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SUMMER WORKSHOP ON FIBRE BUNDLES AND GEOMETRY

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ALGEBRAIC TOPOLOGY

III. Homology and Cohomology

IV. Homotopy Classification of Principal and Vector Bundles

V. Characteristic Classes

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ALGEBRAIC TOPOLOGY III

HOMOLOGY AND COHOMOLOGY

§1 GRADED MODULES

Let R be a commutative ring. Let (R) denote the category of R-modules and R-module homomorphisms.

Definition:

A graded R-module is a $\mathbb{Z}$-indexed collection $(M_n)_{n\in\mathbb{Z}}$ where M_n is an R-module $\forall\, n$. A morphism of graded R-modules $M \xrightarrow{f} N$ is a collection $(M_n \xrightarrow{f_n} N_n)_{n\in\mathbb{Z}}$ of R-module homomorphisms.

Notation

1. Let $Gr(R)$ denote the category of graded R-modules and graded R-module homomorphisms. Let $Gr^{\pm}(R)$ denote the full subcategories whose objects L satisfy $L_n = 0$, $n \lessgtr 0$.

2. (Upper Index Convention): For a graded R-module M, let $M^n = M_{-n}$ $\forall\, n$. Thus $Gr^-(R)$ has as objects graded R-modules M for which $M^n = 0$, $n < 0$. Hence only positive indices need be considered.

There is an identification $(R) \equiv Gr^+(R) \cap Gr^-(R)$

Hom_R and $\otimes_R$

There are two important functors associated to (R)

(a) $\underline{Hom_R}$: $(R)^{op} \times (R) \to (R)$:

$$(M, N) \longmapsto Hom_{(R)}(M,N) \text{ , R-linear maps } M \to N$$

(b) $\underline{\otimes_R}$: $(R) \times (R) \to (R)$:

$$(M, N) \longmapsto M \otimes_R N \text{ , tensor product .}$$

From the universal property of the tensor product, there is an identification:

$$(*) \quad \boxed{Hom_{R}(L \otimes_R M, N) \simeq Hom_R(L, Hom_R(M,N))}$$

Hom_R and $\otimes_R$ may be extended to $Gr(R)$ as follows:

(a) $\underline{Hom_R}$: $Gr(R)^{op} \times Gr(R) \to Gr(R)$:

$$(M, N) \mapsto Hom_R(M,N)$$

$$(Hom_R(M,N))_n = \prod_k Hom_R(M_k, N_{k+n}) ,$$

the R-module of sequences $(f_k)_k$, $M_k \xrightarrow{f_k} N_{k+n}$

In particular, $(Hom_R(M,N))_0 = Hom_{Gr(R)}(M,N)$

(b) $\underline{\otimes_R}$: $Gr(R) \times Gr(R) \to Gr(R)$:

$$(M, N) \longmapsto M \otimes_R N$$

$$(M \otimes_R N)_n = \coprod_k L_k \otimes_R M_{n-k}$$

It is not difficult to check that the identification $(*)$ said holds

Definition

Let L, L', M, M' be graded R-modules and $L \xrightarrow{f} L'$, $M \xrightarrow{g} M'$ of degrees p and q respectively. Define

$$\underline{f \otimes g} \quad L \otimes_R M \to L' \otimes_R M' .$$

$$f \otimes g\,(x \otimes y) = (-1)^{lq} f(x) \otimes g(y)$$

when $x \otimes y \in L_l \otimes M_m$

Definition.

The $\underline{\text{graded dual module}}$ $\underline{\check{M}}$ of a graded R-module M is the graded R-module $Hom_R(M, R)$ where R here denotes the graded R-module

$$\begin{cases} R_n = 0 , & n \neq 0 \\ R_0 = R \end{cases}$$

Thus :
$$\check{M}_n = \prod_k Hom_R(M_k, R_{n+k})$$

$$\cong Hom_R(M_{-n}, R)$$

Using the upper index convention:

$$\check{M}^n = Hom_R(M_n, R)$$

§2: HOMOLOGY AND COHOMOLOGY AXIOMS:

Pairs of Spaces

A *pair of spaces* is a couple (X, A) where A is a subspace of the space X. A *map of pairs* $(X,A) \xrightarrow{f} (Y,B)$ is a map $X \xrightarrow{f} Y$ st $f(A) \subseteq B$, that is, $f \in Map(X,A\,;Y,B)$.
The category of pairs of spaces and pair maps is denoted $(top)_2$. There are inclusions

(a) $(top) \hookrightarrow (top)_2$: $X \longmapsto (X, \phi)$

(b) $(top)_0 \hookrightarrow (top)_2$: $X \longmapsto (X, *)$

Definition:

Two pair maps $f, g : (X,A) \to (Y,B)$ are *homotopic as pair maps* if $\exists$ $h : X \times [0,1] \to Y$ st

1. $h(x,0) = f(x)$; $h(x,1) = g(x)$ $\forall\, x \in X$;
2. $h(A \times [0,1]) \subseteq B$

Thus h gives a path from f to g in $Map(X,A\,;Y,B)$.
The relation is denoted $\simeq_2$ and is an equivalence relation on $Map(X,A\,;Y,B)$. The equivalence classes are denoted by $[X,A\,;Y,B]_2$. Let $[top]_2$ denote the category whose objects are pairs of spaces and whose morphisms are the equivalence classes wrt $\simeq_2$. There are projections and inclusions

$$\begin{tikzcd}
(top) \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\copyright"] & (top)_2 \arrow[d] \arrow[dr, phantom, "\copyright"] & (top)_0 \arrow[l, hook'] \arrow[d] \\
{[top]} \arrow[r, hook] & {[top]_2} & {[top]_0} \arrow[l, hook']
\end{tikzcd}$$

©: natural equivalence

Homology and Cohomology Functors:

Let R be a commutative ring. A *homology (resp. cohomology) theory* is a functor

$$H_* : (top)_2 \times (R) \longrightarrow Gr^+(R)$$

(resp. $H^* : (top)_2^{op} \times (R) \longrightarrow Gr^-(R)$)

satisfying axioms A1 - A4 listed below. Before giving the axioms, it is convenient to introduce some notation.

(a) For $M = R$, denote

$$H_*(X,A) \equiv H_*(X,A\,;R)$$

$$H^*(X,A) \equiv H^*(X,A\,;R)$$

b) For:

$$\begin{array}{ccccc}
(X,A) & & H_*(X,A\,;M) & & H^*(X,A,M) \\
\downarrow f & : & \downarrow H_*(f) = f_* & \text{and} & \uparrow H^*(f) = f^* \\
(Y,B) & & H_*(Y,B\,;M) & & H^*(Y,B,M)
\end{array}$$

(c) Denote:

$$H_*(X;M) \equiv H_*(X,\phi;M)$$
$$H^*(X;M) \equiv H^*(X,\phi;M)$$

(d) If X is a pointed space, the <u>reduced (co) homology</u> of X is

$$\widetilde{H}_*(X;M) \equiv H_*(X,*;M)$$
$$\widetilde{H}^*(X;M) \equiv H^*(X;*;M)$$

The axioms are:

<u>(A1)</u> <u>Homology Axiom:</u>

There is a factorisation of H_* (H^*):

$$\begin{tikzcd} (\mathrm{top})_2 \times (R) \arrow[r, "H_*"] \arrow[d] & Gr^+(R) \\ {[\mathrm{top}]_2} \times (R) \arrow[ur, dashed, "H_*"'] & \end{tikzcd} \qquad \begin{tikzcd} (\mathrm{top})_2^{op} \times (R) \arrow[r, "H^*"] \arrow[d] & Gr^-(R) \\ {[\mathrm{top}]_2^{op}} \times (R) \arrow[ur, dashed, "H^*"'] & \end{tikzcd}$$

<u>(A2)</u> <u>Exactness Axiom:</u>

Given any pair (X,A), consider the inclusions

$$(A,\phi) \xrightarrow{i} (X,\phi) \xrightarrow{j} (X,A)$$

There exists $\underline{\partial}$ of degree -1 ($\underline{\delta}$ of degree $+1$) such the the triangle below is exact.

$$\begin{tikzcd} H_*(A;M) \arrow[r, "i_*"] & H_*(X,M) \arrow[d, "j_*"] \\ & H_*(X,A,M) \arrow[ul, "\partial"] \end{tikzcd} \qquad \begin{tikzcd} H^*(A;M) \arrow[dr, "\delta"'] & H^*(X;M) \arrow[l, "i^*"'] \\ & H^*(X,A;M) \arrow[u, "j^*"'] \end{tikzcd}$$

<u>(A3)</u>: <u>Excision Axiom:</u>

For any pair (X,A), consider the projection:

$$(X,A) \xrightarrow{\pi} (X/A,\ * = [A])$$

π induces an isomorphism:

$$H_*(X,A;M) \xrightarrow{\pi_*} H_*(X/A,*,M) \equiv \widetilde{H}_*(X/A;M) \qquad H^*(X,A;M) \xrightarrow{\pi^*} H^*(X/A,*;M) \equiv \widetilde{H}^*(X/A;M)$$

<u>(A4)</u>: <u>Dimension Axiom:</u>

$$H_*(pt,\phi;M) = \begin{bmatrix} M & \text{, degree } 0 \\ 0 & \text{, else} \end{bmatrix} = H^*(pt,\phi,M)$$

<u>Note</u>:

A (co) homology functor satisfying A1 – A3 is called a generalised (co) homology theory.

<u>Reduced (Co) homology:</u>

<u>1</u> Consider:

$$\begin{tikzcd} (*,\phi) \arrow[dr] \arrow[dd, "id"'] & \\ & (X,\phi) \arrow[dl] \arrow[l, phantom, "\circledcirc"] \\ (*,\phi) & \end{tikzcd}$$

From the functorial properties of H_* (H^*):

and:

$$H_*(X;M) \cong H_*(*,M) \oplus \tilde{H}_*(X;M)$$

$$H^*(X;M) \cong H^*(*;M) \oplus \tilde{H}^*(X,M)$$

These isomorphisms hold in generalised theories.

2. Applying A2 and A3 to a pair (X,A) where $* \in A$ gives the exact triangles

$$\begin{tikzcd} \tilde{H}_*(A;M) \arrow[rr] & & \tilde{H}_*(X,M) \arrow[dl] \\ & \tilde{H}_*(X/A,M) \arrow[ul, "\partial\ (-1)"] & \end{tikzcd} \qquad \begin{tikzcd} \tilde{H}^*(A,M) \arrow[dr, "\delta\ (1)"'] & & \tilde{H}^*(X,M) \arrow[ll] \\ & \tilde{H}^*(X/A,M) \arrow[ur] & \end{tikzcd}$$

3. From 1:

$$\tilde{H}_*(pt;M) = 0$$
$$\tilde{H}^*(pt;M) = 0$$

so, by A1, if X is contractible,

$$\tilde{H}_*(X;M) = 0$$
$$\tilde{H}^*(X;M) = 0$$

Suspension Isomorphisms

Let X be a pointed space and consider the pair $(C(X), X)$. Recall that $X \hookrightarrow C(X) \to C(X)/X = S(X)$. Thus 2. above gives exact triangles

$$\begin{tikzcd} \tilde{H}_*(X,M) \arrow[rr] & & \tilde{H}_*(C(X),M) \arrow[dl] \\ & \tilde{H}_*(S(X),M) \arrow[ul, "\partial"] & \end{tikzcd} \qquad \begin{tikzcd} \tilde{H}^*(X,M) \arrow[dr, "\delta"'] & & \tilde{H}^*(C(X),M) \arrow[ll] \\ & \tilde{H}^*(S(X),M) \arrow[ur] & \end{tikzcd}$$

But $C(X)$ is contractible so by A1 and 3. above

$$\tilde{H}_*(C(X);M) = 0 \quad , \quad \tilde{H}^*(C(X),M) = 0$$

Exactness $\Rightarrow$ ∂, δ are isomorphisms; their inverses are denoted $\sigma_*(X)$ and $\sigma^*(X)$ respectively, called the homology and cohomology suspension isomorphisms:

$$\boxed{\begin{array}{ccc} \tilde{H}_*(X;M) & \underset{\partial\ (-1)}{\overset{\sigma_*(X)\ (+1)}{\rightleftarrows}} & \tilde{H}_*(S(X);M) \\ \tilde{H}^*(S(X);M) & \underset{\delta\ (+1)}{\overset{\sigma^*(X)\ (-1)}{\rightleftarrows}} & \tilde{H}^*(X;M) \end{array}}$$

Application: Homology and Cohomology of Spheres

Consider $S^0 = \{\pm 1\}$. Then

$$H_*(S^0;M) = M \oplus M \text{ in degree zero} \quad 0 \text{ else}$$

and $H^*(S^0;M) = M \oplus M$ in degree zero, 0 else

Hence both $\tilde{H}_*(S^0;M)$ and $\tilde{H}^*(S^0;M)$ are 0 except in degree zero where they are isomorphic to M.

Recall that $S^n = S^n(S^0)$, the n-fold suspension of S^0.

Using the suspension isomorphism

$$\tilde{H}_j(S^n;M) = \begin{cases} M, & j = n \\ 0, & j \neq n \end{cases}$$

$$\tilde{H}^j(S^n;M) = \begin{cases} M, & j = n \\ 0, & j \neq n \end{cases}$$

Using 1. and A4:

$$\boxed{H_j(S^n,M) = \begin{cases} M, & j = 0 \text{ or } n \\ 0, & j \neq 0 \text{ or } n \end{cases} = H^j(S^n,M)}$$

§3 SINGULAR HOMOLOGY AND COHOMOLOGY

Homology and cohomology theories have been constructed using several different methods. The most common are:

1. Singular theory,
2. Čech theory,
3. Sheaf theory (cohomology only),
4. Simplicial theory.

In general, these theories are different. However, they all coincide on certain spaces called CW complexes. Most spaces commonly encountered are CW complexes. Singular theory is defined in the sequel.

Singular Homology

Let Δ_n denote the standard affine n-simplex in $\mathbb{R}^{n+1}$:

$$\Delta_n = \{(t_0, \dots, t_n) \in \mathbb{R}^{n+1} : t_j \geq 0 \ \forall j, \ \sum_{j=0}^{n} t_j = 1\}$$

Define the i^{th} face map on Δ_n to be:

$$\delta_i : \Delta_{n-1} \to \Delta_n : \delta_i(t_0, \dots, t_{n-1}) = (t_0, \dots, t_{i-1}, \underset{(i)}{0}, t_i, \dots, t_{n-1})$$

where $0 \leq i \leq n$.

Lemma:

For $0 \leq i < j \leq n$:

$$\begin{tikzcd} \Delta_{n-2} \arrow[r, "\delta_i"] \arrow[d, "\delta_{j-1}"'] \arrow[dr, phantom, "\circledcirc"] & \Delta_{n-1} \arrow[d, "\delta_j"] \\ \Delta_{n-1} \arrow[r, "\delta_i"'] & \Delta_n \end{tikzcd}$$

The proof is immediate.

Definition:

Let R be a commutative ring and X a topological space. Define $C_q(X; R)$ to be the free R-module with basis $\mathrm{Map}(\Delta_q, X)$.

A basis element, that is, a map $\Delta_q \to X$, is called a singular q-simplex in X.

For $\sigma \in \mathrm{Map}(\Delta_q, X)$, define

$$d\sigma = \sum_{j=0}^{q} (-1)^j (\sigma \circ \delta_j) \quad \in \quad C_{q-1}(X, R)$$

Since $\mathrm{Map}(\Delta_q, X)$ generates $C_q(X;R)$ freely, d extends by linearity to a unique homomorphism.

$$d \quad C_q(X;R) \longrightarrow C_{q-1}(X;R)$$

called the boundary morphism. Moreover:

Lemma:

For $C_q(X;R) \xrightarrow{d} C_{q-1}(X;R) \xrightarrow{d} C_{q-2}(X,R)$, $d^2 = 0$

The proof is a standard calculation using the previous lemma.

For $q < 0$, let $C_q(X, R)$ be the zero R-module and let $d = 0$. Thus $(C_*(X;R), d)$ is a chain complex. Define $H_*(X, R)$ to be the homology of this complex. Thus

$$H_q(X, R) = \frac{\ker\left(C_q(X,R) \xrightarrow{d} C_{q-1}(X,R)\right)}{\operatorname{im}\left(C_{q+1}(X,R) \xrightarrow{d} C_q(X,R)\right)}$$

For an R-module M, define

$$C_*(X; M) = C_*(X, R) \otimes_R M$$

For $A \subseteq X$, define

$$C_*(X, A; M) = \frac{C_*(X, M)}{C_*(A; M)}$$

$C_*(X, A; M)$ is a complex in the difference inherited from X. Define $H_*(X, A, M)$ to be the homology of this complex.

Singular Cohomology

Define the R-module of q-cochains to be the dual module:

$$C^q(X, A, M) = \hom_R(C_q(X, A, R), M)$$

The transpose of

$$C_{q+1}(X, A; R) \xrightarrow{d} C_q(X, A, R) \xrightarrow{d} C_{q-1}(X, A, R)$$

is

$$C^{q+1}(X, A; M) \xleftarrow{\delta} C^q(X, A; M) \xleftarrow{\delta} C^{q-1}(X, A; M)$$

Define the cohomology

$$\underline{H^q(X, A; M)} = \frac{\ker\,(C^q \xrightarrow{\delta} C^{q+1})}{\operatorname{im}\,(C^{q-1} \xrightarrow{\delta} C^q)}$$

Having defined H_* and H^*, it is now necessary to check the functorial properties and to verify that the axioms A1 – A4 are satisfied.

<u>Functoriality</u>:

$$\begin{tikzcd} (X, A) \arrow[d, "f"] \\ (Y, B) \end{tikzcd} \Rightarrow \begin{tikzcd} \mathrm{Map}(\Delta_q, X) \arrow[d] \\ \mathrm{Map}(\Delta_q, Y) \end{tikzcd} \overset{\text{frames}}{\Rightarrow} \begin{tikzcd} C_*(X,R) \arrow[d, "C_*(f)"] \arrow[r, phantom, "\supset"] & C_*(A,R) \arrow[d] \\ C_*(Y,R) \arrow[r, phantom, "\supset"] & C_*(B,R) \end{tikzcd}$$

(map of chain complexes)

Hence

$$\begin{tikzcd} C_*(X,A;M) \arrow[d, "C_*(f)"] \\ C_*(Y,B;M) \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} C^*(X,A,M) \\ C^*(X,A,M) \arrow[u, "C^*(f)"'] \end{tikzcd}$$

maps of (co) chain complexes

General principles $\Rightarrow$

$$\begin{tikzcd} H_*(X,A;M) \arrow[d, "H_*(f) = f_*"] \\ H_*(Y,B;M) \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} H^*(X,A;M) \\ H^*(Y,B,M) \arrow[u, "H^*(f) = f^*"'] \end{tikzcd}$$

and the functor rules are easily verified.

<u>Axioms</u>

<u>A1</u>: Homotopic maps $(X, A) \to (Y, B)$ induce chain homotopic maps on the C_* and C^* complexes and thus isomorphisms on the homology and cohomology.

<u>A2</u>: There are exact sequences of complexes (for a pair (X,A))

$$0 \to C_*(A;M) \to C_*(X;M) \to C_*(X;A;M) \to 0$$

and

$$0 \leftarrow C^*(A,M) \leftarrow C^*(X;M) \leftarrow C^*(X,A;M) \leftarrow 0$$

General principles imply the required exact triangles

<u>A3</u>: See Spanier or Greenberg.

<u>A4</u>: This is easily verified since

$$C_q(pt; R) = \begin{cases} R, & q \geq 0 \\ 0, & q < 0 \end{cases}$$

there being but one q-chain for each $q \geq 0$.

$$\boxed{\therefore\ H_* \text{ and } H^* \text{ are homology and cohomology theories}}$$

§4. CUP PRODUCT

Two products frequently arise in homology and cohomology theories:

1. External Homology Product

$$H_*(X,A;M)\otimes_R H_*(Y,B,N)\to H_*(X\times Y, X\times B\cup A\times Y; M\otimes_R N).$$

2. Cup Product in Cohomology

$$H^*(X,A;R)\otimes_R H^*(X,A;R)\to H^*(X,A;R)$$

$$a\in H^p,\ b\in H^q:\quad a\otimes b\in H^p\otimes H^q\mapsto a\cup b\in H^{p+q}$$

Definition of Cup Product for Singular Cohomology.

Define

$$\lambda:\Delta_p\to\Delta_{p+q}:\quad \lambda(t_0,\dots t_p)=(t_0,\dots t_p,0,\dots 0)$$

$$\mu:\Delta_q\to\Delta_{p+q}:\quad \mu(t_0,\dots t_q)=(0,\dots 0,t_0,\dots t_q)$$

If $\sigma\in Map(\Delta_{p+q},X)$, a singular $p+q$ simplex, then

$$\sigma\lambda\in Map(\Delta_p,X)\quad\text{and}\quad \sigma\mu\in Map(\Delta_q,X).$$

Recall that since $Map(\Delta_n,X)$ is a basis of $C_n(X;R)$, $w\in C^n(X;R)=Hom_R(C_n(X;R),R)$ is determined by its values on $Map(\Delta_n,X)$. Let $u\in C^p(X;R)$, $v\in C^q(X,R)$ and suppose $\sigma\in C^{p+q}(X;R)$. Define

$$(u\cup v)(\sigma)=u(\sigma\lambda)\cdot v(\sigma\mu)$$

For the coboundary morphism

$$\delta: C^{p+q}(X,R)\to C^{p+q+1}(X,R)$$

it is easy to check that:

$$\delta(u\cup v)=(\delta u)\cup v+(-1)^p u\cup(\delta v)$$

In particular, if u,v are cocycles, $\delta u=0$, $\delta v=0$, then $\delta(u\cup v)=0$ so $u\cup v$ is a cocycle. Moreover

$$(u+\delta u')\cup(v+\delta v')=u\cup v+\delta u'\cup v+u\cup\delta v'+\delta u'\cup\delta v'$$

$$=u\cup v+\delta(u'\cup v+u\cup v'+u'\cup\delta v')$$

$$\because\ \delta u=\delta v=0$$

Thus $\cup$ is well-defined on cohomology via

$$[u]\cup[v]=[u\cup v]\quad\forall\,[u]\in H^p,\ [v]\in H^q.$$

The operation

$$\cup\quad H^p(X;R)\otimes_R H^q(X,R)\to H^{p+q}(X,R)$$

is called the cup product. Where there is no ambiguity, $u\cup v$ is denoted uv.

Lemma:

1. The cup product is associative: $a(bc)=(ab)c$
2. The cup product is graded commutative: $ab=(-1)^{pq}ba$

where $a \in H^p$, $b \in H^q$.

3 $H^*(X, R)$ has a unit wrt the cup product.

Corollary

$H^*(X, R)$ is a graded algebra.

Examples:

1. $H^*(S^n, R) = E(x_n)$, the exterior algebra in one generator x_n in degree n.

2. (Application to $\mathbb{R}$-vector bundles)

$H^*(\mathbb{P}_\infty(\mathbb{R}), \mathbb{F}_2) = \mathbb{F}_2[x_1]$, polynomial ring over $\mathbb{F}_2$ in one variable x_1 in degree 1.

$H^*(\mathbb{P}_n(\mathbb{R}), \mathbb{F}_2) = \mathbb{F}_2[x_1] / (x_1^{n+1} = 0)$, a quotient ring.

Moreover, the inclusion

$$\mathbb{P}_m(\mathbb{R}) \hookrightarrow \mathbb{P}_n(\mathbb{R}), \quad m \le n$$

induces an algebra homomorphism

$$H^*(\mathbb{P}_m(\mathbb{R}), \mathbb{F}_2) \longleftarrow H^*(\mathbb{P}_n(\mathbb{R}), \mathbb{F}_2)$$

ie

$$x_1^r \longmapsto \begin{cases} x_1^r, & r \le m \\ 0, & r > m \end{cases}$$

Also $H^*(\mathbb{P}_n(\mathbb{R}), \mathbb{F}_2) = \overset{(0)}{\mathbb{F}_2} \oplus \overset{(1)}{\mathbb{F}_2 x_1} \oplus \cdots \oplus \overset{(n)}{\mathbb{F}_2 x_1^n}$

3 (Application to $\mathbb{C}$-vector bundles)

$H^*(\mathbb{P}_\infty(\mathbb{C}), R) = R[x_2]$, polynomial ring in one variable x_2 in degree 2.

$H^*(\mathbb{P}_n(\mathbb{C}), R) = R[x_2] / (x_2^{n+1} = 0)$, quotient ring.

4 $H^*(\mathbb{P}_\infty(\mathbb{H}), R) = R[x_4]$, polynomial ring in one variable x_4 in degree 4.

$H^*(\mathbb{P}_n(\mathbb{H}), R) = R[x_4] / (x_4^{n+1} = 0)$, quotient ring.

ALGEBRAIC TOPOLOGY IV

HOMOTOPY CLASSIFICATION OF PRINCIPAL AND VECTOR BUNDLES

§1: PRINCIPAL BUNDLES

Definition

1. A topological group is a group G having the structure of a topological space and such that the map

$$G \times G \longrightarrow G : (s,t) \longmapsto st^{-1}$$

is continuous.

2. Let G be a topological group. A space X is called a right G-space if there is a continuous map

$$X \times G \longrightarrow X : (x,g) \mapsto xg$$

satisfying (i) $x1 = x \quad \forall\ x \in X$;

(ii) $(xg)h = x(gh) \quad \forall\ x \in X,\ g,h \in G$.

3. Let X be a right G-space. The orbit of $x \in X$ is the subset $xG \subset X$.

4. Define $x \sim y \iff \exists\ g \in G$ st $xg = y$. Then $\sim$ is an equivalence relation on X. The quotient space of X by $\sim$ is denoted X/G and is called the space of orbits.

5. The stabiliser or isotropy subgroup of $x \in X$ is the subgroup $\{g \in G : xg = x\} \subseteq G$, denoted G_x. In

particular, G is said to act **freely** on X if $G_x = \{1\}$ $\forall x \in X$.

6. Suppose G acts freely on X to the right. Denote:

$$X^* = \{(x, xg) : x \in X, g \in G\} \subset X \times X$$

Define $\underline{T}$: $X^* \to G$: $x_1 T(x_1, x_2) = x_2$.

T is well-defined since the G-action is free.

7. A principal G-space X is a right G-space X st the G-action is free and $T : X^* \to G$ is continuous.

Notes

1. If X is a right G-space then $\forall g \in G$,

$$R_g : X \to X : R_g(x) = xg$$

is a homeomorphism.

2. Every right G-space X is canonically a left G-space and vice versa by

$$\underset{(\text{right})}{x \cdot g} = \underset{(\text{left})}{g^{-1} \cdot x}$$

Definition:

Let G be a topological group. A **principal G-bundle** is a bundle $E \xrightarrow{p} B$ where E is a principal G-space st.

$$\begin{tikzcd} & E \arrow[dl, "\pi"'] \arrow[dr, "p"] \arrow[d, phantom, "\circlearrowleft"] & \\ E/G \arrow[rr, "\cong"] & & B \end{tikzcd}$$

Note:

Explicitly, a principal G-bundle is a right G-space E and a bundle $E \xrightarrow{p} B$ st

(i) $p(x) = p(x') \Leftrightarrow \exists g \in G$ st $xg = x'$. Moreover, g is unique and denoted $T(x, x')$;

(ii) p is a quotient map, that is, $V \subset B$ open $\Leftrightarrow$ $p^{\leftarrow} V \subset E$ open;

(iii) $T : E \times_B E \to E$ is continuous.

Definition:

1. A **morphism of principal G-bundles** $E \xrightarrow{p} B$ and $E' \xrightarrow{p'} B'$ is a pair of maps (u, f) st

$$\begin{tikzcd} E \arrow[r, "u"] \arrow[d, "p"'] \arrow[dr, phantom, "\circlearrowleft"] & E' \arrow[d, "p'"] \\ B \arrow[r, "f"'] & B' \end{tikzcd}$$

and $\quad u(xg) = u(x)g \quad \forall\, x \in E,\ g \in G$

2. When $B = B'$ and $f = id_B$ then u is called a <u>B-morphism</u> of principal G-bundles.

$$\begin{tikzcd} E \arrow[rr, "u"] \arrow[dr, "p"'] & \arrow[d, phantom, "\circlearrowright"] & E' \arrow[dl, "p'"] \\ & B & \end{tikzcd}$$

<u>Lemma</u>

The fibres of a principal G-bundle are each homeomorphic to G.

<u>Proof</u>:

Let $E \xrightarrow{p} B$ be a principal G-bundle, $b \in B$. Choose $x \in p^{-1}(b)$. Define

$$\phi_x : G \to p^{-1}(b) \qquad \phi_x(g) = xg$$

Then ϕ_x is a homeomorphism with inverse:

$$\phi_x^{-1} : p^{-1}(b) \to G \qquad y \longmapsto \tau(x,y)$$

<u>QED</u>

<u>Examples of Principal Bundles</u>

1. <u>Product Bundle</u>:

Let G be a topological group and B a space. Let $E = B \times G$ and $E \xrightarrow{p} B$ $\quad p = p_B$. Define the right G-action

$$E \times G \to E \qquad ((b,g),h) \longmapsto (b, gh)$$

Clearly this action is free. Moreover

$$\tau : E^{\#} \to G \qquad \tau((b,g),(b,h)) = g^{-1}h \text{ , continuous.}$$

Therefore $B \times G \xrightarrow{p} B$ is a principal G-bundle called the <u>product</u> (trivial) <u>principal G-bundle</u> with base-space B.

2. <u>Universal Principal G-Bundle</u>:

Recall the Milnor construction from II.6 which gave a fibration

$$\begin{array}{ccc} G & \to & E_G \\ & & \downarrow \\ & & B_G \end{array}$$

for each topological group G. B_G was the quotient of E_G with respect to the right G-action:

$$E_G \times G \to E_G \qquad ([t_r g_r]_{r \geq 0}, h) \mapsto [t_r(g_r h)]_{r \geq 0}$$

This action is free since:

$$[t_r g_r] = [t_r g_r]\, g \iff$$

$\forall r$, either $t_r = 0$

or $t_r > 0$ and $g_r = g_r g$

But $\sum_r t_r = 1 \Rightarrow$ at least one $t_r \neq 0$ so $g = 1$.

It remains to show that the τ function is continuous $\forall j \geq$ define the open set $W_j = \{x = [t_i, g_i] \in E_G : t_j \neq 0\}$. Then -

$\{(W_j \times W_j) \cap E_G^*\}_{j>0}$ is an open cover of E_G^* because

$(x, y) \in E_G^*$, $x = [t, g_r]$, $y = [s, h_r]$ $\iff$

$[t, g_r]\, g = [s, h_r]$ for $g \in G$ $\iff$

$[t, (g_r g)] = [s, h_r]$

Now $\sum_r t_r = 1 \Rightarrow$ at least one $t_j \gneqq 0$ and so equality implies $t_j = s_j \gneqq 0$. Thus $(x, y) \in (W_j \times W_j) \cap E_G^*$

But τ restricts to

$$\tau| : (W_j \times W_j) \cap E_G^* \to G \; : \; \tau|([t, g_r], [s, h_r]) = g_j^{-1} h_j$$

which is continuous $\forall\, j$. Since the restriction of τ to every set in an open cover is continuous, τ itself is continuous.

Therefore $E_G \to B_G$ is a principal G-bundle called the universal principal G-bundle. The meaning of universal will become clear later.

3. Stiefel Varieties

Let $F = \mathbb{R}, \mathbb{C}, \mathbb{H}$ and suppose F^N has the scalar product:

$$(x \mid y) = \sum_{i=1}^{N} x^i \bar{y}^i$$

Define the Stiefel varieties:

$$V_n'(F^N) = \{ (v_1 \dots v_n) \in \textstyle\times_{i=1}^{n} F^N : v_1 \dots v_n \text{ linearly independent} \}$$

$$\uparrow$$

$$V_n(F^N) = \{ (v_1 \dots v_n) \in \textstyle\times_{i=1}^{n} F^N : v_1 \dots v_n \text{ orthonormal} \}$$

$V_n'(F^N)$ and $V_n(F^N)$ inherit topologies from the inclusions

$$V_n'(F^N),\; V_n(F^N) \subset F^{Nn}$$

In particular, $V_n(F^N)$ is compact. Define the right actions

$$V_n'(F^N) \times GL_n(F) \to V_n'(F^N)$$

$$V_n(F^N) \times U_n(F) \to V_n(F^N)$$

both given by

$$(v_1 \dots v_n)(A) \longmapsto (w_1 \dots w_n)$$

$$\text{where} \quad w_j = \sum_{i=1}^{n} v_i A_{ij}$$

Viewing A as a transition matrix for change of basis in n-dimensional subspaces of F^N, it is seen that these actions are free and that $V_n'(F^N)$ and $V_n(F^N)$ are principal $GL_n(F)$ and $U_n(F)$ spaces respectively. This yields

i) Principal $GL_n(F)$ bundle:

$$\begin{tikzcd} GL_n(F) \arrow[r] & V_n'(F^N) \arrow[d] \\ & G_n(F^N) \end{tikzcd}$$

, Grassmannian

ii) Principal $U_n(F)$ bundle:

$$\begin{tikzcd} U_n(F) \arrow[r] & V_n(F^N) \arrow[d] \\ & G_n(F^N) \end{tikzcd}$$

In particular, for $n = 1$, (11) is the Hopf fibration

$$\begin{array}{ccc} S(F) = U_1(F) & \longrightarrow & S(F^{N+1}) = V_1(F^{N+1}) \\ & & \downarrow \\ & & \mathbb{P}_N(F) = G_1(F^{N+1}) \end{array}$$

and letting $N \to \infty$ gives the Milnor construction for the groups $U_1(F)$:

$$\begin{array}{ccc} U_1(F) & \longrightarrow & S(F^\infty) = V_1(F^\infty) = E_{U_1(F)} \\ & & \downarrow \\ & & \mathbb{P}_\infty(F) = G_1(F^\infty) = B_{U_1(F)} \end{array}$$

The rigidity of the concept of principal bundle is expressed in the following

Lemma:

Let $E \xrightarrow{u} E'$ be a B-morphism of principal G-bundles. Then u is an isomorphism.

Proof:

That u is a bijection follows easily from the definition of principal bundles. The continuity of u^{-1} is deduced from that of T. See FB pg 42 for the complete proof.

QED

Notation

Let $\underline{k_G^1(B)}$ denote the B-isomorphism classes of principal G-bundles over B.

Induced Principal Bundles

Let $E \xrightarrow{p} B$ be a principal G-bundle and suppose $B' \xrightarrow{f} B$ is a continuous map. Form the induced bundle

$$\begin{array}{ccc} E' = B' \times_B E & \xrightarrow{u} & E \\ p' \downarrow & \circledcirc & \downarrow p \\ B' & \xrightarrow{f} & B \end{array} \qquad \begin{array}{l} p'(b', x) = b' \\ u(b', x) = x \end{array}$$

Define the right G-action:

$$E' \times G \to E' : \quad ((b', x), g) \mapsto (b', xg)$$

This is well-defined since

$$f(b') = p(x) = p(xg), \quad G \text{ preserving fibres of } E.$$

The action is free since:

$$(b', x)\, g = (b', x) \iff xg = x \iff g = 1 .$$

Finally, $T' : E'^{\#} \to G : \; T'((b, x_1), (b, x_2)) = T(x_1, x_2)$, continuous.

Therefore, $E' \xrightarrow{p'} B'$ has the structure of a principal G-bundle, called the <u>principal G-bundle induced from $E \xrightarrow{p} B$ via f</u>. E' is denoted by $\underline{f^* E}$.

Moreover, (u, f) is a morphism of principal G-bundles by definition of the G-structure of E'.

<u>Note</u>:

Let
$$\begin{tikzcd} E_1 \arrow[r, "u"] \arrow[d, "p_1"'] \arrow[dr, phantom, "\copyright"] & E_2 \arrow[d, "p_2"] \\ B_1 \arrow[r, "f"] & B_2 \end{tikzcd}$$
be a morphism of principal G-bundles $E_1 \xrightarrow{p_1} B_1$, $E_2 \xrightarrow{p_2} B_2$.

Then
$$\begin{tikzcd} E_1 \arrow[rr, "u"] \arrow[dr, "\omega"] \arrow[dd, "p_1"'] & & E_2 \arrow[dd, "p_2"] \\ & E_2' \arrow[ur, "v"] \arrow[dl, "p_2'"] & \\ B_1 \arrow[rr, "f"'] & & B_2 \end{tikzcd}$$
where $\omega(x) = (p_1(x), u(x))$ is well-defined since:
$$p_2 u(x) \overset{\copyright}{=} f p_1(x)$$
Moreover,
$$\begin{aligned} \omega(x) g &= (p_1(x), u(x)) g \\ &= (p_1(x), u(x)g) \\ &= (p_1(x), u(xg)) \quad (u \text{ morphism}) \\ &= (p_1(xg), u(xg)) \quad (E_1 \text{ fibres preserved by } G) \\ &= \omega(xg). \end{aligned}$$

Therefore ω is a B_1-morphism of principal G-bundles. By the previous lemma, ω is an isomorphism. Thus, up to isomorphism, every morphism of principal G-bundles is an induced bundle morphism.

§2: FIBRE BUNDLES

A fibre bundle is a special type of bundle which is associated to a principal bundle:

Definition:

Let $E \xrightarrow{p} B$ be a principal G-bundle and F a left G-space. Define $\sim$ on $E \times F$ by

$$(x, y) \sim (x', y') \iff \exists\, g \in G \text{ st } x' = xg \,,\; y' = g^{-1}y$$

Then $\sim$ is an equivalence relation. Let $E \times_G F$ denote the quotient space $E \times F/\sim$. Consider:

$$\begin{tikzcd} E \arrow[d, "p"'] & E \times F \arrow[l, "p_E"'] \arrow[r, "\rho"] \arrow[dl, "p'"] & E \times_G F \arrow[dll, dashed, "\pi"] \\ B & & \end{tikzcd}$$

p' is constant on equivalence classes:

$$p'(xg, g^{-1}y) \equiv p(xg) = p(x) \qquad E \xrightarrow{p} B \text{ principal } G\text{-bundle}$$

$$= p'(x, y)$$

Hence the quotient map π is well-defined.

The bundle $E \times_G F \xrightarrow{\pi} B$ is called the <u>fibre bundle associated to the principal G-bundle $E \xrightarrow{p} B$ with fibre F</u>.

This bundle is denoted by <u>$E[F]$</u>

Lemma

The fibres of $E[F]$ are homeomorphic to F.

Proof.

Let $b \in B$ and fix $x \in p^{-1}(b)$. Define

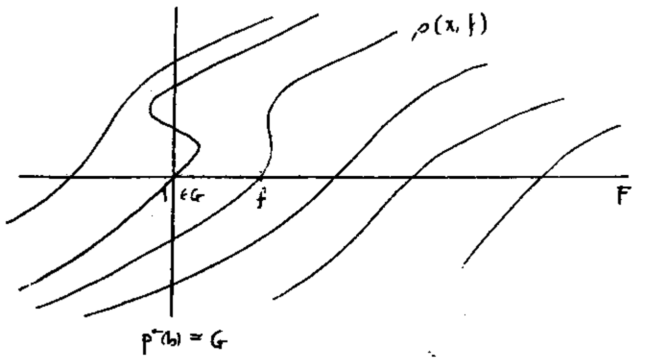
$$\phi_x : F \to \pi^{-1}(b) \qquad \phi_x(f) = \rho(x, f)$$

Then ϕ_x is continuous, bijective and has a continuous inverse:

$$\phi_x^{-1} : \pi^{-1}(b) \to F \;:\; \phi_x^{-1}(\rho(x', f)) = T(x', x) f$$

Picture:

Choice of $x \in p^{-1}(b)$ allows $p^{-1}(b)$ to be identified with G.



Each equivalence class — cuts $1 \times F$ just once.

QED.

Examples.

1. G acts trivially on pt. $E[pt]$ is the bundle $B \xrightarrow{id} B$

2. G acts on G by left multiplication. $E[G]$ may be identified with the original principal G-bundle $E \xrightarrow{p} B$

The most important examples of fibre bundles are:

Vector Bundles

(i) Principal Bundle to Vector Bundle

Let $E \xrightarrow{p} B$ be a principal $Gl_n(K)$-bundle. $Gl_n(K)$ acts on K^n by left matrix multiplication. Form the fibre bundle $E[K^n]$.

$$\begin{tikzcd} E \arrow[d, "p"'] & E \times K^n \arrow[l] \arrow[r, "\rho"] \arrow[dl, "p'"] \arrow[l, phantom, "\odot", shift right=3ex] & E \times_{Gl_n(K)} K^n = E[K^n] \arrow[dll, bend left=10, "\pi"] \arrow[dl, phantom, "\odot"] \\ B & & \end{tikzcd}$$

The fibres of $E[K^n]$ are naturally n-dimensional vector spaces over K. To see this, choose $b \in B$ and fix $x_0 \in p^{-1}(b)$. Define

$$\lambda_1 \rho(x_0, v_1) + \lambda_2 \rho(x_0, v_2) \equiv \rho(x_0, \lambda_1 v_1 + \lambda_2 v_2)$$

This structure is independent of the choice of x_0. Suppose $x_1 \in p^{-1}(b)$. Then $\exists!\, g \in G$ st $x_1 = x_0 g$. Hence:

$$\begin{aligned}
&\rho(x_1, \lambda_1 v_1 + \lambda_2 v_2) = \\
&\quad \rho(x_0 g, \lambda_1 v_1 + \lambda_2 v_2) = \\
&\quad \rho(x_0, g^{-1}(\lambda_1 v_1 + \lambda_2 v_2)) = \\
&\quad \rho(x_0, \lambda_1 g^{-1} v_1 + \lambda_2 g^{-1} v_2) = \qquad \because G \text{ acts linearly on } K^n \\
&\quad \lambda_1 \rho(x_0, g^{-1} v_1) + \lambda_2 \rho(x_0, g^{-1} v_2) = \\
&\quad \lambda_1 \rho(x_1, v_1) + \lambda_2 \rho(x_1, v_2).
\end{aligned}$$

Therefore $E[K^n]$ has the structure of an n-dimensional vector bundle over K, with base space B.

(ii) Vector Bundle to Principal Bundle:

Let $V \xrightarrow{q} B$ be a K-vector bundle of dimension n. Construct the product bundle:

$$\begin{array}{l} \overset{n}{X}_B V = \{ (b; v_1 \dots v_n) \;:\; q(v_1) = \dots = q(v_n) = b \} \\ \big\downarrow q_n \\ B \end{array}$$

and consider the subbundle

$$\overset{n}{X}_B V \supset E = \{ (b; v_1 \dots v_n) \in \overset{n}{X}_B V : v_1 \dots v_n \text{ linearly independent} \}$$

There is a right $Gl_n(K)$ action:

$$E \times Gl_n(K) \to E : ((b; v_1 \dots v_n), A) \longmapsto (b; w_1 \dots w_n)$$

$$w_j = \sum_{i=1}^{n} v_i A_{ij}$$

With respect to this action, E is a principal $GL_n(K)$ bundle called the <u>frame bundle associated to V</u>. Moreover

<u>Lemma</u>

$E[K^n]$ and V are isomorphic as vector bundles.

<u>Proof</u>.

$$\begin{tikzcd} B & & \\ & E\times K^n \arrow[ul] \arrow[r, "\rho"'] \arrow[d, "\psi"] & E\times_{GL_n(K)} K^n = E[K^n] \arrow[ull] \arrow[dl, dashed, "\phi"] \\ & V \arrow[uul] & \end{tikzcd}$$

Define :

$$\psi : E\times K^n \longrightarrow V \qquad \psi\big((b; v_1 \dots v_n),(k_1 \dots k_n)\big) = (b; v)$$

where $v = \sum_i k_i v_i$

ψ is constant on equivalence classes for if $A \in GL_n(K)$:

$$v_i \longmapsto w_j = \sum_i v_i A_{ij}$$

$$k_i \longmapsto l_j = \sum_m A^{-1}_{jm} k_m$$

and:

$$\sum_j l_j w_j = \sum_{i,j,m} v_i A_{ij} A^{-1}_{jm} k_m = \sum_i k_i v_i .$$

Thus ψ factors through $E[K^n]$ via ϕ as shown. The map of vector bundles ϕ is the required isomorphism.

QED

Also :

<u>Lemma</u>:

Let $E \xrightarrow{p} B$ be a principal $GL_n(K)$ bundle. Then the frame bundle associated to $E[K^n]$ and E are isomorphic as principal $GL_n(K)$-bundles.

<u>Proof</u>:

$$E \longleftarrow E\times K^n \xrightarrow{\rho} E\times_{GL_n(K)} K^n = E[K^n]$$

Let $x \in E$. Let $e_i = (0,\dots,\overset{(i)}{1},\dots,0) \in K^n$. Define

$$u : E \longrightarrow \text{Frame } E[K^n] \qquad u(x) = (\rho(x,e_1),\dots,\rho(x,e_n))$$

By definition of the linear structure in the fibre over $p(x)$, $\rho(x,e_1)\dots\rho(x,e_n)$ are linearly independent so u is well-defined.

For $A \in GL_n(K)$,

$$u(x)A = (\rho(x,e_1),\dots\rho(x,e_n))A$$

$$= (w_1,\dots,w_n) \quad \text{where} \quad w_j = \sum_{i=1}^n \rho(x,e_i)A_{ij}$$

But :

$$w_j = \sum_{i=1}^n \rho(x,e_i)A_{ij}$$

$$= \rho(x, \sum_i A_{ij} e_i) \quad \text{again by def}^n \text{ of the linear structure}$$

Hence, by definition of ρ,

$$u(x)A = (\rho(xA,e_1),\dots,\rho(xA,e_n)) = u(xA)$$

u is a morphism of principal $Gl_n(K)$-bundles and so is an isomorphism.

QED

Corollary:

There is an identification:

$$\boxed{k_{Gl_n(F)}(B) \simeq Vec^n_F(B)}$$

Moreover, this identification is preserved by induction, that is if $f : B' \to B$ then:

$$\begin{array}{ccc} k_{Gl_n(F)}(B) & \xrightarrow{f^*} & k_{Gl_n(F)}(B') \\ \wr\wr & © & \wr\wr \\ Vec^n_F(B) & \xrightarrow{f^*} & Vec^n_F(B') \end{array}$$

induction of principal bundles

induction of vector bundles

§3. HOMOTOPY CLASSIFICATION OF PRINCIPAL BUNDLES

The main results of this section are stated for principal bundles with certain mild restrictions.

Definition:

1. A principal G-bundle $E \xrightarrow{p} B$ is trivial if it is B-isomorphic to the product bundle $B \times G \xrightarrow{p_1} B$.

2. A principal G-bundle $E \xrightarrow{p} B$ is locally trivial if $\exists$ open cover $\{U_i\}_{i\in J}$ of B st each $E|_{U_i} \xrightarrow{p|} U_i$ is trivial.

3. A principal G-bundle $E \xrightarrow{p} B$ is numerable if $\exists$ open cover $\{U_i\}_{i\in J}$ of B st each $E|_{U_i} \xrightarrow{p|} U_i$ is trivial and $\exists$ partition of unity subordinate to the cover.

Note:

In particular, if B is paracompact then every locally trivial principal bundle over B is numerable.

Lemma:

A principal G-bundle $E \xrightarrow{p} B$ is trivial iff it admits a global cross section.

Proof:

$\Rightarrow$ E trivial $\Rightarrow$ $E \cong B \times G$ and $\sigma : B \to B \times G$ $\sigma(b) = (b,1)$ is a global cross-section.

$\Leftarrow$ Suppose $\sigma : B \to E$ is a cross-section. Define

$$\phi : B \times G \to E \quad : \quad \phi(b,g) = \sigma(b)g$$

ϕ is a morphism of principal G-bundles and so an isomorphism.

QED

Example

The principal G-bundle $E_G \xrightarrow{p} B_G$ from the Milnor construction is numerable.

Proof:

Use the representation $E_G \ni x = [t_r x_r]_{r \geq 0}$ to define maps

$$E_G \xrightarrow{t_i} [0,1], \quad i \geq 0$$

Since $t_i(x) = t_i(xg)$, t_i factors

$$\begin{tikzcd}
E_G \arrow[r, "t_i"] \arrow[d, "p"'] \arrow[dr, phantom, "\circlearrowleft"] & {[0,1]} \\
B_G \arrow[ur, dashed, "z_i"'] &
\end{tikzcd}$$

Since $\sum_i t_i = 1$, $\sum_i z_i = 1$ also so the sets $U_i = z_i^{\leftarrow}(0,1]$ form an open cover of B_G and $\{z_i\}_{i \geq 0}$ is subordinate to this cover.

Note that $x_i : t_i^{\leftarrow}(0,1] \to G$: $x_i[t_r g_r] = g_i$ is well-defined since $t_i > 0$. Define

$$s_i : t_i^{\leftarrow}(0,1] \to t_i^{\leftarrow}(0,1] \qquad s_i(a) = a\, x_i(a)^{-1}$$

s_i is preserved under the G-action.

$$s_i(ag) = ag\, x_i(ag)^{-1} = ag\,(x_i(a)g)^{-1} = agg^{-1} x_i(a)^{-1} = a\, x_i(a)^{-1} = s_i(a)$$

and therefore factors:

$$\begin{tikzcd}
E_G \arrow[d, "p"'] \arrow[r, hookleftarrow] \arrow[dr, phantom, "\circlearrowleft"] & t_i^{\leftarrow}(0,1] \arrow[r, "s_i"] \arrow[d] \arrow[dr, phantom, "\circlearrowleft"] & t_i^{\leftarrow}(0,1] \\
B_G \arrow[r, hookleftarrow] & U_i \arrow[ur, dashed, "\sigma_i"'] &
\end{tikzcd}$$

Thus σ_i is a cross-section over U_i so $E_G|U_i$ is trivial $\forall\, i \geq 0$.

Therefore, E_G is numerable.

QED

Notation:

Let $k_G(B)$ $(\subset k_G^1(B))$ denote the B-isomorphism classes of principal G-bundles over B. If B p.compt then $k_G(B) = k_G^1(B)$.

Homotopy Classification of Principal Bundles

1. Homotopy Property

Theorem:

Suppose $B' \overset{f}{\underset{g}{\rightrightarrows}} B$ are homotopic and let $E \xrightarrow{p} B$ be a numerable principal G-bundle. Then f^*E and g^*E are isomorphic as principal G-bundles.

Proof.

See FB pg 51

2. Classification

Definition:

A principal G-bundle $\omega.(E \xrightarrow{p} B)$ is universal if ω is numerable and if $\forall$ spaces B'

$$\phi_\omega : [B', B] \to k_G(B') \qquad \phi_\omega([f]) = \text{isomorphism class of } f^*E$$

is an isomorphism.

Note:

A numerable principal G-bundle, $\omega : (E \xrightarrow{p} B)$ is universal iff the following conditions are satisfied:

(a) $\forall$ principal G-bundles $E' \xrightarrow{p'} B'$, $\exists$ $f : B' \to B$ st $f^*E \simeq E'$ (i.e. ϕ_ω surjective);

(b) if $B' \overset{f}{\underset{g}{\rightrightarrows}} B$ with $f^*E \simeq g^*E$, then f and g are homotopic (i.e. ϕ_ω injective).

The fundamental classification theorem is

Theorem.

The principal G-bundle $E_G \xrightarrow{p} B_G$ of the Milnor construction is universal.

Sketch of Proof

For full details, see FB pg 57.

1. For every numerable bundle $E \xrightarrow{\pi} B$, a countable covering of B may be chosen wrt which E is numerable. (FB pg 5?)

2. Consider the numerable principal G-bundle $E \xrightarrow{\pi} B$ where B has an countable open cover $\{U_i\}_{i \geq 0}$ wrt which E is numerable. Let $\sigma_n : U_n \to E$ be cross-sections and let $\{\eta_n\}_{n\geq 0}$ denote the partition of unity subordinate to the cover of B. It is required to define

$$\begin{array}{ccc} E & \xrightarrow{u} & E_G \\ \pi \downarrow & \circledcirc & \downarrow p \\ B & \xrightarrow[f]{} & B_G \end{array} \qquad \text{st } f^*E_G = E$$

The key is to define u for f then follows on taking quotients

Define:

$$u : E \to E_G : u(x) = [t_r g_r]_{r \geq 0}$$

where: $t_r = \lambda_r(p(x))$

and if $t_r = 0$ then $g_r = 1$

or if $t_r \neq 0$ then $g_r = \tau(\sigma_r(p(x)), x)$

Since $\tau(x, yg) = \tau(x,y)g$, it follows that $u(xg) = u(x)g$ so (u, f) is a morphism of principal G-bundles. By a previous remark, $E \simeq f^* E_G$ as principal G-bundles.

QED

Thus, $\forall$ spaces B:

$$\boxed{k_G(B) \simeq [B, B_G]}$$

For paracompact spaces, this provides the first step in the homotopy classification of vector bundles:

$$\boxed{Vec^n_F(B) \simeq k_{GL_n(F)}(B) \simeq [B, B_{GL_n(F)}]}$$

In the next section, a model for $B_{GL_n(F)}$ is constructed.

§4. HOMOTOPY CLASSIFICATION OF VECTOR BUNDLES

1 Restriction of Structure Group:

The following is due to J. Whitehead [ref]

Theorem:

Let $f : X \to Y$ be a map of spaces having the same homotopy types as CW-complexes. Suppose $f_* : \pi_i(X) \to \pi_i(Y)$ is an isomorphism $\forall i$. Then f is a homotopy equivalence.

Application

By the Iwasawa decomposition, $U_n(F) \hookrightarrow GL_n(F)$ is a homotopy equivalence. Hence

$$\begin{array}{ccc} \pi_{i-1}(U_n) & \xrightarrow{\simeq} & \pi_{i-1}(GL_n) \\ \wr\Vert & & \wr\Vert \\ \pi_i(B_{U_n}) & \xrightarrow{\simeq} & \pi_i(B_{GL_n}) \end{array} \qquad \forall i$$

The vertical isomorphisms come from the homotopy exact sequence applied to the fibration $G \to E_G \to B_G$ since E_G is contractible. The isomorphism $\pi_i(B_{U_n}) \xrightarrow{\simeq} \pi_i(B_{GL_n})$ is in fact induced by the inclusion $B_{U_n} \hookrightarrow B_{GL_n}$. By the above theorem, this is a homotopy equivalence. Therefore:

(B para

$$\boxed{\mathrm{Vec}_F^n(B) \simeq [B, B_{GL_n(F)}] = [B, B_{U_n(F)}]} \quad (B \text{ pcm}$$

2. Model for the Classifying Space B_{U_n}

Let G be a topological group. Starting with a fibration

$$\begin{array}{ccc} G & \to & E \\ & & \downarrow \\ & & B \end{array}$$

satisfying mild constraints and having the property that E is contractible, it may be shown that B and B_G have the same homotopy type.

For $U_n(F)$, there are fibrations (FB pg 76):

$$\begin{array}{ccccc} U_m(F) & \longrightarrow & U_{m+n}(F)/U_n(F) & = & V_n(F^{m+n}) \\ & & \downarrow & & \\ & & U_{m+n}(F)/U_m(F)\times U_n(F) & \simeq & G_n(F^{m+n}) \end{array}$$

Let $m \to \infty$.

$$\begin{array}{ccc} U_n(F) & \to & V_n(F^\infty) \quad , \text{ contractible} \\ & & \downarrow \\ & & G_n(F^\infty) \end{array}$$

Hence $G_n(F^\infty)$ and $B_{U_n(F)}$ have the same homotopy type. This gives the explicit homotopy classification of vector bundles over a paracompact space B:

$$\boxed{\mathrm{Vec}_F^n(B) \simeq [B, G_n(F^\infty)]} \quad (B \text{ pcmpt})$$

Note in particular that F-line bundles are classified by the projective space $\mathbb{P}_1(F^\infty)$.

3. Universal Bundle over $G_n(F^\infty)$

The correspondence above may be realised in terms of induced vector bundles by introducing the tautological n-dimensional F-vector bundle γ_n^∞ over $G_n(F^\infty)$, (FB pg 2…). If $[\xi] \in \mathrm{Vec}_F^n(B)$ corresponds to a homotopy class $[f] \in [B, G_n(F^\infty)]$ then ξ and $f^*\gamma_n^\infty$ are isomorphic vector bundles:

$$\begin{array}{ccc} \xi = f^*\gamma_n^\infty & \longrightarrow & \gamma_n^\infty \\ \downarrow & & \downarrow \\ B & \xrightarrow{f} & G_n(F^\infty) \end{array}$$

Thus γ_n^∞ is a universal vector bundle.

ALGEBRAIC TOPOLOGY V

CHARACTERISTIC CLASSES

§ 1 : FIRST STIEFEL-WHITNEY AND CHERN CLASSES

Consider possible connections between the two classes

1. Homotopy Classification of Cohomology Classes

B a CW complex, Π an abelian group $\Rightarrow$

$$H^q(B, \Pi) \simeq [B, K(\Pi, q)]$$

2. Homotopy Classification of Principal G - Bundles:

$$k_G(B) \simeq [B, B_G]$$

Recall that B_G is $K(\Pi, n)$ iff G is $K(\Pi, n-1)$. Thus if G is $K(\Pi, q)$ and B is a CW complex:

$$\boxed{k_G(B) \simeq H^{q+1}(B, \Pi)}$$

Applications:

1. First Stiefel - Whitney Class for $\mathbb{R}$ - Line Bundles

Let $G = \mathbb{Z}_2$, $K(\mathbb{Z}_2, 0)$ space. Hence:

$$k_{\mathbb{Z}_2}(B) \simeq H^1(B, \mathbb{Z}_2)$$

But $\mathbb{Z}_2 \equiv U_1(\mathbb{R})$ and from the previous chapter.

$$Vec^1_{\mathbb{R}}(B) \simeq k_{U_1(\mathbb{R})}(B) \qquad (B \text{ pcompt})$$

Hence (since every CW complex is paracompact)

$$\boxed{\mathrm{Vec}^1_{\mathbb{R}}(B) \simeq H^1(B, \mathbb{Z}_2)} \qquad (B \text{ CW cplx})$$

The first Stiefel-Whitney class of an $\mathbb{R}$-line bundle ξ: $E \xrightarrow{p} B$ is defined to be the cohomology class in $H^1(B, \mathbb{Z}_2)$ corresponding to the isomorphism class $[\xi]$. It is denoted by $\omega_1(\xi)$.

2. First Chern Class for $\mathbb{C}$-Line Bundles

Let $G = S^1$, $K(\mathbb{Z}, 1)$ space. Hence:

$$k_{S^1}(B) \simeq H^2(B, \mathbb{Z})$$

But $S^1 = U_1(\mathbb{C})$ and from the previous chapter

$$\mathrm{Vec}^1_{\mathbb{C}}(B) \simeq k_{U_1(\mathbb{C})}(B) \qquad (B \text{ compt})$$

Hence:

$$\boxed{\mathrm{Vec}^1_{\mathbb{C}}(B) \simeq H^2(B, \mathbb{Z})} \qquad (B \text{ CW cplx})$$

The first Chern class of a $\mathbb{C}$-line bundle ξ: $E \xrightarrow{p} B$ is defined to be the cohomology class in $H^2(B, \mathbb{Z})$ corresponding to the isomorphism class $[\xi]$. It is denoted by $c_1(\xi)$.

Note:

$\mathbb{H}$-line bundles cannot be treated in this fashion since $U_1(\mathbb{H}) \cong S^3$ is not a $K(\pi, n)$-space.

§2 Axioms and Existence of Characteristic Classes

Notation:

Let $F = \mathbb{R}$ or $\mathbb{C}$, $c = \dim_{\mathbb{R}} F$. Denote

$$R_c = \begin{cases} \mathbb{Z}_2 = \mathbb{F}_2, & c = 1 \\ \mathbb{Z}, & c = 2 \end{cases}$$

Denote: $H^{c*}(B, R_c) = \bigoplus_{n \geq 0} H^{cn}(B, R_c)$

Hirzebruch's Axioms for Characteristic Classes of Vector Bundles

Let B be a CW-complex. Characteristic class is a function

$$k_*: \mathrm{Vec}_F(B) \to H^{c*}(B, R_c) \quad \xi \mapsto k_*(\xi)$$

satisfying the following four axioms:

(A1) $k_*(\xi) = k_0(\xi) + k_1(\xi) + \cdots + k_n(\xi)$

where $k_i(\xi) \in H^{ci}(B, R_c)$, $k_0(\xi) = 1$, $k_i(\xi) = 0$ if $i > \dim_F \xi$.

(A2) Naturality

If $B' \xrightarrow{f} B$ then:

$$\begin{tikzcd} \mathrm{Vec}_F(B) \arrow[r, "k_*"] \arrow[d, "f^*"'] \arrow[dr, phantom, "\circlearrowleft"] & H^{c*}(B, R_c) \arrow[d, "f^*"] \\ \mathrm{Vec}_F(B') \arrow[r, "k_*"] & H^{c*}(B', R_c) \end{tikzcd}$$

that is, $k_*(f^*\xi) = f^*(k_*(\xi))$

(A3) : Whitney Sum Formula.

$$k_*(\xi \oplus \zeta) = k_*(\xi) \smile k_*(\zeta)$$

(A4) . Normalisation:

If $\dim_F \xi = 1$ then $k_*(\xi) = 1 + k_1(\xi)$ where $k_1(\xi)$ is the first ($\mathbb{R}$) Stiefel Whitney class or ($\mathbb{C}$) Chern class defined previously.

Note :

An immediate consequence of the axioms is that $k_*(\xi) = 1$ if ξ is a trivial bundle. To see this, note that

$$\xi = f^*\left(\begin{array}{c} F^n \\ \downarrow \\ pt \end{array}\right)$$

and that $H^{c*}(pt, R_c) = \overset{(0)}{R_c}$. (A2) $\Rightarrow k_*(\xi) = 1$.

Existence and Uniqueness of Characteristic Classes

To establish the existence and uniqueness of characteristic classes, two results are necessary, namely the Leray-Hirsch Theorem and the Hirzebruch Borel Splitting Principle.

Leray - Hirsch Theorem

Let $F \to E \xrightarrow{p} B$ be a fibration and suppose that $a_1, \dots a_r$ are classes in $H^*(E, R)$ st for each fibre $f^*(b)$ the images of a_i, ar form a basis of $H^*(F, R)$ over R. Then

1) $p^* : H^*(B, R) \longrightarrow H^*(E, R)$ is a monomorphism;

2) $a_1, \dots a_r$ is a basis of the module $H^*(E, R)$ over the algebra $H^*(B, R)$

Proof.

The proof follows from the spectral sequence of a fibre bundle.

See also FB pg 231.

Hirzebruch - Borel Splitting Principle:

Let ξ be an F-vector bundle of dimension n over B. Then exists a map $f : B' \to B$, called a splitting map, st

1) $f^*\xi = \overset{n}{\underset{i}{\oplus}} \lambda_i$, λ_i a line bundle over B' $\forall i$;

2) $f^* : H^{c*}(B, R_c) \to H^{c*}(B', R_c)$ is a monomorphism.

Proof.

The splitting principle will be proven along with existence of characteristic classes in the next theorem :

<u>Theorem</u>

Characteristic classes exist and are unique

<u>Proof</u>

<u>Step 1</u>: <u>Existence and Splitting Principle</u> (FB Chapter 16)

Let ξ $E \xrightarrow{p} B$ be an n-dimensional F-vector bundle over a CW complex B. Let $E_0 \xrightarrow{p_0} B$ be the bundle whose fibres consist of non-zero vectors in the fibres of ξ.

Define an equivalence relation $\sim$ on the fibres of E_0 by

$p_0^{-1}(b) \ni v, w$: $v \sim w \iff \exists\, \lambda \in F$ st $v = \lambda w$. The quotient bundle $P\xi \xrightarrow{\pi} B$ is called the projective bundle associated to ξ. Pull ξ back to $\pi^*(\xi) \to P\xi$, an n-dimensional vector bundle. There is a canonical subbundle, the line bundle $\lambda_\xi \hookrightarrow \pi^*\xi$ whose fibre over $p \in P\xi$ is just the corresponding line in ξ.

$$\begin{tikzcd}
 & & E_0 \arrow[d, hook] \arrow[dl, dotted] \\
\lambda_\xi \arrow[r, hook] \arrow[dr] & \pi^*\xi \arrow[r] \arrow[d] & E \arrow[d, "p"] \\
 & P\xi \arrow[r, "\pi"] & B
\end{tikzcd}$$

This gives an exact sequence of vector bundles over $P\xi$:

$$0 \to \lambda_\xi \hookrightarrow \pi^*\xi \twoheadrightarrow \zeta_\xi \to 0$$
$$\searrow \quad \downarrow \quad \swarrow$$
$$P\xi$$

where ζ_ξ is the quotient bundle of dimension $n-1$. This sequence splits $\pi^*\xi \cong \lambda_\xi \oplus \zeta_\xi$ (topologically)

Form the dual line bundle $\lambda_\xi^{-1} = \hom(\lambda_\xi, F)$ and let $a = k_1(\lambda_\xi^{-1}) \in H^c(P\xi, R_c)$ denote its first Stiefel-Whitney or Chern class as appropriate.

<u>Claim</u>: Leray-Hirsch Theorem applies to the fibration

$$\begin{tikzcd}
P_{n-1}(F) \arrow[r, "i"] & P\xi \arrow[d, "\pi"] \\
 & B
\end{tikzcd}$$

and the cohomology classes $1, a, a^2, \dots a^{n-1}$.

To see why, note that $\lambda_\xi^{-1} | (P\xi)_b$ is the canonical line bundle over $P_{n-1}(F) \simeq (P\xi)_b$. Also,

$$k_1(\lambda_\xi^{-1} | (P\xi)_b) = i^*(a) \in H^c((P\xi)_b, R_c), \ \tilde{a} \text{ say.}$$

Now $H^*((P\xi)_b, R_c) = R_c[\tilde{a}]/(\tilde{a}^n = 0)$ so $1, \tilde{a}, \tilde{a}^{n-1}$ form a basis of $H^*((P\xi)_b, R_c)$. Therefore Leray-Hirsch applies.

Hence: $H^*(B, R_c) \hookrightarrow H^*(P\xi, R_c)$

and: $H^*(P\xi, R_c) \cong H^*(B,R_c) \oplus H^*(B,R_c)\, a \oplus \cdots \oplus H^*(B,R_c)\, a^{n-1}$

<u>Definition of k_*</u>: The class $a^n \in H^{cn}(P\xi, R_c)$ is a $H^*(B, R_c)$-linear combination of $1, a, \dots, a^{n-1}$. Hence:

$$a^n = -\sum_{i=1}^{n} k_i(\xi)\, a^{n-i}$$, defining k_i

and define $k_0(\xi) = 1$, $k_i(\xi) = 0$, $i > n$. Observe that $k_j(\xi) \in H^{cj}(B, R_c)$ since $a^{n-j} \in H^{c(n-j)}(B, R_c)$.

Splitting Principle: The proof is by induction on $\dim \xi = n$. If $n = 1$, the principle is empty. Assume $n > 1$ and that the Principle holds for bundles of dimension $\leq n-1$. By the above construction, there is a splitting:

$$\begin{tikzcd} \lambda_\xi \oplus \eta_\xi = \pi^*(\xi) \arrow[r] \arrow[d] & E \arrow[d, "p"] \\ \mathbb{P}(\xi) \arrow[r, "\pi"] & B \end{tikzcd}$$

By the Leray-Hirsch theorem, $H^*(\pi)$ is a monomorphism. Let $g: B' \to \mathbb{P}(\xi)$ be a splitting map for η_ξ, η_ξ of dimension $n-1$. Suppose $g^*\eta_\xi = \bigoplus_{2}^{n} \lambda_j$. Let $\lambda_1 = g^*\lambda_\xi$. Then:

$$\begin{tikzcd} f^*\xi = g^*\pi^*\xi = \bigoplus_{1}^{n} \lambda_j \arrow[r] \arrow[d] & \lambda_\xi \oplus \eta_\xi = \pi^*\xi \arrow[r] \arrow[d] & \overset{\xi}{E} \arrow[dd, "p"] \\ B' \arrow[r, "g"] \arrow[drr, dashed, "f"'] & \mathbb{P}(\xi) \arrow[dr, "\pi"] & \\ & & B \end{tikzcd}$$

Moreover $H^*(f) = H^*(g)\, H^*(\pi)$ so $H^*(f)$ is a monomorphism. Thus the f is the required splitting map for ξ.

Step 2. Verification of Axioms

A1 is immediate from the definition of k_*.

A2: Suppose $\xi: E \xrightarrow{p} B$ is an n-dimensional vector bundle and $f: B' \to B$ a map. Denote $\xi' = f^*\xi$. f induces map h of the associated projective bundles.

$$\begin{tikzcd} \xi': f^*E \arrow[r] \arrow[d] & E \arrow[d, "p"] \\ B' \arrow[r, "f"] & B \\ \mathbb{P}\xi' \arrow[u, "\pi'"] \arrow[r, "h"] & \mathbb{P}\xi \arrow[u, "\pi"'] \end{tikzcd}$$

The sequences

$$0 \to \lambda_\xi \to \pi^*(\xi) \to \eta_\xi \to 0 \quad \text{(bundles over } \mathbb{P}\xi)$$

$$\uparrow h$$

$$0 \to \lambda_{\xi'} \to \pi'^*(\xi') \to \eta_{\xi'} \to 0 \quad \text{(bundles over } \mathbb{P}\xi')$$

are exact and the second is induced from the first via h.

Moreover, if $a = k_1(\lambda_\xi^{-1}) \in H^c(\mathbb{P}\xi, R_c)$ then $a' = k_1(\lambda_{\xi'}^{-1}) \in H^c(\mathbb{P}\xi', R_c)$ equals $h^*(a)$. Thus

$$h^*(\text{equation for } a^n) = \text{equation for } a'^n$$

and so $f^*(k_*\xi) = k_*(f^*\xi)$ as required.

A3. See FB pg 238 (Uses splitting principle)

A4: In this case, $\dim \xi = 1$ so $\mathbb{P}(\xi) \xrightarrow{id \equiv \pi} B$, $a = k_1(\xi^{-1})$

Thus $a = k_1(\xi^{-1}) = -k_1(\xi)$

Step 3 Splitting Principle $\Rightarrow$ Uniqueness of k_*:

Let k'_* be any system of characteristic classes. For $\xi \in \mathrm{Vect}_F(B)$ let $f : B' \to B$ be a splitting map with $f^*\xi = \bigoplus_1^n \lambda_j$. Hence:

$$
\begin{aligned}
f^*(k_*\xi) &= && \text{(A2)}\\
k_*(f^*\xi) &= \\
k_*\left(\bigoplus_1^n \lambda_j\right) &= && \text{(A3)}\\
\prod_1^n k_*(\lambda_j) &= && \text{(A1)}\\
\prod_1^n (1 + k_1(\lambda_j)) &= && \text{(A4)} \quad (k, k' \text{ coincide on line bundles})\\
\prod_1^n (1 + k'_1(\lambda_j)) &= \cdots\\
f^*(k'_*\xi)&.
\end{aligned}
$$

f^* monomorphism $\Rightarrow$ $k_*\xi = k'_*\xi \quad \forall\ \xi$

$$\therefore\ k_* = k'_*$$

so characteristic classes are unique.

Notation:

Let $\underline{w_*}$ denote the Stiefel-Whitney classes of $\mathbb{R}$-vector bundles and $\underline{c_*}$ the Chern classes of $\mathbb{C}$-vector bundles.

§3 CALCULATIONS FOR THE UNIVERSAL VECTOR BUNDLES

1 Splitting Principle for the Universal Bundles

Let $\underline{\xi_n}$ denote the universal F-vector bundle of dimension n over $B_{U_n(F)}$. Denote $B_{U_n(F)} \equiv \underline{B_n}$. It is required to construct a splitting map for ξ_n.

Let $T = \times_{i=1}^n U_1(F) \hookrightarrow U_n(F)$ (diagonal matrices)

Then $B_T \hookrightarrow B_n$. Up to homotopy equivalence $B_T \simeq \times_1^n B_1$. Denote $g_n : \times_1^n B_1 \hookrightarrow B_n$. There is an n-dimensional vector bundle ξ_1^n over $\times_1^n B_1$ defined as follows

$$
\begin{array}{ccc}
\times_1^n B_1 & \xrightarrow{p_j} & B_1 \\
\uparrow & & \uparrow \\
p_j^*\xi_1 & \dashrightarrow & \xi_1
\end{array}
\qquad \text{(projection onto } j^{\text{th}} \text{ factor)}
$$

Take $\xi_1^n \equiv \bigoplus_1^n p_j^*\xi_1$. Moreover, $\xi_1^n \simeq g_n^*\xi_n$.

QED

Let $f : B' \to B_n$ be a splitting map for ξ_n; ($\exists f$ by the splitting principle). Then $f^*\xi_n = \bigoplus_1^n \lambda_j$ where λ_j is a line bundle over B'. Since ξ_1 is universal, $\exists\ f_j : B' \to B_1$ st $f_j^*\xi_1 \simeq \lambda_j$. Imbed this $B_1 \hookrightarrow \times_1^n B_1$ as the j^{th} component. Hence:

$$\begin{tikzcd} B' \arrow[r, "f"] \arrow[dr, "(f_1, \dots f_n)"'] & B_n \\ & \overset{n}{\underset{1}{X}} B_1 \arrow[u, "g_n"'] \end{tikzcd}$$

Let $h = g_n \circ (f_1, \dots f_n)$. Then

$$h^* \mathfrak{I}_n \simeq \hat{\oplus}_i \lambda_i = f^* \mathfrak{I}_n$$

By the splitting principle, h and f are homotopic. Therefore:

$$H^*(f) = H^*(h) = H^*(f_1 \dots f_n) \circ H^*(g_n)$$

$H^*(f)$ is injective since f is a splitting map. Hence $H^*(g_n)$ injective. Therefore:

Splitting Map
$$\begin{cases} g_n : \overset{n}{\underset{1}{X}} B_1 \longrightarrow B_n \\ g_n^* \mathfrak{I}_n \simeq \overset{n}{\underset{1}{\oplus}} p_j^* \mathfrak{I}_1 , \quad \oplus \text{ of line bundles} \\ H^*(g_n) \quad \text{monomorphism} \end{cases}$$

Thus g_n is a splitting map for the universal bundle $\mathfrak{I}_n$.

Applications:

a) $F = \mathbb{R}$: $G_1(\mathbb{R}^\infty) \equiv \mathbb{P}_1(\mathbb{R}^\infty)$ is a model for $\overset{n}{\underset{1}{X}} B_1$ so there is a splitting map $\overset{n}{\underset{1}{X}} \mathbb{P}_1(\mathbb{R}^\infty) \to B_{O(n)}$ for $\mathfrak{I}_n$.

b) $F = \mathbb{C}$: $G_1(\mathbb{C}^\infty) \equiv \mathbb{P}_1(\mathbb{C}^\infty)$ is a model for $\overset{n}{\underset{1}{X}} B_1$ so there is a splitting map $\overset{n}{\underset{1}{X}} \mathbb{P}_1(\mathbb{C}^\infty) \to B_{U(n)}$ for $\mathfrak{I}_n$.

2 Cohomology Rings of the Classifying Spaces:

It is required to calculate $H^*(B_{U_n(\mathbb{F})}, R_c)$.

Notation:

Let $\underline{\omega_j} \equiv \omega_j(\mathfrak{I}_n) \in H^j(B_{O(n)}, \mathbb{F}_2)$, universal SW class.

Let $\underline{c_j} \equiv c_j(\mathfrak{I}_n) \in H^{2j}(B_{U(n)}, \mathbb{Z})$, universal Chern class.

Theorem:

1. $H^*(B_{O(n)}, \mathbb{F}_2) = \mathbb{F}_2[\omega_1, \dots \omega_n]$, $\deg \omega_i = i$.
2. $H^*(B_{U(n)}, \mathbb{Z}) = \mathbb{Z}[c_1, \dots c_n]$, $\deg c_i = 2i$.

Proof. (Case 2.)

Denote $x_j \equiv c_j(p_j^* \mathfrak{I}_1)$. Then:

$$\begin{aligned} g_n^*(c_*(\mathfrak{I}_n)) &= c_*(g_n^*(\mathfrak{I}_n)) \\ &= c_*(\overset{n}{\underset{1}{\oplus}} p_j^* \mathfrak{I}_n) \\ &= c_*(p_1^* \mathfrak{I}_n) \cdot \ldots \cdot c_*(p_n^* \mathfrak{I}_n) \\ &= (1 + x_1) \cdot \ldots \cdot (1 + x_n) \\ &= \sum_{k=0}^{n} s_k(x_1, \dots x_n) \end{aligned}$$

where s_k denotes the k^{th} elementary symmetric function. Since $\deg x_j = 2 \; \forall j$, $\deg s_k(x_1, \dots x_n) = 2k$.

Also: $g_n^*(c_*(\mathfrak{I}_n)) = g_n^*(\sum_{k=0}^{n} c_k(\mathfrak{I}_n)) = \sum_{k=0}^{n} g_n^* c_k(\mathfrak{I}_n)$

Comparing elements of like degrees gives

$$g_n^*(c_k) = s_k(x_1, \dots x_n)$$

Now $\quad B_{U(1)} \simeq \mathbb{P}_1(\mathbb{C}^\infty) \quad$ so

$$H^*(B_{U(1)}, \mathbb{Z}) = \mathbb{Z}[x_1]$$

and

$$H^*(\overset{n}{\underset{1}{X}} B_{U(1)}, \mathbb{Z}) = \mathbb{Z}[x_1, \dots x_n]$$

g_n is a splitting map so $H^*(g_n)$ is a monomorphism:

$$g_n^* = H^*(g_n) \; : \; H^*(B_{U(n)}, \mathbb{Z}) \hookrightarrow H^*(\overset{n}{\underset{1}{X}} B_{U(1)}, \mathbb{Z})$$

Therefore:

$$\begin{tikzcd}
H^*(\overset{n}{\underset{1}{X}} B_{U(1)}, \mathbb{Z}) = \mathbb{Z}[x_1 \dots x_n] \arrow[r, phantom, "\supset"] & g_n^* H^*(B_{U(n)}, \mathbb{Z}) \\
\mathbb{Z}[s_1 \dots s_n] \arrow[u, phantom, "\cup"] \arrow[ur, hook] &
\end{tikzcd}$$

$\mathbb{Z}[s_1 \dots s_n]$ = subalgebra generated by elementary symmetric functions

From algebra:

(a) $\mathbb{Z}[s_1, \dots s_n]$ is a polynomial ring.

(b) The symmetric group S_n acts on $\mathbb{Z}[x_1, \dots x_n]$. The subring of invariants is precisely $\mathbb{Z}[s_1, \dots s_n] = \mathbb{Z}[x_1, \dots x_n]^{S_n}$

(c) Permuting variables in $\overset{n}{X} B_{U(1)} \hookrightarrow B_{U(n)}$ may be done by conjugation in $U(n)$ and so leads to the same map up to homotopy.

Thus

$$\begin{tikzcd}
\mathbb{Z}[s_1, \dots s_n] \arrow[d, phantom, "\supseteq"] \arrow[dr, equal, "(b)"] & \\
g_n^* H^*(B_{U(n)}, \mathbb{Z}) \arrow[r, hook] & \mathbb{Z}[x_1, \dots x_n]^{S_n}
\end{tikzcd}$$

whence the required result:

$$H^*(B_{U(n)}, \mathbb{Z}) = \mathbb{Z}[c_1, \dots c_n]$$

since $\quad g_n^*(c_j) = s_j \quad \forall j$. $\qquad$ QED

Note:

The argument in the real case is similar.

Apply this to

$$H^*(B_{O(n)};\mathbb{F}_2) = \mathbb{F}_2[w_1,\dots w_n] \quad , \quad w_i = \text{universal SW class}$$

$$\downarrow \qquad\qquad \Big\downarrow \text{mono}$$

$$H^*(B^n_{O(1)},\mathbb{F}_2) = \mathbb{F}_2[x_1,\dots x_n]$$

$$x_j = w_1(pr_j^*(S_1)) \quad , \quad d^\circ(x_j) = 1$$

$$w_j \longmapsto s_j(x_1,\dots x_n)$$

$$\mathrm{Im} = \mathbb{F}_2[x_1,\dots x_n]^{S_n}$$

Steenrod defined for each pair of spaces (X,A) and dimension i an operation:

$$\underline{Sq^i} \quad H^n(X,A;\mathbb{F}_2) \longrightarrow H^{n+i}(X,A;\mathbb{F}_2)$$

satisfying (all cohomology with $\mathbb{F}_2$ coefficients in the sequel)

① Naturality

$$(X,A) \xleftarrow{\ f\ } (Y,B) \quad \text{a map of pairs}$$

then:

$$\begin{tikzcd} H^n(X,A) \arrow[r, "Sq^i"] \arrow[d, "f^*"] \arrow[dr, phantom, "©"] & H^{n+i}(X,A) \arrow[d, "f^*"] \\ H^n(Y,B) \arrow[r, "Sq^i"'] & H^{n+i}(Y,B) \end{tikzcd}$$

Sq^i fix for fixed n, different spaces

② Commutation with coboundary

$$\begin{tikzcd} H^{n-1}(A) \arrow[r, "\delta"] \arrow[d, "Sq^i"] \arrow[dr, phantom, "©"] & H^n(X,A) \arrow[d, "Sq^i"] \\ H^{n+i-1}(A) \arrow[r] & H^{n+i}(X,A) \end{tikzcd}$$

Sq^i fix for different n

③ (a) $Sq^i = 0$ on $H^n(X,A)$ when $i>n$

(b) $Sq^0 = id$

(c) If $c \in H^i(X,A)$ then $Sq^i(c) = c \cup c = c^2$

[Relation: cohomology class $\xrightarrow{Sq}$ cohomology class]

Theorem:

$\exists!$ family $\{Sq^i\}_{i\geq 0}$ of operations satisfying the above conditions

Suspension Property for Sq^i:

Note: mod 2, squaring is linear. $(x+y)^2 = x^2+y^2 \mod 2$.

Recall: $X \hookrightarrow C(X) \longrightarrow S(X) = C(X)/X$

$$\begin{tikzcd} \tilde{H}^n(X) \arrow[r, "\delta", "\cong"'] \arrow[d, "Sq^i"] & H^{n+1}(C(X),X) \arrow[r, "\cong"] \arrow[d, "Sq^i"] & \tilde{H}^{n+1}(S(X)) \arrow[d, "Sq^i"] \\ \tilde{H}^{n+i}(X) \arrow[r, "\delta", "\cong"'] & H^{n+i+1}(C(X),X) \arrow[r, "\cong"] & \tilde{H}^{n+i+1}(S(X)) \arrow[ll, bend left, "\sigma"] \end{tikzcd}$$

fixes different dimensions

σ, cohomology suspension map

~~Defining~~

$\therefore$

$$\begin{tikzcd} H^0 & H^m & H^i & H^{m+i} & H^{2i} \arrow[from=1-1, to=1-3, bend right, "Sq^i = 0"'] \arrow[from=1-2, to=1-4, bend right, "Sq^i = 0"'] \arrow[from=1-3, to=1-5, bend right, "Sq^i = \text{cup}^2"'] \end{tikzcd}$$

so ① $H^i \xrightarrow{Sq^i} H^{2i}$ is cup square

② what is $H^{i+1}(X) \xrightarrow{Sq^i} H^{2i+1}(X)$?

If $X = S(Y)$ then:

$$\begin{array}{ccc} H^{n+i}(X) & \dashrightarrow & H^{n+i}(X) \\ \| \wr & \circlearrowleft & \| \wr \\ H^i(Y) & \xrightarrow[Sq^i = \text{cup sq.}]{} & H^{2i}(Y) \end{array}$$

Y is called a desuspension of X. If all spaces X could be desuspended, then Sq^i could be described by desuspension and cup squared. Thus to define: *

$$\begin{array}{ccc} H^n(X) & \xrightarrow{Sq^i *} & H^{n+i}(X) \\ \sigma^k \downarrow \wr & © & \downarrow \wr \\ H^i(Y) & \xrightarrow[Sq^i = \text{cup square}]{} & H^{2i}(Y) \end{array} \qquad \begin{array}{l} k = n-i \\ X = S^k Y \end{array}$$

Unfortunately, most spaces do not desuspend (all spheres do). However, observe that all is required is that the desuspension isomorphism exists in certain degrees. There is a type of suspension for which this holds.

<u>Extra Remark</u>

$\exists$ suspension morphism in fibre space theory which allows the above argument to define Sq^i and prove uniqueness to be finished. However, a splitting principle is obtained for Sq^i.

<u>Suspension:</u> (<u>Fibre Space Suspension</u> - for any coefficients, not just $\mathbb{F}_2$)

Recall: $X \hookrightarrow C(X) \longrightarrow C(X)/X = S(X)$.

Let $F \longrightarrow E \xrightarrow{p} B$ be a fibre space with $E \simeq *$

e.g., $\Omega B \longrightarrow EB \longrightarrow B$

i.e. $E \longrightarrow F \longrightarrow P$

Homology $\quad \tilde{H}_*(F) \xleftarrow[\partial]{-1} H_*(E,F) \xrightarrow[p_*]{} \tilde{H}_*(B)$

Cohomology. $\quad \tilde{H}^*(F) \xrightarrow[\delta]{+1} H^*(E,F) \xleftarrow[p^*]{} \tilde{H}^*(B)$

E contractible $\Rightarrow$ δ, ∂ isomorphism

$\therefore \quad \tilde{H}_*(F) \overset{\sigma_*}{\underset{+1}{\dashrightarrow}} \tilde{H}_*(B) \quad$ homology suspension

$\tilde{H}^*(F) \overset{\sigma^*}{\underset{-1}{\dashleftarrow}} \tilde{H}^*(B) \quad$ cohomology suspension

In cohomology over $\mathbb{F}_2$, Sq^i commute with σ^* by the axioms. It is this suspension which is used in defining the family $\{Sq^i\}$.

$$\sigma^* : \tilde{H}^n(B) \longrightarrow \tilde{H}^{n-1}(F)$$

commutes with Sq^i. ~~which~~ This is used to define Sq^i.

Recall that for the first characteristic class, the fact that cohomology was homotopy was used, that is,

$$H^n(X, \Pi) = [X, K(\Pi, n)]$$

It is required to obtain:

$$\begin{array}{ccc} H^n(X, \mathbb{F}_2) & \xrightarrow[\text{natural in } X]{Sq^i} & H^{n+i}(X, \mathbb{F}_2) \\ \| \wr & & \| \wr \\ [X, K(\mathbb{Z}_2, n)] & \longrightarrow & [X, K(\mathbb{Z}_2, n+i)] \end{array}$$

Thus Sq^i should come from a homotopy class of maps

$$K(\mathbb{Z}_2, n) \longrightarrow K(\mathbb{Z}_2, n+i)$$

that is, Sq^i is a member of $[K(\mathbb{Z}_2, n), K(\mathbb{Z}_2, n+i)]$. But

$$[K(\mathbb{Z}_2, n), K(\mathbb{Z}_2, n+i)] \cong H^{n+i}(K(\mathbb{Z}_2, n), \mathbb{Z}_2)$$

Thus, instead of constructing operations, a cohomology class is constructed.

$$Sq^i \in H^{n+i}(K(\mathbb{Z}_2, n), \mathbb{Z}_2)$$

(not original method of Steenrod)

$\therefore$ <u>program</u>: construct $Sq^i \in H^{n+i}(K(\mathbb{Z}_2, n), \mathbb{Z}_2)$

and interpret as cohomology operations using fact:

$$[X, K(\pi, n)] \cong H^n(X, \pi)$$