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SUMMER WORKSHOP ON FIBRE BUNDLES AND GEOMETRY

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ALGEBRAIC TOPOLOGY

I. Topological Preliminaries

II. Homotopy Functors

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ALGEBRAIC TOPOLOGY I

TOPOLOGICAL PRELIMINARIES

§1 : THE SPACES $MAP(X,Y)$

Notation

For $X, Y \in Ob(top)$, denote

$$\underline{Map(X,Y)} \equiv hom_{(top)}(X,Y)$$

For $A \subseteq X$, $B \subseteq Y$, denote

$$\underline{\langle A, B \rangle} \equiv \{ f \in Map(X,Y) : f(A) \subseteq f(B) \}$$

In the sequel, a map will mean a continuous function while a space will mean a topological space.

Definition

Define the space $Map(X,Y)$ to be the set $Map(X,Y)$ endowed with the compact open topology. This topology has an open subbase :

$$\{ \langle K, V \rangle : K \subseteq X \text{ cmpt.}, V \subseteq Y \text{ open} \}.$$

Lemma

Assume $X \neq \phi$. $Map(X,Y)$ is T_2 iff Y is T_2.

Proof:

1 $\Rightarrow$ Let $y_1, y_2 \in Y$, $y_1 \neq y_2$. Define the maps

$$f_i : X \to Y \quad f_i(x) = y_i \quad \forall x \in X$$

Lemma

$Map(X, Y) \xrightarrow{u^*} Map(X', Y)$ is continuous where $u^*(f) = fu$ for $u : X' \to X$

Proof.

Continuity is proven at each point $f \in Map(X,Y)$. Given a subbasic set $\langle K', V \rangle$ in $Map(X', Y)$ st $u^*f \in \langle K', V \rangle$, it suffices to prove that $\exists$ open nbhd of f mapping into $\langle K', V \rangle$ under u^*. K' cmpt $\Rightarrow$ $u(K')$ cmpt $\because$ u cnts. Now $(fu)(K') = u^*f(K') \subset V$ $\because$ $u^*f \in \langle K', V \rangle$. Hence $f \in \langle u(K'), V \rangle$, open nbhd of f. Moreover, if $g \in \langle u(K'), V \rangle$ then $gu \in \langle u^{\leftarrow} u(K'), V \rangle \subseteq \langle K', V \rangle$ so $u^*g \in \langle K', V \rangle$

Q.E.D

2. $Y \xrightarrow{v} Y' \in Map(Y, Y') \Rightarrow Map(X, Y) \xrightarrow{v_*} Map(X, Y')$ by $v_* f \equiv v \cdot f$

that is:

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[dr, "v_* f"'] & Y \arrow[d, "v"] \\ & Y' \end{tikzcd}$$

Lemma:

$Map(X, Y) \xrightarrow{v_*} Map(X, Y')$ is continuous.

Proof

Check continuity pointwise and on subbasic open sets. Let $\langle K, V' \rangle$ be a subbasic set in $Map(X, Y')$ containing $v_* f$. V' open, v cnts $\Rightarrow$ $v^{\leftarrow} V'$ open. Moreover,

$$f(K) \subseteq v^{\leftarrow}((vf)(K)) \subseteq v^{\leftarrow} V' \equiv V$$

so $f \in \langle K, V \rangle$, open nbhd. Finally, if $g \in \langle K, V \rangle$, then $g(K) \subseteq V \Rightarrow (vg)(K) \equiv (v_* g)(K) \subseteq v(V) \subseteq V'$

Q.E.D

Composition and Substitution

Definition (Composition)

Define $c : Map(X, Y) \times Map(Y, Z) \to Map(X, Z)$

$$c(f, g) \equiv g \circ f.$$

Lemma:

If Y is locally compact T_2 then c in the above definition is continuous.

Proof.

Recall that for $K \subset V \subset Y$, K cmpt, V open, the local compactness T_2 of Y implies $\exists$ $K \subset B \subset L \subset V$ st B is open

Let $(f,g) \in \mathrm{Map}(X,Y) \times \mathrm{Map}(Y,Z)$. Let $\langle K, W\rangle$ be open in $\mathrm{Map}(X,Z)$ and containing $c(f,g)$. Thus

$$\underset{\substack{\uparrow \\ \text{open}}}{(gf)(K) \subset W \;\Rightarrow\; g^{\leftarrow} W} \supseteq \underset{\substack{\uparrow \\ \text{cmpt}}}{f(K)} \quad \because f, g \text{ cts.}$$

Y locally cmpt $T_2 \Rightarrow \exists\, f(K) \subset B \subset L \subset g^{\leftarrow} W$ with B open and L cmpt. Thus $f \in \langle K, B\rangle$ and $g \in \langle L, W\rangle$. Thus (f,g) has an open nbhd $\langle K,B\rangle \times \langle L, W\rangle$ in $\mathrm{Map}(X,Y) \times \mathrm{Map}(Y,Z)$.

Moreover, $(f', g') \in \langle K,B\rangle \times \langle L,W\rangle \Rightarrow$

$$f'(K) \subset B \;, \quad g'(L) \subset W$$

$$B \subseteq L \Rightarrow (g'f')(K) \subset g'(B) \subset g'(L) \subset W$$

$$\Rightarrow c(f', g') \in \langle K, W\rangle .$$

Q.E.D

Definition (Subscription)

Define $\sigma : \mathrm{Map}(T,Y) \times T \longrightarrow Y : \sigma(f,t) = f(t)$

Lemma:

If T is locally compact T_2 then σ is continuous.

Proof:

Note that $\forall$ spaces Z, $\mathrm{Map}(pt, Z)$ and Z are homeomorphic

$f_1 \neq f_2$ so $\exists$ T_2 nbhds of f_1, f_2 in $\mathrm{Map}(X,Y)$. Wlog, these may be chosen so:
$$f_i \in U_i = \bigcap_{j_i=1}^{N_i} \langle K_{j_i}, V_{j_i}\rangle , \quad N_i < \infty, \; i = 1,2 .$$
Now $f_i(K_{j_i}) \subseteq V_{j_i} \;\forall j_i \Rightarrow y_i \in V_i = \bigcap_{j_i=1}^{N_i} V_{j_i}$, open in Y.
If $\exists\, p \in V_1 \cap V_2$, then the map

$$f : X \to Y \quad f(x) = p \quad \forall x \in X$$

would be an element of $U_1 \cap U_2$, contradiction. Hence $V_1 \cap V_2 = \phi$ and so V_1, V_2 are T_2 nbhds of y_1, y_2 respectively. Thus Y is T_2.

2. $\Leftarrow$: $f_1, f_2 \in \mathrm{Map}(X,Y)$, $f_1 \neq f_2$. Hence $\exists\, p \in X$ st $f_1(p) \neq f_2(p)$. Choose T_2 nbhds V_1, V_2 of $f_1(p), f_2(p)$ in Y. $\{p\}$ is cmpt in X and, since $V_1 \cap V_2 = \phi$,

$$\langle \{p\}, V_1\rangle \cap \langle \{p\}, V_2\rangle = \phi$$

and $f_i \in \langle \{p\}, V_i\rangle$, $i = 1,2$. Hence $\mathrm{Map}(X,Y)$ is T_2.

Q.E.D

Change of Variable:

1.
$$\begin{array}{c} X \\ \uparrow u \\ X' \end{array} \in \mathrm{Map}(X', X) \Rightarrow \begin{array}{c} \mathrm{Map}(X,Y) \\ \downarrow u^* \\ \mathrm{Map}(X',Y) \end{array} \quad \text{by } u^* f = f \circ u$$

that is

$$\begin{tikzcd} X \arrow[r, "f"] & Y \\ X' \arrow[u, "u"] \arrow[ur, "u^* f"'] \arrow[u, phantom, shift left=4, "\circlearrowleft"] & \end{tikzcd}$$

Moreover.

$$\begin{tikzcd}
\mathrm{Map}(T,Y)\times T \arrow[r, "\sigma"] \arrow[d, phantom, "\cong" rotate=90] \arrow[dr, phantom, "\copyright"] & Y \arrow[d, phantom, "\|"] \\
T\times \mathrm{Map}(T,Y) \arrow[r, dashed] \arrow[d, phantom, "\cong" rotate=90] \arrow[dr, phantom, "\copyright"] & Y \arrow[d, phantom, "\|"] \\
\mathrm{Map}(pt,T)\times \mathrm{Map}(T,Y) \arrow[r, "c"'] & \mathrm{Map}(pt,Y)
\end{tikzcd}$$

so σ coincides with c and by the last lemma c is continuous.

Q.E.D

Note:

Suppose X is compact and (Y,d) is a metric space. Define

$$d^* : \mathrm{Map}(X,Y)\times \mathrm{Map}(X,Y) \longrightarrow \mathbb{R}_0^+ :$$

$$d^*(f,g) = \sup_{x\in X} d(f(x),g(x)) \quad , \text{ metric.}$$

d^* is well-defined since X is cmpt. The metric topology induced by d^* is the compact open topology.

§ 2: BIVARIATE FUNCTIONS   Exponential law

Definition.

Let T, X, Y be topological spaces. Define

$$\underline{\Theta} : \mathrm{Map}(X\times T, Y) \longrightarrow \mathrm{Map}(X, \mathrm{Map}(T,Y))$$

$$[\{\Theta(f)\}(x)](t) = f(x,t)$$

that is, $\{\Theta(f)\}(x) \in \mathrm{Map}(T,Y) \quad \forall\, x \in X$.

$$\begin{array}{ccccc}
X\times T & & X & \ni & x \\
\downarrow f & \mapsto & \downarrow \Theta(f) & & \downarrow \\
Y & & \mathrm{Map}(T,Y) & \ni & \Theta(f)(x)
\end{array}$$

To show that Θ is well-defined, the following lemma is needed:

Lemma:

1. $\forall\, x \in X, \quad \Theta(f)(x) \in \mathrm{Map}(T,Y)$;
2. $\Theta(f) \in \mathrm{Map}(X, \mathrm{Map}(T,Y))$.

Proof.

1. $\Theta(f)(x) = f|_{\{x\}\times T} \Rightarrow$ cts.
2. Fix $x_0 \in X$ and suppose $\Theta(f)(x_0) \in \langle L, V\rangle$, a subbasic open set in $\mathrm{Map}(T,Y)$. $\forall\, t \in L$, f is cts at (x_0,t) so $\exists$ nbd U_t of x_0 and V_t of t with $f(U_t\times V_t) \subset V$ since

$f(x_0,t) \in V$. L is compact so $\exists\ V_{t_1} \dots V_{t_n}$, n open covering L. Define $U = \bigcap_{i=1}^{n} U_{t_i}$, open nbhd of x_0. Then $f(U,L) \subset V$ and $(x,t) \in (U,L) \Rightarrow (x,t) \in (U_{t_i}, V_{t_i})$ for $i \Rightarrow f(x,t) \subset V$. Hence $\Theta(f)(U) \subset \langle L,V \rangle$. Therefore $\Theta(f)$ is cts at x_0 and so is continuous.

Q.E.D

The usefulness of the Θ function derives from the following.

Lemma.

1. Θ is injective;
2. Given T and Y,

   Θ surjective $\forall X \iff$ substitution $\sigma : Map(T,Y) \times T \to Y$ is continuous

3. $\mathrm{im}(\Theta) \xrightarrow{\Theta^{-1}} Map(X \times T, Y)$ is continuous, (well-defined by 1.);
4. Θ is continuous.

Proof.

1. $\Theta(f) = \Theta(g) \Rightarrow \forall\ x \in X,\ t \in T,\ \Theta(f)(x,t) = \Theta(g)(x,t)$

   $\Rightarrow f(x,t) = g(x,t) \quad \forall\ (x,t) \in X \times T$.

   $\Rightarrow f = g$.

   $\therefore\ \Theta$ injective.

2. $\Rightarrow$:

Let $X = Map(T,Y)$ so that

$$Map(Map(T,Y) \times T,\ Y) \xrightarrow{\Theta} Map(Map(T,Y),\ Map(T,Y)),$$ surjective

Consider $id \in Map(Map(T,Y), Map(T,Y))$. By surjectivity, $\exists$ $f \in Map(Map(T,Y) \times T, Y)$ st $\Theta(f) = id$. Thus

$$f(g,t) \underset{\text{def }\Theta}{\equiv} [\Theta(f)(g)](t) \underset{\Theta(f)=id}{=} g(t) \underset{\text{def }\sigma}{\equiv} \sigma(g,t)$$

$\forall\ t \in T,\ \forall\ g \in Map(T,Y)$.

$$\therefore\ \sigma = f$$

so $\sigma \in Map(Map(T,Y) \times T, Y)$ and is thus continuous.

$\Leftarrow$:

Let $g \in Map(X, Map(T,Y))$. Consider

$$\begin{tikzcd}
X \times T \arrow[r, "g \times id_T"] \arrow[dr, dashed, "f"'] & Map(T,Y) \times T \arrow[d, "\sigma"] \arrow[dl, phantom, "\odot"] \\
& Y
\end{tikzcd}$$

g, id_T cts. $\Rightarrow$ $g \times id_T$ cts. σ cts. by hypothesis so

$$f = \sigma \circ (g \times id_T) \quad \text{cts.}$$

$\therefore\ f \in Map(X \times T, Y)$

Finally, $[\Theta(f)(x)](t) = f(x,t) = \sigma \circ (g \times id_T)(x,t) =$

$$\sigma(g(x), t) = [g(x)](t) \quad \forall\ t \in T,\ \forall\ x \in X.$$

Hence $\Theta(f)(x) = g(x) \quad \forall x \in X$

$\Rightarrow \Theta(f) = g$

$\therefore$ Θ surjective.

3.
$$\begin{tikzcd} \mathrm{im}(\Theta) \arrow[r, "\Theta^{-1}"] \arrow[d, hook] & Map(X \times T, Y) \\ Map(X, Map(T,Y)) & \end{tikzcd}$$

Let $K \subset X$ be cmpt, $L \subseteq T$ cmpt, $V \subseteq Y$ open. Then:

$\langle K, \langle L, V\rangle\rangle \subset Map(X, Map(T,Y))$, open;

and $\langle K \times L, V\rangle \subset Map(X \times T, Y)$, open

Thus.

$f \in (\Theta^{-1})^{\leftarrow} \langle K \times L, V\rangle$

$\Leftrightarrow \Theta^{-1} f \in \langle K \times L, V\rangle$

$\Leftrightarrow f \in \Theta \langle K \times L, V\rangle$ $\quad$ Θ bijective to im Θ

But:

$g \in \langle K \times L, V\rangle$

$\Leftrightarrow g(K,L) \subset V$

$\Leftrightarrow (\Theta(g)(K))(L) \subset V$

$\Leftrightarrow \Theta(g) \in \langle K, \langle L, V\rangle\rangle$

$\Rightarrow \Theta \langle K \times L, V\rangle = \langle K, \langle L, V\rangle\rangle$

$\Rightarrow \langle K \times L, V\rangle = \Theta^{-1} \langle K, \langle L, V\rangle\rangle$, open.

$\therefore$ Θ^{-1} is continuous.

4. Chose K, L, V as in 3. Then:

$f \in \Theta^{\leftarrow} \langle K, \langle L, V\rangle\rangle$

$\Leftrightarrow \Theta(f)(K) \subset \langle L, V\rangle$

$\Leftrightarrow \Theta(f)(K)(L) \subset V$

$\Leftrightarrow f(K,L) \subset V$

$\Leftrightarrow f \in \langle K \times L, V\rangle$

$\therefore$ $\Theta^{\leftarrow} \langle K, \langle L, V\rangle\rangle = \langle K \times L, V\rangle$, open

$\Rightarrow \Theta$ continuous.

Q.E.D.

Note:

Let $\{V_\alpha\}_{\alpha \in A}$ be an open subbase for a space Y. Then $Map(X,Y)$ has an open subbase:

$$\{\langle K, V_\alpha\rangle : \alpha \in A,\ K \subseteq X \text{ cmpt}\}$$

§3. POINTED SPACES

Definition

A *pointed space* is a pair (X, x_0) where X is a space and $x_0 \in X$, called the base point. The symbol $*$ is used to denote the base point. A *map of pointed spaces* is a map $X \xrightarrow{f} Y$ where $f(*) = *$. Similarly, a pointed set and maps of pointed sets may be defined. The subspace of $Map(X, Y)$ consisting of $*$ preserving maps is denoted $Map_0(X,Y)$. By definition, $Map_0(X, Y) = \langle \{*\}, \{*\} \rangle$. For $A \subseteq X$, $B \subseteq Y$, the subspace $\langle A, B \rangle$ of $Map(X, Y)$ is denoted by $Map(X, A; Y, B)$.

Definition:

Four categories are defined:

1. (top) : objects topological spaces; morphisms (cts) maps;
2. $(top)_0$: objects pointed spaces; morphisms $*$ preserving maps
3. $(sets)$: objects sets; morphisms functions;
4. $(sets)_0$: objects pointed sets; morphisms $*$-preserving functions

In each case, composition is the usual functional composition.

Note:

If X and Y are pointed spaces, then there is a canonical choice of base point in $Map_0(X, Y)$, namely:

$$f : X \to Y : f(x) = * \quad \forall \; x \in X.$$

Note that this is the unique constant map in $Map_0(X,Y)$.

Definition:

Let X and Y be pointed spaces.

1. Define the *wedge product* (or bouquet) of X and Y to be the subspace $X \times \{*\} \cup \{*\} \times Y$ of $X \times Y$. This is denoted by $X \vee Y$.
2. Define the *smash* (or reduced) *product* of X and Y to be the quotient $(X \times Y)/X \vee Y$. This is denoted by $X \wedge Y$.
3. Let X be a space and $A \subseteq X$. The space formed by *collapsing* A is defined to be the quotient X/A. In particular, $X \wedge Y$ is formed by collapsing $X \vee Y$ within $X \times Y$.

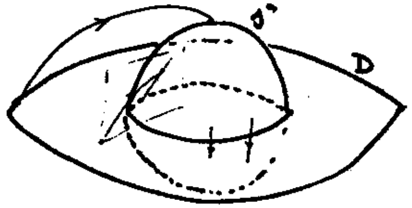
Examples:

1. $X = [0,1]$, $A = \{0,1\} \Rightarrow X/A \approx S^1$
2. Let B^n be a closed ball in $\mathbb{R}^n$ and S^{n-1} its surface. Then:

$$\boxed{B^n / S^{n-1} \cong S^n}$$

To prove this, identify D^n with a closed disc of radius 2 in $\mathbb{R}^{n+1}$, so that $x^{n+1}(D) = \{0\}$ and D is centred at the origin. Define:

$$\psi : D \to S^n : \quad \psi(x) = \begin{cases} (x^1, \dots x^n, -1 + \|x\|) & : \|x\| \le 1 ; \\ x \sqrt{(2-\|x\|)(\|x\|)} + (\|x\|-1)\, e_{n+1} & : 1 < \|x\| < 2 ; \\ (0, \dots 0, 1) & ; \|x\| = 2 . \end{cases}$$



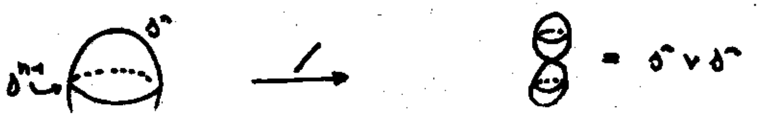
ψ is continuous clearly. Since $\psi(\partial D)$ = single point, ψ induces:

$$\begin{tikzcd} D \arrow[r, "\psi"] \arrow[d, "\pi"'] \arrow[dr, phantom, "\copyright"] & S^n \\ D/\partial D \arrow[ur, "\chi"'] & \end{tikzcd}$$

χ is a homeomorphism $\because$ D compt, π cts, S^n T_2, χ a cts bijection. But $B^n/S^{n-1} \cong D/\partial D$ so the result follows.

3. Let $S^{n-1} \subset S^n$ be the equator. Then:

$$\boxed{S^n / S^{n-1} \cong S^n \vee S^n}$$



4. Let D^n denote the n-hypercube:

$$D^n = \{ (x^1, \dots x^n) \in \mathbb{R}^n : \ -1 \le x_i \le 1 \ \forall \ i \}$$

By 2, $S^n \cong D^n/\partial D^n$. Assign $*_{S^n} = [\partial D^n]$.

Thus
$$S^p \times S^q \cong (D^p/\partial D^p) \times (D^q/\partial D^q)$$
$$= D^p \times D^q \mod R$$
$$= D^{p+q} \mod R$$

where R is the relation:

(1) $\Delta_{D^{p+q}}$

(2) $(x,y) \sim (x',y)$ if $x, x' \in \partial D^p$

(3) $(x,y) \sim (x,y')$ if $y, y' \in \partial D^q$

$S^p \times \{*\}$ imbeds to the equivalence classes of $D^p \times \partial D^q$

$\{*\} \times S^q$ imbeds to the equivalence classes of $\partial D^p \times D^q$

$\{*\} \times \{*\}$ imbeds to $[\partial D^p \times \partial D^q]$

Thus $S^p \wedge S^q = D^p \times D^q \mod (R \cup S)$

where S is the relation:

(1) $\rho \sim \sigma$ if $\rho, \sigma \in (D^p \times \partial D^q) \cup (\partial D^p \times D^q)$

(2) and (3) above are included in S. Hence:

$$S^p \wedge S^q \cong D^{p+q} \mod (\Delta_{D^{p+q}} \cup S)$$

But $\partial(D^{p+q}) = \partial(D^p \times D^q) = \partial D^p \times D^q \cup D^p \times \partial D^q$

Hence $S = \partial D^{p+q} \times \partial D^{p+q}$. Therefore:

$$S^p \wedge S^q = D^{p+q} \bmod (\Delta_{D^{p+q}} \cup (\partial D^{p+q} \times \partial D^{p+q}))$$
$$= S^{p+q} \quad \text{by 2. above}$$

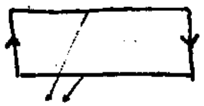
$$\therefore \quad \boxed{S^p \wedge S^q \cong S^{p+q}}$$

5. Let $B = S^1 \times [0,1]$, M = Möbius band. Then:

$$\partial B = S^1 \cup S^1 \quad , \quad \partial M = S^1$$

$B/\partial B$ is homeomorphic to the surface of revolution of a circle about a tangent line.

$M/\partial M$ is homeomorphic to $\mathbb{P}_2(\mathbb{R})$. To see this, recall:

$M \equiv$ 

∂M

Collapse upper and lower edges separately:

The endpoints are automatically identified when forming the band. Hence:

$M/\partial M \simeq$ 

$\simeq$ 2-sphere with antipodal points identified

Lemma

Let A be a closed subspace of a compact T_2 space X.

Then $\quad X/A \cong \alpha(X \setminus A)$.

where α denotes one-point compactification.

Proof:

Let $\quad X \xrightarrow{\pi} X/A$

$$X \setminus A \xhookrightarrow{i} \alpha(X\setminus A) = X \setminus A \cup \{\omega\}$$

and define:

$$\psi \quad X/A \longrightarrow \alpha(X \setminus A) : \psi([x]) = i(x) \ , \ x \notin A$$
$$\psi([x]) = \omega \ , \ x \in A$$

ψ is clearly a bijection. ψ is continuous:

(a) $U \subset \alpha(X\setminus A)$ open, $\omega \notin U \Rightarrow U \subset X \setminus A$ open

$\Rightarrow \psi^{-1} U = [U]$ and $\pi^{-1}[U] = U$

U open in $X \setminus A$, A closed $\Rightarrow U$ open in X

$\therefore \pi(U) = \psi^{-1} U$ open in X/A.

(b) $U \subset \alpha(X \setminus A)$ open, $\omega \in U \Rightarrow U^c \subset X \setminus A$ cmpt.

i cnts. $\Rightarrow U^c$ compact in $X \Rightarrow \pi(U^c)$ closed in X/A $\because$ X/A T_2

Thus $\psi^{-1} U = \pi(U^c)^c$ is open in X/A.

$\therefore \psi$ continuous.

X is cmpt T_2 $\Rightarrow$ $X \setminus A$ locally cmpt T_2 $\Rightarrow \alpha(X \setminus A)$ T_2. Thus ψ is a continuous bijection of a compact space to a T_2 space and so must be a homeomorphism.

Q.E.D.

Lemma:

Let T, X and Y be pointed spaces. Then:

$$\begin{array}{ccc} Map(X \times T, Y) & \xrightarrow{\ \theta\ } & Map(X, Map(T, Y)) \\ \uparrow & © & \uparrow \\ Map(X \times T, X \vee T; Y, *) & \overset{\theta|}{\dashrightarrow} & Map_0(X, Map_0(T, Y)) \end{array}$$

and $\theta^{\leftarrow} Map_0(X, Map_0(T, Y)) = Map(X \times T, X \vee T; Y, *)$

Proof:

1. Suppose $f \in Map(X \times T, X \vee T; Y, *)$. Thus:

$$f(x, t_0) = y_0 = f(x_0, t) \quad \forall x \in X, \forall t \in T.$$

(a) $[\theta(f)(x_0)](t) = f(x_0, t) = y_0 \ \forall t \in T$

$\Rightarrow \theta(f) \in Map_0(X, Map(T, Y))$

$* = x_0$ $\qquad$ $* = $ map $T \xrightarrow{g} Y : g(T) = \{*\}$

(b) $[\theta(f)(x)](t_0) = f(x, t_0) = y_0, \ \forall x \in X$

$\Rightarrow \theta(f)(x) \in Map_0(T, Y) \quad \forall x \in X$

By (a), $\theta(f) \in Map_0(X, Map_0(T, Y))$

$\therefore \theta(Map(X \times T, X \vee T; Y, *)) \subset Map_0(X, Map_0(T, Y))$

2. $f \in \theta^{\leftarrow} Map_0(X, Map_0(T, Y))$

$\Rightarrow \theta(f) \in Map_0(X, Map_0(T, Y))$

(a) $\forall x \in X$, $(\theta(f)(x))(t_0) = y_0$

$\Rightarrow f(x, t_0) = y_0 \ \forall x \in X$

(b) $\theta(f)(x_0) = *$ in $Map_0(T, Y)$ $\Rightarrow \forall t \in T$,

$[\theta(f)(x_0)](t) = y_0$

$\Rightarrow f(x_0, t) = y_0 \quad \forall t \in T.$

(a) and (b) $\Rightarrow$ $f(X \vee T) = \{*\}$

$\therefore f \in Map(X \times T, X \vee T; Y, *)$.

Q.E.D.

Note:

The space formed from X by collapse of a subspace A has a canonical basepoint, namely $[A]$.

Lemma:

$$\begin{array}{ccccc} X & & Map(X, Y) & \longleftrightarrow & Map(X, A; Y, *) \\ \downarrow \pi & \Rightarrow & \uparrow \pi^* & © & \uparrow \pi^* = \pi| \\ X/A & & Map(X/A, Y) & \longleftrightarrow & Map(X/A, Y) \end{array}$$

and $\quad \pi^{\#}: \mathrm{Map}(X, A; Y, *) \to \mathrm{Map}_0(X/A, Y)$

Proof:

$$f \in \mathrm{Map}_0(X/A, Y) \iff f([A]) = y_0 \iff$$
$$(f \circ \pi)(A) = \{y_0\} \iff (\pi^* f)(A) = \{y_0\} \iff$$
$$\pi^* f \in \mathrm{Map}(X, A; Y, *).$$

Q.E.D

Lemma:

Let $\pi^{\#}$ be defined as in the previous lemma. Then:

1. $\pi^{\#}$ is continuous;
2. $\pi^{\#}$ is bijective;
3. If X is cmpt T_2 and A is closed then $\pi^{\#}$ is a homeomorphism.

Proof.

1. $\pi^{\#}$ is cts. $\because$ it is the restriction of π^*, a cts. map.
2. $\pi^{\#}$ is clearly injective. Given any $g \in \mathrm{Map}(X, A; Y, *)$ let $\tilde{g}$ be the quotient of g, that is:

$$\begin{tikzcd} X \arrow[r, "g"] \arrow[d, "\pi"'] & Y \\ X/A \arrow[ur, dashed, "\tilde{g}"'] \arrow[u, phantom, "\copyright"] & \end{tikzcd}$$

Since $g(A) =$ one pt, $\tilde{g}$ is well-defined. It is contd by the properties of the quotient topology. Finally,

$$\pi^{\#} \tilde{g} = \tilde{g} \circ \pi = g$$

$\therefore \pi^{\#}$ surjective.

3. By 1, 2., it suffices to prove that $\pi^{\#}$ is open.

X T_2, A closed $\Rightarrow X/A$ T_2; X cmpt $\Rightarrow X/A$ cmpt.

Let $(K, V) = \langle K, V\rangle \cap \mathrm{Map}_0(X/A, Y)$ with $K \subset X/A$ cmpt and $V \subset Y$ open. It suffices to prove that $\pi^{\#}(K, V)$ is open.

Suppose $g \in \pi^{\#}(K, V)$, $g = \pi^{\#} f$ say. Then $f(K) \subset V$.

K cmpt, X/A T_2 $\Rightarrow$ K closed. π cts $\Rightarrow \pi^{-}K$ closed

X cmpt $\Rightarrow \pi^{-}K$ cmpt. Also $g(\pi^{-}K) \subset V$ so

$g \in \langle \pi^{-}K, V\rangle \cap \mathrm{Map}(X, A; Y, *)$.

Conversely, $g \in \langle \pi^{-}K, V\rangle \cap \mathrm{Map}(X, A; Y, *)$. Then $g = \pi^{\#} f$ say ($\pi^{\#}$ surjective) and $f(K) \subset V$. Hence:

$$\pi^{\#}(K, V) = \langle \pi^{-}K, V\rangle \cap \mathrm{Map}(X, A; Y, *), \text{ open}$$

$\therefore \pi^{\#}$ a homeomorphism.

Q.E.D

Application

Let $X \rightsquigarrow X \times T$, $A \rightsquigarrow X \vee T$, so $X/A \rightsquigarrow X \wedge T$ in the above lemma. Then:

$$\begin{tikzcd}
Map(X \times T,\, Y) \arrow[r, "\theta"] & Map(X,\, Map(T,Y)) \\
Map(X \times T,\, X \vee T;\, Y,\, *) \arrow[u, hook] \arrow[r, "\theta"] \arrow[ur, phantom, "\circlearrowleft"] & Map_0(X,\, Map_0(T,Y)) \arrow[u, hook] \\
Map_0(X \wedge T,\, Y) \arrow[u, "\pi^*"] \arrow[ur, dashed, "\theta"'] \arrow[ur, phantom, "\circlearrowleft", near start] &
\end{tikzcd}$$

$\exists \overset{\theta}{\dashrightarrow}$ $\because$ π^* injective.

Thus there are two important maps:

1. No base point

$$\boxed{\begin{array}{l} Map(X \times T,\, Y) \xrightarrow{\theta} Map(X,\, Map(T,Y)) \\ \theta \text{ a homeomorphism if } T \text{ locally cmpt } T_2. \end{array}}$$

2. With base point

$$\boxed{\begin{array}{l} Map_0(X \wedge T,\, Y) \xrightarrow{\theta} Map_0(X,\, Map_0(T,Y)) \\ \theta \text{ a homeomorphism if } X,\, T \text{ cmpt } T_2. \end{array}}$$

ALGEBRAIC TOPOLOGY II

HOMOTOPY FUNCTORS

§1. HOMOTOPY EQUIVALENCE

Definition.

Let X and Y be spaces. $f, g \in Map(X,Y)$ are <u>homotopic</u> denoted <u>$f \simeq g$</u>, if $\exists\ h \in Map([0,1] \times X, Y)$ st $\forall x \in X$ $h(0,x) = f(x)$ and $h(1,x) = g(x)$. h is called a homotopy of f and g.

The following may easily be verified:

1. Homotopy is an equivalence relation on $Map(X,Y)$.
2. $\Theta(h) \in Map([0,1], Map(X,Y))$, so homotopy may be thought of as a path in $Map(X,Y)$. Denote by h_t the map $\Theta h(t) \in Map(X,Y)$, $t \in [0,1]$. $t \mapsto h_t$ is the path.
3. If Θ is bijective, then

   $f \simeq g \iff \exists$ path from f to g in $Map(X,Y)$.
4. If
$$\begin{tikzcd} X \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & Y \\ X' \arrow[u, "u"] \end{tikzcd}$$
   with $f \simeq g$ then $u^* f \simeq u^* g$ so

   $u^* : Map(X,Y) \to Map(X',Y)$ preserves homotopy classes

5. If
$$\begin{tikzcd} X \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & Y \arrow[d, "v"] \\ & Y' \end{tikzcd}$$
   with $f \simeq g$ then $v_* f = v_* g$ so

   $v_* : Map(X,Y) \to Map(X,Y')$ preserves homotopy classes.
6. If $X \overset{f}{\underset{g}{\rightrightarrows}} Y \overset{f'}{\underset{g'}{\rightrightarrows}} Z$ with $f \simeq g$, $f' \simeq g'$ then $f' \circ f \simeq g' \circ g$.

Definition:

Let X, Y be pointed spaces. $f, g \in Map_0(X,Y)$ are <u>homotopic as maps of pointed spaces</u>, denoted <u>$f \simeq_0 g$</u>, if $\exists\ h \in Map([0,1] \times X, Y)$, a homotopy of f and g with the additional property.

$$h(t, *) = * \quad \forall\, t \in [0,1].$$

Note:

1. h gives a path from f to g in $Map_0(X,Y)$.
2. $\simeq_0$ is an equivalence relation and the above properties hold for $\simeq_0$.

Categories [top] and $[top]_0$

For $X, Y \in Ob(top)$, let $\underline{[X,Y]}$ denote the quotient space of $Map(X,Y)$ modulo the relation of homotopy. Define the category $\underline{[top]}$ as follows

1. $Ob\,[top] = Ob\,(top)$
2. $hom_{[top]}(X,Y) = [X,Y]$
3. id_X in $[top]$ is $[id_X]$
4. Composition in $[top]$ defined by:

$$\begin{tikzcd} X \arrow[r, "{[f]}"] \arrow[dr, "{[g\circ f]}"'] & Y \arrow[d, "{[g]}"] \arrow[dl, phantom, "©"] \\ & Z \end{tikzcd} \qquad [g]\circ[f] = [g\circ f]$$

id and composition are well-defined by the notes above. Thus $[top]$ is a category. Moreover, there is a functor:

$$[\ ] : (top) \rightsquigarrow [top]$$

The category $\underline{[top]_0}$ is obtained similarly from $(top)_0$ via the functor $[\cdot]_0$.

§ 2 : HOMOTOPY FUNCTORS

Of special interest are the following cases:

Definition

Let S^p, Y be pointed spaces, $p \geq 0$. Denote by $\underline{\pi_p(Y)}$ the space $[S^p, Y]_0$.

Multiplication in $\pi_p(Y)$, $p \geq 1$

Let $\alpha, \beta \in \pi_p(Y)$, $p \geq 1$. Suppose $\alpha = [f]_0$, $\beta = [g]_0$. Imbed $S^{p-1} \hookrightarrow S^p$ as a great circle passing through $*_{S^p}$.

Then:

$$S^p \xrightarrow{\pi} S^p/S^{p-1} \cong S^p \vee S^p \quad \text{as before.}$$

Define:

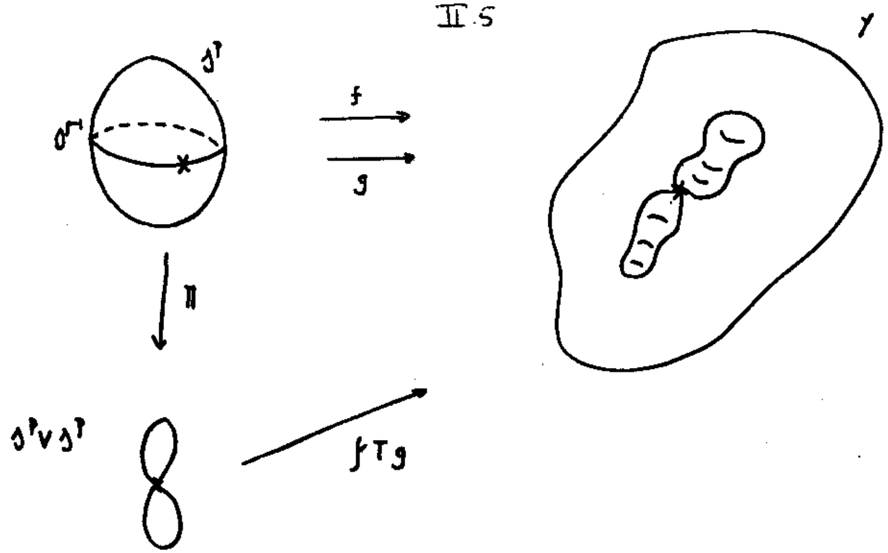
$$f \top g : S^p \vee S^p \to Y$$

$$(f\top g)\,|_{S^p \times \{*\}} = f$$

$$(f\top g)\,|_{\{*\}\times S^p} = g.$$

Note that $f \top g$ is well-defined at $\{*\}\times\{*\}$ since $f(*) = * = g(*)$. Define:

$$\underline{\alpha\cdot\beta} = [(f\top g)\circ\pi]_0 \in \pi_p(Y).$$



To check that the product is well-defined, suppose $f \overset{h}{\simeq}_0 f'$ and $g \overset{k}{\simeq}_0 g'$. Define

$$H : [0,1] \times S^p \longrightarrow Y$$
$$H(t,x) = ((h_t \, T \, k_t) \circ \pi)(x)$$

Then H is continuous;
$$H(0,.) = (h_0 T k_0) \circ \pi = (f T g) \circ \pi$$
$$H(1,.) = (h_1 T k_1) \circ \pi = (f' T g') \circ \pi.$$

and $\forall \, t \in [0,1]$, $H(t, *) = *$.

$$\therefore \quad (f T g) \circ \pi \simeq_0 (f' T g') \circ \pi$$

and $\cdot_p$ is well-defined.

Lemma.

With the above product operation, $\pi_p(Y)$ is a topological group $\forall \, p \geq 1$. For $p \geq 2$, $\pi_p(Y)$ is abelian.

The proof of this lemma may proceed by direct calculation. However, the proof is simpler when some further concepts have been introduced; it is deferred until

Notation.

Let (gr) denote the category of groups and group homomorphisms. Let (ab) denote the category of abelian groups and group homomorphisms.

Definition:

1. Two spaces X, Y have the same homotopy type if they are isomorphic in $[top]$.
2. Two pointed spaces X, Y have the same homotopy type as pointed spaces if they are isomorphic in $[top]_0$.

Note:

X has same homotopy type as $Y \iff \exists \, f \in Map(X,Y)$, $g \in Map(Y,X)$ st $g \circ f \simeq id_X$ $f \circ g \simeq id_Y$ (similarly for pointed spaces)

The following is the fundamental theorem.

Theorem:

1 For $n \geq 1$, π_n is a functor from the category of pointed spaces and continuous maps to the category of groups and group homomorphisms.

2. Moreover, there is a factorisation:

$$\begin{tikzcd}
(top)_0 \arrow[r, "\pi_n"] \arrow[d, "{[\ ]_0}"'] \arrow[dr, phantom, "\copyright"] & (gr) \\
{[top]_0} \arrow[ur, dashed, "\tilde{\pi}_n"'] &
\end{tikzcd}$$

so the functor π_n does not resolve spaces of the same homotopy type.

Proof:

1 It will be proven later that $\forall\ X \in Ob(top)_0$, $\pi_n(X) \in Ob(gr)$. To define π_n on morphisms, consider $X \xrightarrow{v} Y$ in $(top)_0$. Define $\pi_n(v)$ as follows. For $\alpha \in \pi_n(X)$, $\alpha = [f]_0$, say, set:

$$\pi_n(v)(\alpha) = [v \circ f]_0 = [v_* f]_0.$$

This is well-defined because v_* preserves equivalence classes.

To check that $\pi_n(X) \xrightarrow{\pi_n(v)} \pi_n(Y)$ is a group homomorphism

suppose $\alpha, \beta \in \pi_n(X)$ with $\alpha = [f]_0$, $\beta = [g]_0$ say. Then

$$\begin{aligned}
\pi_n(v)\ (\alpha\cdot\beta) &= \pi_n(v)\ [(f \top g)\circ \pi]_0 \\
&= [v_*((f \top g)\circ\pi)]_0 \\
&= [v \circ (f \top g) \circ \pi]_0 \\
&= [((v\circ f) \top (v \circ g))\circ \pi]_0 \qquad \text{by def}^{n} \text{ of } \top \\
&= [v \circ f]_0 \cdot [v \circ g]_0 \\
&= [v_* f]_0 \cdot [v_* g]_0 \\
&= (\pi_n(v)\ \alpha)\cdot(\pi_n(v)\ \beta).
\end{aligned}$$

$$\begin{aligned}
\pi_n(v)\ id_{\pi_n(X)} &= \pi_n(v)\ [i]_0 \qquad \text{where } i : S^n \to X : i(S^n) = 1 \\
&= [v \circ i]_0 \\
&= id_{\pi^n(Y)} \qquad \because\ v\circ i : S^n \to Y : (v\circ i)(S^n) = 1
\end{aligned}$$

2. To check the functorial properties

(a)
$$\begin{tikzcd}
X \arrow[r, "v"] \arrow[dr, "\omega \circ v"'] & Y \arrow[d, "\omega"] \arrow[dl, phantom, "\copyright"] \\
& Z
\end{tikzcd}
\quad \rightsquigarrow \quad
\begin{tikzcd}
\pi_n(X) \arrow[r, "\pi_n(v)"] \arrow[dr, "\pi_n(\omega\circ v)"'] & \pi_n(Y) \arrow[d, "\pi_n(\omega)"] \arrow[dl, phantom, "\copyright"] \\
& \pi_n(Z)
\end{tikzcd}$$

Since:
$$\begin{aligned}
\pi_n(\omega \circ v)\ [f]_0 &= \\
[(\omega\circ v)_* f]_0 &= \\
[\omega_* v_* f]_0 &= \\
\pi_n(\omega)\ [v_* f]_0 &= \\
\pi_n(\omega)\ \pi_n(v)\ [f]_0
\end{aligned}$$

Hence $\quad \Pi_n(w \circ v) = \Pi_n(w) \circ \Pi_n(v)$

(b) $\quad X \xrightarrow{id} X \quad \rightsquigarrow \quad \Pi_n(X) \xrightarrow[= \Pi_n(id_X)]{id_{\Pi_n(X)}} \Pi_n(X)$

since

$$\Pi_n(id_X)\,[f]_\circ =$$
$$[(id_X)_* f]_\circ =$$
$$[id_X \circ f]_\circ =$$
$$[f]_\circ$$

Hence: $\quad \Pi_n(id_X) = id_{\Pi_n(X)}$.

Therefore, $(top)_\circ \xrightarrow{\Pi_n} (gr)$ is a covariant functor.

3. (Factorisation)

Define $[top]_\circ \xrightarrow{\tilde{\Pi}_n} (gr)$ as follows

$$\tilde{\Pi}_n(X) = \Pi_n(X) \quad \forall\ X \in Ob\,[top]_\circ$$
$$X \xrightarrow{[f]_\circ} Y \text{ in } [top]_\circ : \quad \tilde{\Pi}_n\,[f]_\circ = \Pi_n(f)$$

The latter is well-defined since if $f \simeq_\circ f'$ then

$$\tilde{\Pi}_n([f]_\circ)\,[g]_\circ = [f_* g]_\circ$$
$$= [f \circ g]_\circ$$
$$= [f' \circ g]_\circ \qquad f \simeq_\circ f'$$
$$= \tilde{\Pi}_n([f']_\circ)\,[g]_\circ$$

The functorial properties are easily verified. In particular:

$$\tilde{\Pi}_n([f]_\circ \circ [g]_\circ) = \tilde{\Pi}_n([f \circ g]_\circ)$$
$$= \Pi_n(f \circ g)$$
$$= \Pi_n(f) \circ \Pi_n(g)$$
$$= (\tilde{\Pi}_n[f]_\circ) \circ (\tilde{\Pi}_n[g]_\circ).$$

The factorisation $\Pi_n = \tilde{\Pi}_n \circ [\cdot]_\circ$ is immediate from the definition of $\tilde{\Pi}_n$.

Q.E.D

Lemma.

Let X_1, X_2 be pointed spaces. Then $\forall\ n \geq 1$ there is a group isomorphism: $\Pi_n(X_1 \times X_2) \xrightarrow{\approx} \Pi_n(X_1) \times \Pi_n(X_2)$.

Proof.

Let $\quad X_1 \times X_2 \overset{p_1}{\underset{p_2}{\rightrightarrows}} \begin{matrix} X_1 \\ X_2 \end{matrix}$. For any pointed space Y, define

$$\Psi : Map_\circ(Y, X_1 \times X_2) \longrightarrow Map_\circ(Y, X_1) \times Map_\circ(Y, X_2) :$$
$$\Psi(f) = (p_{1*} f,\ p_{2*} f)$$

It is not difficult to check that Ψ is a bijection. Moreover, if $f \simeq_\circ g$ then $p_{i*} f \simeq_\circ p_{i*} g$. Thus Ψ induces a bijection:

$$[Y, X_1 \times X_2]_\circ \overset{\bar{\Psi}}{\longleftrightarrow} [Y, X_1]_\circ \times [Y, X_2]_\circ$$

given by $\tilde{\Psi}[f]_0 = ([p_1 \circ f]_0, [p_2 \circ f]_0)$

Consider the case $Y = S^n$, $n \geq 1$. Thus there is a bijection:

$$\begin{array}{ccc} [S^n, X_1 \times X_2]_0 & \xrightarrow{\tilde{\Psi}} & [S^n, X_1]_0 \times [S^n, X_2]_0 \\ \parallel & & \parallel \\ \pi_n(X_1 \times X_2) & & \pi_n(X_1) \times \pi_n(X_2) \end{array}$$

$\tilde{\Psi}$ is a group isomorphism since:

$$\begin{aligned} \tilde{\Psi}([f]_0 \cdot [g]_0) &= \tilde{\Psi}[(f \tau g) \circ \pi]_0 \\ &= ([p_i \circ ((f \tau g) \circ \pi)]_0)_{i=1,2} \\ &= ([p_i \circ (f \tau g) \circ \pi]_0)_i \\ &= ([(p_i \circ f) \tau (p_i \circ g) \circ \pi]_0)_i \qquad \text{(property of } \tau) \\ &= ([p_1 \circ f]_0 \cdot [p_1 \circ g]_0, [p_2 \circ f]_0 \cdot [p_2 \circ g]_0) \\ &\equiv (\tilde{\Psi}[f]_0) \cdot (\tilde{\Psi}[g]_0). \end{aligned}$$

Q.E.D.

Examples:

1. Homotopy of Spheres:

(a) $\pi_n(S^p) = 0 \quad \forall\ n < p$ (can always project from a point)

(b) $\pi_n(S^n) = \mathbb{Z}$ (degree of a map)

(c) $\pi_{4n-1}(S^{2n}) = \mathbb{Z} \oplus$ finite group (Hopf invariant)

In particular, $\pi_3(S^2) = \mathbb{Z}$.

(d) Serre has proven that in all other cases $\pi_n(S^p)$ is finite abelian group.

2. $\pi_1(\vee^k S^1)$ = free non-abelian group on k generators;

$\pi_1(\times^k S^1)$ = free abelian group on k generators.

§ 3 : CONE, SUSPENSION, PATH SPACE, LOOP SPACE :

Let X, Y, Z be pointed spaces with X and Y cmpt T_2.

Recall the homeomorphisms:

$$\begin{tikzcd}
\mathrm{Map}_0(X \times Y, X \vee Y; Z, *) \arrow[r, "\Theta", "\approx"'] & \mathrm{Map}_0(X, \mathrm{Map}_0(Y,Z)) \\
\mathrm{Map}_0(X \wedge Y, Z) \arrow[u, "\int", "\pi^{\#}"'] \arrow[ur, "\copyright", "\tilde{\Theta}"']
\end{tikzcd}$$

Lemma:

$\tilde{\Theta}$ induces a bijection $[X \wedge Y, Z]_0 \longleftrightarrow [X, \mathrm{Map}_0(Y,Z)]_0$

Proof:

Suppose $f, g \in \mathrm{Map}_0(X \wedge Y, Z)$, $f \overset{h}{\simeq}_0 g$. Define

$$H : [0,1] \times X \to \mathrm{Map}_0(Y,Z) :$$

$$\{H(s,x)\}(y) \equiv (\tilde{\Theta} h_s)(x)(y)$$

$$= h(s, \pi(x,y)).$$

It is easy to check that $\tilde{\Theta}(f) \overset{H}{\simeq}_0 \tilde{\Theta}(g)$.

Similarly, if $f, g \in \mathrm{Map}_0(X, \mathrm{Map}_0(Y,Z))$, $f \overset{h}{\simeq}_0 g$, then $(\tilde{\Theta}^{-1})(h_s)$ provides a homotopy $(\tilde{\Theta}^{-1}) f \simeq_0 (\tilde{\Theta}^{-1}) g$.

Q.E.D

Note:

$X \wedge T$ is canonically isomorphic to $T \wedge X$. Thus:

$$[X \wedge Y, Z]_0 \longleftrightarrow [Y \wedge X, Z]_0 \longleftrightarrow [X, \mathrm{Map}_0(Y,Z)]_0$$

Applications:

1. Consider $[R \wedge S, T]_0 \longleftrightarrow [S, \mathrm{Map}_0(R,T)]_0$ for R, S cmpt T_2. Take $S = S^0$. Then:

$$[R \wedge S^0, T]_0 \longleftrightarrow [S^0, \mathrm{Map}_0(R,T)]_0 \equiv \pi_0(\mathrm{Map}_0(R,T))$$

But: $R \wedge S^0 \approx R$.

Hence, $\boxed{[X,Y]_0 \rightleftarrows \pi_0(\mathrm{Map}_0(X,Y)), \ X \text{ cmpt } T_2.}$ set bijection

2. Consider $R = S^p$, $S = S^q$. Then $S^p \wedge S^q = S^{p+q}$ so

$$[S^{p+q}, T]_0 \rightleftarrows [S^q, \mathrm{Map}_0(S^p, T)]_0$$

In fact, it is not difficult to check that this is a group isomorphism. Thus homotopy of a given space is equivalent to lower homotopy of an associated map space:

$$\boxed{\pi_{p+q}(T) \rightleftarrows \pi_q(\mathrm{Map}_0(S^p, T))}$$ group isomorphism

Definition

1. Fix $X \in Ob(top)_o$. Define

$$\Gamma_X : (top)_o \rightsquigarrow (top)_o$$

by

$$\Gamma_X(Y) = Map_o(X, Y)$$

$$\Gamma_X(Y \xrightarrow{v} Z) = Map_o(X, Y) \xrightarrow{v_*} Map_o(X, Z)$$

Then Γ_X is a covariant functor.

2. Fix $Y \in Ob(top)_o$. Define

$$\Lambda_Y : (top)_o \rightsquigarrow (top)_o$$

by:

$$\Lambda_Y(X) = X \wedge Y$$

$$\Lambda_Y(X \xrightarrow{u} Z) = X \wedge Y \xrightarrow{\tilde{u}} Z \wedge Y$$

where:

$$\begin{tikzcd} X \times Y \arrow[r, "u \times id_Y"] \arrow[d, "\Pi"] \arrow[dr, phantom, "\odot"] & Z \times Y \arrow[d, "\Pi"] \\ X \wedge Y \arrow[r, dashed, "\tilde{u}"] & Z \wedge Y \end{tikzcd}$$

(this is well-defined since u, id_Y preserve $*$)

Then Λ_Y is a covariant functor.

Corollary to Lemma:

Suppose X and T are cmpt T_2. Then: (adjointness relations)

1. $Map_o(\Lambda_T(X), Y) \Rightarrow Map_o(X, \Gamma_T(Y))$

2. $[\Lambda_T(X), Y]_o \Rightarrow [X, \Gamma_T(Y)]_o$

Of particular interest are the following two cases:

1. **Cone and Path Space:**

Let $T = [0,1]$ in the corollary. Define the cone on X as:

$$\underline{C(X)} = \Lambda_{[0,1]}(X) = X \wedge [0,1] \qquad (0 = *)$$

Define the path space on Y as:

$$\underline{E(Y)} = \Gamma_{[0,1]}(Y) = Map_o([0,1], Y)$$

If X is cmpt T_2, the adjointness relations are:

$$\boxed{\begin{array}{ccc} Map_o(C(X), Y) & \rightleftharpoons & Map_o(X, E(Y)) \\ [C(X), Y]_o & \rightleftharpoons & [X, E(Y)]_o \end{array}}$$

2. **Suspension and Loop Space:**

Let $T = S^1$ in the corollary. Define the suspension of X as:

$$\underline{S(X)} = \Lambda_{S^1}(X) = X \wedge S^1$$

Define the loop space on Y as:

$$\underline{\Omega(Y)} = \Gamma_{S^1}(Y) = Map_o(S^1, Y)$$

If X is cmpt. T_2, the adjointness relations are:

$$\boxed{\begin{array}{ccc} \mathrm{Map}_o(S(X), Y) & \cong & \mathrm{Map}_o(X, \Omega(Y)) \\ [S(X), Y]_o & \cong & [X, \Omega(Y)]_o \end{array}}$$

Lemma:

1. There is a canonical inclusion $X \overset{i}{\hookrightarrow} C(X)$ such that Thus

$$X \overset{i}{\hookrightarrow} C(X) \xrightarrow{\pi} C(X)/X \cong S(X)$$

2. There is a canonical bundle:

$$\Omega(Y) \xrightarrow{j} E(Y) \xrightarrow{p} Y$$

where $\Omega(Y)$ is the fibre over the base point, $(p^{\leftarrow}(*) = \Omega(Y))$

Proof.

1. Define $i : X \to C(X)$, $i(x) = \rho(x, 1)$ where

$\rho : X \times [0,1] \to X \wedge [0,1]$ is the projection.

Regard $S^1, * \equiv [0,1]/\{0,1\}, [0]$. Thus

$$S(X) \equiv \frac{X \times S^1}{X \vee S^1} = \frac{X \times S^1}{X \times \{*\} \cup \{*\} \times S^1}$$

$$= \frac{X \times [0,1]}{X \times \{0\} \cup X \times \{1\} \cup \{*\} \times [0,1]}$$

But:

$$C(X) \equiv \frac{X \times [0,1]}{X \times \{0\} \cup \{*\} \times [0,1]}$$

So:

$$C(X)/X \equiv \frac{X \times [0,1]}{X \times \{0\} \cup \{*\} \times [0,1] \cup X \times \{1\}} \equiv S(X)$$

2. Define:

$$p : E(Y) = \mathrm{Map}_o([0,1], Y) \to Y : \quad p(u) = u(1)$$

p is continuous since for open V in Y,

$$p^{\leftarrow} V = \langle \{1\}, V \rangle \cap \mathrm{Map}_o([0,1], Y), \text{ open.}$$

Thus $E(Y) \xrightarrow{p} Y$ is a fibre bundle. Moreover:

$$u \in p^{\leftarrow}(*) \iff u(1) = * \iff u(1) = u(0)$$

$\because 0 = *_{[0,1]}$.

Hence

$$p^{\leftarrow}(*) = \mathrm{Map}([0,1], \{0,1\}; Y, *)$$

$$\simeq \mathrm{Map}_o([0,1]/\{0,1\}, Y)$$

$\because [0,1]$ cmpt T_2, $\{0,1\}$ closed in $[0,1]$

$$= \mathrm{Map}_o(S^1, Y)$$

$$\equiv \Omega(Y).$$

Q.E.D.

Notation:

The proof of the next theorem requires careful notation.

(1) Fix:

$$\begin{array}{ccccc} X \times [0,1] & \xrightarrow{\rho} & C(X) & \xleftarrow{i} & X \\ \downarrow{\scriptstyle id \times \varphi} & \circledcirc & \downarrow{\scriptstyle \pi} & & \\ X \times S^1 & \xrightarrow[\kappa]{} & S(X) = C(X)/X = X \wedge S^1 & & \end{array}$$

where $[0,1] \xrightarrow{\varphi} [0,1]/\{0,1\} = S^1$.

(2) Generally:

$$\mathrm{Map}_0(\Lambda_T(X), Y) \xrightarrow{\Theta} \mathrm{Map}_0(X, \Gamma_T(Y))$$

Note that Θ is always a continuous injection. It is a homeomorphism if X is compact T_2. Consider specific cases.

(a) $T = [0,1]$. Denote Θ by σ in this case:

$$\mathrm{Map}_0(C(X), Y) \xrightarrow{\sigma} \mathrm{Map}_0(X, E(Y))$$

(b) $T = S^1$. Denote Θ by τ in this case:

$$\mathrm{Map}_0(S(X), Y) \xrightarrow{\tau} \mathrm{Map}_0(X, \Omega(Y)).$$

(3) From the previous lemma:

$$X \xrightarrow{i} C(X) \xrightarrow{\pi} S(X)$$
$$\Omega(Y) \xrightarrow{j} E(Y) \xrightarrow{p} Y$$

Theorem:

Let X and Y be pointed spaces.

1 To the map $X \xrightarrow{i} C(X)$ is associated a bundle:

$$\begin{tikzcd} \mathrm{Map}_0(C(X), Y) \arrow[d, "i^*"] \\ \mathrm{Map}_0(X, Y) \end{tikzcd}$$

and the fibre over $*$ is homeomorphic to $\mathrm{Map}_0(S(X), Y)$ via the inclusion $\mathrm{Map}_0(S(X), Y) \xrightarrow{\pi^*} \mathrm{Map}_0(C(X), Y)$

2. To the map $E(Y) \xrightarrow{p} Y$ is associated a bundle:

$$\begin{tikzcd} \mathrm{Map}_0(X, E(Y)) \arrow[d, "p_*"] \\ \mathrm{Map}_0(X, Y) \end{tikzcd}$$

and the fibre over $*$ is homeomorphic to $\mathrm{Map}_0(X, \Omega(Y))$ via the inclusion $\mathrm{Map}_0(X, \Omega(Y)) \xrightarrow{j_*} \mathrm{Map}_0(X, Y)$

3. i^* is a subbundle of p_* via:

$$\begin{tikzcd} \mathrm{Map}_0(C(X), Y) \arrow[rr, "\sigma"] \arrow[dr, "i^*"'] & \arrow[d, phantom, "\text{©}"] & \mathrm{Map}_0(X, E(Y)) \arrow[dl, "p_*"] \\ & \mathrm{Map}_0(X, Y) & \end{tikzcd}$$

Moreover, if X is compact T_2 then σ is a bundle isomorphism.

4. τ gives a map of fibres compatible with the bundle maps:

$$\begin{tikzcd} \mathrm{Map}_0(S(X), Y) \arrow[rr, "\tau"] \arrow[d, "\pi^*"'] & \text{©} & \mathrm{Map}_0(X, \Omega(Y)) \arrow[d, "j_*"] \\ \mathrm{Map}_0(C(X), Y) \arrow[rr, "\sigma"] \arrow[dr, "i^*"'] & \arrow[d, phantom, "\text{©}"] & \mathrm{Map}_0(X, E(Y)) \arrow[dl, "p_*"] \\ & \mathrm{Map}_0(X, Y) & \end{tikzcd}$$

Proof:

(1) i^* is continuous since i is continuous. Thus

$$\begin{tikzcd} \mathrm{Map}_0(C(X),Y) \arrow[d, "i^*"] \\ \mathrm{Map}_0(X,Y) \end{tikzcd}$$

is a bundle. To calculate the fibre over $*$, consider:

$$\begin{aligned} f \in i^{*-1}(*) &\Leftrightarrow (i^*f)(x) = * \quad \forall\, x \in X \\ &\Leftrightarrow f(i(x)) = * \quad \forall\, x \in X \\ &\overset{\text{def of } i}{\Leftrightarrow} f(\rho(x,1)) = * \quad \forall\, x \in X \\ &\Leftrightarrow f \text{ factors as:} \end{aligned}$$

$$\begin{tikzcd} C(X) \arrow[r, "f"] \arrow[d, "\pi"'] & Y \\ C(X)/X = S(X) \arrow[ur, "\bar f"'] \arrow[u, phantom, "\odot", shift left=6] \end{tikzcd} , \quad \pi^*\bar f = f$$

$\therefore \; i^{*-1}(*) \cong \mathrm{Map}_0(S(X), Y)$ via π^*.

(2) p_* is continuous since p is continuous. Thus

$$\begin{tikzcd} \mathrm{Map}_0(X, E(Y)) \arrow[d, "p_*"] \\ \mathrm{Map}_0(X,Y) \end{tikzcd}$$

is a bundle. To calculate the fibre over $*$, consider:

$$\begin{aligned} f \in p_*^{-1}(*) &\Leftrightarrow (p_*f)(x) = * \quad \forall\, x \in X \\ &\overset{\text{def}}{\Leftrightarrow} (f(x))(1) = * \quad \forall\, x \in X \end{aligned}$$

$$\begin{aligned} &\Leftrightarrow f(x) \in \Omega(Y) \quad \forall\, x \in X \\ &\Leftrightarrow f \in \mathrm{Map}_0(X, \Omega(Y)) \qquad \text{(canonical inclusion from definition).} \end{aligned}$$

$\therefore \; p_*^{-1}(*) \cong \mathrm{Map}_0(X, \Omega(Y))$.

(3) σ is known to be a continuous injection. Thus it suffices to prove $\odot$ in the diagram.

$$\begin{tikzcd} \mathrm{Map}_0(C(X),Y) \arrow[rr, "\sigma"] \arrow[dr, "i^*"'] & & \mathrm{Map}_0(X, E(Y)) \arrow[dl, "p_*"] \\ & \mathrm{Map}_0(X,Y) & \end{tikzcd}$$

Note first that σ is defined through the composition:

$$\begin{tikzcd} \mathrm{Map}_0(C(X),Y) \arrow[d, "\rho^* = \rho^*|"'] \arrow[drr, "\sigma"] & & \\ \mathrm{Map}(X \times [0,1], X \vee [0,1]; Y, *) \arrow[rr, "\Theta"] \arrow[urr, phantom, "\odot"] & & \mathrm{Map}_0(X, \underbrace{\mathrm{Map}_0([0,1],Y)}_{=E(Y)}) \end{tikzcd}$$

Thus:

(a) $$\begin{aligned} (\sigma(f)(x))(t) &= (\Theta(\rho^*f)(x))(t) \\ &= (\rho^*f)(x,t) \quad \text{by def}^{n} \text{ of } \Theta \\ &= f(\rho(x,t)). \end{aligned}$$

(b) $(p_* g)(x) = (g(x))(1)$

(c) $(i^*f)(x) = f(i(x)) = f(\rho(x,1))$.

Therefore:

$$((p_* \sigma)(f))(x) = (p_* (\sigma(f)))(x)$$
$$= (\sigma(f)(x))\ (1)$$
$$= f(\rho(x,1))$$
$$= (i^* f)(x) \qquad \forall\, x \in X,\ \forall f \in Map_0(C(X),Y)$$

$\therefore$ ©.

(4) Note first that:

$$\begin{tikzcd} S^1 & & Map_0(S^1, Y) = \Omega(Y) \arrow[d, "\varphi^* = j"] & & Map_0(X, \Omega(Y)) \arrow[d, "(\varphi^*)_* = j_*"] \\ {[0,1]} \arrow[u, "\varphi"] & \Rightarrow & Map_0([0,1], Y) = E(Y) & \Rightarrow & Map_0(X, E(Y)) \end{tikzcd}$$

(a) $((j_* (\tau(f)))(x))(s) =$

$$(\varphi^* (\tau(f)(x)))(s) =$$
$$(\tau(f)(x))\ (\varphi(s)) \stackrel{\bullet}{=}$$
$$f(K(x, \varphi(s))) \qquad \forall\, x \in X,\ s \in [0,1],\ f \in Map_0(S(X),Y)$$

• since τ factors as:

$$\begin{tikzcd} Map_0(S(X), Y) \arrow[d, "k^* = k^*|"'] \arrow[rr, "\tau"] \arrow[drr, phantom, "©"] & & \Omega(Y) \arrow[d, phantom, "\parallel"] \\ Map_0(X \times S^1, X \vee S^1; Y, *) \arrow[rr, "\theta"] & & Map_0(X, Map_0(S^1, Y)) \end{tikzcd}$$

so. $(\tau(f)(x))(r) = (\theta(k^* f)(x))(r) = (k^* f)(x,r)$

$$= f(k(x,r))$$

(b) $((\sigma(\pi^*(f)))(x))(s) =$

$$(\pi^* f)(\rho(x,s)) = \qquad (\sigma \text{ calculated in } s)$$
$$f(\pi\rho(x,s)).$$

But by the construction of $S(X) = X \wedge S^1 = C(X)/X$

$$\pi\rho \stackrel{©}{=} K\circ(id \times \varphi)$$

Hence: $f(\pi\rho(x,s)) =$

$$f(K(x, \varphi(s))) = \quad (\text{by } (a))$$
$$((j_*(\tau(f)))(x))(s), \qquad \forall\, x \in X,\ s \in [0,1],\ f \in Map_0(\ldots)$$

$\therefore$ $\square$ in ©, $j_* \tau = \sigma \pi^*$.

Q.E.D

§4 H-SPACE AND CO-H-SPACE STRUCTURES

Recall the following results

1. If X, Y, Z are pointed spaces, then:

(a)

$$\begin{tikzcd}
\mathrm{Map}_0(X, Y\times Z) \arrow[r, "\cong"] \arrow[d] \arrow[dr, phantom, "\copyright"] & \mathrm{Map}_0(X,Y)\times \mathrm{Map}_0(X,Z) \arrow[d] \\
{[X, Y\times Z]_0} \arrow[r, "\cong"] & {[X,Y]_0 \times [X,Z]_0}
\end{tikzcd}$$

and:

(b)

$$\begin{tikzcd}
\mathrm{Map}_0(X\vee Y, Z) \arrow[r, "\cong"] \arrow[d] \arrow[dr, phantom, "\copyright"] & \mathrm{Map}_0(X,Z)\times \mathrm{Map}_0(Y,Z) \arrow[d] \\
{[X\vee Y, Z]_0} \arrow[r, "\cong"] & {[X,Z]_0 \times [Y,Z]_0}
\end{tikzcd}$$

2. If $X' \xrightarrow{u} X$ is a map of pointed spaces and Y is pointed then

$$\begin{tikzcd}
\mathrm{Map}_0(X,Y) \arrow[r, "u^*"] \arrow[d] \arrow[dr, phantom, "\copyright"] & \mathrm{Map}_0(X',Y) \arrow[d] \\
{[X,Y]_0} \arrow[r, dashed, "u^*"'] & {[X',Y]_0}
\end{tikzcd}$$

3. If $Y \xrightarrow{v} Y'$ is a map of pointed spaces and X is pointed then

$$\begin{tikzcd}
\mathrm{Map}_0(X,Y) \arrow[r, "v_*"] \arrow[d] \arrow[dr, phantom, "\copyright"] & \mathrm{Map}_0(X,Y') \arrow[d] \\
{[X,Y]_0} \arrow[r, dashed, "v_*"'] & {[X,Y']_0}
\end{tikzcd}$$

Definition

1. Let Y be a pointed space and $Y\times Y \xrightarrow{\phi} Y$ a map of pointed spaces. ϕ induces a map

$$\begin{tikzcd}
{[X, Y\times Y]_0} \arrow[r, "\phi_*"] \arrow[dr, phantom, "\copyright"] & {[X,Y]_0} \\
{[X,Y]_0 \times [X,Y]_0} \arrow[u, "\cong"] \arrow[ur, dashed, "\phi_*"'] &
\end{tikzcd}$$

where X is any pointed space. If $\forall$ pointed spaces X, $([X,Y]_0, \phi_*)$ is a group, then ϕ is called a *H-space structure* on Y.

2. Let X be a pointed space and $X \xrightarrow{\psi} X\vee X$ a map of pointed spaces. ψ induces a map:

$$\begin{tikzcd}
{[X\vee X, Y]_0} \arrow[r, "\psi^*"] \arrow[dr, phantom, "\copyright"] & {[X,Y]_0} \\
{[X,Y]_0 \times [X,Y]_0} \arrow[u, "\cong"] \arrow[ur, dashed, "\psi^*"'] &
\end{tikzcd}$$

where Y is any pointed space. If $\forall$ pointed spaces Y, $([X,Y]_0, \psi^*)$ is a group, then ψ is called a *coH-space structure* on X.

The most important examples are:

1. Loop spaces have H-space structures;
2. Suspensions have coH-space structures,
3. Multiplication in topological groups are H-space structures.

These examples are now examined in detail.

H-Space Structure on Loop Spaces

Let X and Y be pointed spaces. Define:

$$\underline{\mu}: \Omega Y \times \Omega Y \to \Omega Y \qquad \mu(f,g)(t) = \begin{cases} f(2t), & 0 \le t \le \tfrac12 \\ g(2t-1), & \tfrac12 \le t \le 1 \end{cases}$$

where $S^1 = [0,1]/\{0,1\}$ and, by abuse of notation, $t \in [0,1]$ denotes $[t] \in S^1$.

μ clearly preserves the base point map $e: S^1 \to Y : e(t) = *$

e is a homotopy identity $\quad \mu(e,f) \simeq_0 f \simeq_0 \mu(f,e) \;\; \forall f \in \Omega$

μ is homotopy associative $\quad \mu(\mu(f,g),h) \simeq_0 \mu(f,\mu(g,h))$.

$f \in \Omega Y$ has a homotopy inverse: $\quad \mu(f,f^{-1}) \simeq_0 e \simeq_0 \mu(f^{-1},f)$

where $f^{-1}: S^1 \to Y : f^{-1}(t) = f(1-t)$.

Thus $\forall X$, $([X,\Omega Y]_0, \mu_*)$ is a group so that μ is a H-space structure on ΩY.

CoH-Space Structure on Suspensions

Let X and Y be pointed spaces. Regarding $S^1 \cong [0,1]/\{0,1\}$ gives

$$S(X) = X \wedge S^1 = \frac{X \times [0,1]}{X \times \{0\} \cup X \times \{1\} \cup \{*\} \times [0,1]}.$$

Let $\rho: X \times [0,1] \to S(X)$ denote the quotient. Define

$$\underline{\nu}: S(X) \to S(X) \vee S(X) \qquad \nu(\rho(x,t)) = \begin{cases} (\rho(x,2t),*), & 0 \le t \le \tfrac12 \\ (*,\rho(x,2t-1)), & \tfrac12 \le t \le 1 \end{cases}$$

where points in $S(X) \vee S(X)$ are represented by $(p,*) \sim (*,p)$

ν preserves base points since ρ preserves basepoints.

$[e]_0$, where $S(X) \xrightarrow{e} Y : e(p) = * \;\forall p$, is an identity in $[S(X),Y]_0$ for the multiplication ν^*.

$[f]_0$ has ν_* inverse $[f^{-1}]_0$ where

$$f^{-1}: S(X) \to Y \qquad f^{-1}(\rho(x,t)) = f(\rho(x,1-t))$$

and ν^* is associative. Hence $([S(X),Y]_0, \nu_*)$ is a group and so ν is a coH-space structure on $S(X)$.

Application:

Let $p \ge 1$. Recall the result $S(S^{p-1}) = S^p$. Thus:

$$[S(S^{p-1}),Y]_0 \equiv [S^p,Y]_0 \underset{\substack{\uparrow \\ \text{equality as sets}}}{=} \pi_p(Y).$$

such that

1. $\exists\; e \in M \ni \forall\; m \in M$,

$$m = e\circ m = m\circ e = e\,\square\, m = m\,\square\, e \,.$$

2. $\forall\; x_1, x_2, y_1, y_2 \in M$,

$$* \quad (x_1 \circ y_1)\,\square\,(x_2 \circ y_2) = (x_1\,\square\, x_2)\circ(y_1\,\square\, y_2)$$

Then $\circ = \square$ and $\forall\; x, y \in M$, $x\circ y = y \circ x$.

Proof:

(i) Set $x_1 = e = y_2$ in $*$:

$$\therefore \quad y_1\,\square\, x_2 = x_2 \circ y_1$$

(ii) Set $x_2 = e = y_1$ in $*$

$$x_1\,\square\, y_2 = x_1 \circ y_2$$

Hence $x\,\square\, y = x\circ y = y\,\square\, x \quad \forall\; x, y \in M$.

Q.E.D.

Application:

Let X and Y be pointed spaces.

(i) $[SX, \Omega Y]_\circ$ inherits a group structure $\circ$ from $SX \xrightarrow{\nu} SX \vee SX$

(ii) $[SX, \Omega Y]_\circ$ inherits another group structure $\square$ from $\Omega Y \times \Omega Y \xrightarrow{\mu} \Omega Y$

(iii) Let $\rho : X \times [0,1] \to SX$ be the projection defined previously. Then $\circ, \square$ are given by:

Now the group multiplication obtained from ν on $S(S^{p-1})$ is easily seen to be the same as the multiplication defined previously in $\pi_p(Y)$. Hence $\pi_p(Y)$ is a group $\forall\; p \geq 1$

Iterated Loops and Suspensions:

Lemma:

Let X be compact T_2. Then the bijection

$$[SX, Y]_\circ \xrightarrow{\theta} [X, \Omega Y]_\circ$$

is a group isomorphism.

Proof:

The result follows easily from © in the diagram

$$\begin{array}{ccc}
\mathrm{Map}_\circ(SX, Y) \times \mathrm{Map}_\circ(SX, Y) & \xrightarrow{\nu^*} & \mathrm{Map}_\circ(SX, Y) \\
\theta\times\theta \Big\downarrow\Big\uparrow \theta^{-1}\times\theta^{-1} & © & \theta \Big\downarrow\Big\uparrow \theta^{-1} \\
\mathrm{Map}_\circ(X, \Omega Y) \times \mathrm{Map}_\circ(X, \Omega Y) & \xrightarrow[\mu_*]{} & \mathrm{Map}_\circ(X, \Omega Y)
\end{array}$$

QED.

Consider the following purely algebraic result:

Lemma:

Let M be a pointed set with two maps

$$\circ, \square : M \times M \to M$$

$$SX \underset{h,\,k}{\overset{f,\,g}{\dashrightarrow}} \Omega Y$$

with

$$[f]_0 \cdot [g]_0 = [k]_0 \qquad k(\rho(x,s))(t) = \begin{cases} f(\rho(x,2s))(t), & 0 \le s \le \\ g(\rho(x,2s-1))(t), & \tfrac12 \le s \end{cases}$$

$$[f]_0 \,\square\, [g]_0 = [h]_0 : \quad h(\rho(x,s))(t) = \begin{cases} f(\rho(x,s))(2t), & 0 \le t \le \\ g(\rho(x,s))(2t-1), & \tfrac12 \le t \end{cases}$$

$\cdot$ $\square$ have a common identity, namely $[e]_0$ where $SX \xrightarrow{e} \Omega Y$ is the base point map. Since s, t are independent, $*$ is seen to be verified. Hence $\cdot = \square$ and the operation is commutative.

As an immediate consequence, if X is compt T_2, then:

$$[S^2X, Y]_0 \overset{\text{lemma}}{\cong} [SX, \Omega Y]_0 \qquad \text{gp op. from } \Omega Y \times \Omega Y \to \Omega$$

$$\Big\| \text{ above}$$

$$[X, \Omega^2 Y]_0 \overset{\text{lemma}}{\cong} [SX, \Omega Y]_0 \qquad \text{gp. op. from } SX \to SX \vee SX$$

Iterating $\quad [S^pX, Y]_0 \cong [X, \Omega^p Y]_0 \quad \forall\ p \ge 1,$

group isomorphism.

In particular:

1. $\forall\ p \ge 1, \quad \pi_p(Y) = [S^p, Y]_0 = [S^p(S^0), Y]_0 = [S^0, \Omega^p Y]_0$

so

$$\boxed{\pi_p(Y) = \pi_0(\Omega^p Y), \quad p \ge 1, \quad \text{group isomorphism}}$$

2. $\forall\ p \ge 2, \quad \pi_p(Y) \equiv [S^p, Y]_0$

$$\cong [S^2(S^{p-2}), Y]_0$$

$$\cong [S(S^{p-1}), \Omega Y]_0, \quad \text{gp isomorphism}$$

But the group law in the last group is abelian. Hence

$$\boxed{\forall\ Y, \quad \pi_p(Y) \ \text{abelian}, \ p \ge 2}$$

<u>H-Space Structure on a Topological Group.</u>

Let G be a topological group. The canonical base point in G is the identity 1. Group multiplication is a H-space structure on G because

(i) $1 \cdot 1 = 1 \Rightarrow$ base point preserved

(ii) $\forall$ pointed spaces X, $[X, G]_0$ is a group by:

$$[f]_0 \cdot [g]_0 = [f \cdot g]_0 \quad \text{where}$$

$$(f \cdot g)(x) = f(x) \cdot g(x) \ \forall\ x \in X$$

There are two group operations on $[S(X), G]_0$.

(i) $\cdot$ from multiplication in G, defined by: (just as above)

$[f]_0 \circ [g]_0 = [h]_0$ $\qquad S(X) \xrightarrow{f,g,h} G$

$$h(\rho(x,t)) = f(\rho(x,t)) \cdot g(\rho(x,t))$$

(ii) $\square$ from $S(X) \xrightarrow{\nu} S(X) \vee S(X)$ defined by

$[f]_0 \,\square\, [g]_0 = [k]_0$: $\qquad S(X) \xrightarrow{f,g,k} G$

$$k(\rho(x,t)) = \begin{cases} f(\rho(x,2t)), & 0 \le t \le \tfrac{1}{2} \\ g(\rho(x,2t-1)), & \tfrac{1}{2} \le t \le 1 \end{cases}$$

* may easily be verified so both $\circ$ and $\square$ give the same group law which is, moreover, abelian. Hence, in particular

$$\boxed{G \text{ a topological group} \Rightarrow \pi_1(G) \text{ abelian}}$$

§5: HOMOTOPY OF LINEAR GROUPS

Underlying Fields

There are three fields of primary interest in topology, namely $\mathbb{R}$, $\mathbb{C}$ and $\mathbb{H}$. They are related by

$$\mathbb{C} = \mathbb{R} + \mathbb{R}i \ , \quad i^2 = -1$$

$$\mathbb{H} = \mathbb{C} + \mathbb{C}j \ , \quad j^2 = -1 \ , \quad ij = -ji = k$$

They may be regarded as real vector spaces of dimensions 1, 2 and 4 respectively

$$\mathbb{C} \ni z = x + iy \ , \quad x, y \in \mathbb{R};$$

$$\mathbb{H} \ni q = x_0 + x_1 i + x_2 j + x_3 k \ , \quad x_0, x_1, x_2, x_3 \in \mathbb{R}$$

A conjugate operation is defined on each:

$x \in \mathbb{R}$, $\quad \bar{x} = x$;

$z = x + iy \in \mathbb{C}$, $\quad \bar{z} = x - iy$;

$q = x_0 + x_1 i + x_2 j + x_3 k \in \mathbb{H}$, $\quad \bar{q} = x_0 - x_1 i - x_2 j - x_3 k$

In each case define $|p|^2 = \bar{p}p$. Then $|p_1 p_2| = |p_1||p_2|$ and:

$x \in \mathbb{R}$: $\quad |x|^2 = x^2$

$z \in \mathbb{C}$: $\quad |z|^2 = x^2 + y^2$

$q \in \mathbb{H}$: $\quad |q|^2 = x_0^2 + x_1^2 + x_2^2 + x_3^2$

Associated Multiplicative Groups

Let $F = \mathbb{R}, \mathbb{C}$ or $\mathbb{H}$. Let $\underline{F^*} = F \setminus \{0\}$. Then F^* is a group where the group law is multiplication. Define

$$F_1^* = \{ f \in F : |f| = 1 \}$$

Since $|f_1 f_2| = |f_1||f_2|$, this is a subgroup of F^*. In particular

$$\mathbb{R}_1^* \simeq S^0$$

$$\mathbb{C}_1^* \simeq S^1$$

$$\mathbb{H}_1^* \simeq S^3$$

Note that the last group is non-abelian.

Linear Groups

1 General Linear Groups

Let $F = \mathbb{R}, \mathbb{C}$ or $\mathbb{H}$. The GL groups may be defined in two ways:

(a) Let $M_n(F)$ denote the algebra of $n \times n$ matrices over F. Let $M_n(F)$ have the topology induced from its identification with $F^{n \times n}$. Then $\underline{GL_n(F)}$ is the open subset of $M_n(F)$ consisting of non-singular matrices. $GL_n(F)$ is a topological group with respect to matrix multiplication

For any $m \geq 1$, there is an injection

$$GL_n(F) \hookrightarrow GL_{n+m}(F)$$

via

$$A \longmapsto \begin{bmatrix} A & 0 \\ 0 & I_{m \times m} \end{bmatrix}$$

(b) Let F^n have the canonical structure of a vector sp over F. Then define

$$\underline{GL_n(F)} = \mathrm{Aut}_F(F^n)$$

Since $\mathbb{R} \subset \mathbb{C} \subset \mathbb{H}$, there are injections

$$GL_n(\mathbb{H}) = \mathrm{Aut}_{\mathbb{H}}(\mathbb{H}^n) \hookrightarrow \mathrm{Aut}_{\mathbb{C}}(\mathbb{H}^n) = \mathrm{Aut}_{\mathbb{C}}(\mathbb{C}^{2n}) = GL_{2n}(\mathbb{C})$$

and

$$GL_n(\mathbb{C}) = \mathrm{Aut}_{\mathbb{C}}(\mathbb{C}^n) \hookrightarrow \mathrm{Aut}_{\mathbb{R}}(\mathbb{C}^n) = \mathrm{Aut}_{\mathbb{R}}(\mathbb{R}^{2n}) = GL_{2n}(\mathbb{R}).$$

Notes

1. (a) and (b) are related by representing $A \in \mathrm{Aut}_F(F^n)$ by a matrix wrt the canonical basis in F^n.

2. $GL_1(F) \simeq F^*$, isomorphism of topological groups.

2 Special Linear Groups

Let $F = \mathbb{R}$ or $\mathbb{C}$. There is a group homomorphism

$$\det : GL_n(F) \rightarrow GL_1(F)$$

(since $\det(AB) = \det A \det B$). Define

$$\underline{SL_n(F)} \equiv \ker(\det) = \{A \in GL_n(F) : \det A = 1\}$$

Note that there is no canonical way to define det in $GL_n(\mathbb{H})$ since $\mathbb{H}$ is a non-commutative field.

3 Unitary Groups

Let $F = \mathbb{R}, \mathbb{C}$ or $\mathbb{H}$. Define the scalar product:

$$(\cdot|\cdot) : F^n \times F^n \rightarrow F : \quad (x|y) = \sum_{i=1}^{n} \bar{x}_i y_i$$

Define $\underline{U_n(F)}$ to be the subgroup of $GL_n(F) \equiv \mathrm{Aut}_F(F^n)$ leaving $(\cdot|\cdot)$ invariant. Thus

$$A \in U_n(F) \iff (Ax|Ay) = (x|y) \quad \forall\, x,y \in F^n.$$

In particular:

$F = \mathbb{R}$: $U_n(\mathbb{R})$ is denoted $\underline{O(n)}$, called the <u>orthogonal group</u>

$F = \mathbb{C}$: $U_n(\mathbb{C})$ is denoted $\underline{U(n)}$, called the <u>unitary group</u>

$F = \mathbb{H}$: $U_n(\mathbb{H})$ is denoted $\underline{Sp(n)}$, called the <u>symplectic group</u>

4 Special Unitary Groups

Let $F = \mathbb{R}$ or $\mathbb{C}$ and define $\underline{SU_n(F)} \equiv SL_n(F) \cap U_n(F)$

Then

$F = \mathbb{R}$: $SU_n(F)$ is denoted $\underline{SO(n)}$, called the <u>special orthogonal group</u>.

$F = \mathbb{C}$: $SU_n(F)$ is denoted $\underline{SU(n)}$, called the <u>special unitary group</u>

Notes

1 All the groups defined above are, in fact, Lie groups. Thus there are five important families of Lie groups which arise from linear algebra:

(i) $GL_n(\mathbb{R}) \supset O(n)$

(ii) $GL_n(\mathbb{C}) \supset U(n)$

(iii) $GL_n(\mathbb{H}) \supset Sp(n)$

(iv) $SL_n(\mathbb{R}) \supset SO(n)$

(v) $SL_n(\mathbb{C}) \supset SU(n)$

The groups in the rhs are, in fact, maximal compact subgroups. It will be shown that they determine the homotopy of the larger groups.

2. There are group isomorphisms

$$Sp(1) \cong S^3 \equiv \mathbb{H}_1^* \cong SU(2)$$

Iwasawa Decomposition:

A fundamental theorem from the theory of Lie groups is the following:

Iwasawa Decomposition Theorem:

Given a Lie group G, there exist subgroups $K \subset G$ compact and $V \subset G$ st $V \cong \mathbb{R}^k$ fs k as spaces and st

$$G = V K = K V \quad \text{as spaces},$$

that is, $K \times V \longrightarrow G \quad (A,B) \mapsto AB$, is a homeomorphism.

Moreover, V decomposes as $V = A \cdot N = N \cdot A$ where N is nilpotent and A is abelian.

Notes

1. K is a maximal compact subgroup of G.
2. For the list on the previous page, if G is a group in the left column then K is the corresponding group

Application to Homotopy Theory

Let G be a Lie group with Iwasawa decomposition

$$G = K V$$

Since V is contractible, $\pi_*(V) = 0$. Therefore:

$$G = K \times V, \quad \text{homeomorphism}$$

$$\Rightarrow \pi_p(G) = \pi_p(K \times V)$$
$$= \pi_p(K) \times \pi_p(V)$$
$$= \pi_p(K) \qquad \forall\ p \geq 0$$

$$\therefore \quad \boxed{\pi_*(G) = \pi_*(K)}$$

Thus the homotopy of G reduces to that of a compact group.

Examples:

1. $\pi_*(GL_n(\mathbb{R})) = \pi_*(O(n))$
2. $\pi_*(GL_n(\mathbb{C})) = \pi_*(U(n))$
3. $\pi_*(GL_n(\mathbb{H})) = \pi_*(Sp(n))$
4. $\pi_*(SL_n(\mathbb{R})) = \pi_*(SO(n))$
5. $\pi_*(SL_n(\mathbb{C})) = \pi_*(SU(n))$

Iwasawa Decomposition for $GL_n(F)$ and $SL_n(F)$

Let $F = \mathbb{R}$ or $\mathbb{C}$. The proof applies with some modification to the case $F = \mathbb{H}$, where only $GL_n(\mathbb{H})$ is considered.

1 Let $E(F^n)$ denote the set of bases of F^n and $E^\circ(F^n)$ the set of orthonormal bases of F^n with respect to the scalar product $(\cdot\,|\,\cdot)$ defined above. Let e denote the standard ordered o.n basis of F^n,

$$e_j = (0, \dots \overset{(j)}{1}, \dots 0)$$

Taking coordinates wrt e, $E(F^n)$ and $E^\circ(F^n)$ may be identified with subspaces of $M_n(F)$.

Regard $GL_n(F) \subset M_n(F)$. There is a right action:

$$E(F^n) \times GL_n(F) \longrightarrow E(F^n) : (b, A) \longmapsto bA :$$

$$b = (b_1 \dots b_n), \quad A = (a_{ij}),$$

$$(bA)_j = \sum_{i=1}^{n} b_i a_{ij}$$

Note that:

(a) Given $b, b' \in E(F^n)$, $\exists!\, A \in GL_n(F)$ st $b' = bA$.

Thus, fixing $b \in E(F^n)$ gives a homeomorphism

$$GL_n(F) \longrightarrow E(F^n) : A \longmapsto bA.$$

(b) If $A \in GL_n(F)$ and $b \in E^\circ(F^n)$, then:

$$bA \in E^\circ(F^n) \iff A \in U_n(F)$$

2 Fix $M \in GL_n(F)$. Let $u = eM$. Apply the Gram Schmidt process to u: this gives an orthogonal basis v and an orthonormal basis w defined by

$$v_1 = u_1$$

$$w_1 = v_1 / \|v_1\|$$

$$\left.\begin{aligned} v_j &= u_j - \sum_{i=1}^{j-1} (u_j | w_i)\, w_i \\ w_j &= v_j / \|v_j\| \end{aligned}\right] \quad 2 \le j \le n$$

where $\|v\|^2 = (v|v)$.

The formulae show that:

(a) $u = vN$ where $N = \begin{bmatrix} 1 & & * \\ & \ddots & \\ 0 & & 1 \end{bmatrix}$

(b) $v = wA$ where: $A = \begin{bmatrix} a_1 & & 0 \\ & \ddots & \\ 0 & & a_n \end{bmatrix}$

$$a_i = \|v_i\| > 0$$

Now $w, e \in E^\circ(F^n)$ so $\exists!\, K \in U_n(F)$ st

(c) $w = eK$

Combining (i) – (v)

$$u = vN = wAN = eKAN$$

But $\quad u = eM$

$$\therefore \; eM = eKAN$$

$$\Rightarrow \; M = KAN$$

3. Thus $\quad GL_n(F) = U_n(F) \cdot A_n \cdot N_n(F)$

where A_n is the group of diagonal $n \times n$ matrices with real strictly positive diagonal elements and $N_n(F)$ is the group of $n \times n$ upper triangular matrices over F with 1's along the diagonal.

4. Lemma: A_n and $N_n(F)$ are contractible

Proof

(a) $A_n \cong \mathbb{R}_+^n \cong \mathbb{R}^n$ as spaces; hence A_n is contractible.

(b) Define $\quad h : [0,1] \times N_n(F) \to N_n(F)$:

$$h(t, X) = I + t(X - I)$$

Then h is a contraction of $N_n(F)$ to the point I. Clearly $N_n(F) = F^{\frac{n(n-1)}{2}}$ as spaces.

QED.

Note:

$\ln : A_n \to \mathbb{R}^n$ is a group isomorphism.

5. Lemma $\quad U_n(F)$ is compact

Proof

The homeomorphism $GL_n(F) \Rightarrow E_n(F)$ restricts:

$$\begin{tikzcd} F^{n\times n} \supset GL_n(F) \arrow[r, "\cong"] & E(F^n) \subset F^{n\times n} \\ U_n(F) \arrow[u] \arrow[r, "\cong"] \arrow[ur, phantom, "©"] & E^0(F^n) \arrow[u] \end{tikzcd}$$

$E^0(F^n)$ is a closed bounded subset of a Euclidean space and so is compact. Thus $U_n(F)$ is compact.

Q.E.D.

6. This completes the Iwasawa decomposition of $GL_n(F)$.

7. A similar procedure may be adopted for the case $SL_n(F)$, $F = \mathbb{R}, \mathbb{C}$. Then

$$SL_n(F) = SU_n(F) \cdot SA_n \cdot N_n(F)$$

where $\quad SA_n = \{ A \in A_n : \det A = 1 \}$

$$= \{ \operatorname{diag}(a_1 \ldots a_n) \in A_n : a_n^{-1} = \prod_{i=1}^{n-1} a_i \}$$

Using this second characterisation, there is a topological group isomorphism

$$\ln : SA_n \Rightarrow \mathbb{R}^{n-1} \qquad \text{so} \qquad SA_n \text{ is contractible}$$

$SU_n(F)$ is compact since $U_n(F) \xrightarrow{\det} F$ is continuous and $U_n(F)$ is compact.

3. Hence, the Iwasawa decompositions are:

$$\boxed{\begin{array}{ll} GL_n(F) = U_n(F)\cdot A_n\cdot N_n(F) & : F = \mathbb{R}, \mathbb{C}, \mathbb{H} \\ SL_n(F) = SU_n(F)\cdot SA_n\cdot N_n(F) & F = \mathbb{R}, \mathbb{C} \end{array}}$$

Homotopy of Linear Groups:

Lemma.

The maps $U_n(F) \overset{i}{\hookrightarrow} GL_n(F)$

and $SU_n(F) \overset{i}{\hookrightarrow} SL_n(F)$

are homotopy equivalences.

Proof.

Decompose $GL_n(F)$, $SL_n(F) = K \cdot A \cdot N$

For $x \in GL_n(F)$, $SL_n(F)$; $x = k\cdot a\cdot n$ and set $r(x) = k$.

Thus $r : GL_n(F), SL_n(F) \longrightarrow U_n(F), SU_n(F)$

Clearly $ri = id$. Let

$$t \mapsto h_t : A_n, SA_n, N_n(F) \longrightarrow A_n, SA_n, N_n(F)$$

be contractions. Then $ir \simeq id$ via:

$$t \mapsto k_t : \quad k_t(x) = k_t(k\cdot a\cdot n)$$
$$= k\cdot h_t(a)\cdot h_t(n).$$

Corollary

The homotopy theory of $GL_n(F)$ and $SL_n(F)$ reduces to that of compact groups.

$$\boxed{\begin{array}{ll} \Pi_*(GL_n(F)) = \Pi_*(U_n(F)) & : F = \mathbb{R}, \mathbb{C}, \mathbb{H} \\ \Pi_*(SL_n(F)) = \Pi_*(SU_n(F)) & F = \mathbb{R}, \mathbb{C} \end{array}}$$

§ 6 : FIBRATIONS

Definition:

A bundle $E \xrightarrow{p} B$ satisfies the homotopy lifting property for a space W if for all maps

$$\begin{tikzcd} W \times \{0\} \arrow[r, "f"] \arrow[d, hook] \arrow[dr, phantom, "\circlearrowleft"] & E \arrow[d, "p"] \\ W \times [0,1] \arrow[r, "h"] & B \end{tikzcd}$$

then $\exists$ map k st :

$$\begin{tikzcd} W \times \{0\} \arrow[r, "f"] \arrow[d, hook] & E \arrow[d, "p"] \\ W \times [0,1] \arrow[r, "h"'] \arrow[ur, "k"] & B \end{tikzcd}$$

k is called a lifting of h

Definition:

A bundle $E \xrightarrow{p} B$ is called a fibration if it satisfies the homotopy lifting property for all spaces W.

Notation:

$f : E \xrightarrow{p} B$ is a fibration where E, B are pointed

$$\begin{tikzcd} F \arrow[r, hook] & E \arrow[d, "p"] \\ & B \end{tikzcd}$$

where $F = p^{-1}(x)$.

Theorem:

Let $E \xrightarrow{p} B$ be a bundle and suppose there is a numerable open covering $\{U_i\}_{i \in J}$ of B st $\forall\ i \in J$, $p^{-1}(U_i) \xrightarrow{p|} U_i$ is a fibration. Then $E \xrightarrow{p} B$ is a fibration.

Corollary

It will be shown below that any trivial bundle is a fibration. By the theorem it follows that any bundle which may be trivialised over a numerable open cover is a fibration.

Examples:

1. Trivial Bundle

Suppose:

$$\begin{tikzcd} W \times \{0\} \arrow[r, "f"] \arrow[d, hook, "i"] \arrow[dr, phantom, "\circlearrowleft"] & B \times F \arrow[r, "p_F"] \arrow[d, "p_B"] & F \\ W \times [0,1] \arrow[r, "h"] & B & \end{tikzcd}$$

Define $k : W \times [0,1] \longrightarrow B \times F$: $k(\omega, t) = (p_F f(\omega), h(\omega, t))$

Then $ki = f$, $p_B k = h$ as required.

2 Path Space Fibration

Let Y be a pointed space and consider the bundle $E(Y) \xrightarrow{p} Y$

Suppose

$$\begin{tikzcd} W \times \{0\} \arrow[r, "f"] \arrow[d, hook, "i"] \arrow[dr, phantom, "©"] & E(Y) \arrow[d, "p"] \\ W \times [0,1] \arrow[r, "h"] & Y \end{tikzcd}$$

Define $k : W \times [0,1] \longrightarrow E(Y)$.

$$k(w,t) : [0,1] \longrightarrow Y$$

$$k(w,t)(s) = \begin{cases} f(x)\,(s(1+t)), & s(1+t) \leq 1 \\ h(x,\ s(1+t)), & s(1+t) > 1 \end{cases}$$

Then k is a lifting of h and a homotopy of f as required. This example is called the path-space fibration.

$$\begin{tikzcd} \Omega(Y) \arrow[r] & E(Y) \arrow[d, "p"] \\ & Y \end{tikzcd}$$

3 Suppose $\begin{tikzcd} F \arrow[r] & E \arrow[d, "p"] \\ & B \end{tikzcd}$ is a fibration.

Then $\begin{tikzcd} \Omega(F) \arrow[r] & \Omega(E) \arrow[d, "\pi"] \\ & \Omega(B) \end{tikzcd}$ is also a fibration

where π is the map induced by p $\pi(\omega) = p\omega$

To see this, suppose

$$\begin{tikzcd} W \times \{0\} \arrow[r, "f"] \arrow[d, hook, "i"] \arrow[dr, phantom, "©"] & \Omega(E) \arrow[d, "\pi"] \\ W \times [0,1] \arrow[r, "h"] & \Omega(B) \end{tikzcd}$$

This induces

$$\begin{tikzcd} S^1 \times W \times \{0\} \arrow[r, "f'"] \arrow[d, hook, "i'"] \arrow[dr, phantom, "©"] & E \arrow[d, "p"] \\ S^1 \times W \times [0,1] \arrow[r, "h'"'] & B \end{tikzcd}$$

where $f'(s,w) = f(w)(s)$ and $h'(s,w,t) = h(w,t)(s)$

Since $E \xrightarrow{p} B$ is a fibration, there is a lifting k' of h':

$$\begin{tikzcd} S^1 \times W \times \{0\} \arrow[r, "f'"] \arrow[d, hook, "i'"] & E \arrow[d, "p"] \\ S^1 \times W \times [0,1] \arrow[r, "h'"'] \arrow[ur, "k'"] & B \end{tikzcd}$$

Define $k : W \times [0,1] \longrightarrow \Omega(E)$.

$$k(w,t) : S^1 \longrightarrow E. \quad k(w,t)(s) = k'(s,w,t)$$

Then

$$\begin{tikzcd} W \times \{0\} \arrow[r, "f"] \arrow[d, hook, "i"] & \Omega(E) \arrow[d, "\pi"] \\ W \times [0,1] \arrow[r, "h"'] \arrow[ur, "k"] & \Omega(B) \end{tikzcd}$$

as required.

The next theorem shows why fibrations are important for calculations in homotopy theory:

Theorem (Long Exact Homotopy Sequence)

Let
$$\begin{tikzcd} F \arrow[r, "i"] & E \arrow[d, "p"] \\ & B \end{tikzcd}$$
be a fibration.

Then $\forall\ n \geq 1$, $\exists$ homomorphisms $\partial : \Pi_n(B) \to \Pi_{n-1}(F)$ s.t. the following sequence is exact:

$$\cdots \xrightarrow{\partial} \Pi_n(F) \xrightarrow{i_*} \Pi_n(E) \xrightarrow{p_*} \Pi_n(B) \xrightarrow{\partial} \Pi_{n-1}(F) \xrightarrow{i_*} \cdots$$
$$\cdots \xrightarrow{\partial} \Pi_1(F) \xrightarrow{i_*} \Pi_1(E) \xrightarrow{p_*} \Pi_1(B) \xrightarrow{\partial}$$
$$\Pi_0(F) \xrightarrow{i_*} \Pi_0(E) \xrightarrow{p_*} \Pi_0(B)$$

Sketch of Proof

Using the fact $\Pi_n(Y) = \Pi_{n-1}(\Omega Y)$, proven in , and example 3 above, it suffices to construct ∂ for the case:

$$\Pi_1(B) \xrightarrow{\partial} \Pi_0(F)$$

and to check exactness.

To construct ∂ in this case, let $[u] \in \Pi_1(B)$. u induces a map $\tilde{u}$:

$$\begin{tikzcd} pt \times \{0\} \arrow[r, "f"] \arrow[d, hook, "i"] \arrow[dr, phantom, "\copyright"] & E \arrow[d, "p"] \\ pt \times [0,1] \arrow[r] & B \end{tikzcd}$$

with $\tilde{u}(pt, 0) = \tilde{u}(pt, 1)$

$\tilde{u}$ lifts to v

$$\begin{tikzcd} & E \arrow[d, "p"] \\ pt \times [0,1] \arrow[ur, "v"] \arrow[r, "\tilde{u}"'] & B \end{tikzcd}$$

so $v(pt, 1) \in p^{-1}(*) = F$. Define $\omega : S^0 \to F : \begin{cases} \omega(1) = * \\ \omega(-1) = v(pt, 1) \end{cases}$

and set $\partial[u] = [\omega]$. Then ∂ is the required homomorphism.

QED

Applications

i. Trivial Bundle

Suppose
$$\begin{tikzcd} F \arrow[r, "i"] & E \arrow[d, "p"] \\ & B \end{tikzcd}$$
is any fibration which admits a cross-section s. Then

$$ps = id_B \Rightarrow p_* s_* = id$$

so p_* is surjective:

$$\cdots \to \Pi_n(E) \underset{p_*}{\overset{s_*}{\leftrightarrows}} \Pi_n(B) \xrightarrow{\partial} \Pi_{n-1}(F) \xrightarrow{i_*} \Pi_{n-1}(B) \to \cdots$$

$\therefore$ $\ker \partial = \operatorname{im} p_* = \Pi_n(B) \Rightarrow \partial$ is the zero hom

$\therefore$ $\operatorname{im} \partial = 0 = \ker i_* \Rightarrow i_*$ injective

Thus the long exact sequence splits into short exact sequences

$$0 \to \Pi_n(F) \xrightarrow{i_*} \Pi_n(E) \xrightarrow{p_*} \Pi_n(B) \longrightarrow 0, \ n \geq 1$$

whence $\pi_n(E) = \pi_n(F) \oplus \pi_n(B)$, $n \geq 1$

In particular, any trivial bundle admits a cross-section so that the above argument applies.

2. Fibrations with Contractible Total Spaces

Suppose
$$\begin{array}{ccc} F & \longrightarrow & E \\ & & \downarrow \\ & & B \end{array}$$
is a fibration in which the total space E is contractible. Thus $\pi_*(E) = 0$ and the long exact sequence yields the isomorphisms

$$\pi_n(B) \xrightarrow{\partial} \pi_{n-1}(F) \quad , \quad n \geq 1$$

Two important examples of this situation are:

(a) Homotopy of S^1

Consider the fibration

$$\begin{array}{ccc} \mathbb{Z} & \longrightarrow & \mathbb{R} \\ & & \downarrow p \\ & & S^1 \end{array} \qquad p(t) = e^{2\pi i t}$$

Thus $\pi_n(S^1) \cong \pi_{n-1}(\mathbb{Z})$, $n \geq 1$. Therefore:

$$\boxed{\begin{array}{l} \pi_1(S^1) = \mathbb{Z} \\ \pi_i(S^1) = 0 \ , \ i \neq 1 \end{array}}$$

(b) Path Space Fibrations

Consider the path-space fibration of a pointed space Y:

$$\begin{array}{ccc} \Omega(Y) & \longrightarrow & E(Y) \\ & & \downarrow \\ & & Y \end{array}$$

$E(Y)$ is contractible, for example by

$$h : E(Y) \times [0,1] \to E(Y)$$

$$h(u,t) : [0,1] \to Y \qquad h(u,t)(s) = u(st)$$

The homotopy exact sequence then gives the previous result

$$\pi_n(Y) \underset{\partial}{\cong} \pi_{n-1}(\Omega Y)$$

(which was actually used to prove the long exact sequence)

Hopf Fibrations and Projective Spaces

Projective spaces may be introduced in a number of ways.

Definition

1. Let F be a field and consider the F-vector space F^{n+1}. Define $\underline{\mathbb{P}_n(F)}$ to be the set of one-dimensional subspaces of F^{n+1}.

2. Let F be a topological field and let F^* be the group $F - \{0\}$ with multiplication in F the group operation. F^* acts

on $F^{n+1} \setminus \{0\}$ by scalar multiplication. Define

$$\underline{\mathbb{P}_n(F)} = (F^{n+1} \setminus \{0\}) / F^*, \quad \text{quotient space}$$

A point $y \in \mathbb{P}_n(F)$ has homogeneous coordinates $y_0 : y_1 : \dots : y_n$ where $y_j \in F \;\forall j$, y_j not all zero and $y \sim y'$ iff $\exists\, \lambda \in F^*$ st $y_j = \lambda y_j' \;\forall j$

3. Let $F = \mathbb{R}, \mathbb{C}$ or $\mathbb{H}$. Let

$$\underline{F_1^*} = \{ f \in F \;\; |f| = 1 \}, \quad \text{group wrt multiplication}$$

Define $\underline{S(F^{n+1})} = \{ x \in F^{n+1} : \|x\| = 1 \}$.

Scalar multiplication gives a group action of F_1^* on $S(F^{n+1})$

$$S(F^{n+1}) \times F_1^* \to S(F^{n+1}) : (x, f) \longmapsto fx$$

Define

$$\underline{\mathbb{P}_n(F)} = S(F^{n+1}) / F_1^*.$$

<u>Hopf Fibrations</u>

Using definition 3. above, the Hopf fibrations are:

$$\begin{array}{ccc} F_1^* & \longrightarrow & S(F^{n+1}) \\ & & \downarrow \\ & & \mathbb{P}_n(F) \end{array}$$

Recall that

(a) $\left| \begin{array}{l} S(\mathbb{R}^{n+1}) = S^n \\ \mathbb{R}_1^* = S^0 = \{\pm 1\} \end{array} \right.$ (b) $\left| \begin{array}{l} S(\mathbb{C}^n) = S^{2n+1} \\ \mathbb{C}_1^* = S^1 \end{array} \right.$ (c) $\left| \begin{array}{l} S(\mathbb{H}^n) = S^{4n+3} \\ \mathbb{H}_1^* = S^3 = SU(2) \end{array} \right.$

Thus the Hopf fibrations are:

(a)
$$\begin{array}{ccc} S^0 & \longrightarrow & S^n \\ & & \downarrow \\ & & \mathbb{P}_n(\mathbb{R}) \end{array}$$

(b)
$$\begin{array}{ccc} S^1 & \longrightarrow & S^{2n+1} \\ & & \downarrow \\ & & \mathbb{P}_n(\mathbb{C}) \end{array}$$

(c)
$$\begin{array}{ccc} S^3 & \longrightarrow & S^{4n+3} \\ & & \downarrow \\ & & \mathbb{P}_n(\mathbb{H}) \end{array}$$

<u>Higher Projective Spaces $n > 1$</u>

Apply the long exact homotopy sequence to the above fibrations when $n > 1$

(a)
$$\begin{array}{ccccccc} \pi_1(S^n) & \longrightarrow & \pi_1(\mathbb{P}_n(\mathbb{R})) & \xrightarrow{\partial} & \pi_0(S^0) & \longrightarrow & \pi_0(S^n) \\ \wr\wr & & & & \wr\wr & & \wr\wr \\ 0 \;\because n>1 & & & & \mathbb{Z}_2 & & 0 \end{array}$$

$$\therefore \; \pi_1(\mathbb{P}_n(\mathbb{R})) \cong \mathbb{Z}_2$$

$i > 1$
$$\begin{array}{ccccccc} \pi_i(S^0) & \longrightarrow & \pi_i(S^n) & \longrightarrow & \pi_i(\mathbb{P}_n(\mathbb{R})) & \xrightarrow{\partial} & \pi_{i-1}(S^0) \\ \wr\wr & & & & & & \wr\wr \\ 0 & & & & & & 0 \end{array}$$

$$\therefore \; \pi_i(\mathbb{P}_n(\mathbb{R})) \cong \pi_i(S^n), \quad i > 1$$

(b) $$\begin{array}{ccccccc} \pi_2(S^{2n+1}) & \longrightarrow & \pi_2(\mathbb{P}_n(\mathbb{C})) & \xrightarrow{\partial} & \pi_1(S^1) & \longrightarrow & \pi_1(S^{2n+1}) \longrightarrow \\ \wr\mid & & & & \wr\mid & & \wr\mid \\ 0 \ \because \text{ odd sphere} & & & & \mathbb{Z} & & 0 \quad n>1 \end{array}$$

$$\begin{array}{ccc} \pi_1(\mathbb{P}_n(\mathbb{C})) & \xrightarrow{\partial} & \pi_0(S^1) \\ & & \wr\mid \\ & & 0 \end{array}$$

$$\therefore \quad \pi_1(\mathbb{P}_n(\mathbb{C})) = 0 \ , \quad \pi_2(\mathbb{P}_n(\mathbb{C})) \simeq \mathbb{Z} \quad , n>1$$

$i>2$: $$\begin{array}{ccccccc} \pi_i(S^1) & \longrightarrow & \pi_i(S^{2n+1}) & \longrightarrow & \pi_i(\mathbb{P}_n(\mathbb{C})) & \xrightarrow{\partial} & \pi_{i-1}(S^1) \\ \wr\mid & & & & & & \wr\mid \\ 0 & & & & & & 0 \end{array}$$

$$\therefore \quad \pi_i(\mathbb{P}_n(\mathbb{C})) \simeq \pi_i(S^{2n+1}) \quad , \ i>2 \ , n>1$$

(c) Little is known about the homotopy of the fibre S^3. One result is

$$\pi_4(\mathbb{P}_n(\mathbb{H})) \simeq \pi_3(S^3) = \mathbb{Z} \ , \quad n>1$$

Summary

For $n>1$

$$\left[\begin{array}{l} \pi_1(\mathbb{P}_n(\mathbb{R})) = \mathbb{Z}_2 \\ \pi_i(\mathbb{P}_n(\mathbb{R})) = \pi_i(S^n) \ , \quad i \neq 1 \end{array}\right.$$

$$\left[\begin{array}{l} \pi_2(\mathbb{P}_n(\mathbb{C})) = \mathbb{Z} \\ \pi_i(\mathbb{P}_n(\mathbb{C})) = \pi_i(S^{2n+1}) \ , \quad i \neq 1 \end{array}\right.$$

Projective Lines $n=1$

For the case $n=1$, there are homeomorphisms

$$\mathbb{P}_1(\mathbb{R}) \simeq S^1 \ ; \quad \mathbb{P}_1(\mathbb{C}) \simeq S^2 \ ; \quad \mathbb{P}_1(\mathbb{H}) = S^4$$

This yields the fibrations

(a) $\mathbb{R}$: $S^0 \longrightarrow S^1 \to S^1$ (just double covering)

(b) $\mathbb{C}$: $S^1 \longrightarrow S^3 \to S^2$

(c) $\mathbb{H}$: $S^3 \longrightarrow S^7 \to S^4$

There is another Hopf fibration, called the exceptional Hopf fibration, which is related to the Cayley numbers $\mathbb{K}$:

(d) $\mathbb{K}$: $S^7 \longrightarrow S^{15} \to S^8$

Application of the homotopy exact sequence gives b)

$$\begin{array}{ccccccc} \pi_2(S^3) & \longrightarrow & \pi_2(S^2) & \xrightarrow{\partial} & \pi_1(S^1) & \longrightarrow & \pi_1(S^3) \\ \wr\wr & & & & \wr\wr & & \wr\wr \\ 0 & & & & \mathbb{Z} & & 0 \end{array}$$

$\therefore \quad \pi_2(S^2) = \mathbb{Z}$ as asserted earlier.

More importantly, for $i \geq 3$

$$\begin{array}{ccccccc} \pi_i(S^1) & \longrightarrow & \pi_i(S^3) & \longrightarrow & \pi_i(S^2) & \xrightarrow{\partial} & \pi_{i-1}(S^1) \\ \wr\wr & & & & & & \wr\wr \\ 0 & & & & & & 0 \end{array}$$

$$\therefore \quad \underline{\pi_i(S^3) = \pi_i(S^2)}, \quad i \geq 3$$

Thus S^2 and S^3 have the same homotopy except at $i=2$.

In particular $\pi_3(S^2) = \mathbb{Z}$.

c) and (d)

It may be shown that $S^3 \xrightarrow{i} S^7$, $S^7 \xrightarrow{i} S^{15}$ are homotopic to constant maps. Hence $i_* = 0$ and the long exact sequences yield the short exact sequences

$$0 \longrightarrow \pi_n(S^7) \longrightarrow \pi_n(S^4) \xrightarrow{\partial} \pi_{n-1}(S^3) \longrightarrow 0$$

$$0 \longrightarrow \pi_n(S^{15}) \longrightarrow \pi_n(S^8) \xrightarrow{\partial} \pi_{n-1}(S^7) \longrightarrow 0$$

$(n \geq 1)$.

Infinite Dimensional Projective Spaces

Let $F = \mathbb{R}, \mathbb{C}$ or $\mathbb{H}$. Since

$$\hookrightarrow S(F^n) \hookrightarrow S(F^{n+1}) \hookrightarrow \cdots \hookrightarrow \bigcup_{n \geq 1} S(F^n) = S^\infty,$$

there are Hopf fibrations

$$\begin{array}{ccc} F_1^* & \longrightarrow & S^\infty \\ & & \downarrow \\ & & \mathbb{P}_\infty(F) \end{array}$$

where $\mathbb{P}_\infty(F) = S^\infty / F_1^*$ through the induced action of F_1^* on S^∞. For any fixed i, $\pi_i(S(F^n)) = 0$ for n sufficiently large. Hence $\pi_*(S^\infty) = 0$. The long exact sequence then gives the isomorphisms.

$$\pi_n(\mathbb{P}_\infty(F)) \xrightarrow{\partial} \pi_{n-1}(F_1^*), \quad n \geq 1.$$

Since $\mathbb{R}_1^* = S^0$, $\mathbb{C}_1^* = S^1$, $\mathbb{H}_1^* = S^3$, there are:

$$\boxed{\begin{array}{ll} \pi_1(\mathbb{P}_\infty(\mathbb{R})) = \mathbb{Z}_2 \ ; & \pi_i(\mathbb{P}_\infty(\mathbb{R})) = 0, \ i \neq 1 \ ; \\ \pi_2(\mathbb{P}_\infty(\mathbb{C})) = \mathbb{Z} \ ; & \pi_i(\mathbb{P}_\infty(\mathbb{C})) = 0, \ i \neq 2 \ ; \\ \pi_n(\mathbb{P}_\infty(\mathbb{H})) = \pi_{n-1}(S^3), & n \geq 1. \end{array}}$$

$\mathbb{P}_\infty(\mathbb{R})$ and $\mathbb{P}_\infty(\mathbb{C})$ are used in the classification of line bundles

Milnor's construction

By a method analogous to that producing the Hopf fibrations via infinite dimensional projective spaces, Milnor's construction assigns to a topological group G a fibration

$$\begin{array}{ccc} G & \to & E_G \\ & & \downarrow \\ & & B_G \end{array}$$

where the total space E_G is contractible

Definition

Let X and Y be topological spaces. The join of X and Y, denoted $X * Y$, is the space

$$(X \times [0,1] \times Y)/\sim$$

where $\sim$ is the equivalence relation

$$(x, t, y) \sim (x', t', y') \iff t = t'$$

and if $t = t' > 0$ then $x = x'$
and if $t = t' < 1$ then $y = y'$.

Example:

$$S^p * S^q = S^{p+q+1}$$

so $\overset{n}{*} S^0 = S^{n-1}$ $\qquad \overset{n}{*} S^q = S^{n(q+1)-1}$

$$\overset{n}{*} S^1 = S^{2n-1}$$

A useful notation describing joins may be established as follows.

Let Δ_n be the standard n-simplex:

$$\Delta_n = \{(t_0, \dots t_n) \subset \mathbb{R}^{n+1} \quad t_i \geq 0 \ \forall i, \ \sum_{i=0}^{n} t_i = 1\}$$

Then $X_0 * X_1 \dots * X_n$ is the quotient of $\Delta_n \times X_0 \times \dots \times X_n$ by the relation $\sim$

$$(t_0, \dots t_n; x_0, \dots x_n) \sim (t_0', \dots t_n'; x_0', \dots x_n')$$

$\iff t_j = t_j' \ \forall j$ and if $t_j = t_j' > 0$ then $x_j = x_j'$

Denote the equivalence class of $(t_0, \dots t_n; x_0, \dots x_n)$ by

$$[t_0 x_0, t_1 x_1, \dots t_n x_n]$$

Thus

$$[t_0 x_0, \dots t_n x_n] = [t_0' x_0', \dots t_n' x_n']$$

$\iff t_j = t_j' \ \forall j$ and if $t_j > 0$ then $x_j = x_j'$

Construction of E_G:

Let G be a topological group. Denote:

$$\underline{E_G(n)} = \overset{n+1}{*} G$$

There is an inclusion

$$E_G(n) \hookrightarrow E_G(n+1) : [t_0 x_0, \dots t_n x_n] \mapsto [t_0 x_0, \dots t_n x_n, 0]$$

any gp el $\cdot$ $t_{n+1} = 0$

Denote: $\underline{E_G} = \bigcup_{n \geq 0} E_G(n)$

Using the inclusions, $x \in E_G$ may be represented by

$$[t_0x_0, t_1x_1, \cdots], \quad t_i \geq 0, \quad x_i \in G \ \forall i,$$

where all but a finite number of the t_i are zero and $\sum_{i=0}^{\infty} t_i =$

G acts on E_G via:

$$E_G \times G \to E_G : ([t_kx_k]_k, g) \mapsto [t_k(x_kg)]_k$$

In fact, G acts similarly on each $E_G(n)$ and the inclusions preserve the actions. Let $\underline{B_G(n)} \equiv E_G(n)/G$

Denote by $\underline{B_G}$ the quotient space E_G/G. This gives a fibration:

$$\begin{array}{ccc} G & \longrightarrow & E_G \\ & & \downarrow \\ & & B_G \end{array}$$

$E_G \to B_G$ is called the <u>universal bundle of G</u> and B_G is called the <u>classifying space</u> for the group G.

Moreover, E_G is contractible via:

$$h : E_G \times [0,1] \to E_G \ : \ h([t_kx_k], s) = [(st_k)x_k]_k$$

Thus application of the long exact sequence yields:

$$\boxed{\pi_n(G) \cong \pi_{n-1}(B_G)}$$

<u>Relation to Hopf Fibrations.</u>

Let $F = \mathbb{R}, \mathbb{C}, \mathbb{H}$. Then

$$G = F_1^*$$

$$E_G(n) = S(F^{n+1}) \quad : \quad E_G = S^\infty \qquad \text{action} = \text{scalar multiplication}$$

$$B_G(n) \equiv E_G(n)/G = \mathbb{P}_n(F)$$

$$B_G = \mathbb{P}_\infty(F).$$

and, as before,

$$\pi_n(F_1^*) \cong \pi_{n-1}(\mathbb{P}_\infty(F))$$

Explicitly

$$B_{\mathbb{Z}_2} = B_{S^0} = \mathbb{P}_\infty(\mathbb{R})$$

$$B_{S^1} = \mathbb{P}_\infty(\mathbb{C})$$

$$B_{SU(2)} = B_{S^3} = \mathbb{P}_\infty(\mathbb{H})$$