$

The corresponding Green's formula linking problem (I_ε) with problem (II_ε) is the following

$$\int_{Q_\varepsilon} y_\varepsilon(t; v_\varepsilon)\, \psi_\varepsilon \, dx\, dt = \int_0^T d_\varepsilon(t; \psi_\varepsilon)\, v_\varepsilon(t)\, dt \quad \forall \psi_\varepsilon \in L^2(Q_\varepsilon)$$
$$\forall v_\varepsilon \in L^2(0,T).$$

Then it is easy to see by duality that theorem 1 is the direct consequence of the following

Theorem 2: As $\varepsilon \to 0$, if

$$\psi_\varepsilon \text{ (extended by } 0) \longrightarrow \psi \quad \text{in } L^2(Q) \text{ strongly}$$
$$\text{(resp. weakly)},$$

then

$$d_\varepsilon(t; \psi_\varepsilon) \longrightarrow \varphi(0, t; \psi) \text{ in } L^2(0,T) \text{ strongly (resp. weakly)},$$

$$\hat{\varphi}_\varepsilon(t; \psi_\varepsilon) \; (\varphi_\varepsilon \text{ extended by } d_\varepsilon(t)) \longrightarrow \varphi(t; \psi)$$
$$\text{in } L^\infty(0,T; H_0^1(\Omega)) \text{ strongly (resp. * weakly)},$$

$$\tilde{\varphi}_\varepsilon(t; \psi_\varepsilon) \; (\varphi_\varepsilon \text{ extended by } 0) \longrightarrow \varphi(t; \psi)$$
$$\text{in } L^\infty(0,T; L^2(\Omega)) \text{ strongly (resp. * weakly)},$$

$$\frac{\partial \tilde{\varphi}_\varepsilon}{\partial t}(t; \psi_\varepsilon) \longrightarrow \frac{\partial \psi}{\partial t}(t; \psi) \text{ in } L^2(Q) \text{ strongly (resp. weakly)}.$$

To prove theorem 2, we have to establish some a priori estimates uniformly with respect to ε for φ_ε and for $d_\varepsilon(t)$. The essential point in the demonstration is to use the following elliptic lemma which is also important in applications.

<u>Lemma</u>: Let ρ_ε be the solution of problem

$$\begin{cases} A\rho_\varepsilon = 0 & \text{in } \Omega_\varepsilon, \\ \rho_\varepsilon = c_\varepsilon \text{ (unknown constant) on } \Gamma_\varepsilon, \\ \int_{\Gamma_\varepsilon} \frac{\partial \rho_\varepsilon}{\partial \nu_A} ds = 1, \\ \rho_\varepsilon = 0 & \text{on } \Gamma, \end{cases}$$

then as $\varepsilon \to 0$, ρ_ε (extended by c_ε or by 0 in B_ε) converges strongly in $L^2(\Omega)$ to the solution ρ of

$$\begin{cases} A\rho = \delta(x) & \text{in } \Omega, \\ \rho = 0 & \text{on } \Gamma. \end{cases}$$

Moreover, there exists two constants d_1 and d_2 independent of ε such that

$$0 < d_2 \leq \varepsilon c_\varepsilon \leq d_1.$$

III. <u>Applications</u>

1. <u>Application of the lemma in mechanics</u> (cf. [4])

Let's consider the torsion problem of a cylindrical rod R_ε with a multiple connected cross section Ω_ε. It is well-known that in order to determine the corresponding stress function, it is sufficient to solve the following problem

$$\begin{cases} -\Delta \varphi_\varepsilon(x,y) = 1 & \text{in } \Omega_\varepsilon \quad \left(\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) \\ \varphi_\varepsilon = c_\varepsilon \text{ (unknown constant) on } \Gamma_\varepsilon, \\ \int_{\Gamma_\varepsilon} \frac{\partial \varphi_\varepsilon}{\partial \nu} ds = V_\varepsilon \text{ (the area of } B_\varepsilon), \\ \varphi_\varepsilon = 0 & \text{on } \Gamma. \end{cases}$$

At the same time, for a cylindrical rod R with a simply connected cross section Ω, the corresponding problem is

$$\begin{cases} -\Delta \varphi(x,y) = 1 & \text{in } \Omega, \\ \varphi = 0 & \text{on } \Gamma. \end{cases}$$

Since $V_\varepsilon \to 0$ as $\varepsilon \to 0$, according to the lemma, if the cross section B_ε of the hole is small enough, for the torsion problem the stress function corresponding to R_ε is approximate to that one corresponding to R and this hole may be neglected approximately. In other words, if a cylindrical rod has a very small cylindrical hole, it is not at all dangerous for the torsion problem.

2. Application of theorem 1 in the exploitation of oil fields.

In the exploitation of oil fields one often changes the wells (holes) by pointwise source terms at the right-hand side of the equation. Precisely speaking, if the function of pressure p_ε of the oil is independent of the coordinate z, then it satisfies the following parabolic problem in a domain with holes:

$$\begin{cases} \beta \dfrac{\partial p_\varepsilon}{\partial t} - \left(\dfrac{\partial}{\partial x}\left(\lambda \dfrac{\partial p_\varepsilon}{\partial x}\right) + \dfrac{\partial}{\partial y}\left(\lambda \dfrac{\partial p_\varepsilon}{\partial y}\right)\right) = 0 & \text{in } Q_\varepsilon, \\ p_\varepsilon = p_0 \text{ (constant)} & \text{on } \Sigma, \\ p_\varepsilon = c_\varepsilon(t) \text{ (unknown function of } t) \text{ and} & \\ \displaystyle\int_{\Gamma_\varepsilon} \lambda \frac{\partial p_\varepsilon}{\partial \nu}\, ds = F(t) \text{ (given function of } t), & \text{on } \Sigma_\varepsilon, \\ p_\varepsilon(x,0) = p_0 & \text{in } \Omega_\varepsilon. \end{cases}$$

Instead of determining this function p_ε directly from the previous problem, one often solves the following problem with a pointwise source term at the right-hand side of the equation but in a domain Q without holes:

$$\begin{cases} \beta \dfrac{\partial p}{\partial t} - \left(\dfrac{\partial}{\partial x}\left(\lambda \dfrac{\partial p}{\partial x}\right) + \dfrac{\partial}{\partial y}\left(\lambda \dfrac{\partial p}{\partial y}\right)\right) = F(t)\,\delta(x) & \text{in } Q, \\ p = p_0 & \text{on } \Sigma, \\ p(x,0) = p_0 & \text{in } \Omega. \end{cases}$$

Because the diameter of B_ε is much smaller than the diameter of Ω, theorem 1 shows that this method gives a good approximation.

3. Application of theorem 1 in certain problems of optimal control

The problem of optimal control we are interested in is to minimize the cost function given by

$$J(v) = N\int_0^T v^2 dt + \int_\Omega |y(T;v) - z_d|^2 dx,$$

Where $y(t;v)$ is the solution of problem (I), z_d is given in $L^2(\Omega)$ and N is given positive. In order the cost function make sense we have to define $J(v)$ on a subspace of $L^2(0,T)$ as follows:

$$\mathcal{U} = \{ v \mid v(t) \in L^2(0,T),\ y(T;v) \in L^2(\Omega) \}$$

provided with the graph norm (cf. [2].[8]).

It is easy to see that there exists a unique element $u_0 = u_0(t) \in \mathcal{U}$ such that

$$J(u_0) = \inf_{v \in \mathcal{U}} \dot{J}(v).$$

On the other hand, for the cost function

$$J_\varepsilon(v) = N\int_0^T v^2(t)\, dt + \int_{\Omega_\varepsilon} |y_\varepsilon(T;v) - z_d|^2 dx$$

there exists a unique element $u_\varepsilon = u_\varepsilon(t) \in L^2(0,T)$ such that

$$J_\varepsilon(u_\varepsilon) = \inf_{v \in L^2(0,T)} J_\varepsilon(v).$$

Using the preceding results and the Rockafellar's duality theorem, we have (cf. [5].[2])

<u>Theorem 3</u>: As $\varepsilon \to 0$, one has

$$J_\varepsilon(u_\varepsilon) \to J(u_0),$$

$$u_\varepsilon \to u_0 \quad \text{in } L^2(0,T) \text{ strongly}.$$

IV. Various remarks

1. The spheroid B_ε can be replaced by a small star domain around the origin. Besides, we can also perturb the choice of B_ε by translation, i.e. B_ε can be replaced by

$$\widetilde{B_\varepsilon} = b_\varepsilon + B_\varepsilon$$

with $\lim_{\varepsilon \to 0} b_\varepsilon = 0$ in Ω.

2. The preceding results stay valid if the boundary condition on Σ is changed by any one of the following boundary conditions:

1°. $\frac{\partial y}{\partial \nu_A} + \lambda(x)\, y = 0 \qquad$ on Σ

or

2°. $y = c(t)$ (unknown function of t) and

$$\int_\Gamma \left(\frac{\partial y}{\partial \nu_A} + \lambda(x) y\right) ds = 0 \quad \text{for a.e. } t \in (0,T), \text{ on } \Sigma,$$

where $\frac{\partial y}{\partial \nu_A}$ is the conormal derivative with the unit normal $\vec{\nu}$ oriented towards the exterior of Ω and $\lambda(x) \geq 0$ is a smooth function on Γ.

3. We can also obtain a similar result for the hyperbolic case (cf. [6])

$$\begin{cases} \frac{\partial^2 y}{\partial t^2} + Ay = v(t)\,\delta(x) & \text{in } Q, \\ y = 0 & \text{on } \Sigma, \\ y(x,0) = 0, \ \frac{\partial y}{\partial t}(x,0) = 0 & \text{in } \Omega \end{cases}$$

and for the pseudoparabolic case (cf. [7])

$$\begin{cases} (1+A)\frac{\partial y}{\partial t} + Ay = v(t)\,\delta(x) & \text{in } Q, \\ y = 0 & \text{on } \Sigma, \\ y(x,0) = 0 & \text{in } \Omega, \end{cases}$$

but in both cases we have to suppose that the operator A is self-adjoint, i.e.

$$A^* = A.$$

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