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BOUNDARY VALUE PROBLEMS AND FREE BOUNDARY PROBLEMS FOR QUASILINEAR HYPERBOLIC-PARABOLIC COUPLED SYSTEMS

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1. INTRODUCTION.

Here we handle with so-called quasilinear hyperbolic-parabolic coupled systems which often occur in applications. Roughly speaking, in this kind of system a part of equations formulate a first order quasilinear hyperbolic system with respect to certain unknown functions $u = \{u_1,\ldots,u_n\}$, another part of equations a second order quasilinear parabolic system with respect to the remainder of unknown functions $v = \{v_1,\ldots,v_m\}$, and these two parts are nonlinearly coupled each other. For instance, the system of motion for a compressible viscous, heat-conductive fluid[1], the system of radiation hydrodynamics[2], the system of motion of viscoelastic materials[3] etc. are of this kind.

The initial value problem with smooth initial data has been studied by several authors. For example, for the system of motion for a compressible viscous, heat-conductive fluid in 3-dimensional case, J. Nash[4] and N. Itaya[5][6] have proved the existence and the uniqueness of the local smooth solution. Recently, A. Matsumura and T. Nishida[7][14] have even proved the corresponding global existence theorem for the small initial data.

For the quasilinear hyperbolic-parabolic coupled system, the boundary value problems, especially the free boundary problems are more important in applications, because the latter is concerned with determining the corresponding discontinuous solution which can describe, for instance, the behaviour of a fluid containing a radiation shock in radiation hydrodynamics. But for the boundary value problems, especially for the free boundary problems, we can only find certain results in some special cases even for one-dimensional

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case (for instance, A. Tani[8] has discussed the mixed initial-boundary value problems for the system of compressible viscous, heat-conductive fluids in a cylinderical domain with a special Dirichlet boundary condition: the velocity $\vec{u} = 0$ and the absolute temperature $T = T_1(t,x)$; A. V. Kazhikhov and V. V. Shelukhin[9] have considered the corresponding one-dimensional initial-boundary value problem with the boundary data:

$$u(t,0) = u(t,1) = T_x(t,0) = T_x(t,1) = 0, \quad t \geq 0;$$

Moreover, A Tani[15] has also studied a free boundary value problem for compressible viscous fluid motion etc.). So it is worthwhile to carry out a systematic research on this subject.

In what follows we shall concentrate our attention on the boundary value problems and the free boundary problems for the following general types of quasilinear hyperbolic-parabolic coupled systems in one-dimensional case:

Type I:

$$\sum_{j=1}^{n} \zeta_{\ell j}(t,x,u,v)\left(\frac{\partial u_j}{\partial t} + \lambda_\ell(t,x,u,v,v_x)\frac{\partial u_j}{\partial x}\right) = \mu_\ell(t,x,u,v,v_x) \quad (\ell = 1,\ldots,n) \qquad (1.1)$$

$$\frac{\partial v}{\partial t} - a(t,x,u,u_x,v,v_x)\frac{\partial^2 v}{\partial x^2} = b(t,x,u,u_x,v,v_x) , \qquad (1.2)$$

where $v = (v_1,\ldots,v_m)^T$ is a vector function and a is a diagonal matrix: $a = \mathrm{diag}(a_1,\ldots,a_m)$. On the domain under consideration, we suppose that

$$\det|\zeta_{\ell j}| \neq 0$$

and

$$a_i > 0 \qquad (\lambda = 1,\ldots,m) .$$

In this system, (1.1) is hyperbolic with respect to $u = (u_1,\ldots,u_n)^T$ (under the characteristic form with the characteristic directions $\frac{dx}{dt} = \lambda_\ell$ $(\ell = 1,\ldots,n)$), (1.2) is parabolic with respect to v and (1.1), (1.2) are nonlinearly coupled each other.

Type II.

$$\sum_{j=1}^{n} \zeta_{\ell j}(t,x,u,v)\left(\frac{\partial u_j}{\partial t} + \lambda_\ell(t,x,u,v,v_x)\frac{\partial u_j}{\partial x}\right) = \zeta_\ell(t,x,u,v)\left(\frac{\partial v}{\partial t} + \lambda_\ell(t,x,u,v,v_x)\frac{\partial v}{\partial x}\right) + \mu_\ell(t,x,u,v,v_x) \quad (\ell = 1,\dots,n) \tag{1.3}$$

$$\frac{\partial v}{\partial t} - a(t,x,u,v,v_x)\frac{\partial^2 v}{\partial x^2} = b(t,x,u,v,v_x) \ , \tag{1.4}$$

in which the coefficients a and b don't depend on u_x, but on the right-hand side of (1.3) there is an additional term which denotes the directional derivative of v along the characteristic direction $\frac{dx}{dt} = \lambda_\ell$.

Our goal is to discuss various kinds of boundary value problems and of free boundary problems for these systems in a class of smooth functions or piecewise smooth functions and give a condition of local solvability in order to obtain the corresponding existence and uniqueness theorem. The results obtained by us can be applied to many practical cases and imply an affirmative answer for a conjecture given by C. M. Dafermos[3] about the incomplete parabolic damping.

2. EXAMPLES.

1. System of motion for a compressible viscous, heat-conductive fluid.

In one-dimensional case the system can be written as follows

$$\frac{\partial \rho}{\partial t} + u\frac{\partial \rho}{\partial x} = -\rho\frac{\partial u}{\partial x} \ . \tag{2.1}$$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = \frac{1}{\rho}\frac{\partial}{\partial x}\left(\mu\frac{\partial u}{\partial x}\right) - \frac{1}{\rho}\frac{\partial p}{\partial x} + f \ , \tag{2.2}$$

$$\frac{\partial T}{\partial t} + u\frac{\partial T}{\partial x} = \frac{1}{\rho T S_T}\frac{\partial}{\partial x}\left(\lambda\frac{\partial T}{\partial x}\right) + \frac{\mu}{\rho T S_T}\left(\frac{\partial u}{\partial x}\right)^2 + \frac{\rho S_\rho}{S_T}\frac{\partial u}{\partial x} \ . \tag{2.3}$$

where t : time, x : spatial coordinate, ρ : density $(\rho > 0)$, u : velocity,

p : pressure, T : absolute temperature $(T > 0)$, S : entropy, μ : coefficient of viscosity $(\mu > 0)$, λ : coefficient of heat conduction $(\lambda > 0)$, f : outer force which is a given function of (t,x), and p, S, μ, λ are given functions of (ρ, T).

It is easy to see that in this coupled system (2.1) is a single first order (hyperbolic) equation for ρ, (2.2), (2.3) is a second order parabolic system for (u,T). So, this system is of the following form of quasilinear hyperbolic-parabolic coupled systems:

$$\sum_{j=1}^{n} \zeta_{\ell j}(t,x,u,v)\left(\frac{\partial u_j}{\partial t} + \lambda_\ell\left(t,x,u,v,\frac{\partial v}{\partial x}\right)\frac{\partial u_j}{\partial x}\right) = \mu_\ell\left(t,x,u,v,\frac{\partial v}{\partial x}\right) \quad (\ell = 1,\dots,n) \ ,$$
$$\frac{\partial v}{\partial t} - a\left(t,x,u,\frac{\partial u}{\partial x},v\right)\frac{\partial^2 v}{\partial x^2} = b\left(t,x,u,\frac{\partial u}{\partial x},v,\frac{\partial v}{\partial x}\right) \ , \tag{*}$$

in which λ_ℓ and μ_ℓ are affine functions of $r = \frac{\partial v}{\partial x}$, a doesn't depend on $\frac{\partial v}{\partial x}$ but b does. Obviously, (*) is a special case of the system of type (I).

2. System of radiation hydrodynamcis.

In order to determine the motion of a fluid with very high temperature we have to consider the hydrodynamics in the presence of a radiation field. For the one-dimensional unsteady flow, under the diffusion approximation the corresponding system of radiation hydrodynamics can be written in Lagrangian representation as the following conservation law:

$$\frac{\partial \tau}{\partial t} - \frac{\partial u}{\partial x} = 0 \ ,$$

$$\frac{\partial u}{\partial t} + \frac{\partial (p + p_\nu)}{\partial x} = 0 \ ,$$

$$\frac{\partial\left(e + \frac{u^2}{2} + \tau E_\nu\right)}{\partial t} + \frac{\partial\left\{u(p + p_\nu) - D\rho\frac{\partial E_\nu}{\partial x}\right\}}{\partial x} = 0$$

in which

$\tau = \frac{1}{\rho}$: specific volume,

$p = R\rho T$: pressure, R = constant > 0,

$p_\nu = \frac{a}{3} T^3$: [illegible] $\frac{4\sigma}{[illegible]}$, c : light speed, σ = constant > 0,

$e = \frac{RT}{\gamma - 1}$: inner energy, γ : adiabatic exponent,

$E_\nu = 3p_\nu = aT^4$: radiation energy,

$D = \frac{\ell c}{3}$, $\ell = AT^\alpha$ (A, $\alpha > 0$ constants): Rosseland mean free path.

Taking (u,ρ,T) as unknown functions, the system can be written as

$$\sqrt{RT}\left(\frac{\partial\rho}{\partial t} + \rho\sqrt{RT}\frac{\partial\rho}{\partial x}\right) + \rho\left(\frac{\partial u}{\partial t} + \rho\sqrt{RT}\frac{\partial u}{\partial x}\right) = -\rho\left(RP + \frac{16}{3}\frac{\sigma}{c}T^3\right)\frac{\partial T}{\partial x}, \tag{2.4}$$

$$\sqrt{RT}\left(\frac{\partial\rho}{\partial t} - \rho\sqrt{RT}\frac{\partial\rho}{\partial x}\right) - \rho\left(\frac{\partial u}{\partial t} - \rho\sqrt{RT}\frac{\partial u}{\partial x}\right) = \rho\left(RP + \frac{16}{3}\frac{\sigma}{c}T^3\right)\frac{\partial T}{\partial x}, \tag{2.5}$$

$$\begin{aligned}\left(\frac{R}{\gamma-1} + \frac{16\sigma}{c}\frac{T^3}{\rho}\right)\frac{\partial T}{\partial t} - \frac{16A\sigma}{3}\rho T^{3+\alpha}\frac{\partial^2 T}{\partial x^2} - \frac{16A\sigma}{3}T^{3+\alpha}\frac{\partial\rho}{\partial x}\frac{\partial T}{\partial x} \\ - \frac{16A\sigma}{3}(3+\alpha)\rho T^{2+\alpha}\left(\frac{\partial T}{\partial x}\right)^2 + \left(\frac{16\sigma}{3c}T^4 + \rho RT\right)\frac{\partial u}{\partial x} = 0.\end{aligned} \tag{2.6}$$

It is clear that (2.6) is a single second order parabolic equation for T, and (2.4), (2.5) is a first order quasilinear hyperbolic system for ρ and u (with the characteristic directions $\frac{dx}{dt} = \lambda_{1,2} = \pm\rho\sqrt{RT}$), then (2.4)-(2.6) is also a special case of system (*).

3. System of one-dimensional viscoelastic materials of the rate type

$$\begin{aligned} &u_t - v_x = 0, \\ &v_t + p(u)_x = v_{xx} \end{aligned} \tag{2.7}$$

and system of one-dimensional thermoviscoelastic materials

$$\begin{aligned} &u_t - v_x = 0, \\ &v_t + p(u,\theta)_x = v_{xx}, \\ &\left[e(u,\theta) + \frac{v^2}{2}\right]_t + [p(u,\theta)v]_x - [vv_x]_x = \theta_{xx} \quad (e'_\theta > 0) \end{aligned} \tag{2.8}$$

are both a single first order (hyperbolic) equation for u coupled by a parabolic equation or system respectively.

System of one-dimensional thermoelastic materials

$$\begin{aligned} &u_t - v_x = 0, \\ &v_t + p(u,\theta)_x = 0, \quad (p'_u < 0), \\ &\left[e(u,\theta) + \frac{v^2}{2}\right]_t + [p(u,\theta)v]_x = \theta_{xx} \quad (e'_\theta > 0) \end{aligned} \tag{2.9}$$

is a hyperbolic system for u and v coupled by a single parabolic equation for θ.

These systems are of the form (*).

4. System of a model of nerve impulse propagation

$$\begin{aligned} &u_t = r(x)u_{xx} + F_0(u,w), \\ &w_t = G(u,w) \end{aligned} \tag{2.10}$$

and system of reaction-diffusion

$$\begin{aligned} &\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + f(u,v), \\ &\frac{\partial v}{\partial t} = g(u,v) \end{aligned} \tag{2.11}$$

are obviously of the form (*).

5. Moreover, certain higher order equations can be also reduced to a hyperbolic-parabolic coupled system, for instance, we consider the following problem (see J. M. Greenberg[10], J. M. Greenberg, R. C. MacCamy and V. J. Mizel[11], also see J. L. Lions[12]):

$$\begin{aligned} &\frac{\partial^2 u}{\partial t^2} - E\left(\frac{\partial u}{\partial x}\right)\frac{\partial^2 u}{\partial x^2} - \lambda\frac{\partial}{\partial t}\frac{\partial^2 u}{\partial x^2} = 0, \quad 0 < x < 1, \quad t > 0 \quad (\lambda > 0), \\ &u(0,t) = u(1,t) = 0, \\ &u(x,0) = u_0(x), \ \frac{\partial u}{\partial t}(x,0) = u_1(x). \end{aligned} \tag{2.12}$$

Taking

$$v = \frac{\partial u}{\partial t}, \quad w = \frac{\partial u}{\partial x}$$

as new unknown functions, this problem is equivalent to the following one

$$\frac{\partial u}{\partial t} = v, \ \frac{\partial w}{\partial t} = \frac{\partial v}{\partial x}, \ \frac{\partial v}{\partial t} - E(w)\frac{\partial w}{\partial x} - \lambda\frac{\partial^2 v}{\partial x^2} = 0 ,$$

$$v(0,t) = v(1,t) = 0 , \tag{2.13}$$

$$u(x,0) = u_0(x), \ w(x,0) = u_0'(x), \ v(x,0) = u_1(x)$$

in which the first two equations formulate a hyperbolic system for u and w and the last one is parabolic for v, so this system is of the form (*), too.

Now we shall point out that in many cases by means of adding certain new unknown functions some problems for system (*) can be equivalently reduced to a corresponding problem for the system of type (II), for which the existence and uniqueness theorem seems somewhat easier to prove.

<u>Example 1</u>: Consider Cauchy problem:

$$\sum_{j=1}^{n} \zeta_{\ell j}(t,x,u,v)\left(\frac{\partial u_j}{\partial t} + \lambda_\ell(t,x,u,v,v_x)\frac{\partial u_j}{\partial x}\right) = \mu_\ell(t,x,u,v,v_x), \quad (\ell = 1,\dots,n) , \tag{2.14}$$

$$\frac{\partial v}{\partial t} - a(t,x,u,u_x,v)\frac{\partial^2 v}{\partial x^2} = b(t,x,u,u_x,v,v_x) , \tag{2.15}$$

$$t = 0: u = \varphi(x), \ v = \psi(x) , \tag{2.16}$$

in which λ_ℓ and $\mu_\ell (\ell = 1,\dots,n)$ are affine functions of $r = \frac{\partial v}{\partial x}$. Set

$$w_j = \frac{\partial u_j}{\partial x} \quad (j = 1,\dots,n) ,$$

differentiating (2.14) with respect to x and using equation (2.15), we can prove that u,v and w satisfy the following Cauchy problem

$$\sum_{j=1}^{n} \zeta_{\ell j}(t,x,u,v)\left(\frac{\partial u_j}{\partial t} + \lambda_\ell(t,x,u,v,v_x)\frac{\partial u_j}{\partial x}\right) = \mu_\ell(t,x,u,v,v_x) ,$$

$$\sum_{j=1}^{n} \zeta_{\ell j}(t,x,u,v)\left(\frac{\partial w_j}{\partial t} + \lambda_\ell(t,x,u,v,v_x)\frac{\partial w_j}{\partial x}\right) = \overline{\zeta}_\ell(t,x,u,w,v)\left(\frac{\partial v}{\partial t} + \lambda_\ell\frac{\partial v}{\partial x}\right)$$

$$+ \overline{\mu}_\ell(t,x,u,w,v,v_x), \quad (\ell = 1,\dots,n) , \tag{2.17}$$

$$\frac{\partial v}{\partial t} - a(t,x,u,w,v)\frac{\partial^2 v}{\partial x^2} = b(t,x,u,w,v,v_x) ,$$

$$t = 0 : u = \varphi(x), \ w = \varphi'(x), \ v = \psi(x)$$

in which

$$\overline{\zeta}_\ell(t,x,u,w,v) = \left(\frac{\partial \mu_\ell}{\partial r} - \sum_{j=1}^{n}\zeta_{\ell j}\frac{\partial \lambda_\ell}{\partial r}w_j\right)/a(t,x,u,w,v), \quad \left(r \text{ denotes } \frac{\partial v}{\partial x}\right) ,$$

$$\overline{\mu}_\ell(t,x,u,w,v,r) = \frac{\partial \mu_\ell}{\partial x} + \sum_{k=1}^{n}\frac{\partial \mu_\ell}{\partial u_k}w_k + \frac{\partial \mu_\ell}{\partial v}\frac{\partial v}{\partial x} - \sum_{j=1}^{n}\zeta_{\ell j}w_j\left(\frac{\partial \lambda_\ell}{\partial x} + \sum_{k=1}^{n}\frac{\partial \lambda_\ell}{\partial u_k}w_k\right.$$

$$\left. + \frac{\partial \lambda_\ell}{\partial v}\frac{\partial v}{\partial x}\right) - \sum_{j=1}^{n}\left(\frac{\partial \zeta_{\ell j}}{\partial x} + \sum_{k=1}^{n}\frac{\partial \zeta_{\ell j}}{\partial u_k}w_k + \frac{\partial \zeta_{\ell j}}{\partial v}\frac{\partial v}{\partial x}\right)\sum_{k=1}^{n}\zeta^{jk}\mu_k - \overline{\zeta}_\ell\lambda_\ell v_x - b\overline{\zeta}_\ell$$

are determined by the coefficients and (ζ^{jk}) is the inverse matrix of $(\zeta_{\ell j})$.

Conversely, if (u,v,w) is the solution of problem (2.17), then we can prove that (u,v) is the solution of the original problem (2.14)-(2.16) and $w = \frac{\partial u}{\partial x}$.

<u>Example 2</u>: Taking

$$s = \frac{\partial w}{\partial x}$$

as an unknown function, problem (2.13) is equivalent to the following problem of type (II):

$$\frac{\partial u}{\partial t} = v,\ \frac{\partial w}{\partial t} = \frac{\partial v}{\partial x},\ \frac{\partial s}{\partial t} = \frac{1}{\lambda}\left(\frac{\partial v}{\partial t} - E(w)s\right),\ \frac{\partial v}{\partial t} - E(w)s - \lambda\frac{\partial^2 v}{\partial x^2} = 0\ ,$$

$$v(0,t) = v(1,t) = 0\ , \tag{2.18}$$

$$u(x,0) = u_0(x),\ w(x,0) = u_0'(x),\ s(x,0) = u_0''(x),\ v(x,0) = u_1(x)\ .$$

Here, we can find out that on the right-hand side of the third equation, there is a directional derivative of v along the characteristic direction $\frac{dx}{dt} = 0$ and that $a = \lambda$, $b = E(w)s$ don't depend on $\left(\frac{\partial u}{\partial x}, \frac{\partial w}{\partial x}, \frac{\partial s}{\partial x}\right)$.

Hence, in order to explain our results and methods, in what follows we shall take as an example the second initial-boundary value problem for the system of type (II). All other kinds of problems (such as the Cauchy problem, the first initial-boundary value problem, the initial-boundary value problem with interface etc.) can be discussed in a similar way and the similar results for the system of type (I) hold true, too.

3. SECOND INITIAL-BOUNDARY VALUE PROBLEMS.

On a rectangular domain

$$R(\delta) = \{(t,x) \mid 0 \le t \le \delta,\ 0 \le x \le 1\} \tag{3.1}$$

we consider the second initial-boundary value problem for the system of type (II):

$$\sum_{j=1}^{n} \zeta_{\ell j}(t,x,u,v)\left(\frac{\partial u_j}{\partial t} + \lambda_\ell(t,x,u,v,v_x)\frac{\partial u_j}{\partial x}\right) = \zeta_\ell(t,x,u,v)\left(\frac{\partial v}{\partial t} + \lambda_\ell(t,x,u,v,v_x)\frac{\partial v}{\partial x}\right) + \mu_\ell(t,x,u,v,v_x),\quad (\ell = 1,\dots,n)\ , \tag{3.2}$$

$$\frac{\partial v}{\partial t} - a(t,x,u,v,v_x)\frac{\partial^2 v}{\partial x^2} = b(t,x,u,v,v_x)\ . \tag{3.3}$$

Without loss of generality, the initial conditions may be written as

$$t = 0 : u = v = 0\ . \tag{3.4}$$

Moreover we can suppose that

$$a(0,x,0,0,0) \equiv 1 \tag{3.5}$$

(Otherwise, use the transformation of independent variables

$$\bar{x} = \int_0^x \frac{d\xi}{\sqrt{a(0,\xi,0,0,0)}}\ )$$

and that

$$b(0,x,0,0,0) \equiv 0\ , \tag{3.6}$$

$$\zeta_{\ell j}(0,x,o,o) \equiv \delta_{\ell j} = \begin{cases} 1, & \ell = j \\ 0, & \ell \neq j \end{cases}\ ; \tag{3.7}$$

(to this end, it is sufficient to introduce the transformation of unknown functions

$$\bar{v} = v - tb(0,x,0,0,0)\ ,$$

$$\bar{u}_\ell = \sum_{j=1}^{n} \zeta_{\ell j}(0,x,0,0)u_j)\ .$$

Under the hypothesis (3.7), the u_ℓ $(\ell = 1,\dots,n)$ are called the diagonal variables.

The boundary conditions are as follows:

$$\text{on } x = 1,\quad u_r = G_r(t,u,v)\quad (r = 1,\dots,h;\ h \le n)\ , \tag{3.8}$$

$$\frac{\partial v}{\partial x} = F_+(t,u,v)\ ; \tag{3.9}$$

$$\text{on } x = 0,\quad u_{\hat{s}} = \hat{G}_{\hat{s}}(t,u,v)\quad (\hat{s} = k+1,\dots,n;\ k \ge 0)\ , \tag{3.10}$$

$$\frac{\partial v}{\partial x} = F_-(t,u,v)\ . \tag{3.11}$$

Here the boundary conditions for v are of Neumann type, so this problem is called the second initial-boundary value problem.

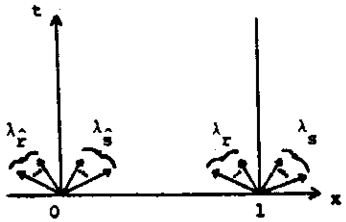
We assume that the following conditions are satisfied:

(1). <u>Conditions of orientability:</u>

$$\lambda_r(0,1,0,0,0) < 0, \quad \lambda_s(0,1,0,0,0) > 0 \qquad \begin{pmatrix} r = 1,\dots,h \\ s = h+1,\dots,n \end{pmatrix}, \tag{3.12}$$

$$\lambda_{\hat r}(0,0,0,0,0) < 0, \quad \lambda_{\hat s}(0,0,0,0,0) > 0 \qquad \begin{pmatrix} \hat r = 1,\dots,k \\ \hat s = k+1,\dots,n \end{pmatrix}. \tag{3.13}$$

As usual, the characteristic directions are called departing characteristic directions on the boundary, if as long as time increases, they go towards the interior of the domain.



Thus, on the boundary, the number of boundary conditions for u is equal to the number of departing characteristic directions. For example, on $x = 1$ the number of boundary conditions for u is equal to h, the number which appears in (3.12).

(2) <u>Conditions of compatibility:</u>

$$G_r(0,0,0) = 0, \quad \hat G_{\hat s}(0,0,0) = 0 \quad (r = 1,\dots,h;\ \hat s = k+1,\dots,n), \tag{3.14}$$

$$\frac{\partial G_r}{\partial t}(0,0,0) + \sum_{j=1}^{n} \frac{\partial G_r}{\partial u_j}(0,0,0)\mu_j(0,1,0,0,0) = \mu_r(0,1,0,0,0),$$
$$\frac{\partial \hat G_{\hat s}}{\partial t}(0,0,0) + \sum_{j=1}^{n} \frac{\partial \hat G_{\hat s}}{\partial u_j}(0,0,0)\mu_j(0,0,0,0,0) = \mu_{\hat s}(0,0,0,0,0), \tag{3.15}$$
$$(r = 1,\dots,h;\ \hat s = k+1,\dots,n),$$

$$F_{\pm}(0,0,0) = 0. \tag{3.16}$$

(3) <u>Conditions of smoothness:</u> the coefficients of the system and the boundary conditions are suitably smooth. For simplicity, we omit the detail here.

By means of certain a priori estimations for the solutions of the heat equation and of the linear hyperbolic system, using an iteration method and the Leray-Schauder fixed point theorem, we have proved the following

<u>Theorem:</u> Under the preceding hypotheses, suppose further that the following conditions are satisfied:

$$\det\left|\delta_{rr'} - \frac{\partial G_r}{\partial u_{r'}}(0,0,0)\right| \neq 0 \qquad (r, r' = 1,\dots,h),$$
$$\det\left|\delta_{\hat s\hat s'} - \frac{\partial \hat G_{\hat s}}{\partial u_{\hat s'}}(0,0,0)\right| \neq 0 \qquad (\hat s, \hat s' = k+1,\dots,n), \tag{3.17}$$

i.e. the boundary conditions may be rewritten as

$$\text{on } x = 1, \quad u_r = H_r(t, u_s, v) \quad (r = 1,\dots,h;\ s = h+1,\dots,n)$$
$$\frac{\partial v}{\partial x} = F_+(t,u,v), \tag{3.18}$$

$$\text{on } x = 0, \quad u_{\hat s} = H_{\hat s}(t, u_{\hat r}, v) \quad (\hat r = 1,\dots,k;\ \hat s = k+1,\dots,n)$$
$$\frac{\partial v}{\partial x} = F_-(t,u,v). \tag{3.19}$$

then, the second initial-boundary value problem admits a unique local classical solution on $R(\delta)$ where $\delta > 0$ is suitably small.

4. <u>IDEAS OF THE PROOF.</u>

1. A priori estimations for the solutions of the second initial-boundary value problem of heat equations:

$$\frac{\partial v}{\partial t} = \frac{\partial^2 v}{\partial x^2} + b(t,x),$$
$$t = 0:\ v = 0, \tag{4.1}$$
$$x = 0:\ \frac{\partial v}{\partial x} = \varphi_1(t); \quad x = 1:\ \frac{\partial v}{\partial x} = \varphi_2(t).$$

Suppose that on the domain $R(\delta_0)$, $\varphi_i(t) \in C^1$, $\varphi_i(0) = 0$ $(i = 1,2)$ and $b(t,x) \in C^{\frac{\alpha}{2},\alpha}$ $(0 < \alpha < 1)$, where

$C^{\beta,\alpha}$ = Hölder space of functions f such that f is Hölder continuous with respect to t and to x with the exponents β and α respectively $(0 < \alpha, \beta \leq 1)$,

then it is well known[13] that problem (4.1) admits a unique classical solution v on $R(\delta_0)$ with

$$v(t,x) = \int_0^t \int_0^1 N(t,x;\tau,\xi)b(\tau,\xi)d\xi d\tau + \int_0^t N(t,x;\tau,1)\varphi_2(\tau)d\tau$$

$$- \int_0^t N(t,x;\tau,0)\varphi_1(\tau)d\tau \ ,$$

$$\frac{\partial v}{\partial x}(t,x) = \int_0^t \int_0^1 \frac{\partial N(t,x;\tau,\xi)}{\partial x} b(\tau,\xi)d\xi d\tau + \int_0^t \frac{\partial N(t,x;\tau,1)}{\partial x} \varphi_2(\tau)d\tau$$

$$- \int_0^t \frac{\partial N(t,x;\tau,0)}{\partial x} \varphi_1(\tau)d\tau \ , \tag{4.2}$$

$$\frac{\partial^2 v}{\partial x^2}(t,x) = \int_0^t \int_0^1 \frac{\partial^2 N}{\partial x^2}(t,x;\tau,\xi)(b(\tau,\xi) - b(\tau,x))d\xi d\tau + \int_0^t N(t,x;\tau,1)\dot{\varphi}_2(\tau)d\tau$$

$$- \int_0^t N(t,x;\tau,0)\dot{\varphi}_1(\tau)d\tau \ ,$$

$$\frac{\partial v}{\partial t}(t,x) = \int_0^t \int_0^1 \frac{\partial N}{\partial t}(t,x;\tau,\xi)(b(\tau,\xi) - b(t,\xi))d\xi d\tau + \int_0^1 N(t,x;0,\xi)b(t,\xi)d\xi$$

$$+ \int_0^t N(t,x;\tau,1)\dot{\varphi}_2(\tau)d\tau - \int_0^t N(t,x;\tau,0)\dot{\varphi}_1(\tau)d\tau \ ,$$

in which

$$\dot{\varphi}_i(\tau) = \frac{d}{d\tau}\varphi_i(\tau) \qquad (i = 1,2) \ ,$$

$$N(t,x;\tau,\xi) = \sum_{n=-\infty}^{\infty} [G_0(t,x;\tau,2n+\xi) + G_0(t,x;\tau,2n-\xi)] \tag{4.3}$$

is the Neumann function for the second initial-boundary value problem of the heat equation and

$$G_0(t,x;\tau,\xi) = \frac{1}{2\sqrt{\pi(t-\tau)}} e^{-\frac{(x-\xi)^2}{4(t-\tau)}} \quad (t > \tau) \tag{4.4}$$

is the fundamental solution of the heat equation.

Moreover, on $R(\delta_0)$ $v \in \bar{C}^{2+\alpha}$ $(0 < \alpha < 1)$, where

$$\bar{C}^{2+\alpha} = \{f \mid f, \frac{\partial f}{\partial x}, \frac{\partial f}{\partial t}, \frac{\partial^2 f}{\partial x^2} \text{ continuous}, \ \frac{\partial f}{\partial x} \in C^{\frac{1+\alpha}{2},1}, \ \frac{\partial f}{\partial t}, \frac{\partial^2 f}{\partial x^2} \in C^{\frac{\alpha}{2},\alpha}\} . \tag{4.5}$$

On $R(\delta)$, $\forall\, \delta$, $0 < \delta \leq \delta_0$, introduce the following norms:

$$\|f\| = \sup_{(t,x) \in R(\delta)} |f(t,x)| \ ,$$

$$H_t^{\beta}[f] = \sup_{\substack{(t_1,x),(t_2,x) \\ \in R(\delta)^2}} \frac{|f(t_1,x)-f(t_2,x)|}{|t_1-t_2|^{\beta}}, \quad H^{\alpha}[f] = \sup_{\substack{(t,x_1),(t,x_2) \\ \in R(\delta)^2}} \frac{|f(t,x_1)-f(t,x_2)|}{|x_1-x_2|^{\alpha}} , \tag{4.6}$$

$$H^{\alpha}[f] = H_t^{\frac{\alpha}{2}}[f] + H_x^{\alpha}[f] \qquad (0 < \alpha, \beta \leq 1) .$$

Using the preceding expressions and the property of the fundamental solution, after a long calculation we have obtained the following three a priori estimations on $R(\delta)$ $\forall\, \delta$, $0 < \delta \leq \delta_0$:

$$1°. \quad \|v\| \underset{\text{def}}{\equiv} \|v\| + \|\frac{\partial v}{\partial x}\| \leq C_1(\delta^{1/2}\|b\| + \|\varphi\|) \quad \text{or} \quad \|v\| \leq C_1(\delta^{1/2}\|b\| + \delta\|\dot{\varphi}\|) \ ; \tag{4.7}$$

$$2°. \quad \|v\|_1 \underset{\text{def}}{\equiv} \|v\| + \|\frac{\partial v}{\partial t}\| + \|\frac{\partial^2 v}{\partial x^2}\| + H_t^{1/2}[\frac{\partial v}{\partial x}]$$
$$\leq C_2(\|b\| + \delta^{\frac{\alpha}{2}} H_x^{\alpha}[b] + \delta^{1/2}\|\varphi\|_1) \qquad (\|\varphi\|_1 = \|\varphi\| + \|\dot{\varphi}\|) \ ; \tag{4.8}$$

$$3°. \quad \|v\|_2 \underset{\text{def}}{\equiv} \|v\|_1 + H_t^{\frac{1+\alpha}{2}}[\frac{\partial v}{\partial x}] + H^{\alpha}[\frac{\partial v}{\partial t}] + H^{\alpha}[\frac{\partial^2 v}{\partial x^2}] \leq C_3(\|b\| + H^{\alpha}[b] + \|\varphi\|_1) . \tag{4.9}$$

in which C_i $(i = 1,2,3)$ signify constants depending only on δ_0.

2. A priori estimations for the solutions of the following initial-boundary value problem of first order linear hyperbolic systems:

$$\left.\begin{aligned} &\sum_{j=1}^{n} \zeta_{\ell j}(t,x)\left(\frac{\partial u_j}{\partial t} + \lambda_\ell(t,x) \frac{\partial u_j}{\partial x}\right) = \zeta_\ell(t,x)\left(\frac{\partial v}{\partial t} + \lambda_\ell(t,x) \frac{\partial v}{\partial x}\right) + \mu_\ell(t,x) \\ &\qquad\qquad\qquad\qquad\qquad\qquad (\ell = 1,\dots,n)\ , \\ &t = 0 : u = 0\ , \\ &x = 1 : \sum_{j=1}^{n} \zeta_{rj}(t,1)u_j = \psi_r(t) \quad (r = 1,\dots,h\ ;\ h < n)\ , \\ &x = 0 : \sum_{j=1}^{n} \zeta_{\hat{s}j}(t,0)u_j = \hat{\psi}_{\hat{s}}(t) \quad (\hat{s} = k + 1,\dots,n;\ k > 0)\ , \end{aligned}\right\} \tag{4.10}$$

in which $v = v(t,x)$ is a given C^1 function and we suppose that on $R(\delta_0)$

$$\det|\zeta_{\ell j}(t,x)| \neq 0 \tag{4.11}$$

and

$$\zeta_{\ell j}(0,x) = \delta_{\ell j}\ . \tag{4.12}$$

We suppose further that the following conditions are satisfied:

1°. <u>Conditions of orientability:</u>

$$\begin{aligned} &\text{on } x = 1,\ \lambda_r(t,1) < 0,\ \lambda_s(t,1) > 0 \quad (r = 1,\dots,h;\ s = h + 1,\dots,n)\ , \\ &\text{on } x = 0,\ \lambda_{\hat{r}}(t,0) < 0,\ \lambda_{\hat{s}}(t,0) > 0 \quad (\hat{r} = 1,\dots,k;\ \hat{s} = k + 1,\dots,n)\ . \end{aligned} \tag{4.13}$$

2°. <u>Conditions of compatibility:</u>

$$\begin{aligned} &\psi_r(0) = 0,\ \ \hat{\psi}_{\hat{s}}(0) = 0\ , \\ &\zeta_r(0,1)\left(\frac{\partial v}{\partial t}(0,1) + \lambda_r(0,1)\frac{\partial v}{\partial x}(0,1)\right) + \mu_r(0,1) = \dot{\psi}_r(0), \quad \left(\begin{matrix} r=1,\dots,h \\ \hat{s} = k + 1,\dots,n \end{matrix}\right) \\ &\zeta_{\hat{s}}(0,0)\left(\frac{\partial v}{\partial t}(0,0) + \lambda_{\hat{s}}(0,0)\frac{\partial v}{\partial x}(0,0)\right) + \mu_{\hat{s}}(0,0) = \dot{\hat{\psi}}_{\hat{s}}(0)\ . \end{aligned} \tag{4.14}$$

3°. <u>Conditions of smoothness.</u>

Usually, for the initial-boundary value problem of first order linear hyperbolic systems, the term on the right-hand side of equations should be assumed to be continuous as well as its first derivative with respect to x. In the present case, since $v(t,x)$ is

a C^1 function, $\frac{\partial v}{\partial t} + \lambda_\ell(t,x)\frac{\partial v}{\partial x}$ is only continuous. But, noticing that it is the directional derivative of v along the characteristic curve $\frac{dx}{dt} = \lambda_\ell(t,x)$, we can integrate by parts this term $\zeta_\ell\left(\frac{\partial v}{\partial t} + \lambda_\ell \frac{\partial v}{\partial x}\right)$ when we integrate the system along the characteristic curve, then we can prove as usual that problem (4.10) admits a unique classical solution u on $R(\delta_0)$ with $u \in C^1$ or $u \in C^{1+\beta} = \{f | f \in C^1, \frac{\partial f}{\partial t}, \frac{\partial f}{\partial x} \in C^{\beta,\beta}\}$ under the different hypotheses of smoothness respectively.

Introduce the following classes of functions:

$$\begin{aligned} \Gamma_0 &= \left\{\lambda_\ell,\ \zeta_\ell,\ \mu_\ell,\ \frac{1}{\lambda_r(t,1)},\ \frac{1}{\lambda_{\hat{s}}(t,0)}\right\}\ , \\ \Gamma_1 &= \left\{\zeta_{\ell j},\ \frac{\partial \zeta_{\ell j}}{\partial t},\ \frac{\partial \zeta_{\ell j}}{\partial x},\ \zeta_\ell,\ \frac{\partial \zeta_\ell}{\partial t},\ \frac{\partial \zeta_\ell}{\partial x},\ \lambda_\ell,\ \frac{1}{\det|\zeta_{\ell j}|}\right\}\ , \\ \Gamma_2 &= \Gamma_1 \cup \left\{\frac{\partial \lambda_\ell}{\partial x},\ \mu_\ell,\ \frac{\partial \mu_\ell}{\partial x},\ \frac{1}{\lambda_r(t,1)},\ \frac{1}{\lambda_{\hat{s}}(t,0)}\right\} \\ &(\ell,j = 1,\dots,n;\ \ r = 1,\dots,h;\ \ \hat{s} = m + 1,\dots,n) \end{aligned} \tag{4.15}$$

and the following norms of functions on $R(\delta)$ $(0 < \delta < \delta_0)$

$$\begin{aligned} \|u\|_1 &= \|u\| + \left\|\frac{\partial u}{\partial t}\right\| + \left\|\frac{\partial u}{\partial x}\right\|\ , \\ \|u\|_1^* &= \|u\| + \left\|\frac{\partial u}{\partial t}\right\| + \varepsilon\left\|\frac{\partial u}{\partial x}\right\|\ , \\ \|u\|_{1+\beta} &= \|u\|_1 + H_*^\beta\left[\frac{\partial u}{\partial t}\right] + H_*^\beta\left[\frac{\partial u}{\partial x}\right]\ , \\ \|u\|_{1+\beta}^* &= \|u\|_1^* + H_t^\beta\left[\frac{\partial u}{\partial t}\right] + \varepsilon\left(H_x^\beta\left[\frac{\partial u}{\partial t}\right] + H_*^\beta\left[\frac{\partial u}{\partial x}\right]\right)\ , \end{aligned} \tag{4.16}$$

where

$$H_*^\beta[f] = H_x^\beta[f] + H_t^\beta[f] \tag{4.17}$$

and the constant $\varepsilon > 0$ will be suitably chosen later on. By means of the integral relations satisfied by $u(t,x)$ and by $\frac{\partial u}{\partial t}, \frac{\partial u}{\partial x}$ respectively, after a long calculation we have established three a priori estimations on $R(\delta)$ $\forall\ \delta_0 > \delta > 0$ as follows:

$1°.$

$$\|u\| \le (1 + K_1\delta)\|\psi\| + (H_0 + K_1\delta)\|v\| + K_1\delta\|\mu\| \tag{4.18}$$

in which constant K_1 depends only on the norm $\|\Gamma_1\|$ on $R(\delta_0)$ and

$$H_0 = 2 \sup_{\substack{\ell=1,\dots,n \\ (t,x)\in R(\delta_0)}} |\zeta_\ell(t,x)| .$$

$2°.$

$$\|u\|_1^* \le (1 + d_0^{-1}\varepsilon + K_2\delta^\beta)\|\psi\| + (K_0 + K_2\delta)(1 + \|v\|_1) \tag{4.19}$$

provided that λ_ℓ, μ_ℓ are Hölder continuous with respect to t with the exponent β, in which $d_0 = \min\limits_{\substack{1\le r\le h \\ k+1\le \hat{s}\le n}} \{-\lambda_r(0,1), \lambda_{\hat{s}}(0,0)\}$, K_0 depends only on the norm $\|\Gamma_0\|$ on $R(\delta_0)$

and K_2 depends only on the norm $\|\Gamma_2\|$ and $H_t^\beta[\Gamma_0]$ on $R(\delta_0)$.

$$3°.\quad \|u\|_{1+\beta}^* \le (1 + 2\, d_0^{-1}\varepsilon + d_0^{-2}\varepsilon + K_2\delta^\beta)H_t^\beta(\psi) + (K_2 + K_3\delta)(1 + \|\psi\| + \|v\|_{1+\beta}) \tag{4.20}$$

provided that all the functions in Γ_2 are Hölder continuous with respect to t and x with the exponent β, where K_3 depends only on $\|\Gamma_2\|$ and $H_*^\beta[\Gamma_2]$ on $R(\delta_0)$.

3. Introduce the following sets of functions on $R(\delta)$:

$$\textstyle\sum_*(\delta) = \left\{(u,v) \mid u \in C^1,\ v \in C^1,\ \frac{\partial^2 v}{\partial x^2} \in C^0,\ u(0,x) = v(0,x) = 0\right\},$$

$$\begin{aligned} \textstyle\sum_1(\delta) = \Big\{(u,v) \mid\ & u \in C^{1+\frac{\alpha}{2}},\ v \in C^{-2+\alpha},\ u(0,x) = v(0,x) = 0, \\ & \frac{\partial u_j}{\partial t}(0,x) = \mu_j(0,x,0,0,0),\ \frac{\partial v}{\partial t}(0,x) = 0\Big\}, \end{aligned} \tag{4.21}$$

$$\begin{aligned} \textstyle\sum(\delta) = \Big\{(u,v) \mid (u,v) \in \sum_1(\delta),\ & \|u\| \le A_0,\ \|u\|_1^* \le A_1,\ \|u\|_{1+\frac{\alpha}{2}}^* \le A_2, \\ & \|v\| \le B_0,\ \|v\|_1 \le B_1,\ \|v\|_2 \le B_2\Big\}, \end{aligned}$$

where A_i, B_i $(i = 0,1,2)$ are positive constants to be chosen later with $A_0 \le A_1 \le A_2$, $B_0 \le B_1 \le B_2$.

For any $(\tilde{u},\tilde{v}) \in \sum_1(\delta)$, according to the preceding points we can define an iterative operator $(u,v) = T(\tilde{u},\tilde{v})$ by means of the following linear problem

$$\begin{aligned} &\sum_{j=1}^n \zeta_{\ell j}(t,x,\tilde{u},\tilde{v})\Big(\frac{\partial u_j}{\partial t} + \lambda_\ell\big(t,x,\tilde{u},\tilde{v},\frac{\partial\tilde{v}}{\partial x}\big)\frac{\partial u_j}{\partial x}\Big) \\ &\quad = \zeta_\ell(t,x,\tilde{u},\tilde{v})\Big(\frac{\partial\tilde{v}}{\partial t} + \lambda_\ell\big(t,x,\tilde{u},\tilde{v},\frac{\partial\tilde{v}}{\partial x}\big)\frac{\partial\tilde{v}}{\partial x}\Big) + \mu_\ell\big(t,x,\tilde{u},\tilde{v},\frac{\partial\tilde{v}}{\partial x}\big) \quad (\ell = 1,\dots,n), \\ &\frac{\partial v}{\partial t} - \frac{\partial^2 v}{\partial x^2} = b\big(t,x,\tilde{u},\tilde{v},\frac{\partial\tilde{v}}{\partial x}\big) + \big[a\big(t,x,\tilde{u},\tilde{v},\frac{\partial\tilde{v}}{\partial x}\big) - 1\big]\frac{\partial^2\tilde{v}}{\partial x^2}, \\ &t = 0:\ u = 0,\ v = 0, \\ &x = 1:\ \sum_{j=1}^n \zeta_{rj}(t,1,\tilde{u}(t,1),\tilde{v}(t,1))u_j = G_r(t,\tilde{u}(t,1),\tilde{v}(t,1)) \\ &\qquad + \sum_{j=1}^n (\zeta_{rj}(t,1,\tilde{u}(t,1),\tilde{v}(t,1)) - \delta_{rj})\tilde{u}_j(t,1) \equiv \psi_r(t),\quad r = 1,\dots,h, \\ &\qquad \frac{\partial v}{\partial x}(t,1) = F_+(t,\tilde{u}(t,1),\tilde{v}(t,1)) \equiv \varphi_+(t), \\ &x = 0:\ \sum_{j=1}^n \zeta_{\hat{s}j}(t,0,\tilde{u}(t,0),\tilde{v}(t,0))u_j = \hat{G}_{\hat{s}}(t,\tilde{u}(t,0),\tilde{v}(t,0)) \\ &\qquad + \sum_{j=1}^n (\zeta_{\hat{s}j}(t,0,\tilde{u}(t,0),\tilde{v}(t,0)) - \delta_{\hat{s}j})\tilde{u}_j(t,0) \equiv \psi_{\hat{s}}(t),\quad \hat{s} = k+1,\dots,n, \\ &\qquad \frac{\partial v}{\partial x}(t,0) = F_-(t,\tilde{u}(t,0),\tilde{v}(t,0)) \equiv \varphi_-(t). \end{aligned} \tag{4.22}$$

For the time being, we suppose that

$$\sum_{j=1}^n \Big|\frac{\partial G_r}{\partial u_j}(0,0,0)\Big| < 1,\quad \sum_{j=1}^n \Big|\frac{\partial \hat{G}_{\hat{s}}}{\partial u_j}(0,0,0)\Big| < 1. \tag{4.23}$$

Then, using the preceding a priori estimations, we can choose a small constant $\varepsilon > 0$ and constants $A_0, A_1, A_2, B_0, B_1, B_2$ such that the operator T maps $\sum(\delta)$ into itself, if $\delta > 0$ is suitably small. Because $\sum(\delta)$ is a nonempty convex, closed, compact subset of the Banach space $\sum_*(\delta)$ provided the norm

$$\|(u,v)\|_* = \|u\|_1 + \|v\|_1 + \left\|\frac{\partial^2 v}{\partial x^2}\right\|$$

and T is a continuous mapping from $\sum(\delta)$ into itself in this space, according to Leray-Schauder fixed point theorem this operator $(u,v) \to T(\tilde{u},\tilde{v})$ has a fixed point (u,v) which is the solution of the original quasilinear problem on $R(\delta)$. The uniqueness of the solution can be proved as usual by means of the corresponding a priori estimations.

4. In order to finish the proof, we have to point out that the contraction condition (4.23) can be realized under hypothesis (3.17). In fact, under this hypothesis the boundary conditions can be written as (3.18), (3.19). Then, introducing a transformation of unknown functions

$$\bar{u}_i = I_i(x) u_i \qquad (i = 1,\dots,n)$$

and multiplying the ℓ-th equation of (3.2) by $I_\ell(x)$, where

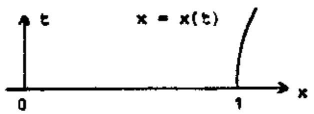
$$I_i(x) = a_i x + b_i(1 - x)$$

with $a_s = b_{\hat{r}} = 1$, $a_r = b_{\hat{s}} > 0$ small enough, it is easy to see that the problem for $(\bar{u}, v)$ satisfies the corresponding contraction condition (4.23) as well as all the hypothesis of the theorem, so the theorem is proved.

5. SECOND FREE BOUNDARY PROBLEMS.

On a domain

$$R(\delta) = \{(t,x) \mid 0 \le t \le \delta,\ 0 \le x \le x(t)\},$$



where $x = x(t)$ is an unknown boundary curve, we consider the following second initial-boundary value problem with free boundaries (for simplicity, called the second free boundary problem):

$$\sum_{j=1}^{n} \zeta_{\ell j}(t,x,u,v)\left(\frac{\partial u_j}{\partial t} + \lambda_\ell(t,x,u,v,v_x)\frac{\partial u_j}{\partial x}\right) = \zeta_\ell(t,x,u,v)\left(\frac{\partial v}{\partial t} + \lambda_\ell(t,x,u,v,v_x)\frac{\partial v}{\partial x}\right)$$

$$+ \mu_\ell(t,x,u,v,v_x) \quad (\ell = 1,\dots,n),$$

$$\frac{\partial v}{\partial t} - a(t,x,u,v,v_x)\frac{\partial^2 v}{\partial x^2} = b(t,x,u,v,v_x), \tag{5.1}$$

$t = 0:\ u = v = 0,$

$x = 0:\ u_{\hat{s}} = \hat{G}_{\hat{s}}(t,u,v) \quad (\hat{s} = k+1,\dots,n;\ k \ge 0),$

$\frac{\partial v}{\partial x} = F_-(t,u,v),$

$x = x(t):\ u_r = G_r(t,x,u,v) \quad (r = 1,\dots,h;\ h \le n),$

$\frac{\partial v}{\partial x} = F_+(t,x,u,v)$

and

$$\frac{dx(t)}{dt} = D(t,x,u,v,v_x), \quad x(0) = 1,$$

which is an ordinary equation to determine the free boundary $x = x(t)$ in the procedure of solution.

This kind of problem can be met in the motion of the fluid with radiation shocks.

We assume once more that the corresponding conditions of orientability, of compatibility and of smoothness hold true. For instance, we assume

$$\lambda_r(0,1,0,0,0) < D(0,1,0,0,0),\ \lambda_s(0,1,0,0,0) > D(0,1,0,0,0),$$
$$\lambda_{\hat{r}}(0,0,0,0,0) < 0,\ \lambda_{\hat{s}}(0,0,0,0,0) > 0 \tag{5.2}$$
$$(r = 1,\dots,h,\ s = h+1,\dots,n;\ \hat{r} = 1,\dots,k,\ \hat{s} = k+1,\dots,n).$$

Here the essential difficulty consists in the presence of the free boundary curve, but, using the transformation of independent variables

$$\bar{t} = t, \quad \bar{x} = \frac{x}{x(t)}, \tag{5.3}$$

the domain $R(\delta)$ is reduced to the domain

$$\overline{R}(\delta) = \{(\overline{t},\overline{x}) \mid 0 \le \overline{t} \le \delta,\ 0 \le \overline{x} \le 1\}$$

with fixed boundaries, now the coefficients of the system and the boundary conditions depend on $x(\overline{t})$ such that they are certain operators of (u,v). That is to say, we obtain a second initial-boundary value problem in functional form as follows (where $(\overline{t},\overline{x})$ is again replaced by (t,x)):

$$\sum_{j=1}^{n} \zeta_{\ell j}(t,x|u,v)\left(\frac{\partial u_j}{\partial t} + \lambda_\ell(t,x|u,v)\frac{\partial u_j}{\partial x}\right) = \zeta_\ell(t,x|u,v)\left(\frac{\partial v}{\partial t} + \lambda_\ell(t,x|u,v)\frac{\partial v}{\partial x}\right)$$

$$+ \mu_\ell(t,x|u,v) \qquad (\ell = 1,\dots,n) ,$$

$$\frac{\partial v}{\partial t} - a(t,x|u,v)\,\frac{\partial^2 v}{\partial x^2} = b(t,x|u,v) ,$$

$$t = 0: \quad u = v = 0 , \tag{5.4}$$

$$x = 1: \quad u_r = G_r(t|u,v) ,$$

$$\frac{\partial v}{\partial x} = F_+(t|u,v) ,$$

$$x = 0: \quad u_{\hat{s}} = \hat{G}_{\hat{s}}(t|u,v) ,$$

$$\frac{\partial v}{\partial x} = F_-(t|u,v) ,$$

where

$$\zeta_{\ell j}(t,x|u,v) = \zeta_{\ell j}(t,x(t)x,u,v) ,$$

$$\lambda_\ell(t,x|u,v) = \left(\lambda_\ell\left(t,x(t)x,u,v,\ \frac{\partial v}{\partial x}\cdot\frac{1}{x(t)}\right) - x'(t)x\right)/x(t) ,$$

$$\zeta_\ell(t,x|u,v) = \zeta_\ell(t,x(t)x,u,v) ,$$

$$\mu_\ell(t,x|u,v) = \mu_\ell\left(t,x(t)x,u,v,\ \frac{\partial v}{\partial x}\cdot\frac{1}{x(t)}\right) , \tag{5.5}$$

$$a(t,x|u,v) = a\left(t,x(t)x,u,v,\ \frac{\partial v}{\partial x}\cdot\frac{1}{x(t)}\right)/x^2(t) ,$$

$$b(t,x|u,v) = b\left(t,x(t)x,u,v,\ \frac{\partial v}{\partial x}\cdot\frac{1}{x(t)}\right) + \frac{x'(t)x}{x(t)}\frac{\partial v}{\partial x} ;$$

$$G_r(t|u,v) = G_r(t,x(t),u,v) ,$$

$$\hat{G}_{\hat{s}}(t|u,v) = \hat{G}_{\hat{s}}(t,u,v) , \tag{5.6}$$

$$F_+(t|u,v) = F_+(t,x(t),u,v)\cdot x(t) ,$$

$$F_-(t|u,v) = F_-(t,u,v)\cdot x(t)$$

and $x = x(t)$ is defined by

$$\frac{dx(t)}{dt} = D\left(t,x(t),u(t,1),v(t,1),\ \frac{\partial v}{\partial x}(t,1)\cdot\frac{1}{x(t)}\right) \tag{5.7}$$

$$x(0) = 1 .$$

For the second initial-boundary value problem in functional form we can prove that the situation is similar to the second initial-boundary value problem, then we can obtain the corresponding condition of solvability for the original second free boundary problem as follows:

$$\det\left|\delta_{\hat{s}\tilde{s}} - \frac{\partial \hat{G}_{\hat{s}}}{\partial u_{\tilde{s}}}(0,0,0)\right| \ne 0, \qquad (\hat{s},\tilde{s} = k+1,\dots,n)$$

$$\det\left|\delta_{r\tilde{r}} - \frac{\partial G_r}{\partial u_{\tilde{r}}}(0,1,0,0)\right| \ne 0 \qquad (r,\tilde{r} = 1,\dots,h) , \tag{5.8}$$

i.e. the boundary conditions can be written as

$$
\begin{aligned}
x = 0: \quad & u_{\hat{s}} = \hat{H}_{\hat{s}}(t,u_{\hat{r}},v) \qquad (\hat{r} = 1,\dots,k;\ \hat{s} = k+1,\dots,n)\ , \\
& \frac{\partial v}{\partial x} = F_{-}(t,u,v)\ , \\
x = 1: \quad & u_r = H_r(t,x,u_s,v) \qquad (r = 1,\dots,h;\ s = h+1,\dots,n) \\
& \frac{\partial v}{\partial x} = F_{+}(t,x,u,v)\ .
\end{aligned}
\tag{5.9}
$$

6. VARIOUS REMARKS.

1. In the case where the given boundary $x = x(t)$ is the k-th characteristic curve, we can consider $x = x(t)$ as a free boundary with the condition

$$\frac{dx}{dt} = \lambda_k\left(t,x,u,v,\frac{\partial v}{\partial x}\right),\quad x(0) = 1\ . \tag{6.1}$$

Using the preceding transformation we obtain again a second mixed initial-boundary value problem in functional form.

2. In a similar way we can also solve the following problems:

1°. the Cauchy problem;

2°. the first mixed problem with the boundary conditions:

$$
\begin{aligned}
x = 0: \quad & u_{\hat{s}} = \hat{G}_{\hat{s}}(t,u,v) \qquad (\hat{s} = k+1,\dots,n) \\
& v = F_{-}(t,u) \\
x = 1: \quad & u_r = G_r(t,u,v), \qquad (r = 1,\dots,h) \\
& v = F_{+}(t,u)
\end{aligned}
\tag{6.2}
$$

and the corresponding first free boundary problem.

3°. the problem with the interface $x = 0$ on the domain

$$R(\delta) = R_{+}(\delta) \cup R_{-}(\delta)$$

with

$$
\begin{aligned}
R_{-}(\delta) &= \{(t,x) \mid 0 < t < \delta,\ -1 < x < 0\}\ , \\
R_{+}(\delta) &= \{(t,x) \mid 0 < t < \delta,\ 0 < x < 1\}\ ,
\end{aligned}
\tag{6.3}
$$

for the following system

$$
\begin{aligned}
\sum_{j=1}^{n} \zeta_{\ell j}^{(\pm)}(t,x,u^{(\pm)},v^{(\pm)})&\left(\frac{\partial u_j^{(\pm)}}{\partial t} + \lambda_\ell^{(\pm)}\left(t,x,u^{(\pm)},v^{(\pm)},\frac{\partial v^{(\pm)}}{\partial x}\right)\frac{\partial u_j^{(\pm)}}{\partial x}\right) \\
&= \zeta_\ell^{(\pm)}(t,x,u^{(\pm)},v^{(\pm)})\left(\frac{\partial v^{(\pm)}}{\partial t} + \lambda_\ell^{(\pm)}\left(t,x,u^{(\pm)},v^{(\pm)},\frac{\partial v^{(\pm)}}{\partial x}\right)\frac{\partial v^{(\pm)}}{\partial x}\right) \\
&+ \mu_\ell^{(\pm)}\left(t,x,u^{(\pm)},v^{(\pm)},\frac{\partial v^{(\pm)}}{\partial x}\right) \qquad (\ell = 1,\dots,n)\ , \\
\frac{\partial v^{(\pm)}}{\partial t} - a^{(\pm)}\left(t,x,u^{(\pm)},v^{(\pm)},\frac{\partial v^{(\pm)}}{\partial x}\right)&\frac{\partial^2 v^{(\pm)}}{\partial x^2} = b^{(\pm)}\left(t,x,u^{(\pm)},v^{(\pm)},\frac{\partial v^{(\pm)}}{\partial x}\right)
\end{aligned}
$$

on $R^{\pm}(\delta)$ respectively,

$$
\begin{aligned}
t = 0: \quad & u^{(\pm)} = v^{(\pm)} = 0\ , \\
x = 0: \quad & u_{\hat{s}}^{(+)} = \hat{G}_{\hat{s}}(t,u^{(\pm)},v^{(\pm)}) \qquad \hat{s} = k+1,\dots,n\ , \\
& u_r^{(-)} = G_r(t,u^{(\pm)},v^{(\pm)}) \qquad r = 1,\dots,h\ , \\
& v^{(+)} = v^{(-)} + f(t)\ , \\
& \frac{\partial v^{(+)}}{\partial x} = \alpha(t,u^{(\pm)},v^{(\pm)})\,\frac{\partial v^{(-)}}{\partial x} + g(t,u^{(\pm)},v^{(\pm)}) \qquad (\alpha > 0)\ ; \\
x = \pm 1: \quad & \text{convenient boundary conditions}
\end{aligned}
\tag{6.4}
$$

with the following hypotheses:

$$
\begin{aligned}
&\lambda_{\hat{r}}^{(+)}(0,0,0,0,0) < 0,\quad \lambda_{\hat{s}}^{(+)}(0,0,0,0,0) > 0,\quad \hat{r} = 1,\dots,k;\quad \hat{s} = k+1,\dots,n\ , \\
&\lambda_r^{(-)}(0,0,0,0,0) < 0,\quad \lambda_s^{(-)}(0,0,0,0,0) > 0,\quad r = 1,\dots,h;\quad s = h+1,\dots,n\ .
\end{aligned}
\tag{6.5}
$$

4°. The problem with free interface $x = x(t)$ can be similarly discussed, too. This time the conditions on $x = x(t)$ are the following:

$$x = x(t) : \quad u_{\hat{s}}^{(+)} = \hat{G}_{\hat{s}}(t,x,u^{(\pm)},v^{(\pm)}), \qquad \hat{s} = k+1,\ldots,n .$$

$$u_r^{(-)} = G_r(t,x,u^{(\pm)},v^{(\pm)}), \qquad r = 1,\ldots,h .$$

$$v^{(+)} = v^{(-)} + T(t,x); \tag{6.6}$$

$$\frac{\partial v^{(+)}}{\partial x} = \alpha(t,x,u^{(\pm)},v^{(\pm)})\frac{\partial v^{(-)}}{\partial x} + g(t,x,u^{(\pm)},v^{(\pm)})$$

and

$$\frac{dx}{dt} = D\left(t,x,u^{(\pm)},v^{(\pm)},\frac{\partial v^{(\pm)}}{\partial x}\right), \quad x(0) = 0 \tag{6.7}$$

with the following hypotheses:

$$\lambda_{\hat{r}}^{(+)}(0,0,0,0,0) < D(0,0,0,0,0) \equiv D(0), \quad \lambda_{\hat{s}}^{(+)}(0,0,0,0,0) > D(0) ,$$

$$\lambda_r^{(-)}(0,0,0,0,0) < D(0), \quad \lambda_s^{(-)}(0,0,0,0,0) > D(0) \tag{6.8}$$

$$(\hat{r} = 1,\ldots,k; \quad \hat{s} = k+1,\ldots,n; \quad r = 1,\ldots,h; \quad s = h+1,\ldots,n) .$$

For the system of conservation laws, these conditions (6.6), (6.7) and (6.8) (with $h = k - 1$) can be obtained from the corresponding Rankine-Hugoniot's conditions and the corresponding entropy condition respectively.

Similarly, we can consider the problem with the characteristic interface $x = x(t)$, too.

7. APPLICATION TO A CONJECTIVE GIVEN BY C. M. DAFERMOS.

The conjective given by C. M. Dafermos[3] is that incomplete parabolic damping can preserve the smoothness of smooth initial data but is incapable of smoothening rough initial data. For the system of one-dimensional viscoelastic materials of the rate type, he has verified that this conjecture is true.

Now, using the preceding results we can consider this conjecture in general case. Indeed, the system with incomplete parabolic damping is a hyperbolic-parabolic coupled system, since the problem with the free interface (or characteristic interface) $x = x(t)$ is well-posed (under the corresponding conditions of solvability), if the (rough) initial data are piecewise smooth with a discontinuity at the origin $x = 0$, satisfying the corresponding conditions of compatibility (i.e. corresponding Rankine-Hugoniot's conditions for conservation laws) and the corresponding conditions of orientability (i.e. corresponding entropy condition), then the local solution is also piecewise smooth with a discontinuity on $x = x(t)$, because the corresponding conditions of solvability can be checked in many concrete cases. Thus, incomplete parabolic damping is incapable of smoothening rough initial data, that is to say, the second part of this conjecture is true. On the other hand, for smooth initial data, according to the preceeding results, the solution remains smooth locally in time, so the first part of this conjecture is true at least in a local sense. As to the corresponding global existence theorem, we have to discuss the concrete system and the problem seems yet open.

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