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34100 TRIESTE (ITALY) - P.O.B. 586 - MIRAMARE - STRADA COSTIERA 11 - TELEPHONES: 224281/2/3/4/5 6
CABLE: CENTRATOM - TELEX 460392-I

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SUMMER WORKSHOP ON FIBRE BUNDLES AND GEOMETRY

(5 - 30 July 1982)

AN INTRODUCTION TO VECTOR BUNDLES

J.-M. LEMAIRE

Institut de Mathématiques
et Sciences Physique
Université de Nice
Parc Valrose
06034 Nice Cedex
France

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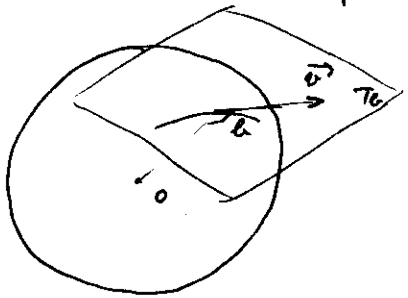
An Introduction to Vector Bundles

J.M. LEMAIRE

§1. Motivation

1.1 An example

Consider the unit sphere S^2 in 3-space $\mathbb{R}^3$. It is well-known that the tangent plane T_b at a point b of S^2 is the set of all possible velocity vectors at b of parametrized curves drawn on S^2 and passing through b.



Thus, at each point b of S^2 is attached a 2 dim'l vector space T_b (over the real numbers) and clearly enough, T_b "depends continuously" on b. We want to give a precise definition of what "continuously" means here.

idea a/ one may consider the set $G_2(\mathbb{R}^3)$ of all 2 dim'l planes in $\mathbb{R}^3$ – G stands for Grassmann, who is credited with first considering this set. We should define a reasonable topology on $G_2(\mathbb{R}^3)$ such that the map

$$\begin{aligned} : S^2 &\longrightarrow G_2(\mathbb{R}^3) \\ b &\longmapsto T_b \end{aligned}$$

is continuous: this is the most simple-minded way of defining a family of vector spaces parametrized by S^2.

idea b/ It is probably less straightforward, but more fruitful. Consider the disjoint union

$$TS^2 = \coprod_{b \in S^2} T_b$$

of all tangent planes to S^2, and the map

$$p : TS^2 \longrightarrow S^2$$

which maps each vector in T_b to b. In order to specify that if b is close to b', then T_b "is close" to $T_{b'}$, we may give TS^2 a certain topology such that the map p is continuous.

Actually both points of view are interesting and complementary to each other:

$p : TS^2 \to S^2$ is the *tangent bundle* of S^2

$g : S^2 \to G_2(\mathbb{R}^3)$ is a *Gauss map* for this bundle

1.2. Let us compare a/ and b/ for a *constant* family of vector spaces parametrized by a space B. Viewpoint a/ yields the constant map

$$B \longrightarrow \{V\}$$

which assigns V to every b in B. But viewpoint b/ yields the projection

$$p_1 : B \times V \longrightarrow B$$

onto the first factor. Now the reasonable topology one can put on $B \times V$ is the product topology.

Back to our initial example, take $B = S^2$ and $V = \mathbb{R}^3$. Then TS^2 appears as a subset of $S^2 \times \mathbb{R}^3$, namely

$$TS^2 = \{ (b, v) \in S^2 \times \mathbb{R}^3 \mid b \perp v \}$$

and $p : TS^2 \to S^2$ is the restriction of the first projection p_1 to TS^2:

$$\begin{tikzcd} TS^2 \arrow[rr, hook, "\text{incl.}"] \arrow[dr, "p"'] & & S^2 \times \mathbb{R}^3 \arrow[dl, "p_1"] \\ & S^2 & \end{tikzcd}$$

It is therefore natural to put on TS^2 the induced topology of the subspace TS^2 of $S^2 \times \mathbb{R}^3$. We here obtain an instance of what we shall call a subbundle of a product bundle.

1.3 Observe that all the above depends on the imbedding $S^2 \subset \mathbb{R}^3$: if we slightly deform the imbedding, the map $S^2 \to G_2(\mathbb{R}^3)$ will be deformed accordingly, but the tangent bundle $TS^2 \to S^2$ should not change up to isomorphism – because this tangent bundle can be defined abstractly, regardless of the embedding.

We now have to set up the appropriate definitions, so that we can pass easily from one viewpoint to the other, in such a way that homotopies of Gauss maps will correspond to isomorphisms of (vector) bundles –

§2. Bundle constructions and vocabulary

2.1 To start with, a bundle is nothing but a map $p : E \to B$, and the use of the word "bundle" is meant to emphasize the partition of E into the inverse images of the points of B

$$E = \coprod_{b \in E} p^{-1}(b)$$

$p^{-1}(b)$ is called the fibre over b
E is the total space
B is the base space

and it is sometimes convenient to use a different letter, such as ξ to denote the data (E, p, B)

2.2 Definition. If $\xi = (E, p, B)$, $\xi' = (E', p', B)$ are two bundles over B, a bundle map (or a B-map) is a map $f : E \to E'$ such that $p' \circ f = p$:

$$\begin{tikzcd} E \arrow[rr, "f"] \arrow[dr, "p"'] & & E' \arrow[dl, "p'"] \\ & B & \end{tikzcd}$$

The two bundles ξ, ξ' are isomorphic (more precisely B-isomorphic) if there exist bundle maps f and f' such that $f \circ f' = id_{E'}$, $f' \circ f = id_E$

$$\begin{tikzcd} E \arrow[rr, shift left, "f"] \arrow[dr, "p"'] & & E' \arrow[ll, shift left, "f'"] \arrow[dl, "p'"] \\ & B & \end{tikzcd}$$

More generally, if $g : B' \to B$ is a map and $\xi' = (E', p', B')$, $\xi = (E, p, B)$ are bundles, a bundle map over is a commutative square:

$$\begin{tikzcd} E' \arrow[r, "f"] \arrow[d, "p'"] & E \arrow[d, "p"] \\ B' \arrow[r, "g"] & B \end{tikzcd}$$

Particular instances of bundle maps are inclusions of subbundles

$$\begin{tikzcd} E' \arrow[r, hook] \arrow[dr, "p/E = p'"'] & E \arrow[d, "p"] \\ & B \end{tikzcd}$$

or restricted bundles

$$\begin{tikzcd} p^{-1}(B') \arrow[r, hook] \arrow[d, "p/p^{-1}(B')"'] & E \arrow[d, "p"] \\ B' \arrow[r, hook] & B \end{tikzcd}$$

If $\xi = (E, p, B)$, we will simply denote by $\xi|B'$ the restricted bundle $(p^{-1}(B'), p|p^{-1}(B'), B')$

2.3. Definition. let $p : E \to B$, $f : B' \to B$ be maps. The fibre product of p and f is the subspace $B' \times_B E$ of $B' \times E$ defined by
$$B' \times_B E = \{ (b', e) \in B' \times E \mid f(b') = p(e) \}$$
we have the commutative square

$$\begin{tikzcd} B' \times_B E \arrow[r, "p_2"] \arrow[d, "p_1"] & E \arrow[d, "p"] \\ B' \arrow[r, "f"] & B \end{tikzcd}$$

where p_1 and p_2 are the two projections, restricted to $B' \times_B E$.

If we think of $\xi = (E, p, B)$ as a bundle, then $(B' \times_B E, p_1, B')$ is denoted $f^*(\xi)$ as a bundle:

2.4 Definition: $f^*(\xi)$ is the bundle induced from ξ by f. One also says that $f^*(\xi)$ is the pullback of ξ.

2.5 Observations (left to be checked by the reader)

a/ The fibres of $f^*(\xi)$ over $b' \in B'$ and of ξ over $f(b') \in B$ are "the same".

b/ If ξ is (isomorphic to) a product bundle, so is $f^*\xi$.

c/ Compare $g^*(f^*\xi)$ and $(f \circ g)^*(\xi)$

d/ Let $i : A \hookrightarrow B$ be the inclusion of a subspace. Compare $i^*\xi$ and ξ / A.

e/ Assume $f : B' \to B$ maps every $b' \in B$ to a single point $b_0 \in B$. Compare:
$$f^*\xi \quad \text{and} \quad B' \times p^{-1}(b_0)$$

2.6 Definitions

- a bundle over B is trivial if it is B-isomorphic to a product bundle
- a bundle ξ over B is locally trivial if B admits an open covering $\mathcal{U}$ such that, for every $U \in \mathcal{U}$, ξ/U is trivial.

More generally one can define locally isomorphic bdles in the same way.

Observe that for a connected B, a bundle over B is locally trivial iff it is locally isomorphic to a trivial bundle (this is not a completely trivial statement!)

2.7. Example

The Möbius strip and the cylinder, projected onto the middle circle, provide an example of two locally isomorphic, non isomorphic bdls

2.8. Cross-sections

A cross-section (or simply a section) of a bdle $\xi : (E, p, B)$ is a continuous map $s : B \to E$ such that $p \circ s = id_B$.

To be checked:

- identify a section of a product bundle with a map from the base into the fibre.
- if ξ has a section, so has $f^*\xi$: construct it.
- interpret sections of the Möbius bdle above as antiperiodic functions on $\mathbb{R}$. Show that the Möbius bdle has no section which is everywhere non-zero.

§3. Vector bundles

3.1. According to idea b/ in the first paragraph, a vector bundle should be a bundle of the form

$$E = \coprod_{b \in B} F_b \longrightarrow B$$

where F_b is a (real or complex) vector space for each b, and E is given a suitable topology. The least we can assume on this topology is that addition of vectors is continuous where it is defined (i.e. for couples of vectors lying in the same fibre), and scalar multiplication is also continuous. We shall call such a bundle a vector space over B: in other words

3.2 Definition: a K-vector space over B is a bundle $\xi : E \xrightarrow{p} B$ together with continuous maps

$$+ : E \times_B E \longrightarrow E \quad \text{(addition)}$$

$(K = \mathbb{R}$ or $\mathbb{C})$ $$\odot : K \times E \longrightarrow E \quad \text{(scalar mult.)}$$

which satisfy the obvious axioms analogous to the vector space ones.

A bundle map $f : E \to E'$ between two vector spaces over B is linear if the restriction to each fibre is linear

$$\forall b \in B, \quad f|p^{-1}(b) \in \mathcal{L}(p^{-1}(b), p'^{-1}(b)).$$

Natural examples of vector spaces over a space which actually arise satisfy the local tri... condition we now introduce: moreover, this latter condition ensures a neat classification theorem (at least if the base is compact)

3.3. Definition. A vector bundle over B is a vector space over B which is locally linearly trivial.

To wit, observe that the restriction of a vector space over B to a subspace of B is again a vector space over this subspace. Def 3.3. therefore means that there exists a covering $\mathcal{U}$ of B by open sets such that, for every $U \in \mathcal{U}$, we have a linear isomorphism φ_U

$$\begin{tikzcd} p^{-1}(U) \arrow[rr, "\cong", "\varphi_U"'] \arrow[dr, "p"'] & & U \times K^n \arrow[dl, "p_1"] \\ & U & \end{tikzcd} \quad (K = \mathbb{R}, \mathbb{C})$$

where linear means that

$\forall b \in B$, $\varphi_U \mid p^{-1}(b)$ is a linear isomorphism of $p^{-1}(b)$ to K^n.

3.4. observations

(3.4.1) The dimension of the fibre of a vector bundle is a locally constant function on B. Therefore if B is connected, we can speak of the rank of a vector bdle over B, which is the dimension of any fibre.

(3.4.2) If ξ is a vector bdle over B and $f: B' \to B$ is a map, $f^*\xi$ is a v.b. over B' which has the same rank. Indeed, if $\mathcal{U}$ is a trivializing covering of B for ξ, its inverse image by f is a trivializing covering of B' for $f^*\xi$.

3.5. Examples

3.5.1 The tangent bdle of a manifold: by construction (see the Diff. Geom. course), any atlas for the manifold trivializes the tangent bdle.

3.5.2 The tautological bundle over a grassmanian

Let $G_n(\mathbb{R}^m)$ be the set of all n-dim'l vector subspaces of $\mathbb{R}^m$ (the same holds for $\mathbb{C}$ of course)

We topologize $G_n(\mathbb{R}^m)$ as follows

Let $V'_n(\mathbb{R}^m)$ be the set of n-tuples

$$V'_n(\mathbb{R}^m) = \left\{ (v_1, \dots, v_n) \in (\mathbb{R}^m)^n \,\middle/\, \begin{array}{l}\text{the } v_i\text{'s are}\\ \text{lin. independent}\end{array} \right\}$$

Clearly enough, $V'_n(\mathbb{R}^m)$ is an open subset of $(\mathbb{R}^m)^n = \mathbb{R}^{mn}$, because the condition on the v_i's is equivalent to one the $n \times n$ minor determinants of the matrix of the v_i's being non-zero, which is an open condition.

There is a map

$$\pi: V'_n(\mathbb{R}^m) \longrightarrow G_n(\mathbb{R}^m)$$

which assigns to each system (v_i) its span

Now we give $G_n(\mathbb{R}^m)$ the finest topology so that π is continuous. In other words, a subset $U \subset G_n(\mathbb{R}^m)$ is open iff $\pi^{-1}(U)$ is open, or more naively, two n dim'l subspaces in $\mathbb{R}^m$ are near if they admit bases which are near (in the usual topology for $(\mathbb{R}^m)^n = \mathbb{R}^{mn}$)

An alternative approach to the Grassmann manifold is to consider the Stiefel manifold $V_n(\mathbb{R}^m)$ of orthonormal n-frames in $\mathbb{R}^m$, that is

$$V_n(\mathbb{R}^m) = \{ (v_1, \dots, v_n) \in (\mathbb{R}^m)^n \mid \langle v_i, v_j \rangle = \delta_{ij} \}$$

Observe that $V_n(\mathbb{R}^m) \subset V_n'(\mathbb{R}^m)$ and is closed and bounded in $\mathbb{R}^{mn}$, and therefore compact. The restriction of π to $V_n(\mathbb{R}^m)$ is a continuous surjection, therefore $G_n(\mathbb{R}^m)$ is compact (check it is Hausdorff!). See the diff. geom course for the diff. manifold structure on $G_n(\mathbb{R}^m)$.

[One may also look upon $G_n(\mathbb{R}^m)$ as the orbit space $V_n'(\mathbb{R}^m)/GL(n;\mathbb{R})$, or $V_n(\mathbb{R}^m)/O(n)$. Note $V_n(\mathbb{R}^m) = O(m)/O(m-n)$ and $V_n'(\mathbb{R}^m) = GL(m;\mathbb{R})/B$, where B is the group of matrices of the form:

$$\left(\begin{array}{c|c} \begin{matrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{matrix} & 0 \\ \hline \begin{matrix} \times & \cdots & \times \\ \vdots & & \vdots \\ \times & \cdots & \times \end{matrix} & GL(m-n) \end{array}\right)$$

].

Let us now define the tautological bundle γ_n^m (or canonical, or universal). This is the subbundle of the trivial bundle $G_n(\mathbb{R}^m) \times \mathbb{R}^m$ defined by

$$E_n(\mathbb{R}^m) = \{ (H, v) \in G_n(\mathbb{R}^m) \times \mathbb{R}^m \mid v \in H \}$$

$$\begin{array}{ccc} E_n(\mathbb{R}^m) & & (H,v) \\ p \downarrow \gamma_n^m & & \downarrow \\ G_n(\mathbb{R}^m) & & H \end{array}$$

It is tautological in the sense that its fibre over $H \in G_n(\mathbb{R}^m)$ is the vector space H !

Observe that the total space $E_n(\mathbb{R}^m)$ is the graph of the relation $\ni$ in $G_n(\mathbb{R}^m) \times \mathbb{R}^m$.

3.5.3. Proposition. γ_n^m is locally trivial.

Proof: Let $H_0 \in G_n(\mathbb{R}^m)$, and let

$$U_{H_0} = \{ H \in G_n(\mathbb{R}^m) \mid H \cap H_0^\perp = 0 \}$$

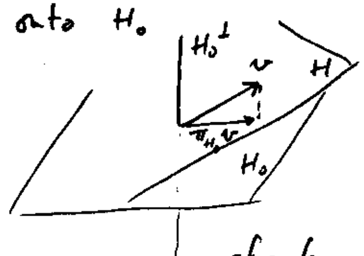
U_{H_0} is the set of those n dim'l subspaces which project isomorphically onto H_0 by the orthogonal projection along $H_0^\perp$ onto H_0.

Check that U_{H_0} is an open neighbourhood of H_0. Now define a U_{H_0}-isomorphism of bundles

$$\begin{tikzcd} p^{-1}(U_{H_0}) \arrow[rr, "\varphi_{H_0}"] \arrow[dr] & & U_{H_0} \times H_0 \arrow[dl] \\ & U_{H_0} & \end{tikzcd}$$

by $\varphi_{H_0}(H, v) = (H, \pi_{H_0} v)$

where $v \in H$ and π_{H_0} is the orth. projection onto H_0



Note $\varphi_{H_0}^{-1}$ can be defined by

$$\varphi_{H_0}^{-1}(H, w) = (H, v)$$

where $\{v\} = (w + H_0^\perp) \cap H$

check continuity ---

3.5.4. Remark: Let $e_1, \dots, e_m$ be the canonical basis for $\mathbb{R}^m$. For each subset $I \subset \{1, \dots, m\}$ with n elements, consider the span H_I of the v_i's with $i \in I$. Show that the $\binom{m}{n}$ open set U_{H_I} cover $G_n(\mathbb{R}^m)$ and trivialize γ_n^m.

3.6. Morphisms of vector bundles.

These are nothing but bundle morphisms (over the same base or over a map between bases) which are linear (see 3.2.). However, local triviality ensures some nice properties, such as:

3.6.1 Proposition: let $f : E \to E'$ be a linear map of vector bundles over B. Then f is an isomorphism iff the restriction of f to each fibre is a linear isomorphism.

Proof: left to the reader: first define the map f^{-1} and check continuity using local triviality and the fact that the map $GL(n,\mathbb{R}) \to GL(n,\mathbb{R})$ which sends a matrix to its inverse is continuous (warning: this has to be carefully looked at when dealing with infinite dim'l bdles) -

Actually a particular case of 3.6.1 is

3.6.2 Remark: a morphism of the trivial bundle $B \times \mathbb{R}^m$ into $B \times \mathbb{R}^n$ is nothing but a continuous map $B \to \mathcal{L}(\mathbb{R}^m, \mathbb{R}^n)$. It is an isomorphism iff the image lies in $Iso(\mathbb{R}^m, \mathbb{R}^n)$ ($m = n$ then)

3.6.3 Proposition. Let

$$\begin{tikzcd} E \arrow[r, "f"] \arrow[d, "\xi"'] & E' \arrow[d, "\xi'"] \\ B \arrow[r, "g"] & B' \end{tikzcd}$$

be a map of vector bundles over the map g. Then f induces a map of v.b. over B

$$\begin{tikzcd} E \arrow[rr, "\bar{f}"] \arrow[dr, "\xi"'] & & B \times_{B'} E' \arrow[dl, "g^*\xi"] \\ & B & \end{tikzcd}$$

and the latter is an isomorphism iff f is a linear isomorphism when restricted to any fibre of ξ.

Proof: The first part amounts to understanding $f^*\xi$ and the second follows from 3.6.1.

§4. The first classification theorem

We use the above material to show that any vector bundle which imbeds as a subbundle of a trivial bundle is actually a pullback of some tautological bundle (hence the name "universal" for these bundles).

Prop 4.1. Let $\xi = (E, p, B)$ be a subbundle of rank n of the product bundle $B \times \mathbb{R}^m$. Consider the map

$$g = g(\xi) : B \longrightarrow G_n(\mathbb{R}^m)$$

defined by

$$g(b) = p^{-1}(b) \subset \mathbb{R}^m$$

then g is continuous and ξ is canonically isomorphic to $g^*(\gamma_n^m)$.

Proof. continuity being a local property, it is enough to check it when ξ is trivial. One can then choose sections $s_1, \dots, s_n$ of ξ such that $\forall b \in B$, $(s_1(b), \dots, s_n(b))$ is a basis of $p^{-1}(b)$. These define a continuous map $s : B \to V_n'(\mathbb{R}^m)$ such that $\pi s = g$. Continuity of g then follows from the definition of the topology of $G_n(\mathbb{R}^m)$. The isomorphism $\xi \cong g^*(\gamma_n^m)$ follows immediately from 3.6.3. -

We already mentioned that an open covering $\mathcal{U}$ of B trivializes a bundle ξ over B if $\xi|U$ is trivial for every $U \in \mathcal{U}$. We shall say that a bundle has finite type if there exists a finite trivializing covering for this bundle. Note that any v.b. over a compact base is of finite type. We can now state the first classification theorem for vector bundles:

Theorem 4.2. Let B be paracompact. The following are equivalent for a vector bundle ξ of rank n over B:

(i) ξ is of finite type

(ii) There exists a trivial bundle of some rank N such that ξ is isomorphic to a subvector bundle of this trivial bundle

(iii) there exists some integer N and a map $g : B \to G_n(\mathbb{R}^N)$ such that $\xi = g^*(\gamma_n^N)$

Proof (ii) $\Rightarrow$ (iii) is 4.1

(iii) $\Rightarrow$ (i) because γ_n^N is of finite type and (3.4.2).

The proof of (i) $\Rightarrow$ (ii) uses a partition of unity (as might be expected from the hypothesis on B) and the following remark.

Remark (4.2.1). (ii) is equivalent to

(ii)' there exists a continuous map

$$f : E = E(\xi) \longrightarrow \mathbb{R}^N$$

such that f is a linear injection when restricted to each fibre $p^{-1}(b) \subset E$

Proof of 4.2.1 (ii) $\Rightarrow$ (ii)'

Let

$$\begin{tikzcd} & E \arrow[r, hook, "i"] \arrow[d, "\xi'"] & B \times \mathbb{R}^N \arrow[dl] \\ E \arrow[ur, "h"] \arrow[r] & B & \end{tikzcd}$$

be the situation described by (ii)

if $q : B \times \mathbb{R}^N \to \mathbb{R}^N$ is the second projection, then the map $q \circ i \circ h$ meets the requirements of (ii)'.

(ii)' $\Rightarrow$ (ii) The map

$$\begin{tikzcd} E \arrow[rr, "F"] \arrow[dr] & & B \times \mathbb{R}^N \arrow[dl] \\ & B & \end{tikzcd}$$

defined by $F(e) = (p(e), f(e))$ is an injection of vector bundles. Check the image is a vector bundle using the idea in (3.5.3) and apply (3.6.1)

Proof of 4.2, (i) $\Rightarrow$ (ii)'.

We construct a map $f : E(\xi) \to \mathbb{R}^N$ which satisfies (ii)'.

Let λ_i be a partition of unity subordinate to a finite trivializing covering $U_1, \dots, U_s$ for ξ. That is, $\lambda_i : B \to \mathbb{R}$ is continuous, the support of λ_i is contained into U_i, and $\Sigma \lambda_i = 1$

Define $f_i : E(\xi) \longrightarrow (\mathbb{R}^n)^s$ as follows: let $\varphi_i : p^{-1}(U_i) \longrightarrow U_i \times \mathbb{R}^n$ be a trivialization of $\xi|U_i$, and $q_i : p^{-1}(U_i) \to \mathbb{R}^n$ be φ_i composed with the projection onto $\mathbb{R}^n$. Then set

$$f_i(e) = \underbrace{(0, 0, \dots, \underset{i\text{-th place}}{\lambda_i(pe)\, q_i(e)}, 0 \dots, 0)}_{\in (\mathbb{R}^n)^s} \quad \text{if } e \in p^{-1}(U_i)$$

$$f_i(e) = 0 \quad \text{if } e \notin p^{-1}(U_i).$$

In other words, if we set $f = \Sigma f_i$, we have

$$f(e) = (\lambda_1(pe)\cdot q_1(e), \lambda_2(pe) q_2(e), \dots, \lambda_s(pe) q_s(e))$$

and this is well-defined, continuous (by standard properties of continuous functions) and linear and injective on fibres (because the q_i's are so and one of the $\lambda_i(pe)\, q_i(e)$ should be non zero for some i and $q_i(e) \neq 0$. This concludes the proof of (4.2).

Before we state and prove the second (homotopy) classification theorem, we gather some additional material.

§4. More about sections; the bundle Hom.

4.1 Remark: Let $\xi : E \xrightarrow{p} B$ be a vector bundle. The set $\Gamma(B,\xi) = \Gamma(\xi)$ of all (continuous) sections of p has a natural structure of vector space.

Indeed, if t, s are sections ($ps = pt = id_B$), one sets

$$(s+t)(b) = s(b) + t(b) \in p^{-1}(b)$$

$$\forall \lambda \in K, \quad \lambda . s(b) = \lambda . s(b) \in p^{-1}(b)$$

Note that a section over $A \subset B$ of ξ is nothing but a section of the bundle $\xi|_A$.

Continuous sections of a vector bundle have the following very nice extension property if the base is compact:

4.2 Lemma. Let $A \hookrightarrow B$ be a closed subspace of the compact(*) space B. Then any section of $\xi|_A$ extends to a section of ξ.

Proof: Let's first assume that ξ is trivial, with trivialization $\varphi : E(\xi) \xrightarrow{\cong} B \times \mathbb{R}^n$. Then the sections of ξ (resp. $\xi|_A$) are in 1-1 correspondence with the continuous maps $B \to \mathbb{R}^n$ (resp. $A \to \mathbb{R}^n$). Now lemma 4.2 is just a restatement of Tietze's extension property for compact spaces (see any textbook of general topology, e.g. Dugundji's).

In the general case, choose a trivializing covering $U_0, \dots, U_s$ for ξ, and let s_i be any extension to U_i of $s|_{A \cap U_i}$, where s is the given section defined over A. Let (λ_i) be a partition of unity with $\operatorname{supp} \lambda_i \subset U_i, \forall i$. Now $\sum_{i=0}^{s} \lambda_i s_i$ is well defined as a section of ξ over B and $\Sigma \lambda_i s_i|_A = (\Sigma \lambda_i) s = s$.

[This proof extends to v.b. with a differentiable structure, but not to v.b. with an analytic (algebraic) structure: the maximum principle prevents the existence of analytic partitions of unity!!]

We now proceed to interpret the set of morphisms from a v.b. ξ to another v.b. η as the set of sections of a certain bundle.

(*) actually this holds for any paracompact (e.g. metric) space.

Assume first that ξ and η are trivial bundles: a v. b. map

$$\begin{tikzcd} B \times \mathbb{R}^m \arrow[rr, "f"] \arrow[dr] & & B \times \mathbb{R}^n \arrow[dl] \\ & B & \end{tikzcd}$$

is nothing but a <u>continuous</u> map

$$B \longrightarrow \mathcal{L}(\mathbb{R}^m, \mathbb{R}^n)$$

Note that we here use the fact that a map

$$F: B \times \mathbb{R}^m \longrightarrow \mathbb{R}^n$$

is continuous iff the map

$$\hat{F}: B \longrightarrow \mathcal{L}(\mathbb{R}^m, \mathbb{R}^n)$$
$$b \longmapsto (v \mapsto F(b,v))$$

is continuous. This holds because $\mathbb{R}^m$ is locally compact. Again some care will be necessary in order to deal with infinite dim'l vector bundles! Ch. 1 in D. Husemoller's notes provides a complete reference for this.

Now a continuous $B \to \mathcal{L}(\mathbb{R}^m, \mathbb{R}^n)$ is a section of the product bundle

$$B \times \mathcal{L}(\mathbb{R}^m, \mathbb{R}^n) \longrightarrow B$$

Assume now that ξ and η are trivial, and let $\varphi: E(\xi) \to B \times \mathbb{R}^m$, $\psi: E(\eta) \to B \times \mathbb{R}^n$ be trivializations. We can define a bijection

$$\coprod_{b \in B} \mathcal{L}(p^{-1}(b), q^{-1}(b)) \xrightarrow[\alpha(\varphi,\psi)]{} B \times \mathcal{L}(\mathbb{R}^m, \mathbb{R}^n)$$

by

$$\alpha(\varphi,\psi)(u_b) = \left(b, (\psi|q^{-1}(b)) \circ u_b \circ (\varphi|p^{-1}(b))^{-1}\right)$$

$\forall\, u_b \in \mathcal{L}(p^{-1}(b), q^{-1}(b))$.

I claim that if we put on $\coprod_{b \in B} \mathcal{L}(p^{-1}(b), q^{-1}(b))$ the unique topology such that $\alpha(\varphi,\psi)$ is a homeomorphism, this topology <u>does not</u> depend upon the choice of the trivializations φ, ψ. Indeed, if φ', ψ' are other trivializations, the reader can define a homeomorphism β of $B \times \mathcal{L}(\mathbb{R}^m, \mathbb{R}^n)$ onto itself such that

$$\alpha(\varphi', \psi') = \beta \circ \alpha(\varphi, \psi)$$

Finally, let ξ and η be any vector bundles over B, and choose a common trivializing cover $\mathcal{U}$ for ξ and η. Then the above procedure yields a well-defined topology on each subset

$$\forall U \in \mathcal{U}, \quad \coprod_{b \in U} \mathcal{L}(p^{-1}(b), q^{-1}(b))$$

and these topologies agree on the intersections because different trivializations yield the same topology. Therefore we have defined a topology on

$$\coprod_{b \in B} \mathcal{L}(p^{-1}(b), q^{-1}(b))$$

(a subset V of $\coprod_{b \in B}$ is open iff it meets each $\coprod_{b \in U}$, $U \in \mathcal{U}$ along an open set.)

and the projection onto B defines a vector bundle, whose fibre over b is the vector space $\mathcal{L}(p^{-1}(b), q^{-1}(b))$ and admits $\mathcal{U}$ as a trivializing covering.

This bundle will be denoted

$$\mathcal{H}om(\xi, \eta)$$

Now the following should be clear:

4.3. observation: the vector space of morphisms from ξ into η is in 1-1 correspondence (i.e. canonically isomorphic) with the v.s. of sections of the bundle $\mathcal{H}om(\xi,\eta)$. We shall write

$$\Gamma(\mathcal{H}om(\xi,\eta)) = \mathrm{Hom}(\xi,\eta)$$

(Notice the typographic difference: $\mathcal{H}om$ is a vector bundle over B, while Hom is a vector space!)

More generally, we have

$$\Gamma(A;\mathcal{H}om(\xi,\eta)) = \mathrm{Hom}(\xi|_A, \eta|_A).$$

4.4. comment: The above method for defining the bundle $\mathcal{H}om(\xi,\eta)$ actually extends to any naturally defined functor on vector space: one can thus define the bundles

$$\xi\oplus\eta,\quad \xi\otimes\eta,\quad S^n(\xi) \text{ (n-th symmetric power)}$$
$$\Lambda^n(\xi) \text{ (n-th exterior power)}$$

and so on.

4.5. Application: Lemma: Let B be compact (and connected) (more generally paracompact) and ξ,η be vector bundles over B. Assume there exists a closed subset $A\subset B$ such that

$$\xi|_A \cong \eta|_A$$

Then there exists an open subset $U\supset A$ such that $\xi|_U \cong \eta|_U$

Proof of 4.5. Let $h:\xi|_A \xrightarrow{\cong} \eta|_A$ be this isomorphism. It corresponds to a section of $\mathcal{H}om(\xi,\eta)$ over A. Extend this section over B by 4.2. We thus obtain a morphism $\bar{h}:\xi\to\eta$. Now the set of those $b\in B$ such that $\bar{h}_b$ is an isomorphism from $p^{-1}(b)$ onto $q^{-1}(b)$ ($\bar{h}_b = \bar{h}|p^{-1}(b)$) is the required open set U: observe that U is locally defined by the condition $\det \bar{h}_b \neq 0$ which is an open condition. //.

§5. the homotopy classification theorem.

We are now ready to prove the precise statement we hinted at in 1.3. The key fact is the following.

Theorem 5.1. Let X, Y be spaces, X compact, f_0, f_1 be continuous maps from X to Y. Let ξ be a vector bundle over Y. Then, if f_0 and f_1 are homotopic, the bundles

$$f_0^*\xi \quad\text{and}\quad f_1^*\xi$$

are isomorphic.

Proof. First recall (see D. Husemoller's notes, ch II for details), that f_0, f_1 are homotopic if there exists

$$F: X\times[0,1] \longrightarrow Y$$

such that $F|X\times\{0\} = f_0$, $F|X\times\{1\} = f_1$.

Here are some notations:

$$\begin{tikzcd} X \times [0,1] \arrow[r, "F"] \arrow[d, shift right, "\pi"'] & Y \\ X \arrow[u, shift right, "j_t"'] \arrow[ur, "f_t"'] & \end{tikzcd}$$

j_t is the inclusion of X onto $X \times \{t\}$, $t \in [0,1]$

$f_t = F \circ j_t$: this agrees for $t = 0, 1$ with the previous notation.

$\pi : X \times [0,1] \to X$ is the first projection

Note $\pi \circ j_t = id_X$ for all t.

Now consider the bundles

$$F^* \xi \quad \text{and} \quad \pi^* f_t^* \xi$$

Clearly their restrictions to $X \times \{t\}$ are isomorphic since

$$j_t^* F^* \xi = (F \circ j_t)^* \xi = f_t^* \xi = j_t^* \pi^* f_t^* \xi$$

because $j_t^* \pi^* = (\pi \circ j_t)^* = id$

By 4.5, there exists a neighbourhood U of $X \times \{t\}$ in $X \times [0,1]$ such that

$$F^* \xi |_U \cong \pi^* f_t^* \xi |_U$$

Now X being compact, U contains a "slice" $X \times]t-\varepsilon, t+\varepsilon[$. Therefore:

$$\forall s \in]t-\varepsilon, t+\varepsilon[, \quad j_s^* F^* \xi = f_s^* \xi \cong j_s^* \pi^* f_t^* \xi \cong f_t^* \xi$$

This means that the isomorphism class of the bundle $f_t^* \xi$ is a locally constant function of t.

Now let $J_0 = \{ t \in [0,1] \mid f_0^* \xi \cong f_t^* \xi \}$

We have $0 \in J_0$ and we just proved J_0 is open. Now if $t \notin J_0$, the same proof shows that $t' \notin J_0$ for t' close enough to t. Therefore J_0 is non-empty, open and closed in $[0,1]$ and so $J_0 = [0,1]$. In particular $1 \in J_0$ and

$$f_0^* \xi \cong f_1^* \xi \quad //.$$

Let us make the following interpretation of 5.1. Consider the "set" (we ignore set theoretic difficulties in this course) of all n dim'l vector bundles over B, let it be $Vect_n(B)$

We have a map (of sets!)

$$Map(B, G_n(\mathbb{R}^N)) \xrightarrow{\theta^N} Vect_n(B)$$

defined by:

$$\forall g, \quad (g : B \to G_n(\mathbb{R}^N)) \mapsto g^*(\gamma_n^N)$$

Result 5.1 says that this map induces a map, for which we abusively use the same notation:

$$[B, G_n(\mathbb{R}^N)] \xrightarrow{\theta^N} Vect_n(B)$$

from the set of homotopy classes of maps from B to $G_n(\mathbb{R}^N)$ into the set (it's OK now!) of isomorphism classes of n-dim'l vector bundles over B.

Observe that this θ^N does not significantly depend on N, because if

$$N < N' \quad i_N^{N'} : G_n(\mathbb{R}^N) \to G_n(\mathbb{R}^{N'})$$

is the inclusion induced by the injection $\mathbb{R}^N \subset \mathbb{R}^{N'}$

given by the first N coordinates, then one immediately sees that

$$(i_N^{N'})^*(\gamma_n^{N'}) = \gamma_n^N$$

Thus the diagram:

$$(5.2)\qquad \begin{tikzcd}
{[B, G_n(\mathbb{R}^N)]} \arrow[d, "i_N^{N'}\circ ?"'] \arrow[dr, "\theta^N"] & \\
{[B, G_n(\mathbb{R}^{N'})]} \arrow[r, "\theta^{N'}"'] & \mathrm{Vect}_n(B)
\end{tikzcd}$$

where the vertical map is composition with $i_N^{N'}$, is commutative.

In order to get the appropriate description of the set $\mathrm{Vect}_n(B)$, which should not depend on the "imbedding dimension N", we introduce the space:

$$G_n(\mathbb{R}^\infty) = \bigcup_N G_n(\mathbb{R}^N)$$

together with the union topology: a subset $A \subset G_n(\mathbb{R}^\infty)$ is closed iff it meets every $G_n(\mathbb{R}^N)$ along a closed set. We will see later that $G_n(\mathbb{R}^\infty)$ has a natural structure of locally finite CW-complex. Let us observe the following for the moment: let S be a subset of $G_n(\mathbb{R}^\infty)$ such that $S \cap (G_n(\mathbb{R}^N) - G_n(\mathbb{R}^{N-1}))$ contains one point at most for each N, and exactly one for infinitely many N's. Then S is closed and not compact in $G_n(\mathbb{R}^\infty)$. Using this observation, it is easy to see that any compact subset in $G_n(\mathbb{R}^\infty)$ is actually contained in $G_n(\mathbb{R}^N)$ for some N. Therefore, using (5.2) and the above, the maps θ^N assemble to a well-defined map, <u>if B is compact</u>*:

$$\theta : [B, G_n(\mathbb{R}^\infty)] \longrightarrow \mathrm{Vect}_n(B)$$

(*) Although the method used here does not extend to non-compact spaces, Thm 5.3 below extends to any paracompact space B (See Husemoller's notes p. IV.27).

We can now state the result we have been aiming at in this set of notes:

Theorem 5.3 If B is compact
$$\theta : [B, G_n(\mathbb{R}^\infty)] \longrightarrow \mathrm{Vect}_n(B)$$
is a bijection.

Proof: Surjectivity comes from 4.2 and the fact that any bundle over a compact B has finite type. Injectivity is proven in the next lemma

Lemma 5.4. Let
$$B \overset{g_0}{\underset{g_1}{\rightrightarrows}} G_n(\mathbb{R}^N)$$
be maps such that $g_0^*(\gamma_n^N) \cong g_1^*(\gamma_n^N)$.
Then $\qquad i_N^{2N}\circ g_0 \simeq i_N^{2N}\circ g_1$
In other words, maps into $G_n(\mathbb{R}^N)$ which induce isomorphic bundles from γ_n^N become homotopic in $G_n(\mathbb{R}^{2N})$.

Proof of 5.3. From remark 4.2.1, the statement $g_0^*(\gamma_n^N) \cong g_1^*(\gamma_n^N)$ can be restated as follows: if ξ is any bundle in the isomorphism class of $g_0^*(\gamma_n) \cong g_1^*(\gamma_n)$, there exists maps
$$f_0, f_1 : E(\xi) \longrightarrow \mathbb{R}^N$$
such that f_i, $i=0,1$ is linear and injective on the fibres of ξ, and more precisely, such that f_i defines a bundle map over g_i:

$$\begin{array}{ccc} E(\xi) & \xrightarrow{\bar f_i} & E_n(\mathbb{R}^N) \\ p\downarrow & & \downarrow \gamma_n^N \\ B & \xrightarrow{g_i} & G_n(\mathbb{R}^N) \end{array}$$

with $\bar f_i(e) = (g_i(pe), f_i(e))$, $g_i(b) = f_i(p^{-1}(b)) \subset \mathbb{R}^N$

Consider now the homotopy

$$\tilde F: E(\xi) \times [0,1] \longrightarrow \mathbb{R}^N \times \mathbb{R}^N = \mathbb{R}^{2N}$$
$$(e,t) \longmapsto ((1-t)f_0(e),\ t.f_1(e))$$

For each $t \in [0,1]$, the map $\tilde f_t = F / E(\xi) \times \{t\}$ is linear and injective on the fibres from $E(\xi)$ to $\mathbb{R}^{2N}$. It therefore defines

$$\tilde g_t : B \longrightarrow G_n(\mathbb{R}^{2N})$$

by $\tilde g_t(b) = \tilde f_t(p^{-1}(b)) \subset \mathbb{R}^{2N}$

Observe that $\tilde g_0 = \iota_N^{2N} \circ g_0$ and

$$\tilde g_1 = \iota_N'^{2N} \circ g_1$$

where ι_N^{2N} (resp $\iota_N'^{2N}$) are the inclusions $G_n(\mathbb{R}^N) \subset G_n(\mathbb{R}^{2N})$ defined by the first (resp. the last) N coordinates in $\mathbb{R}^{2N}$. Therefore we proved

$$\iota_N^{2N} \circ g_0 \sim \iota_N'^{2N} \circ g_1$$

and we are done if we observe that $\iota_N^{2N} \sim \iota_N'^{2N}$ (using the fact that composition of maps behaves well with homotopy). This last observation can be achieved by the following homotopy between the two inclusions of $\mathbb{R}^N$

$$j_t : \mathbb{R}^N \times [0,1] \longrightarrow \mathbb{R}^N$$

defined by

$$j_t(x_1, -, x_N) = (tx_1, -, tx_N, (1-t)x_1, -, (1-t)x_N)$$

Indeed j_t induces a map

$$i_t : G_n(\mathbb{R}^N) \hookrightarrow G_n(\mathbb{R}^{2N})$$

by $i_t(H) = j_t(H)$ (image of the subspace)

and $i_1 = \iota_N^{2N}$, $i_0 = \iota_N'^{2N}$. //.

Examples 5.5, and comments.

5.5.1. Any vector bundle over a contractible compact space B is trivial, because $[B, G_n(\mathbb{R}^\infty)]$ has only one element. Observe – if this has not been done before – that θ maps the trivial map onto the trivial class: recall: 2.5 e/ -

5.5.2. Thm 5.2 extends readily to complex vector bundles: (or even quaternionic ones)

$$[B, G_n(\mathbb{C}^\infty)] \xrightarrow{\cong} Vect_n^{\mathbb{C}}(B)$$

5.5.3 It should be pointed out that $[X, Y]$ stands for homotopy classes with no requirement on base-points, while homotopy groups are defined using homotopies which fix a base point. Here are some examples where this difference is not relevant(*):

$$Vect_1^{\mathbb{R}}(S^1) = [S^1, \mathbb{R}P(\infty)] = \pi_1(\mathbb{R}P(\infty)) = \mathbb{Z}/2\mathbb{Z}$$

the two classes are the trivial one an the Mo"bius real line bundle over the circle.

$$Vect_1^{\mathbb{C}}(S^2) = [S^2, \mathbb{C}P(\infty)] = \pi_2(\mathbb{C}P(\infty)) = \mathbb{Z}$$

complex line bundles over the Riemann sphere (complex projective line) are classified by an

(*) there is no difference if Y is simply connected – This is the case for $G_n(\mathbb{C}^N)$ or the 2-fold orientation cover of $G_n(\mathbb{R}^N)$.

integer: it will be shown later in the course that this integer is the first Chern class of the line bundle (provided one makes the appropriate sign convention: the most usual one is $c_1(\gamma_1) = -1$).

Let us conclude these notes by a remark for the algebraic geometers: over S^2 the classification of holomorphic complex line bundles and topological line bundles coincide: both are given by c_1, the first Chern class. But the classification for rank $n > 1$ holomorphic bundles over S^2 is much finer than the topological one: the theorem of Birkhoff-Grothendieck asserts that any holomorphic v. bundle of rank n splits holomorphically as a direct sum of line bundles: thus hol. v. bundles over S^2 are classified by increasing sequences of n integers. In contrast, one has $[S^2, G_n(\mathbb{C}^\infty)] = [S^2, \mathbb{CP}(\infty)] = \mathbb{Z}$, thus two complex vector bundles over S^2 are isomorphic (topologically) iff they have the same Chern class.

In general, the determination of the set $[B, G_n(K^\infty)]$, $K = \mathbb{R}, \mathbb{C}$ is very difficult and involves the full use of the algebraic topologist's toolkit. For instance, Atiyah and Rees have proven that rank 2 complex vector bundles over the complex projective space $\mathbb{CP}(3)$ are classified by their Chern classes c_1 and c_2 and a tricky invariant which takes values in $\mathbb{Z}/2\mathbb{Z}$, and is defined using real and symplectic K-theory –