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SUMMER WORKSHOP ON FIBRE BUNDLES AND GEOMETRY

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DIFFERENTIAL GEOMETRY

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Parallelism.

Theorem. Let P be a principal bundle on M and w a connection form on P. If $K = dw + (w,w)$ vanishes on M, then P is associated to a representation of the fundamental group of M in G.

Sketch of Proof. We will first deal with the local situation. Let U be a simply connected open subset of M. Then using a local section of P we may as well assume that w is a form on U, satisfying $dw + [w,w] = 0$. Then we claim that on a smaller open set U there is a function $\varphi : U \to G$ such that $\varphi^*\alpha = w$ where α is the Maurer-Cartan form on G. In fact, we will construct the graph of φ as follows. Consider in $U \times G$, the form $p_1^*w - p_2^*\alpha$. The distribution given by $\{X : (p_1^*w - p_2^*\alpha)(X) = 0\}$ is checked to be integrable using the Maurer-Cartan equations $d\alpha + [\alpha,\alpha] = 0$ and the assumption $dw + [w,w] = 0$. Hence by Frobenius theorem, there exists a maximal integral submanifold for this distribution. Taking this to be the graph of a function φ, we obtain the desired map. In fact, if the section of P over U is modified by φ, we will ~~in fact~~ have $s^*w = 0$. Moreover, it is easy to see that if φ, ψ are two G-valued functions on a connected U with $\varphi^*\alpha = \psi^*\alpha$, then $\varphi(x) = \psi(x)\cdot g$ for some $g \in G$. Hence if we have a covering of M by open sets U_i, sections $s_i : U_i \to P$ and functions $\varphi_i : U_i \to G$ satisfying $\varphi_i^*\alpha = s_i^* w$, then it follows that for every i,j with $U_i \cap U_j \neq \emptyset$, (which we may assume connected) the functions φ_i and φ_j differ by a group element g_{ij}. This shows that P is given by constant transition functions g_{ij} and it is easy to see that this implies our assertion.

Corollary 1. If P is a principal bundle on the unit interval, and w is a connection, then P may be trivialised over U such that w becomes trivial.

Proof. We notice that the curvature form of w is locally a 2-form on the 1-dimensional manifold and hence 0. By the theorem above we may conclude that there exists a section $s : I \to P$ such that $s^*w = 0$. This shows that with the trivialisation of $P = I \times G$ given by the section s, w is $p_2^*\alpha$, where α is the Maurer Cartan form.

Corollary 2. Let P be a principal G-bundle w a connection form, and $\gamma : I \to M$ a curve. If $\xi \in P$ with $\pi\xi = \gamma(0)$, then there exists a lift $\tilde{\gamma} : I \to P$ of γ such that $\tilde{\gamma}(0) = \xi$, and $\tilde{\gamma}^*w = 0$.

Proof. On the pull back bundle γ^*P, we have the pull back connection form w'. By Corollary 1, there exists a section s of γ^*P such that $s^*w' = 0$. This may be interpreted as the required lift.

If M is a manifold and we have a connection in a tensor bundle, then the above considerations show that given any curve and a tensor at the initial point, it can be carried along the curve. Thus in particular, a connection on the tangent bundle $T(M)$ enables one to talk of tangent vectors of M at two points of a curve being parallel. But this notion of parallelism is not absolute in the sense that it does not enable one to say that a vector at m is parallel to a vector at m'. The concept depends on the curve joining m and m'.

Getting back to the case of a principal G-bundle, if γ is a loop at a point m, then an initial frame ξ gives rise to a lift of γ and hence an end frame ξ' at the same point. In general, $\xi' \neq \xi$ and would be related by an element of the group G. Namely, there exists $g \in G$ with $\xi' = \xi g$. Then g is said to be the <u>holonomy</u> of the connection along the loop γ. As the loop γ is allowed to vary at a fixed point, the various elements obtained, starting with a fixed frame ξ form a group called the <u>holonomy group</u>. If the loops are only allowed to vary over homotopically trivial ones, then the corresponding group is called the <u>restricted holonomy group</u>.

Let P be a principal G-bundle, w a connection on P and $\xi \in P$. Consider the set P' of extremities of all horizontal paths in P starting at ξ. Let $\xi' \in P'$. Then over a locally trivial neighbourhood of $\pi\xi'$, we may take n linearly independent G-invariant horizontal vector fields $X_1, \ldots, X_n$ mapping on $\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}$ (where $(x_1, \ldots x_n)$ is a coordinate system at $\pi\xi'$. We have a map of $I \times \mathbb{R}^n$ into P' given by $(t, a_i) \to \varphi_t^{(a)}(\xi_o)$ where $\varphi_t^{(a)}$ is the flow corresponding to the vector field $\Sigma a_i X_i$. Since $\Sigma a_i X_i$ is horizontal, the curve $t \to \varphi_t^{(a)}(\xi_o)$ is also horizontal, proving that $\varphi_1^{(a)}(\xi) \in P'$. The map $(a) \to \pi\ \varphi_1^{(a)}(\xi)$ has as differential the map $(a_i) \to \Sigma a_i \frac{\partial}{\partial x_i}$ and is hence of maximal rank. In other words, we have a section over a neighbourhood of $\pi\xi'$ with image P'. Hence the transition functions for P with respect to such a family of local trivialisations will have images in the holonomy group at a point $\xi' \in P'$. Since ξ can

be joined to ξ' by a horizontal path, we can see easily that the holonomy group at ξ is the same as that at ξ'. Thus we have

<u>Proposition</u>. If $P' \subset P$ consist of extremeties of horizontal paths of a connection w on P starting from a fixed point ξ, local sections may be found with image in P' and the transition functions taking values in the holonomy group at ξ.

<u>Theorem</u> (Cartan-Ambrose-Singer) The restricted holonomy group is a Lie group whose Lie algebra is the subspace of the Lie algbera of the structure group spanned by the value K(X,Y) of the curvature form.

<u>Sketch of Proof</u>. Using the above proposition, one might replace P by P' so that without loss of generality, one may assume that any two points in P can be connected by a horizontal path, and that G = restricted holonomy group. One checks easily that the subspace V generated by K(X,Y) is actually a Lie subalgebra. Take the set of all vector fields on P such that the value of the connection form w is contained in V. One proves that this gives an integrable distribution. If P' is a maximal integral submanifold through ξ for this distribution, then by definition, horizontal paths through ξ are actually integral for this distribution.

Linear connections

Let M be a differentiable manifold. A connection on the tangent bundle $T(M)$ is called a <u>linear connection</u>.

Let $\xi = (m, v) \in T(M)$, where $m \in M$ and v a tangent vector at m. Using the connection, we can define a vector $G_\xi \in T_\xi(T(M))$ (horizontal lift) such that $T_\xi(\pi)(G_\xi) = v$. This vector field is called a <u>geodesic vector field</u> and the associated flow φ_t , <u>the geodesic flow</u>. The orbit of ξ under this flow projects onto a curve on M , starting from m with v as its vector at m. This is called the <u>geodesic from</u> m <u>with initial vector</u> v.

<u>Proposition</u>. The map $\xi \to \varphi_1(\xi)$ is a diffeomorphism of a neighbourhood of $0 \in T_m(M)$ with a neighbourhood of m in M , where $m = \pi\xi$.

<u>Proof</u>. Let U be a coordinate neighbourhood in M with $(x^1,\dots,x^n)$ as coordinate. Then on $\pi^{-1}(U)$, we have the coordinate system given by $(x^1,\dots,x^n, y_1,\dots,y_n)$, where any tangent vector at $(x^1,\dots,x^n)$ has the expression $\Sigma\, y_i \frac{\partial}{\partial x_i}$. Then one checks that the geodesic vector field G is given by $\Sigma\, y_i \frac{\partial}{\partial x_i} - \Sigma\, \Gamma^k_{ij} y^i y^j \frac{\partial}{\partial y^k}$ where Γ^k_{ij} are determined by $\nabla_{\frac{\partial}{\partial x_i}} \frac{\partial}{\partial x_j} = \sum_k \Gamma^k_{ij} \frac{\partial}{\partial x^k}$.

The differential at 0 of the map $(y^1,\dots,y^n) \to \varphi_1(x^1,\dots,x^n,y^1,\dots,y^n)$ can be computed to be $\frac{\partial}{\partial y^i} \to \frac{\partial}{\partial x^i}$ so that by inverse function theorem it follows that φ_1 gives a local diffeomorphism as claimed.

Now the geodesic vector field on $T(M)$ gives rise only to a local group of local automorphism so that in general geodesics cannot be continued indefinitely. We say a connection is <u>complete</u> if the geodesic vector field has a global flow.

Universal Connection

Let G be a Lie group and H a closed subgroup. Then one may actually show a) that H is a Lie subgroup b) G/H is a differentiable manifold, and c) $G \to G/H$ is a principal H-bundle. (In the cases where we will actually apply these, these facts may be directly verified). More generally, if H, H' are closed subgroups of G with H normal in H', then $G/H \to G/H'$ is a principal H'/H bundle. Consider the particular cases $G = O(n)$, $H' = O(r) \times O(n-r)$ and $H = O(n-r)$ and $G = U(n)$, $H' = U(r) \times U(n-r)$ and $H = U(n-r)$. Here H' is imbedded in G as matrices of the form $\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$, while H is imbedded in G as matrices of the form $\begin{pmatrix} I & 0 \\ 0 & B \end{pmatrix}$. Then we may identify G/H with the Stiefel manifold of orthogonal r-frames in $\mathbb{R}^n$ or $\mathbb{C}^n$, while the space G/H' may be identified [with the Grassmannian of r dimensional subspaces] of $\mathbb{R}^n$ or $\mathbb{C}^n$. In fact, the former identification maps any $X \in G$ on the first r column vectors. The latter associates to any $X \in G$ the subspace generated by the first r column vectors. The space of orthogonal r-frames is simply the space of (n,r) matrices Y such that $Y^*Y = I_r$. The action of H'/H may also be described as right multiplication by an (r,r) matrix g with $g^*g = I$. This bundle is called the <u>universal</u> bundle in view of the following.

<u>Theorem</u>. Given any principal $GO(r)$ (or $U(r)$)-bundle P on a manifold of dimension $\leq m$, there exists a equivariant map of P into the universal G-bundle (inducing a <u>classifying</u> map of M into the Grassmannian). Here the universal bundle means the above mentioned bundle for sufficiently

large ambient n depending on m.

<u>Sketch of Proof</u>. Let (U_i) be a covering of M such that P is trivial over each U_i. Then clearly we may map a section of P over U_i onto a constant matrix $\in G$ and extend this map to P as a G-equivariant map into the universal bundle. Now one uses the

<u>Lemma</u>. Given a covering $(U_i)_{i \in I}$, there exists a refinement $(V_i)_{i \in I}$ and a partition $I = I_o \cup \cdots \cup I_n$ such that $V_i \cap V_j = \phi$ if i and j belong to the same I_k. If U_i are such that P/U_i is trivial, then it is equally true that P/W_ℓ is also trivial, where $W_\ell = \bigcup_{i \in I_\ell} V_i$. Thus we have found an open covering $W_o, \ldots, W_n$ with P/W_ℓ is trivial. Choose a partition of unity with respect to the covering (W_ℓ) in the sense that $\Sigma \varphi_\ell^2 = 1$ and support $\varphi_\ell \subset W_\ell$. Then the map

$$f : x \mapsto \begin{pmatrix} \varphi_o & f_o \\ & \cdot \\ & \cdot \\ \varphi_n & f_n \end{pmatrix}$$

is easily seen to be a global differentiable map into the space of $((n+1)r, r)$ matrices. Here $f_o, \ldots, f_n$ are the local G-equivariant maps $P|U_i \to G$. Moreover, it is clear that $f^*f = \sum_{i=0}^{n} \varphi_i^2 f_i^* f_i = (\Sigma \varphi_i^2) I = I$. It is also obvious that f is a G-equivariant map into the universal bundle.

Now on the Stiefel bundle, we have a connection form given by $\omega_o = A^* dA$. Since $A^*A = I$, it follows that this connection has values in the Lie algebra of the orthogonal (or unitary) group. Moreover it is invariant under the action of the orthogonal (or unitary) group in n variables acting on the Stiefel manifold on the left. It is easily verified that this is a connection form. We will call this the <u>universal connection</u>. This is justified by the

<u>Theorem</u>. Given a principal G-bundle P on a manifold of dimension m and a connection form w P there exists a G-equivariant map φ of P into the Stiefel bundle such that $\varphi^* w_o = w$, for sufficiently large ambient n (depending only on m).

Even when the bundle P is trivial, this requires proof, for the connection could be nontrivial. Once the problem is solved locally, fortunately the use of partition of unity as in the previous theoem automotically takes care of the requirement about the pullback of the universal connection being w. The local problem reduces to the following. Given a matrix form w on an open set in $\mathbb{R}^n$ with values in the Lie algebra of G, (namely w is skew symmetric or skew Hermitian), one has to find an $(n,1)$-matrix valued function φ such that $\varphi^* d\varphi = w$ and $\varphi^* \varphi =$ Identity. We will not give the general proof, but indicate the idea by dealing with the case of $G = U(1)$. In this case $\varphi = (\varphi_1, \dots, \varphi_n)$ has values in $\mathbb{C}^n$ and we need $\Sigma \bar{\varphi}_k \, d\varphi_k = w$ and $\Sigma |\varphi_k|^2 = 1$. If we take $\varphi_k = r_k e^{i\theta_k}$, then

$$\bar{\varphi}_k \, d\varphi_k = r_k e^{-i\theta_k} \cdot (dr_k \, e^{i\theta_k} + i r_k \, e^{i\theta_k} d\theta_k)$$

$$= r_k \, dr_k + i r_k^2 \, d\theta_k$$

and $\Sigma \bar{\varphi}_k \varphi_k = \Sigma r_k^2 = 1$.

Thus we need to find functions r_k and θ_k such that $\Sigma r_k^2 = 1$ and $\Sigma r_k \, dr_k + i\Sigma \, dr_k^e \, d\theta_k = iw$. But $\Sigma r_k dr_k = 0$ if $\Sigma r_k^2 = 1$. Hence these conditions are equivalent to $\Sigma r_k^2 = 1$ and $\Sigma r_k^2 \, d\theta_k = w$. Since the integer n is at our disposal, we may write $w = i\sum_{k=1}^{n} f_k \, dX_k$ in local coordinates. By shifting a scalar to x_k, we may assume $f_k^2 < \frac{1}{n}$ then one may define $r_k = f_k$, $\theta_k = x_k$ and $f_{n+1} = -\sqrt{1 - \sum_{i=1}^{n} f_i^2}$, $\theta_{n+1} = 0$. This completes the proof. The general case is similar, but one has to take some care in view of noncommutativity.

Manifolds with additional structures.

In differential geometry, one considers differentiable manifolds endowed with some additional structure. This additional structure consists usually of a tensor field with some prescribed property. We will see some examples of this sort.

1. Riemannian metric.

If M is a manifold and g is a symmetric positive definite bilinear form on the tangent bundle of M, we say g is a Riemannian structure on M. More generally if we replace the positive definiteness postulated above by the weaker notion 'nondegeneracy', we get what is called a pseudo Riemannian structure.

The Euclidean space $\mathbb{R}^n$ has a natural Riemannian structure, namely, for any two tangent vectors $\Sigma a_i \frac{\partial}{\partial x_i}$, $\Sigma b_i \frac{\partial}{\partial x_i}$ at a point x, the scalar product is defined to be $\Sigma a_i b_i$. Another way of saying this is the following. Let V be a finite dimensional vector space over $\mathbb{R}$. Then any scalar product $(\, , \,)$ on the vector space V gives rise to a Riemannian structure on the manifold V. In fact at any point $v \in V$, the tangent space can be identified canonically with V so that a scalar product on it is readily available. This can be further generalised to the case of any Lie group G. The tangent space at any $g \in G$ is canonically identified with the Lie algebra $\mathfrak{g}$ of G. Hence any scalar product on $\mathfrak{g}$ gives rise to a Riemannian structure on G. This may be referred to as a left invariant Riemannian structure on G. One can of course define a right invariant Riemannian structure also in a similar way.

Clearly a submanifold of a Riemannian manifold acquires a natural Riemannian structure. (This is of course not true for pseudo Riemannian structures since the restriction to a subspace of a nondegenerate bilinear form may well be degenerate).

2. Symplectic structure.

Let w be an alternating 2-form on T which is nondegenerate. We call w an almost symplectic structure on M. The existence of such a form implies that the dimension of M is even. A very interesting situation in which such a structure arises is when the manifold is the total space of the cotangent bundle of a manifold N. Let then $M = T^*(N)$ and $\pi : M \to N$ be the natural projection. Any point of m consists of a point $n \in N$ and a differential α at n. If $X \in T_m(M)$ then $T_m(\pi)(X)$ is a tangent vector at the point $\pi m = n$ of N. We have thus a scalar $(T_m(\pi)(X), \alpha)$ associated to X. Since this is linear in X, it is clear that we get a canonical 1-form on $T^*(M)$. We will now get the local expression for this form. Let $(x_1, \ldots, x_n)$ be a coordinate system in an open set U of N. Then $\pi^{-1}(U)$ may be identified with $U \times \mathbb{R}^n$ by mapping $((x),(y)) \in U \times \mathbb{R}^n$ onto the differential $\Sigma y_i \, dx_i$ at the point $(x_1, \ldots, x_n)$. A basis for tangent vectors at any point $m \in \pi^{-1}(U)$ is obtained by $\frac{\partial}{\partial x_i}$, $\frac{\partial}{\partial y_i}$ under this identification. The canonical form referred to above associates to $\Sigma a_i \frac{\partial}{\partial x_i} + \Sigma b_i \frac{\partial}{\partial y_i}$ the scalar $(\Sigma a_i \frac{\partial}{\partial x_i}, \Sigma y_i dx_i) = \Sigma a_i y_i$ where (y_i) are coordinates of m. In other words, we obtain the differential form $\beta = \Sigma y_i dx_i$. Incidentally, this also shows that the canonical form is differentiable. Now $d\beta$ is a 2-form whose local expression is $\sum_{i=1}^{n} dy_i \wedge dx_i$. If we take the basis $\frac{\partial}{\partial x_i}, \frac{\partial}{\partial y_i}$ for the tangent space at m, the matrix of this bilinear form is $\begin{pmatrix} 0 & -1 \\ I & 0 \end{pmatrix}$ showing that it is nondegenerate as well. Thus we have a natural almost symplectic structure on $T^*(N)$.

__Definition__. An almost symplectic structure w is said to be __symplectic__ (or locally Hamiltonian) if $dw = 0$. It is globally Hamiltonial if there exists a 1-form β such that $d\beta = w$.

__Remark__. Notice that if $dw = 0$, locally w can be written as $d\beta$ for some 1-form β, by the Poincare lemma.

The canonical structure on $T^*(M)$ is by construction globally Hamiltonian. Then we have

__Theorem__ (__Darboux__). If β is a 1-form such that $d\beta$ is a nondegenerate skew symmetric 2-form, then locally coordinate $(X_1,\dots,X_n, Y_1,\dots,Y_n)$ can be chosen so that $\beta = \Sigma Y_i \, d X_i$.

Unlike the Riemannian situation, the above statement shows that any two symplectic structures are locally alike.

3. __Almost complex structure__

An automorphism J of the tangent bundle satisfying $J^2 = -1$ is called an __almost complex__ structure. A complex manifold, M is a

manifold in which the maps of an atlas are homeomorphism onto open sets of C^n and the transition maps as in 1) are holomorphic. We claim that M has a natural almost complex structure. In fact this would follow if we show a) that open set in C^n have a natural almost complex structure and b) complex analytic isomorphisms of one open set in C^n onto another preserved the almost complex structures. But assertion a) is trivial, since the tangent space has a natural identification with C^n and multiplication by $i = \sqrt{-1}$, gives rise to the required automorphism J. On the other hand, if $\varphi : U_1 \to U_2$ is a holomorphic map, in order to check that for any $x \in U_1$, $T_x(\varphi) : T_x(U_1) \to T_{\varphi(x)}(U_2)$ preserves the automorphism J, it is enough to show the same thing for the functions $p_i \varphi$ where p_i are the coordinate functions on U_2. Since the problem is local, we are reduced to the following situation. Let $f = \varphi + i\psi$ be a holomorphic function of $(z_1,\dots,z_n)$ with $z_\ell = x_\ell + iy_\ell$. Then

$$JT_x(f)(\frac{\partial}{\partial x_\ell}) = J(\frac{\partial\varphi}{\partial x_\ell}, \frac{\partial\psi}{\partial x_\ell}) = (-\frac{\partial\psi}{\partial x_\ell}, \frac{\partial\varphi}{\partial x_\ell}) \quad \text{and}$$

$$JT_x(f)(\frac{\partial}{\partial y_\ell}) = J(\frac{\partial\varphi}{\partial y_\ell}, \frac{\partial\psi}{\partial y_\ell}) = (-\frac{\partial\psi}{\partial y_\ell}, \frac{\partial\psi}{\partial y_\ell}) .$$

On the other hand,

$$T_x(f)\, J(\frac{\partial}{\partial x_\ell}) = T_x(f)(\frac{\partial}{\partial y_\ell}) = (\frac{\partial\varphi}{\partial y_\ell}, \frac{\partial\psi}{\partial y_\ell})$$

$$\text{and} \quad T_x(f)\, J(\frac{\partial}{\partial y_\ell}) = -T_x(f)\,(\frac{\partial}{\partial x_\ell}) = (-\frac{\partial\varphi}{\partial x_\ell}, -\frac{\partial\psi}{\partial x_\ell})$$

Hence what we need to verify is that

$$\frac{\partial\psi}{\partial x_\ell} = -\frac{\partial\varphi}{\partial y_\ell} \quad \text{and} \quad \frac{\partial\varphi}{\partial x_\ell} = \frac{\partial\psi}{\partial y_\ell} .$$

But these are the Cauchy-Riemann criterion for a differentiable function to be homomorphic. Thus we have shown

Proposition. Every complex manifold has a natural almost complex structure. Moreover a differentiable map $f : M \to N$ of complex manifolds is holomorphic if and only if $T_m(f)$ commutes with the automorphism J for every $m \in M$.

We will investigate the consequence of the complex structure on the deRham complex. If M has an almost complex structure, then we call a vector v in $T_m(M) \otimes \mathbb{C}$, of type (1,0) (resp. (0,1)) if $J_v = i_v$ (resp. $J_v = -iv$). Since $J^2 = -1$, the eigen spaces, $T^{1,0}$, $T^{0,1}$ corresponding to the eigenvalues $\pm i$, span $T_m(M) \times \mathbb{C}$. If M is a complex manifold with local coordinates $(z_1,\ldots,z_n) = (x_1,y_1,\ldots,x_n,y_n)$, then $T_m(M)$ has a basis consisting of $\left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial y_1}, \ldots, \frac{\partial}{\partial x_n}, \frac{\partial}{\partial y_n}\right)$. We also have $J\left(\frac{\partial}{\partial x_\ell}\right) = \frac{\partial}{\partial y_\ell}$ and $J\left(\frac{\partial}{\partial y_\ell}\right) = -\frac{\partial}{\partial x_\ell}$ so that $\frac{\partial}{\partial x_\ell} - i\frac{\partial}{\partial y_\ell}$ and $\frac{\partial}{\partial x_\ell} + i\frac{\partial}{\partial y_\ell}$ generate $T^{1,0}$ and $T^{0,1}$ respectively. We write $\frac{\partial}{\partial z_\ell} = \frac{\partial}{\partial x_\ell} - i\frac{\partial}{\partial y_\ell}$ and $\frac{\partial}{\partial \bar{z}_\ell} = \frac{\partial}{\partial x_\ell} + i\frac{\partial}{\partial y_\ell}$. Clearly $\left(\frac{\partial}{\partial z_\ell}, \frac{\partial}{\partial \bar{z}_\ell}\right)$ form a $\mathbb{C}$-basis for $T_m(M) \otimes \mathbb{C}$.

The decomposition $T(M) \otimes \mathbb{C} = T^{1,0} + T^{0,1}$ induces a decomposition $\mathcal{E}^r(M) \times \mathbb{C} = \sum_{p+q=r} \mathcal{E}^{p,q}(M)$ of the space of differential forms of degree r. We may say that a differential form w is of type (p,q) if $w(X_1,\ldots,X_r) = 0$ whenever more than p of the X_i's are of type (1,0) or more than q of the X_i's are of type (0,1). Again

in its local expression, w is of type (p,q) if and only if it is of the form

$$\sum_{I,J} f_{I,J}\, dz_I \wedge d\bar{z}_J = \sum_{\substack{i_1 < \cdots < i_p \\ j_1 < \cdots < j_q}} f_{i_1\ldots i_p, j_1 \ldots j_q}\, dz_{i_1} \wedge \cdots \wedge dz_{i_p} \wedge d\bar{z}_{j_1} \wedge \cdots \wedge d\bar{z}_{j_q}.$$

We may extend the exterior derivative d over complex differential forms $\mathbb{C}$-linearly. Then it is easy to see that

$$d\left(\Sigma f_{I,J}\, dz_I \wedge d\bar{z}_J\right) = \Sigma\left(\pm \frac{\partial f_{I,J}}{\partial z_i} dz_{i \cup I} \wedge d\bar{z}_J \pm \frac{\partial f_{I,J}}{\partial \bar{z}_j} dz_I \wedge d\bar{z}_{j \cup J}\right)$$

This shows that if w is of type (p,q) then dw is a sum of two forms, one of type $(p+1,q)$ and another, of type $(p,q+1)$. These two forms may be denoted $d_z w$ and $d_{\bar{z}} w$. The equation $d^2 = 0$ translates into $d_z^2 = 0$, $d_{\bar{z}}^2 = 0$ and $d_z d_{\bar{z}} + d_{\bar{z}} d_z = 0$.

Definition. The complex

$$0 \longrightarrow \mathcal{E}^{p,0} \xrightarrow{d_{\bar{z}}} \mathcal{E}^{p,1} \xrightarrow{d_{\bar{z}}} \mathcal{E}^{p,2} \longrightarrow \cdots$$

is called the Dolbeault complex of the complex manifold M. Its cohomology are called the Dolbeault cohomology.

The 0th cohomology of the Dolbeault complex consists of $(p,0)$ forms α satisfying $d_{\bar{z}}\alpha = 0$. Locally these are of the form

$$\alpha = \Sigma f_I\, dz_I$$

so that $$d_{\bar{z}}\, \alpha = \sum \frac{\partial f_I}{\partial \bar{z}_j} d\bar{z}_j \wedge dz_I.$$

The condition $d_{\bar{z}}\alpha = 0$ is thus equivalent to saying that $\partial f_I / \partial \bar{z}_j$ for every I or what is the same, all f_I are holomorphic.

Definition. A form of type $(p,0)$ is said to be a holomorphic p-form if it is locally expressible as $\Sigma f_I \, dZ_I$ where f_I are holomorphic.

4. Hermitian structure.

Suppose M has an almost complex structure and also a Riemannian structure g. Then we say g is Hermitian if $g(JX, JY) = g(X,Y) + ig(X,JY)$, then h is a Hermitian metric on the tangent space considered as a $\mathbb{C}$-vector space where multiplication by i is defined as J. For,

$$\begin{aligned} h(JX,Y) &= g(JX,Y) + ig(JX,JY) \\ &= ig(X,Y) - g(X,JY) \\ &= ih(X,Y) \end{aligned}$$

and

$$\begin{aligned} h(X,JY) &= g(X,JY) - ig(X,Y) \\ &= -ih(X,Y) . \end{aligned}$$

Notice that the Riemannian metric is uniquely determined by h or equally well by the alternating form $\Omega(X,Y) = g(X,JY)$. It is clear that Ω is nondegenerate so that we have a natural almost symplectic structure associated to g and J.

Definition. A complex manifold with a Hermitian structure is said to be Kahler if the associated alternating form Ω is closed.

The importance of Kahler manifolds is due to fact that all nonsingular projective algebraic varieties are Kahlerian. By this we mean complex submanifolds of the complex projective space $P\mathbb{C}^n$ are

Kahlerian. In fact, since it is obvious that any complex submanifold of a Kahler manifold acquires a natural Kahler structure, we have only to check that $P\mathbb{C}^n$ is Kahlerian. If $(Z_0,Z_1,\ldots,Z_n)$ is a system of homogeneous coordinates, then the form obtained by piecing together the forms on $U_k=\{Z_k\neq 0\}$ having the expression $i\, d''_z d'_z \log \sum_{j=0}^{n} |x_k^{-1} x_j|^2$ is seen to be the alternating form associated to a Kahler metric on $P\mathbb{C}^n$.

Gauss Bonnet Theorem.

Let X be a compact, oriented Riemannain manifold. Then the tangent bundle of oriented orthogonal frames is a principal SO(n)-bundle. The Riemannian connection gives rise to a curvature form which according to the Chern Weil Theory, gives rise to various closed forms on X, whose de Rham cohomology classes are topological invariants of the tangent bundle. One particular form which may be obtained thus is the following.

Consider the space so(n) of Skew symmetric endomorphisms of an oriented vector space V with a metric g. Then the determinant which is a polynomial function on all endomorphisms, when restricted to so(n) becomes a perfect square. The orientation on V may be used to define a square root, called the Pfaffian of the skew symmetric endomorphism. In fact, since all eigenvalues are purely imaginary, we have a decomposition $V = \sum_{\alpha \geq 0} V_\alpha$, where V_α is the space on which the transformation has 2 eigenvalues $i\alpha$ and $-i\alpha$. We define (Pfoff) $= \pm\, \alpha^{\frac{1}{2}\dim V_\alpha}$. The sign is ± 1 according as the orientation on V determined by the decomposition $V = \Sigma(V_\alpha^+ + V_\alpha^-)$ coincides with the given one or not. By definition it is clear that $(\text{Pfaff})^2 = \text{Det}$. It is easy to see that if A is a skew symmetric transformation and P is an orthogonal transformation then $\text{Pfaff } A = (\text{Pfaff})PAP^{-1}$.

Now the substitution of a curvature form of a principal SO(n)-bundle gives rise to a closed 2n-form w on the base. In particular, if M is a compact, oriented, Riemannian manifold of dimension n even, then $\int_M (\text{Pf } K)$ gives a number. This is independent of the Riemannian connection, since we know that $\text{Pf } K - \text{Pf } K' = d$ for some (n-1) form α and $\int_M d\alpha = 0$, by Stokes' formula.

Since this number is intrinsically associated to M, it is not surprising that it should have other geometric meanings. The Gauss Bonnet theorem identifies this number as the 'Euler characteristic' of M. We will give a geometric definition of the latter and then identify the two definitions.

Let X be a vector field on M. A point $m \in M$ is a singular point of X if $X_m = 0$. At any singular point, consider the map $v \xrightarrow{H_X} [Y,X]_m$ where $v \in T_m(M)$ and Y is any vector field with $Y_m = v$. To see this makes unambiguous sense, we have to verify that if $Y_m = 0$, then $[Y,X]_m = 0$. This is of course the case since $X_m = 0$. In fact, if we have locally $X = \Sigma f_i \frac{\partial}{\partial x_i}$, $Y = \Sigma g_i \frac{\partial}{\partial x_i}$ with $f_i(n) = g_i(m) = 0$, then $[Y,X] = \sum_i \sum_j (g_j \frac{\partial f_i}{\partial x_i} - f_j \frac{\partial g_i}{\partial x_i})\frac{\partial}{\partial x_i}$. Hence $[Y,X]_m = 0$. We say the singularity m is nondegenerate if the endomorphism $H_X : T_m(X) \to T_m(M)$ is invertible. Moreover its index is defined to be ± 1 according as $\det H_X \gtrless 0$. In local coordinates, the transformation H_X is given by the matrix $(\frac{\partial f_i}{\partial x_j})$, according to the calculation above.

Lemma. There exists a nondegenerate vector field. In fact, there exists a nondegenerate vector field such that locally near any singular point, it has the expression $\Sigma x_i \frac{\partial}{\partial x_i}$ if the index is positive and $\sum_{i=1}^{n-1} x_i \frac{\partial}{\partial x_i} - x_n \frac{\partial}{\partial x_n}$, if the index is negative.

We now take a vector field as in the Lemma and choose a convenient linear connection on M. If $U_1,\ldots,U_r$ are disjoint closed neighbourhoods of the critical points, then on $B - U_i$, the vector field is nonsingular so that we have a decomposition $T(M) = 1 + W$, on this set. Choose a connection in W and treat it as a linear connection. On the other hand on open neighbourhoods $V_i \subset U_i$, we may take Riemannian connections and choose a partition of unity to piece up all these connections to a single linear connection on M. If we substitute its curvature in the Pfaffian, it becomes zero on $M - U_i$. Thus the required form is supported in V_i. Suppose these forms are w_i. Then $\int_M w = \Sigma \int_{V_i} w_i$. Thus we are reduced to the following situation. Let U be an open neighbourhood of 0 in $\mathbb{R}^n$ and X a nondegenerate vector field with only singularity at 0 in U. We may also assume that X has the form $\Sigma x_i \frac{\partial}{\partial x_i}$ or $\sum_{i=1}^{n-1} x_i \frac{\partial}{\partial x_i} - x_n \frac{\partial}{\partial x_n}$. Taking a reduction of the tangent bundle to the orthogonal group compatible with the splitting of the tangent bundle given by the vector field outside a neighbourhood of the singular point. Compute $\int w$ where w is the form obtained by substituting the curvature form in the Pfaffian.

In any case, the above recipe says that the local contributions depend only on the nature of X in a neighbourhood of the singular point. Thus when X is of the special type mentioned, it follows that there are only two constants to be computed. This could be done by direct calculation. But one might also argue as follows. On the

n-dimensional torus T^n, consider the vector field $\sum_i \sin \theta_i \frac{d}{d\theta_i}$. Now since T^n is a group, it is parallelisable and hence one can use a flat metric and see that $\int w = 0$. On the other hand, it has 2^n singular points, which can be divided into two sets of 2^{n-1} each, where it takes the one or the other form. This shows that the constants involved are the negatives of each other for the two types mentioned. Finally, one has only to check that the constant involved is nonzero for some manifold with a vector field which is of the form $\Sigma x_i \frac{\partial}{\partial x_i}$ near all singular points. For example, S^n, n even is such a manifold. This completes the proof of Gauss Bonnet theorem. For any compact oriented manifold of even dimension, the number $\int w$, where w is the form obtained by substituting the curvature form in the Pfaffian is, upto a constant, the Euler characteristic of M. The latter may be taken to mean the sum with proper signs of the singularity of any special vector field on M.

Riemannian connection

Let M be a Riemannian manifold and g the Riemannian metric. We wish to construct a linear connection on M which has no torsion, and which preserves g under parallel translation. The latter condition is equivalent to the requirement that $\nabla_X g = 0$. Explicitly,

$$g(\nabla_X Y, Z) + g(Y, \nabla_X Z) - Xg(Y, Z) = 0$$

for all vector fields X,Y,Z .

Let us assume that in local coordinates, we have $g = \sum g_{ij}\, dx^i dx^j$ and $\nabla_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j} = \sum \Gamma^k_{ij} \frac{\partial}{\partial x^k}$. Then in the above equality we may take $X = \frac{\partial}{\partial x^i}$, $Y = \frac{\partial}{\partial x^j}$, $Z = \frac{\partial}{\partial x^k}$ to obtain

$$g\Big(\sum \Gamma^\ell_{ij} \frac{\partial}{\partial x^\ell}, \frac{\partial}{\partial x^k}\Big) + g\Big(\frac{\partial}{\partial x^j}, \sum \Gamma^\ell_{ik} \frac{\partial}{\partial x^\ell}\Big) - \frac{\partial}{\partial x^i} g\Big(\frac{\partial}{\partial x^j}, \frac{\partial}{\partial x^k}\Big) = 0$$

i.e. $$\sum_\ell \Gamma^\ell_{ij}\, g_{\ell,k} + \sum_\ell \Gamma^\ell_{ik}\, g_j - \frac{\partial g_{j,k}}{\partial x^i} = 0 .$$

Write three such equations by cyclically permuting i,j,k , adding the first two and substracting the last to get

$$\sum_\ell \Gamma^\ell_{ij}\, g_{\ell,k} + \sum \Gamma^\ell_{jk}\, g_{\ell i} - \sum \Gamma^\ell_{ki}\, g_{\ell j} + \sum_\ell \Gamma^\ell_{ik}\, g_{j\ell} + \sum_\ell \Gamma^\ell_{ji}\, g_{k\ell} - \sum \Gamma^\ell_{kj}\, g_{i\ell}$$
$$= \frac{\partial g_{j,k}}{\partial x^i} + \frac{\partial g_{k,i}}{\partial x^j} - \frac{\partial g_{i,j}}{\partial x^k} .$$

The condition that the torsion of ∇ is 0 implies that

$\nabla_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j} = \nabla_{\frac{\partial}{\partial x^j}} \frac{\partial}{\partial x^i}$, or what is the same $\Gamma^k_{ij} = \Gamma^k_{ji}$.

Hence the above equation yields

$$2 \sum \Gamma^\ell_{ij}\, g_{k,\ell} = \frac{\partial g_{j,k}}{\partial x^i} + \frac{\partial g_{k,i}}{\partial x^j} - \frac{\partial g_{i,j}}{\partial x^k} .$$

Since (g_{ij}) is invertible, this determines Γ^k_{ij} completely and we may as well define Γ^k_{ij} by this equation and check that the linear connection defined by it has no torsion and preserves g. Although this was done locally, the uniqueness ensures that they would coincide on the overlaps, thereby giving a global connection. Thus we have

Theorem. Any Riemannian structure on M gives rise to a unique connection which preserves the metric and has torsion 0. Whenever one takes of a geodesic, parallel translate, etc. of a Riemannian manifold, it is this Riemannian connection that one has in mind.

Let us now consider a differentiable manifold M imbedded in $\mathbb{R}^n$. The Euclidean metric on $\mathbb{R}^n$ gives M a Riemannian structure. It is easy to see how the covariant differentiation of vector fields is defined following the Riemannian connection. In fact, given a vector field X on M , let us extend it locally a neighbourhood of M in a neighbourhood in M as a vector field $\tilde{X}$ of $\mathbb{R}^N$. If v is a tangent vector at $m \in M$, then treating v as a vector in $\mathbb{R}^N$, we may define the covariant derivative of $\tilde{X}$. However the resulting vector at m may not be tangential to M . But using the metric we may project it down to $T_m(M)$. This is easily checked to be the covariant differentiation as we have defined.

We have on the other hand, a natural differentiable map of M into the Grassmannian of r-dimensional subspaces of $\mathbb{R}^n$. This is called the Gaussian map. On the Grassmannian we have seen that there is a natural $O(r)$ bundle. Since any orthonormal tangent frame at $m \in M$ gives rise to an r-orthonormal tangent frame in $\mathbb{R}^n$, we get a map of the principal tangent bundle P into the Stiefel bundle. Now the universal connection on the Stiefel bundle gives rise to a connection on P. It is of course natural to expect that this is actually the Riemannian connection. In fact, it is clear that the trivial $O(n)$ bundle on the Grassmannian is obtained by taking the Maurer Cartan form on $O(n)$ and pulling it back on $Gr \times O(n)$. Now the natural projection of the Lie algebra $so(n) \twoheadrightarrow so(r) \times so(n-r)$, gives rise to a $so(r) \times so(n-r)$-valued form. One checks directly the universal connection on Grass is obtained by taking the trivial $O(n)$ connection and projecting down to an $O(r)$ connection form. Thus we have only to check that on M, if we take the trivial $O(n)$-bundle (restriction of the tangent bundle on $\mathbb{R}^n$ to M) and project it down to an $O(r)$-connection on M, then we obtain the Riemannian connection. This is only a restatement of the alternative description of covariant differentiation of an imbedded manifold. Thus we have

Theorem (Gauss' theorema egregium).

Let M be a submanifold of $\mathbb{R}^n$. If φ is the Gauss map $M \to \mathrm{Grass}_r(\mathbb{R}^n)$, then $\varphi^* K$ is the curvature form for the induced

Riemannian metric on M, where K is the curvature form of the universal connection on $\mathrm{Grass}_r(\mathbb{R}^n)$. In particular, if M is an oriented hypersurface in $\mathbb{R}^n$, the Gauss map may be considered to be a map into S^{n-1}. The universal bundle is the tangent bundle of S^{n-1} and the universal connection is the Riemannian connection of S^{n-1}. Thus the Riemannian curvature on M is the pullback by φ of the Riemannian curvature on S^{n-1}.