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MINIWORKSHOP ON STRONG ELECTRON CORRELATIONS
"Disorder and Interaction in Quantum Systems and Their Classical Analogs"

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"Pairing and Non-abelian Quantum Hall States"

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These are preliminary lecture notes, intended only for distribution to participants.

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Pairing and Non-abelian Quantum Hall States

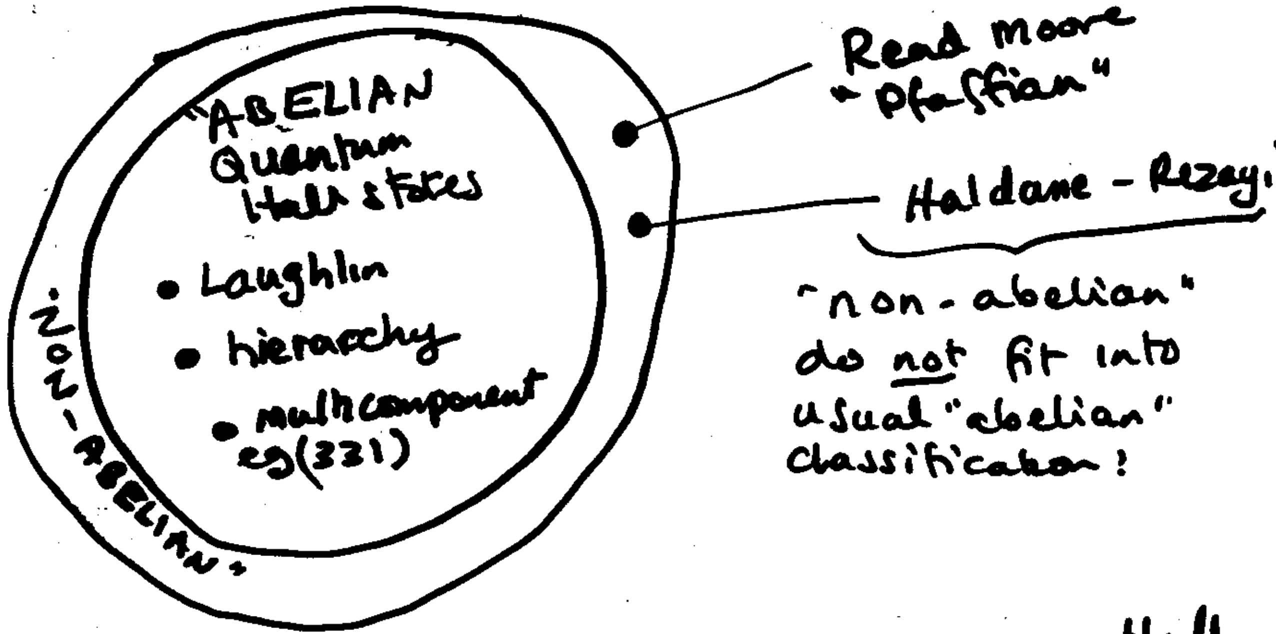
- Some unfinished business in classification of quantum Hall states

- Pairing states – analogs of BCS – but with subtle new effects – "Non abelian statistics"

- recent work:

  Read + Rezayi
  Nayak + Wilczek
  Read + Milanovic.
  (see cond-mat)

Quantum Hall states*



• The <u>Topological order</u> of quantum Hall states is often (usually) classified by an effective <u>Abelian Chern Simons</u> "Topological field Theory" (cf. <u>Wen</u>)

• **but some exceptions are "paired Hall states!"**

→ are these physical? where do they fit in?

* by "states" I mean ground states of some explicit and "reasonable" Hamiltonian!

- Outline of Abelian theory.

(NB Microscopically different theories can have the same topological order (eg spin singlet, spin polarized $\nu = 2/3$).)

$\underline{\underline{K}}$ is an integer-valued symmetric matrix (non-singular)

$\underline{Q}$ is an integer-valued vector

$$\sigma^H = \frac{e^2}{h}(Q, K^{-1}Q)$$

$$(-1)^{Q_i} = (-1)^{K_{ii}}$$

(generalized "odd-denominator" selection rule

$$S^{\text{eff}} = \frac{1}{2\pi}\int d^2x\, dt \left(\frac{1}{2}\sum_{ij} \hbar K_{ij}\, \epsilon^{\mu\nu\lambda} a_{i\mu}\partial_\nu a_{j\lambda} - e\sum_i Q_i A_\mu \partial_\nu a_{i\lambda} \right)$$

CS gauge field

effective action.

electromagnetic gauge field.

Properties of Abelian QHE Theories.

* vortices $\underline{m}_i \leftarrow$ integer vector

$$e^{i\theta_{ij}} = e^{2\pi i(\underline{m}_i, \underline{\underline{K}}^{-1}_{ij} \underline{m}_j)}$$ relative statistics

$$Q_i = e\,(\underline{Q}, \underline{\underline{K}}^{-1} \underline{m}_i)$$

$$e^{i\theta_i} = e^{i\pi(\underline{m}_i, \underline{\underline{K}}^{-1} m_i)}$$

* edge theory

$$S = \frac{1}{2\pi}\int dx\,dt \sum_{i,j} K^{-1}_{ij} \cdot \frac{1}{2}\, \partial_x \varphi_i\, \partial_t \varphi_j - \int dt\, H^{\text{eff}}$$

$$H^{\text{eff}} = \frac{1}{4\pi}\sum_{ij} V_{ij}\, \partial_x \varphi_i\, \partial_x \varphi_j$$

if K is $n \times n$, get n boson modes (with different velocities)

"$c = n$" (conformal anomaly) (= # of independent boson modes)

Wen, MPM Fisher, Fröhlich, Zee, etc have written extensively on this $(\underline{\underline{K}}, \underline{Q})$ Abelian C-S Theory.

results include

"degeneracy of Surface of genus g" $= |\det K|^g$

⋮

- The paired Hall states (Read Moore) (Haldane - Rezayi) are incompressible Hall states that are exact ground states of apparently "reasonable" Hamiltonians YET DO NOT FIT IN TO THE ABELIAN SCHEME

∴ $\boxed{\text{IT is incomplete!}}$

PRL (1988)

- **Haldane - Rezayi**

— historically the first.

— proposed for the $\nu = 5/2 = 2 + 1/2$ QHE.

— spin singlet.

$$\Psi(\{z_i, \alpha_i \beta_i\})$$

(latest form!)

$$= \prod_{i<j} (z_i - z_j)^2 \, \mathrm{Pf}\left(\frac{\alpha_i \beta_j - \beta_i \alpha_j}{(z_i - z_j)^2} \right) \prod_i e^{-\frac{1}{4}|z_i|^2/\ell^2}.$$

"Pfaffian"

$$\left.\begin{array}{l} \alpha_i = \uparrow \\ \beta_i = \downarrow \end{array}\right\}$$ spin variables

$$\boxed{(\mathrm{Pf}\,\underline{\underline{A}})^2 = \det \underline{\underline{A}}}$$ for an (even dimensional) antisymmetric matrix $\underline{A}$

$$\mathrm{Pf}\,A = \frac{1}{\sqrt{\cdots}} \sum_{P=1}^{2N!} (-1)^P \prod_{i=1}^{N} A_{2i,\,P(2i-1)}$$ 2N X 2N matrix.

NB ($\det A = 0$ if A is odd x odd, Antisymmetric) older forms of H-R

$$\Psi = \prod_{i<j}(z_i - z_j)^2 \det \frac{1}{z_i^{\uparrow} - z_j^{\downarrow}} \ \mathrm{Per} \frac{1}{z_i^{\uparrow} - z_j^{\downarrow}} \prod_i e^{-1/4|z_i|^2/\ell^2}$$

$$\equiv \prod_{i<j}(z_i - z_j) \det \frac{1}{(z_i^{\uparrow} - z_j^{\downarrow})^2} \prod_i e^{-1/4|z_i|^2/\ell^2}$$

$$= \Psi_{331} \cdot \mathrm{per} \frac{1}{z_i^{\uparrow} - z_j^{\downarrow}} \prod_i e^{-1/4|z_i|^2/\ell^2}$$

where Ψ_{331}

$$= \prod_{i<j}(z_i^{\uparrow} - z_j^{\uparrow})^3 (z_i^{\downarrow} - z_j^{\downarrow})^3 (z_i^{\uparrow} - z_j^{\downarrow}) \prod_i e^{-1/4|z_i|^2/\ell^2}$$

note: $\underbrace{\mathrm{Pf}\begin{bmatrix} 0 & M \\ -M^T & 0 \end{bmatrix}}_{\substack{\text{antisymmetric} \\ 2N \times 2N \\ \text{matrix}}} = \det \underset{\substack{N \times N \\ \text{matrix}}}{M}.$

Read Moore $(\nu = 1/2)$ Nucl. Phys B 330 (1991) 362

$$\Psi = \prod_{i<j}(z_i - z_j)^2 \, \mathrm{Pf}\left(\frac{1}{z_i - z_j}\right) \prod_i e^{-1/4|z_i|^2/\ell^2}$$

* degeneracy on torus = 6 = 2×3
("expect" 8 = 2×4)

* edge modes: $c = 1 + 1/2$ ← Majorana
one { charged boson field
{ neutral (Majorana) fermion (!) } both chiral in same sense

definitely **not** in Abelian QHE class

* conformal field theory of edge related to <u>Ising model</u>

* RM predict non-Abelian statistics
$\left(U(1) \times \frac{SU(2)_1 \times SU(2)_1}{SU(2)_2} \right.$ GKO construction $\left. \right)$
(initially disputed by Greiter Wen Wilczek, now confirmed by Nayak + Wilczek)

* the "3" in 6 = 2×3 torus degeneracy "comes from" the 3 "conformal blocks" of Ising model conformal field theory

Hamiltonians

Ψ_{HR} exact for "pseudo potential"†

$V_m = 0 \quad m \neq 1$
$V_1 > 0$ } "Hollow core model"

Ψ_{Rm} exact for three-body interaction *

Ψ_{331} exact for $V_1^{\uparrow\uparrow} = V_1^{\downarrow\downarrow} > 0$, $V_0^{\uparrow\downarrow} > 0$
(others = 0)

* $$H^{eff}_{Read\text{-}Moore} = \sum_{i<j<k} \nabla_i^2 \nabla_j^2 \, \delta^2(r_i - r_k)\, \delta^2(r_j - r_k)$$

† pseudo potential V_m = energy of a pair of particles in state
(relative angular momentum m). $(z_1 - z_2)^m (z_1 + z_2)^n e^{-1/4 |z_1|^2 + |z_2|^2}$

Vortices in Read-Moore state.

(a) ~~[illegible]~~ add flux at a point ($N\phi$ flux quanta)

$$\Psi = \prod_i (z_i - z_0)^{N\phi} \prod_{i<j} (z_i - z_j)^2 \; \mathrm{Pf}_{ij}\left(\frac{1}{z_i - z_j}\right) \times \underbrace{\prod e^{-\frac{1}{4}|z|^2/\ell^2}}_{\text{will drop this factor!}}$$

$$\equiv \prod_{i<j} (z_i - z_j)^2 \; \mathrm{Pf}\left(\frac{(z_i - z_0)^{N\phi}(z_j - z_0)^{N\phi}}{z_i - z_j}\right)$$

Now put vortices at different places. This is different when done inside the Pfaffian.

$$\Psi = \prod_{i<j} (z_i - z_j)^2 \; \mathrm{Pf}\left(\frac{\Phi(z_i, z_j; \{Z_n\})}{z_i - z_j}\right)$$

← 2n vortex coords.

$$\Phi(z, z'; \{Z_n\}) = \frac{1}{2n!} \sum_{P=1}^{2n!} \prod_{i=1}^{n} (z - Z_i)(z' - Z_{i+n})$$

Symmetric in the vortices

more generally, take

$$\Psi = \prod_{i<j}(z_i - z_j)^2 \, \underset{ij}{Pf}\left(\frac{\Phi^{m}(z_i, z_j; \{z_m\})}{z_i - z_j} + \underbrace{\chi^{m+1}(z_i, z_j)}_{\text{antisym. polynomial}} \right)$$

Now

$$\underbrace{Pf^{(2N)}(\chi^{(m)}(z, z'))}_{\text{antisymmetric polynomial}} = \underbrace{\prod_{i<j}(z_i - z_j)\, \phi(z_1 \dots z_{2N})}_{\text{antisymmetric polynomial}}$$

degree m in each variable

degree $\geq 2n-1$ in each variable

$$Pf^{(2N)}(\chi^{m}(z, z')) = 0 \quad \text{if} \quad \boxed{m < 2n-1}$$

($2N \times 2N$ matrix)

$$\boxed{Pf^{(2N)}(A + \lambda \chi^{m}) \equiv \sum_{p=0}^{p \leq \frac{m+1}{2}} \lambda^{p} B_p(A, \chi)}$$

only low orders of p!

Linear combinations

$$\sum_\alpha a_\alpha \, Pf^{(2N)}\left(\frac{\Phi^{m}(z,z';\{Z\})}{z-z'} + \chi_{\alpha}^{(m-1)}(z,z') \right)$$

m = # of added flux quanta $Z_1 \cdots Z_m$

$$= A \sum_{p=0}^{p \le \frac{m}{2}} \underbrace{\psi^{(m)}(z_{2(N-p)+1} \cdots z_{2N})}_{\text{antisymmetric polynomials ("Broken pairs")}} \; Pf^{(2(N-p))} \underbrace{\frac{\Phi(z,z',\{Z\})}{z-z'}}$$

cf. Read + Milanovic

vortex positions specified in pairing function

Count:

For Fixed Vortex* locations, There are 2^{n-1} states (!)

(Read + Moore see this abstractly)

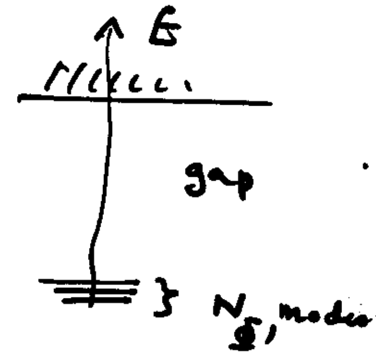
also Read + Milanovic, Read + Rezayi, Wilczek + Nayak

adiabatic motion of vortices mixes these states
→ non Abelian statistics (!)
("Berry Phase" → Unitary matrix

* $\Phi = \frac{h}{2e}$ vortices ("half vortices")

"Physical interpretation"

* add N_Φ flux quanta, get $2N_\Phi$ vortices + N_Φ fermionic "zero modes"

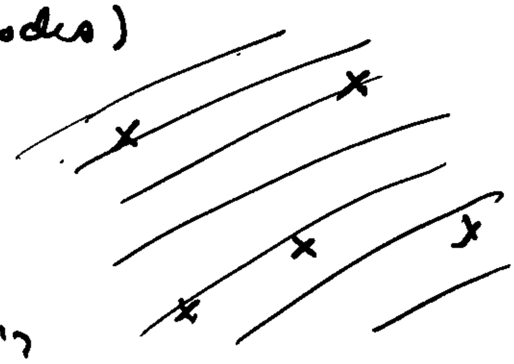


* populate with "broken pairs" ("Bogoliubov excitations)
* n_f such fermions.

  n_f is either even or odd $(-1)^{n_f} = (-1)^N$

  $$\rightarrow \quad 2^N = \underset{n_f \text{ even}}{2^{n-1}} + \underset{n_f \text{ odd}}{2^{n-1}}$$

* adiabatic motion of vortices generates a Bogoliubov Transformation in this space (?) (of zero modes)



* The zero modes are not localized on particular vortices.

  "Bogoliubov-de Gennes"?

- Ψ_{331} and Pfaffian (TL Ho PRL 95)

Ψ_{331} is an Abelian state. but

$$\Psi_{331} = \prod_{i<j}(z_i - z_j)^2 \cdot \det \frac{1}{z_i^{\uparrow} - z_j^{\downarrow}} \quad \text{(cauchy)}$$

$$= \prod_{i<j}(z_i - z_j)^2 \, \mathrm{Pf}_{ij}\left(\frac{\alpha_i \beta_j + \beta_i \alpha_j}{z_i - z_j}\right)$$

$S^z = 0$ member of $S = 1$ pair!

$$\Psi_{\text{Read moore}} = \prod_{i<j}(z_i - z_j)^2 \, \mathrm{Pf}\left(\frac{\gamma_i \gamma_j}{z_i - z_j}\right)$$

Can continuously interpolate $\Psi_{331} \longrightarrow \Psi_{\text{read moore}}$
Abelian non-abelian

- but must consider Hamiltonian.
- Pf states are even in Layer interchange
- H_{331} has zero energy states that are odd in Layer interchange

Torus degeneracy

e	o	tot	
6 +	2 =	8	Ψ_{331}
6 +	0 =	6	Ψ_{RM}.

edge modes $c = 1 + 1$ Ψ_{331}
$c = 1 + \frac{1}{2}$ Ψ_{RM}.

Conclusions

- Classification of quantum Hall state topological order based on "Abelian Chern-Simons theories" (with $c=1$ edge modes) is <u>not</u> complete.
- Ψ_{HR}, Ψ_{RM} are explicit examples of "non-abelian" states, exact states of "not-unreasonable" Hamiltonians.

* additional fermionic degrees of freedom as well as vortices.
? Bogoliubov de-Gennes analogs?
Broken Pairs].

- Physical realizations? $\nu=5/2$?
- **<u>Also</u>** $\nu=1$ Bosons appear to be in a Ψ_{RM}-type state! (not composite fermion "Fermi liquid")
(numerical evidence)