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WORKSHOP ON NONLINEAR CONTROL AND CONTROL OF CHAOS

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***"Controller/Observer Design for (Chaotic)
Nonlinear Control Systems"***

presented by:

Henk Nijmeijer
University of Twente
Department of Applied Mathematics
P.O. Box 217
7500 AE Enschede
Netherlands

Linear control

$$\begin{cases} \dot{x} = Ax + Bu \\ y = Cx \end{cases}, \quad x(0) = x_0$$

$$x = x(t) \in \mathbb{R}^n$$

$$u = u(t) \in \mathbb{R}^m$$

$$y = y(t) \in \mathbb{R}^p$$

state

input/control $\neq$ parameters

output/measurement

$A : (n, n)$ -matrix

$B : (n, m)$ -matrix

$C : (p, n)$ -matrix

$$x_0 \in \mathbb{R}^n$$

initial state

$u \in \mathcal{U}$: admissible controls

piecewise contin.

- linear, time-invariant, ...
- 'example'
- discrete-time

(2)

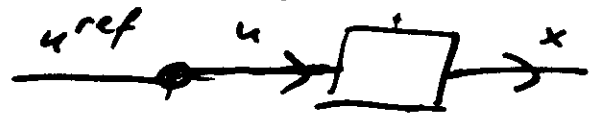
Use the control-possibilities as to influence the system in a desirable way.

* Different control objectives possible, e.g.

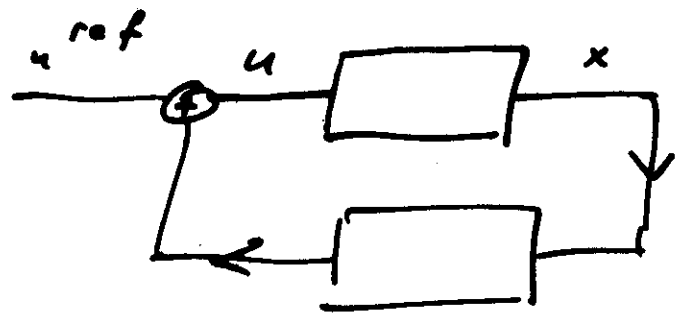
- Stabilization
- tracking
- optimal control

* What information is used in the control

- 'nothing' $u = u^{ref}(t)$ open-loop



- state feedback



- output feedback

closed loop

STATIC STATE FEEDBACK

$$u = Fx + Gv + u^{ref}$$

F : (m, n) -matrix , G : $(m \times m)$ -matrix

v : new control signal

STATIC OUTPUT FEEDBACK

(3)

$$u = Ky + \{Gv + u^{ref}\}$$

K : (m, p) -matrix

$$\dot{x} = (A + BF)x + BGv + Bu^{ref}$$

or

$$\dot{x} = (A + BKC)x + BGv + Bu^{ref}$$

Dynamic state/output feedback

$$\begin{cases} \dot{z} = Pz + Qx + Rv \\ u = Sz + Tx + Uv + u^{ref} \end{cases}, z \in \mathbb{R}^v$$

P, Q, R, S, T, U appropri. dim.

closed loop (with output feedback)

$$\begin{pmatrix} \dot{x} \\ \dot{z} \end{pmatrix} = \begin{pmatrix} A + BTC & BS \\ QC & P \end{pmatrix} \begin{pmatrix} x \\ z \end{pmatrix} + \begin{pmatrix} BU \\ R \end{pmatrix} v + \begin{pmatrix} B \\ 0 \end{pmatrix} u^{ref}$$

Controllability $\dot{x} = Ax + Bu$

(4)

Given any $x_0, x_1 \in \mathbb{R}^n$, any $T > 0$

$\exists \bar{u} \in \mathcal{U} \quad \bar{u}: [0, T] \rightarrow \mathbb{R}^m \quad \text{s.t.}$

$$x(T, 0, x_0, \bar{u}) = x_1$$



Thm: $\Sigma(A, B)$ controllable iff

$$\text{rk} \underbrace{[B; AB; \dots; A^{n-1}B]}_{(n, nm)\text{-matrix}} = n$$

($m=1$: invertibility condition)

Stabilization at $x=0$

Is it possible to steer asymptotically
($\approx$ exponentially!) any x_0 to 0?

① open loop, finite time : controllability

Closed loop

$$u = Fx$$

$$\dot{x} = (A + BF)x$$

asympt. stable

Sufficient condition: $\Sigma(A, B)$ controllable

STRONGER:

Σ controllable $\iff$ "pole-assignability"

Given any n symmetric (w.r.t. real axis) points in $\mathbb{C}$, say $\lambda_1, \dots, \lambda_n \Rightarrow$
 $\exists F$ s.t. $\sigma(A + BF) = \{\lambda_1, \dots, \lambda_n\}$
 $\uparrow$
real

What if Σ not controllable?

$$R = \text{Im}(B; AB; \dots; A^{n-1}B), \quad R \subset \mathbb{R}^n$$

$$AR \subset R \quad (\text{Cayley-Hamilton})$$

$$\Rightarrow A \cong \begin{pmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{pmatrix} \quad R = \begin{pmatrix} * \\ 0 \end{pmatrix}$$

$$\text{stabilizability} \iff \sigma(A_{22}) \subset \mathbb{C}^-$$

Observability

$$\Sigma \begin{cases} \dot{x} = Ax + Bu \\ y = Cx \end{cases}, \quad x(0) = x_0$$

Σ observable if given $(u(t)), y(t), t \in [0, T]$
can we uniquely determine x_0
(and thus $x(t), t \geq 0$)

$$\mathcal{N} = \ker \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix} \quad \begin{array}{l} (n, p)\text{-matrix} \\ p=1 \text{ } (n, n)\text{-matrix} \end{array}$$

Thm Σ observable $\Leftrightarrow \mathcal{N} = \{0\}$

Asymptotic reconstruction of $x(t)$

$$\begin{aligned} \text{observer } \dot{\hat{x}} &= A\hat{x} + K(\hat{y} - y) & \hat{y} &= C\hat{x} \\ e &\triangleq x - \hat{x} & K &: (n, p)\text{-matrix} \\ \dot{e} &= (A - KC)e & \hat{x}(0) &= \hat{x}_0 \text{ arbitrary} \end{aligned}$$

(7)

Thus, if $A-KC$ asympt. stable
 then $e \rightarrow 0 \iff \hat{x}(t) \rightarrow x(t)$
 (exponentially)

Observability $\implies \exists K$ s.t.

$A-KC$ asympt stable

Observability $\iff \forall n$ symm. points in $\mathbb{C}$
 $\lambda_1, \dots, \lambda_n \exists K$ s.t.

$$\sigma(A-KC) = \{\lambda_1, \dots, \lambda_n\}$$

Detectability : If $\mathcal{N} \neq \{0\}$

$$A\mathcal{N} \subset \mathcal{N} \rightarrow$$

$$A \cong \begin{pmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{pmatrix}$$

$$\mathcal{N} = \text{sp} \left\{ \begin{pmatrix} * \\ 0 \end{pmatrix} \right\}$$

$$\sigma(A_{22}) \subset \mathbb{C}^- \iff \exists K \text{ s.t.}$$

$A-KC$ asympt stable

(8)

$$\Sigma \begin{cases} \dot{x} = Ax + Bu \\ y = Cx \end{cases}$$

Desired to obtain ^{dynamic} output feedback controller that stabilizes Σ

Thm Σ stabilizable & detectable

$\Leftrightarrow \exists$ _{dynamic} output feedback that stabilizes Σ .

F s.t. $u = Fx \rightsquigarrow (A+BF)$ as. stable

$$\downarrow$$

$$u = F\hat{x}$$

$$\dot{\hat{x}} = A\hat{x} + K(\hat{y} - y) + BF\hat{x}$$

$(A-KC)$ as. stable

$$e = x - \hat{x}$$

$$\begin{cases} \dot{x} = (A+BF)x - BF e \\ \dot{e} = 0 \quad (A-KC)e \end{cases} \begin{pmatrix} A+BF & -BF \\ 0 & A-KC \end{pmatrix}$$

Separation principle

Tracking

$$\begin{cases} \dot{x} = Ax + Bu \\ y = Cx \end{cases}$$

Tracking problem: Design a controller that asymptotically steers

$$y(t) \rightarrow y_d(t)$$

↑ Given, 'smooth'

Assume $p=1$ and $m=1$

(y is 1 dimensional)
(u is 1 dimensional)

$$\begin{cases} \dot{x} = Ax + bu \\ y = Cx \end{cases}$$

$$y = Cx$$

$$\dot{y} = CAx + cbu$$

Either $cb = 0$

or $cb \neq 0$

STOP

$$\ddot{y} = CA^2b + \underbrace{CA}b u$$

Either $CAb = 0$

or $CAb \neq 0$

stop

Let $p = \min_k CA^{k+1}b \neq 0$

relative degree
order

p is minimal number of derivative, (10)
of y s.t.

$y^{(p)}$ explicitly depends on u

$$y^{(p)} = CA^p x + CA^{p-1} b u$$

Note Either $p \leq n$ or $p = \infty$

Output independent
of u .

A possible tracking controller

$$u = -(CA^{p-1}b)^{-1}CA^p x + (CA^{p-1}b)^{-1}v$$

with

$$v = y_d^{(p)} - \alpha_{p-1} e^{(p-1)} - \dots - \alpha_0 e$$

$$e \triangleq y - y_d$$

$$\Rightarrow e^{(p)} + \alpha_{p-1} e^{(p-1)} + \dots + \alpha_0 e = 0$$

$$e_1 = e, e_2 = \dot{e}, \dots, e_p = e^{(p-1)} \quad (11)$$

$$\dot{e}_1 = e_2$$

⋮

$$\dot{e}_{p-1} = e_p$$

$$\dot{e}_p = -\alpha_0 e_1 - \dots - \alpha_{p-1} e_{p-1}$$

$$\begin{pmatrix} e_1 \\ \vdots \\ e_p \end{pmatrix}' = \underbrace{\begin{pmatrix} 0 & 1 & & 0 \\ & \ddots & \ddots & \\ & & 1 & 0 \\ 0 & & & 0 \end{pmatrix}}_{\bar{A}} \begin{pmatrix} e_1 \\ \vdots \\ e_p \end{pmatrix} + \underbrace{\begin{pmatrix} 0 \\ \vdots \\ 1 \end{pmatrix}}_{\bar{b}} (-\alpha_0 \dots -\alpha_{p-1}) \begin{pmatrix} e_1 \\ \vdots \\ e_p \end{pmatrix}$$

$(\bar{A}, \bar{b})$ controllable $\Rightarrow$

$\bar{F} = (-\alpha_0 \dots -\alpha_{p-1})$ s.t. $\bar{A} + \bar{b} \bar{F}$ as stable

Is this a 'good' controller?

Not always

$$\underline{Ex} \quad \begin{cases} \dot{x}_1 = -x_2 + u \\ \dot{x}_2 = x_1 + u \end{cases}$$

$$y = x_1$$

$$\begin{cases} \dot{y} = -x_2 + u \\ y_d \end{cases} \quad \left\{ \begin{array}{l} u = x_2 + \dot{y}_d + \alpha e \end{array} \right.$$

$$\Rightarrow \dot{e} = -\alpha e \quad \leadsto \quad e \rightarrow 0 \quad \text{exponentially!}$$

$$\text{But } x_2(t) \leadsto ? \quad \left\{ \begin{array}{l} \dot{x}_2 = x_2 \\ \dot{y}_d = 0, \alpha = 1 \end{array} \right.$$

$$x_2 \rightarrow \infty$$

Although the error-dynamics are (linear) exponentially stable, the unobservable part may become unstable.

This does not occur if

$\rho = n$ There is no other dynamical part

or if

$$\boxed{p < n}$$

the system is minimum-phase

$$\text{Let } u = -(cA^{p-1}b)^{-1}cA^p x \quad (v=0)$$

$$\dot{x} = Ax + Bu$$

$$\text{Assume } y(0) = \dot{y}(0) = \dots = y^{(p-1)}(0) = 0$$

$$\text{Then } y(t) = 0 \text{ for all } t$$

Σ is minimum-phase if
the resulting dynamics (
of dimension $n-p$ are)

are exponentially stable

Example 2nd order system

(14)

$$J \ddot{q} + F \dot{q} = u \quad (+ T(t))$$

$$J > 0, F \geq 0$$

$$\begin{pmatrix} q_1 \\ q_2 \end{pmatrix}' = \begin{pmatrix} 0 & 1 \\ 0 & -F/J \end{pmatrix} \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} + \begin{pmatrix} 0 \\ 1/J \end{pmatrix} u$$

$$y = q_1$$

Desired $y = q_{1d}(t)$

$p=2$! (=dim !)

$$\Rightarrow u = J \ddot{q}_{1d} + F q_2 - k_2 \dot{e} - k_1 e$$

$$e = q_1 - q_{1d}$$

$$J(\ddot{e} + k_2/J \dot{e} + k_1/J e) = 0$$

Alternative controller

ass. stable
iff $k_1 > 0, k_2 > 0$

$$u = \underbrace{J \ddot{q}_{1d} + F q_{1d}}_{\text{feedforward}} - \underbrace{k_d \dot{e}}_{\text{derivative}} - \underbrace{k_p e}_{\text{proportional}}$$

$$\Rightarrow J \ddot{e} + (F + k_d) \dot{e} + k_p e = 0$$

feedforward ($u^{ref}(t)$!)

ass. stable
iff $k_p > 0, k_d > 0$

Including observer

$$J\ddot{q} + F\dot{q} = u$$

$$u = J\ddot{q}_d + F\dot{q}_d - K_d \underset{\uparrow}{\dot{\hat{e}}} - K_p \underset{\uparrow}{\hat{e}}$$

$$\underbrace{\text{observer}} \left\{ \begin{array}{l} \dot{\hat{e}} = w + 2J^{-1}K_d(e - \hat{e}) - J^{-1}Fe \\ \dot{w} = 2J^{-1}K_p(e - \hat{e}) \end{array} \right.$$

Closed loop system

$$\left\{ \begin{array}{l} J\ddot{e} + (F + K_d)\dot{e} + K_p e = K_d \ddot{\hat{e}} + K_p \dot{\hat{e}} \\ J\ddot{\hat{e}} + K_d \dot{\hat{e}} + K_p \hat{e} = -K_d \dot{e} - K_p e \end{array} \right.$$

as. stable if $0 < K_p < J^{-1}K_d^2$

Many alternatives

Always observable

Nonlinear 2nd order control systems

Robustness / Adaptive Control

LYAPUNOV CONTROL IN ROBOTIC SYSTEMS: TRACKING REGULAR AND CHAOTIC DYNAMICS

HENK NIJMEIJER*, HARRY BERGHUIS**

Lyapunov-type controllers for trajectory tracking of rigid robot manipulators are described. The so-called passivity-based controllers exploit the desired physical energy of the robot system. A discussion about tracking control in the absence of complete state information and model knowledge then leads to practical stability. It is shown that these Lyapunov-type controllers can also be used in other mechanical systems as e.g. the controlled Duffing equation. In that case the controller can be used as a tool for creation or annihilation of chaotic dynamics.

1. Introduction

Over the last decade a lot of research has been done on designing sophisticated control strategies for rigid robot manipulators, see e.g. (Berghuis, 1993; Ortega and Spong, 1989; Spong and Vidyasagar, 1989) and references therein. In particular, for the tracking control problem of a rigid robot manipulator one may distinguish several controller schemes. Perhaps the best known method is the so-called computed torque controller (see Spong and Vidyasagar, 1989), which is essentially based upon feedback linearization of the robot model (cf. Nijmeijer and van der Schaft, 1990). Despite its mathematical elegance and simplicity the computed torque controller in robotics does not incorporate the physical nature of the manipulator involved. Therefore, more recently tracking controllers that are using the robot's physical structure, have been developed. These so-called *passivity-based* controllers are constructed by the idea of reshaping the energy of the manipulator in such a way so as to fulfil the control objective, see (Ortega and Spong, 1989; Takegaki and Arimoto, 1981) and others. Essentially, this energy-shaping philosophy induces a Lyapunov-type of controller design and in particular the stable tracking performance of the system in closed-loop is shown via an often tedious but in itself direct Lyapunov-function analysis. An interesting feature of these passivity-based schemes is that they are all built with a linear state feedback (Proportional-Derivative (PD)-feedback in the manipulator's position) which makes these schemes quite attractive in practice, see (Spong and Vidyasagar, 1989). In practical situations the inherent velocity-feedback – the derivative-feedback of the manipulator's position – may not be desirable because velocity-measurements are impossible (or corrupted by noise). In that case the tracking controller needs to

* Dept. of Applied Mathematics, University of Twente, P.O. Box 217, 7500 AE Enschede, the Netherlands, e-mail: h.nijmeijer@math.utwente.nl

** Hollandse Signaalapparaten B.V., Dept. RDT-R&S-MFE-SER, P.O. Box 42, 7550 GD Hengelo, the Netherlands, e-mail: h.berghuis@hgl.signaal.nl

be fed with a velocity-estimate obtained from a velocity-observer. This methodology has been developed using Lyapunov-type arguments in (Berghuis and Nijmeijer, 1993) and will be described in Section 2. Since in practice there will always be uncertainty in the dynamic model for the robot manipulator, a further question studied in Section 2 is how our analysis would extend to the case where model uncertainties are incorporated. A solution to this problem without using any model knowledge is given that yields practical stability of the closed-loop system, see also (Berghuis and Nijmeijer, 1994).

Given the physical basis for the aforementioned tracking controller (and the tracking controller-observer combination) it should not be surprising that this sort of controller is of use in many other physical systems. In this paper we focuss on the Duffing equation

$$\ddot{x} + p_0 \dot{x} + p_1 x + p_2 x^3 = a \cos(\omega t) \quad (1)$$

The above equation was introduced in 1918 by Duffing to describe a certain non-linear oscillator with a cubic stiffness term. We note that we concentrate here on (1) but we could have treated the well-known driven van der Pol equation in a similar way. The interest of (1) – or the driven van der Pol equation – is that for certain parameter values of p_0 , p_1 , p_2 , a and ω the dynamics (1) is complex and may include chaotic motion, see e.g. (Guckenheimer and Holmes, 1983). Equation (1) in itself is not controlled, but we will study its controlled version, in that we add in the right-hand side of (1) a control function u (which in principle can be physically realized in the system). The reason to do so is that in this way we are able to study the tracking control of the Duffing equation. We will show in Section 3 that the tracking controller-observer design for the controlled equation (1) parallels the developments of Section 2. Even more, like we have seen for the manipulator dynamics, also in this case we obtain practical stability of the closed-loop system without model knowledge (and thus, for instance, with unknown parameters p_0, p_1, p_2 and a). In particular, the given tracking control strategies that are again of Lyapunov-type, enable us to annihilate any complex or chaotic dynamics of (1) while tracking towards any desired trajectory. The latter has recently been studied extensively by e.g. Chen and Dong (1993a; 1993b) and Nijmeijer and Berghuis (1994).

It should be noted that the strong analogy between the tracking control schemes in robot manipulators and the controlled Duffing equation is not very surprising from a mathematical point of view since both systems are feedback equivalent to a second order linear system (cf. Nijmeijer and van der Schaft, 1990). The surprising point is that this also holds without using model knowledge. One consequence of what we have done is that a robot system may track any trajectory of the Duffing eqn. (1) and thus we may track periodic, complex and chaotic signals. In other words, the Lyapunov-type controller methods used in the tracking control of robot manipulators can also be employed as “a route towards chaos” and conversely as a methodology in “controlling chaos”. Both subjects have received a lot of attention in the literature, see e.g. the survey-paper by Chen and Dong (1993b) and references therein, Ott *et al.* (1990) and Singer *et al.* (1991). It is certainly not the purpose of the present paper to repeat much on the control creation of chaos. It is remarkable in our opinion that

the physically motivated Lyapunov-type control schemes used in robotics are also of relevance in the area of creation and annihilation of chaotic motion. In Section 4 we illustrate explicitly how chaotic motion tracking can be established by injecting a chaotic trajectory of the Duffing equation as the desired robot trajectory. In fact, it follows what kind of desired trajectory the overall system will follow (periodic, chaotic ...) depending on the initial condition of the Duffing equation. To illustrate our ideas we give a few simulation examples where the Lyapunov-type controller-observer with or without model knowledge is used to create a periodic, or chaotic motion for the robot system. Of course, in the case when model knowledge is not used the creation of periodic or chaotic motion is again up to any prescribed degree of accuracy.

We conclude this paper with some concluding remarks in Section 5.

2. Motion Control of Rigid Robots

Consider the dynamics of an n degrees-of-freedom (DOF) revolute joint rigid robot system

$$M(q)\ddot{q} + C(q, \dot{q})\dot{q} + G(q) = \tau \quad (2)$$

where $M(q)$ is the positive definite inertia matrix $[n \times n]$, $C(q, \dot{q})\dot{q}$ is the Coriolis and centripetal torques $[n \times 1]$, $G(q)$ is the gravitational torque $[n \times 1]$, and τ is the control input $[n \times 1]$. The matrix $C(q, \dot{q})$ is defined via the Christoffel symbols (see Ortega and Spong, 1989), which implies that $M(q) - 2C(q, \dot{q})$ is skew symmetric. For the revolute joint system we have (e.g. Spong and Vidyasagar, 1989)

$$0 < M_m \leq \|M(q)\| \leq M_M \quad \forall q \in \mathbb{R}^n \quad (3)$$

$$\|C(q, x)\| \leq C_M \|x\| \quad \forall q, x \in \mathbb{R}^n \quad (4)$$

$$\|G(q)\| \leq G_M \quad \forall q \in \mathbb{R}^n \quad (5)$$

where in (3), (4), (5) and in the sequel the norm of a vector x is defined as

$$\|x\| = \sqrt{x^T x} \quad (6)$$

and the norm of a matrix A as

$$\|A\| = \sqrt{\lambda_{\max}(A^T A)} \quad (7)$$

with $\lambda_{\max}(\cdot)$ denoting the maximum eigenvalue. Moreover, similarly to (3), for any symmetric positive definite matrix $A(x)$ and for all x , A_m and A_M denote the minimum and maximum eigenvalue of $A(x)$, respectively, if they exist.

To let system (2) follow an arbitrary smooth reference trajectory, various model-based control methods have been developed. Among many references we mention (Craig, 1988; Kelly and Salgado, 1994; Khosla and Kanade, 1988; Koditschek, 1989; Ortega and Spong, 1989; Paden and Panja, 1988; Sadegh and Horowitz, 1990; Slotine and Li, 1987; Wen and Bayard, 1988). One of the most useful controllers is given by

the *Desired Compensation Control Law* (DCCL). This controller belongs to the class of passivity-based controllers, and is described by (cf. Wen and Bayard, 1988)

$$\tau = M(q_d)\ddot{q}_d + C(q_d, \dot{q}_d)\dot{q}_d + G(q_d) - K_d\dot{e} - K_p e \quad (8)$$

where $q_d(t)$ represents the desired motion, $e \equiv q - q_d$ is the tracking error, and K_d and K_p are positive definite diagonal matrices $[n \times n]$. The DCCL consists of two parts: a linear state feedback part (also known as PD feedback) and a model-based compensation part that is computed along the reference trajectory. This way of model compensation is attractive for two reasons. Firstly, it permits off-line calculation of the computationally expensive model-based components, and, consequently, it preserves the simplicity of the PD controller. Secondly, the use of clean reference signals instead of noise-corrupted sensor data in the model compensation allows us to enhance the tracking accuracy, as can be concluded from several experimental studies, see e.g. (Leahy and Whalen, 1991; Whitcomb *et al.*, 1993; Berghuis, 1993).

Let us make the following assumption on the reference input.

Assumption 1. *The desired trajectory signal is bounded, i.e.*

$$V_M = \sup_t \|\dot{q}_d(t)\| < \infty \quad (9)$$

$$A_M = \sup_t \|\ddot{q}_d(t)\| < \infty \quad (10)$$

In addition, we assume

Assumption 2. *The controller gains K_p and K_d in (8) are related as*

$$K_p = \lambda K_d \quad (11)$$

where λ is a positive scalar.

In the sequel, we exploit the robot model properties (e.g. Kelly and Salgado, 1994).

$$\|M(x)z - M(y)z\| \leq k_M \|x - y\| \|z\| \quad (12)$$

$$\|C(x, v)w - C(y, z)w\| \leq k_C \|x - y\| \|z\| \|w\| + C_M \|v - z\| \|w\| \quad (13)$$

$$\|G(x) - G(y)\| \leq k_G \|x - y\| \quad (14)$$

Then Proposition 1 can be proved (see also Kelly and Salgado, 1994).

Proposition 1. *Consider the closed-loop system (2), (8) under Assumptions 1 and 2. Define*

$$x_1^T = [\dot{e}^T, (\lambda e)^T] \quad (15)$$

and assume that $\|x_1(0)\|$ represents an upper bound on the initial error state $x_1(0)$. Then under the condition

$$K_{d,m} > \lambda M_M + 2C_M V_M + 2\lambda^{-1} k_1 \quad (16)$$

where $k_1 \equiv k_M A_M + k_C V_M^2 + k_G$, the closed-loop system (2), (8) is asymptotically stable with guaranteed region of attraction

$$B = \left\{ x_1 \in \mathbb{R}^{2n} \mid \|x_1\| < \sqrt{\frac{\lambda M_m}{18 K_{d,M}}} \left[\frac{K_{d,m} - \lambda M_M - 2 C_M V_M - 2 \lambda^{-1} k_1}{C_M} \right] \right\} \quad (17)$$

Moreover, region (17) can be enlarged arbitrarily by increasing K_d ; i.e. system (2), (8) is semi-globally asymptotically stable.

Proof. The closed-loop dynamics (2), (8) are equal to

$$M(q)\ddot{e} + C(q, \dot{q})\dot{e} + K_d s_1 = \Delta Y(\cdot) \quad (18)$$

where

$$\begin{aligned} \Delta Y(\cdot) = & \left(M(q_d)\ddot{q}_d - M(q)\ddot{q}_d \right) + \left(C(q_d, \dot{q}_d)\dot{q}_d \right. \\ & \left. - C(q, \dot{q})\dot{q}_d \right) + \left(G(q_d) - G(q) \right) \end{aligned} \quad (19)$$

$$s_1 = \dot{e} + \lambda e \quad (20)$$

Using (3), (4), (5), (9), (10), (12), (13), (14) we obtain

$$\begin{aligned} \|\Delta Y(\cdot)\| & \leq (k_M A_M + k_C V_M^2 + k_G) \|e\| + C_M V_M \|\dot{e}\| \\ & = k_1 \|e\| + C_M V_M \|\dot{e}\| \end{aligned} \quad (21)$$

Let us consider the candidate Lyapunov function

$$V_1(e, \dot{e}) = \frac{1}{2} s_1^T M(q) s_1 + \frac{1}{2} e^T (2 \lambda K_d - \lambda^2 M(q)) e \quad (22)$$

As shown in Appendix, $V_1(e, \dot{e})$ satisfies

$$\frac{1}{2} P_m \|x_1\|^2 \leq V_1(e, \dot{e}) \leq \frac{1}{2} P_M \|x_1\|^2 \quad (23)$$

where $P_m = \frac{1}{3} M_m$ and $P_M = 6 \lambda^{-1} K_{d,M}$.

The time-derivative of $V_1(e, \dot{e})$ along (18) satisfies

$$\begin{aligned} \dot{V}_1(e, \dot{e}) = & -\dot{e}^T (K_d - \lambda M(q)) \dot{e} - (\lambda e)^T K_d (\lambda e) \\ & + (\dot{e} + \lambda e)^T \Delta Y(\cdot) + \dot{e}^T C(q, \dot{q}) (\lambda e) \end{aligned} \quad (24)$$

where Assumption 2 and the skew symmetry of $\dot{M}(q) - 2C(q, \dot{q})$ has been used. Employing (21) and using the sum of perfect squares, an upper bound on $\dot{V}_1(e, \dot{e})$ is given by

$$\begin{aligned} \dot{V}_1(e, \dot{e}) \leq & - \left(K_{d,m} - \lambda M_M - 2 C_M V_M - \frac{1}{2} \lambda^{-1} k_1 - C_M \|\lambda e\| \right) \|\dot{e}\|^2 \\ & - \left(K_{d,m} - C_M V_M - \frac{3}{2} \lambda^{-1} k_1 \right) \|\lambda e\|^2 \end{aligned} \quad (25)$$

This implies

$$\dot{V}_1(e, \dot{e}) \leq -(K_{d,m} - \lambda M_M - 2C_M V_M - 2\lambda^{-1} k_1) \|x_1\|^2 + C_M \|x_1\|^3 \quad (26)$$

Hence, from (26) $\dot{V}_1(e, \dot{e})$ is locally negative definite. This, together with the positive definiteness of $V_1(e, \dot{e})$, implies that the closed-loop system (2), (8) is locally asymptotically stable (cf. Khalil, 1992). Finally, because the region of attraction (17) can be arbitrarily enlarged by increasing K_d we obtain semi-global asymptotic stability (see (Teel and Praly, 1994) for a similar definition). ■

The DCCL in (8) yields semi-global asymptotic stability. By introducing a *non-linear PD action* $-K_f \|e\|^2(\dot{e} + \lambda e)$ in the control loop it is possible to establish global asymptotic stability, see (Sadegh and Horowitz, 1990).

In practice, there will always be uncertainty in the dynamical model (2). This leads us to the natural question: what can be said about the stability issue in the presence of model errors? To this end, consider a simple case in which the robot is controlled by just linear PD feedback, i.e. (8) without model knowledge. So

$$\tau = -K_d \dot{e} - K_p e \quad (27)$$

Then the next proposition can be proven (Qu and Dorsey, 1991), which essentially states that linear high-gain state-feedback guarantees the robot system to follow any reference trajectory with bounded error.

Proposition 2. *Consider the PD-feedback controller (27) under Assumptions 1 and 2. Then the closed-loop system (2), (27) is semi-globally uniformly ultimately bounded or practically stable under some suitably selected (high-gain) condition on the derivative controller gain K_d .*

Proof (Main steps). The closed-loop dynamics (2), (27) are equal to those of (18) except for an additional term at the right-hand side representing the model-based feedforward component in (8), i.e.

$$\Delta Z = M(q_d) \ddot{q}_d + C(q_d, \dot{q}_d) \dot{q}_d + G(q_d) \quad (28)$$

According to (3), (4), (5), (9), (10) and (28) can be bounded as

$$\begin{aligned} \|\Delta Z\| &\leq M_M A_M + C_M V_M^2 + G_M \\ &\equiv k_2 \end{aligned} \quad (29)$$

Consequently, the bound on $\dot{V}_1(e, \dot{e})$ in (26) changes into

$$\dot{V}_1(e, \dot{e}) \leq 2k_2 \|x_1\| - (K_{d,m} - \lambda M_M - 2C_M V_M - 2\lambda^{-1} k_1) \|x_1\|^2 + C_M \|x_1\|^3 \quad (30)$$

Hence, the time-derivative of the Lyapunov function is negative definite in an annulus of a certain width around the origin. Therefore, the closed-loop system is locally uniformly ultimately bounded (cf. Chen and Leitmann, 1987; Qu and Dorsey, 1991). ■

One characteristic feature of the above state-feedback controllers is that they require both position *and* velocity measurements. In practice, however, this requirement is generally not fulfilled. Hence, it makes sense to develop a controller that preserves the attractive implementation properties of the DCCL but which only employs position feedback. For this purpose, consider the following modification to (8) (cf. Kaneko and Horowitz, 1994; Berghuis and Nijmeijer, 1993):

$$\text{Controller } \left\{ \tau = M(q_d)\ddot{q}_d + C(q_d, \dot{q}_d)\dot{q}_d + G(q_d) - K_d\dot{\hat{e}} - K_p\hat{e} \right. \quad (31)$$

$$\text{Observer } \left\{ \begin{array}{l} \dot{\hat{e}} = \dot{w} + L_d(e - \hat{e}) \\ \dot{w} = L_p(e - \hat{e}) \end{array} \right. \quad (32)$$

where $\hat{e}$ represents the estimated tracking error, $[\dot{\hat{e}}^T \hat{e}^T]^T$ is the estimated error state, and L_p and L_d are positive definite diagonal matrices $[n \times n]$. The estimated error state is generated by a *linear* observer, which only requires the position error e as input. Next, the state estimate is injected in the control loop.

Let us make the following assumption on the structure of the observer gains:

Assumption 3. L_d and L_p can be written as

$$L_d = (\ell_d + \lambda)I \quad (33)$$

$$L_p = \lambda\ell_d I \quad (34)$$

where $\ell_d > 0$ is a scalar and λ is as before (see (11)).

Then we have

Proposition 3. Consider the closed loop (2), (31), (32) under Assumptions 1–3. Define

$$x_2^T = [\dot{e}^T, (\lambda e)^T, \dot{\hat{e}}, (\lambda \hat{e})^T] \quad (35)$$

where $\tilde{e} \equiv e - \hat{e}$, and assume that $\|x_2(0)\|$ is an upper bound on $x_2(0)$. If

$$K_{d,m} > \lambda M_M + 4C_M V_M + 3\lambda^{-1}k_1 \quad (36)$$

$$\ell_d > 2M_m^{-1}K_{d,M} \quad (37)$$

then the closed loop system is semi-globally asymptotically stable with guaranteed region of attraction

$$B = \left\{ x_2 \in \mathbb{R}^{4n} \mid \|x_2\| < \sqrt{\frac{\lambda M_m}{450K_{d,M}}} \left[\frac{K_{d,m} - \lambda M_M - 4C_M V_M - 3\lambda^{-1}k_1}{C_M} \right] \right\} \quad (38)$$

Proof. Introduce

$$s_2 = \dot{\bar{e}} + \lambda \bar{e} \quad (39)$$

then the closed-loop dynamics (2), (31), (32) are described by

$$M(q)\ddot{e} + C(q, \dot{q})\dot{e} + K_d(s_1 - s_2) = \Delta Y(\cdot) \quad (40)$$

$$M(q)s_2 + C(q, \dot{q})s_2 + \ell_d M(q)s_2 + K_d(s_1 - s_2) = \Delta Y(\cdot) + C(q, \dot{q})(s_2 - \dot{e}) \quad (41)$$

where $\Delta Y(\cdot)$ is defined in (19). Consider a candidate Lyapunov function

$$V(e, \dot{e}, \bar{e}, \dot{\bar{e}}) = V_1(e, \dot{e}) + V_2(\bar{e}, \dot{\bar{e}}) \quad (42)$$

with $V_1(e, \dot{e})$ as in (22), and $V_2(\bar{e}, \dot{\bar{e}})$ equal to

$$V_2(\bar{e}, \dot{\bar{e}}) = \frac{1}{2} s_2^T M(q) s_2 + \frac{1}{2} \bar{e}^T (2\lambda K_d) \bar{e} \quad (43)$$

As before, $V(e, \dot{e}, \bar{e}, \dot{\bar{e}})$ satisfies

$$\frac{1}{2} P_m \|x_2\|^2 \leq V(e, \dot{e}, \bar{e}, \dot{\bar{e}}) \leq \frac{1}{2} P_M \|x_2\|^2 \quad (44)$$

where $P_m = \frac{1}{3} M_m$ and $P_M = 6\lambda^{-1} K_{d,M}$.

Evaluating the time-derivative of $V(\cdot)$ along (40), (41), using bound (21) on $\Delta Y(\cdot)$, and employing the sum of perfect squares, the proof can be completed along the lines of the proof of Proposition 1. ■

As can be seen by comparing Propositions 1 and 3, both the state- and output-feedback solution to DCCL can be treated in a unified way. Using the same mathematical machinery essentially the same stability result is established. As a natural consequence, we also obtain the next proposition, in analogy to Proposition 2. For the proof we refer to Berghuis and Nijmeijer (1994).

Proposition 4. *Consider the linear estimated state-feedback controller*

$$\tau = K_d \dot{\bar{e}} - K_p \bar{e} \quad (45)$$

where the error state is determined with the linear observer proposed in (32). Suppose Assumptions 1–3 are satisfied. Then the closed-loop system (2), (45) is semi-globally uniformly ultimately bounded under suitably selected (high-gain) conditions on K_d and ℓ_d .

Remark 1. Proposition 4 was given under the condition that both Assumption 2 and Assumption 3 hold, which in particular implies that in (11) and (33) the same parameter λ appears. This is in fact not at all necessary in that we may replace (11) and (33) as $K_p = \lambda_1 K_d$, $L_d = (\ell_d + \lambda_2)I$ and $L_p = \lambda_2 \ell_d I$ for different parameters λ_1 and λ_2 , and the result of Proposition 4 is still true, but the proof becomes slightly

more involved. This extra freedom of selecting the controller-observer gains will be used in the simulations of Section 4.

Hence, the highly non-linear robot dynamics can be stabilized around any bounded reference trajectory by a linear output-feedback controller. In the next section it is shown that the presented results in robot control can also be used in other physical systems, in particular the Duffing dynamics.

3. Feedback Control of Duffing Equation

The Duffing equation describes a specific non-linear circuit or a pendulum moving in a viscous medium, and is given as (see (1))

$$\ddot{x} + p_0\dot{x} + p_1x + p_2x^3 = a \cos(\omega t) \quad (46)$$

where $p_0 > 0$, $p_1 > 0$, p_2 , a and ω are known constants. Depending on the choice of these constants it is known that solutions of (46) exhibit periodic, almost periodic and chaotic behaviour cf. (Chen and Dong, 1993a; 1993b; Guckenheimer and Holmes, 1983).

It is our purpose to discuss a controlled version of (46). For this we consider

$$\ddot{x} + p_0\dot{x} + p_1x + p_2x^3 = u + a \cos(\omega t) \quad (47)$$

where $u(\cdot)$ is the *physical control input*. The general problem that we want to study is whether we are able to find a suitable feedback controller

$$u = k(x, \dot{x}, x_d, t) \quad (48)$$

such that for the closed-loop system (47), (48) the solution $x(t)$ asymptotically converges to a desired trajectory $x_d(t)$, $t \geq 0$. Here $x_d(t)$ may represent *any* smooth and bounded time-function, including fixed points or periodic orbits.

The crucial observation is that from a control point of view the dynamics of the controlled Duffing equation (47) is essentially the same as that of a one-DOF robot system. The main difference are the linear and cubic term in x , but these terms do not cause any problems in both the control design and the stability analysis, as will be shown below.

Assume we want the system to follow the reference $x_d(t)$. For this purpose, we define in analogy with (8) the control input as

$$u = \ddot{x}_d + p_0\dot{x}_d + p_1x_d + p_2x_d^3 - a \cos(\omega t) - K_d\dot{e} - K_p e + v \quad (49)$$

$$v = 3p_2xx_d e \quad (50)$$

with $e \equiv x - x_d$, and $K_d > 0$, $K_p > 0$ scalar. The compensation term v is introduced in order to deal with the cubic term in (47). As before, let us take (cf. (11))

$$K_p = \lambda K_d \quad (51)$$

and $\lambda > 0$ scalar. Then we can prove:

Proposition 5. *Let K_p satisfy (51). Then under the condition*

$$K_d > \max(\lambda, -p_1\lambda^{-1}) \quad (52)$$

the closed-loop system (47), (49), (50) is globally asymptotically stable.

Proof. The closed-loop system (47), (49), (50) is described by

$$\ddot{e} + (p_0 + K_d)\dot{e} + (p_1 + K_p)e + p_2e^3 = 0. \quad (53)$$

In correspondence to (22), consider the candidate Lyapunov function

$$V_1(e, \dot{e}) = \frac{1}{2}s_1^2 + \frac{1}{2}((p_1 + \lambda K_d) + \lambda(p_0 + K_d) - \lambda^2)e^2 + \frac{1}{4}p_2e^4 \quad (54)$$

Because of (52) and $p_0 > 0$, $V_1(e, \dot{e})$ is positive definite. Along (53), $\dot{V}_1(e, \dot{e})$ equals

$$\dot{V}_1(e, \dot{e}) = -(p_0 + K_d - \lambda)\dot{e}^2 - \lambda(p_1 + \lambda K_d)e^2 - \lambda p_2e^4 \quad (55)$$

Under (52) we have $\dot{V}_1(e, \dot{e})$ globally negative definite in $(e, \dot{e})$. Consequently, the closed-loop system (53) is globally asymptotically stable (cf. Khalil, 1992). ■

Controller (49), (50) allows us to steer the Duffing equation towards an arbitrary reference trajectory $x_d(t)$. Hence, the chaotic behaviour the uncontrolled Duffing dynamics may display is completely annihilated by feedback control. Now, suppose that $x_d(t)$ represents a (stable or unstable) equilibrium motion of the uncontrolled dynamics, i.e.

$$\ddot{x}_d + p_0\dot{x}_d + p_1x_d + p_2x_d^3 = a \cos(\omega t) \quad (56)$$

Then by combining (49), (50) and (56) we have:

Corollary 1. *If the desired motion $x_d(t)$ satisfies (56), then the controller*

$$u = -K_d\dot{e} - K_p e + 3p_2x_d x_{de} \quad (57)$$

guarantees that the Duffing equation asymptotically converges under assumption (51) and condition (52) towards (56) in a global sense.

It is rather straightforward to show that PD-feedback (27) allows the controlled Duffing dynamics (47) to follow any bounded reference trajectory with bounded error. In particular, under high-gain PD-control semi-global stability of the closed-loop can be shown, in analogy to Proposition 2. As discussed in (Nijmeijer and Berghuis, 1994), for (47) also output-feedback type of controllers can be developed that yield global asymptotic stability. Here we will concentrate on the model-independent linear estimated state-feedback controller (45), and analyze its stability properties when it

is used to control the Duffing equation. In particular, consider

$$\text{Controller } \begin{cases} u = -K_d \dot{e} - K_p e \end{cases} \quad (58)$$

$$\text{Observer } \begin{cases} \dot{\hat{e}} = w + 2K_d(e - \hat{e}) \\ \dot{w} = 2K_p(e - \hat{e}) \end{cases} \quad (59)$$

Since measuring $\dot{x}$ in the controlled Duffing equation (47) might be difficult and noise-sensitive, the controller-observer combination (58), (59) seems very attractive. We can prove the following result.

Proposition 6. *Consider Duffing equation (47) under robust linear output-feedback control (51), (58), (59). Then the closed-loop dynamics is locally uniformly ultimately bounded for K_d sufficiently large.*

Proof. The error dynamics (47), (51), (58), (59) is given by

$$\ddot{e} + p_0 \dot{e} + p_1 e + p_2 e^3 + K_d s_1 = K_d s_2 + \Delta W - 3p_2 x x_d e \quad (60)$$

$$\ddot{\tilde{e}} + K_d s_2 = -K_d s_1 + \Delta W - 3p_2 x x_d e - p_0 \dot{\tilde{e}} - p_1 \tilde{e} - p_2 \tilde{e}^3 \quad (61)$$

where

$$\Delta W(\cdot) = a \cos(\omega t) - \ddot{x}_d - p_0 \dot{x}_d - p_1 x_d - p_2 x_d^3 \quad (62)$$

Take the Lyapunov function candidate

$$V(e, \dot{e}, \tilde{e}, \dot{\tilde{e}}) = V_1(e, \dot{e}) + V_2(\tilde{e}, \dot{\tilde{e}}) \quad (63)$$

where $V_1(e, \dot{e})$ as in (54) and

$$V_2(\tilde{e}, \dot{\tilde{e}}) = \frac{1}{2} s_2^2 + \frac{1}{2} (2\lambda K_d - \lambda^2) \tilde{e}^2 \quad (64)$$

The time-derivative of $V(\cdot)$ along (60), (61) equals to

$$\begin{aligned} \dot{V}(e, \dot{e}, \tilde{e}, \dot{\tilde{e}}) &= -(p_0 + K_d - \lambda) \dot{e}^2 - (\lambda^{-1} p_1 + K_d) (\lambda e)^2 - \lambda p_2 e^4 \\ &\quad - (K_d - \lambda) \dot{\tilde{e}}^2 - K_d (\lambda \tilde{e})^2 \\ &\quad + (\dot{e} + \lambda e + \dot{\tilde{e}} + \lambda \tilde{e}) (\Delta W - 3p_2 x_d^2 e - 3p_2 x_d e^2) \\ &\quad - (\dot{\tilde{e}} + \lambda \tilde{e}) (p_0 \dot{e} + p_1 e + p_2 e^3) \end{aligned} \quad (65)$$

For the bounded reference trajectory we define

$$P_M = \sup_t \|x_d(t)\|, \quad V_M = \sup_t \|\dot{x}_d(t)\|, \quad A_M = \sup_t \|\ddot{x}_d(t)\| \quad (66)$$

Then a bound on $\dot{V}(\cdot)$ is given by

$$\dot{V}(\cdot) \leq a_1 \|x_2\| - (K_d - a_2) \|x_2\|^2 + a_4 \|x_2\|^4 \quad (67)$$

where x_2 as in (35), and the constants a_i , $i = 1, 2, 4$, are given by

$$a_1 \equiv 4(|a| + A_M + p_0 V_M + |p_1| P_M + p_2 P_M^3) \quad (68)$$

$$a_2 \equiv \max(\lambda, -\lambda^{-1} p_1) + 2p_0 + 2\lambda^{-1} |p_1| + 18\lambda^{-1} V_M^2 p_2 \quad (69)$$

$$a_4 \equiv 8\lambda^{-3} p_2 \quad (70)$$

Thus, $\dot{V}(\cdot)$ is negative definite in an annulus around the origin, whose width can be enlarged with K_d . As shown in (Chen and Leitmann, 1987) (see also Qu and Dorsey, 1991), this implies that the closed-loop system is locally uniformly ultimately bounded. This completes the proof. ■

4. A Route to Chaos

To generate chaos in robot systems, we consider the simplest case, the dynamics of a one-DOF robot, i.e.

$$m\ell^2 \ddot{q} + mg\ell \sin(q) = \tau \quad (71)$$

where $m > 0$ and $\ell > 0$ scalar. Let us take the robust linear output-feedback controller (cf. (32), (45))

$$\text{Controller } \begin{cases} \tau = -K_d \dot{e} - K_p e \end{cases} \quad (72)$$

$$\text{Observer } \begin{cases} \dot{\hat{e}} = w + L_d(e - \hat{e}) \\ \dot{w} = L_p(e - \hat{e}) \end{cases} \quad (73)$$

The reference trajectory is assumed to satisfy the Duffing characteristics

$$\ddot{q}_d + p_0 \dot{q}_d + p_1 q_d + p_2 q_d^3 = a \cos(\omega t) \quad (74)$$

Now, by selecting different sets of parameters in (74) and initial conditions for (74) the controlled robot system can be forced to display periodic, almost periodic and chaotic behavior. Because we consider the robust controller, the actual state trajectory of the robot system follows the prespecified reference with bounded error. This error bound can be arbitrarily decreased by enlarging the controller gains K_d .

To illustrate our results we simulated the one-DOF robot (71) to track a periodic and chaotic trajectory of the Duffing equation (74). This was done by using the controller-observer combination (72), (73). The robot characteristics were chosen as

$$m = 0.1 \text{ kg}, \quad \ell = 2 \text{ m} \quad (75)$$

The Duffing parameters from (74) were selected either as

$$p_0 = 0.4, \quad p_1 = -1.1, \quad p_2 = 1.0, \quad \omega = 1.8, \quad a = 0.62 \quad (76)$$

for generating a periodic orbit or

$$p_0 = 0.4, \quad p_1 = -1.1, \quad p_2 = 1.0, \quad \omega = 1.8, \quad a = 1.8 \quad (77)$$

for generating a chaotic trajectory (cf. Chen and Dong, 1993). The controller-observer gains in (72), (73) were chosen in both cases as (see Remark 1)

$$K_d = 10, \quad \lambda = 5 \quad (78)$$

$$\ell_d = 30, \quad \lambda_2 = 0.3 \quad (79)$$

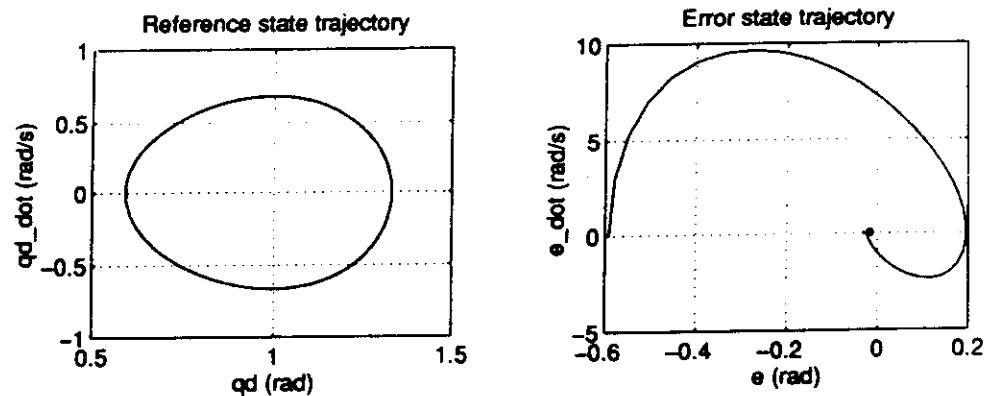


Fig. 1. Results for period-one solution of Duffing equation (74).

The resulting Duffing trajectories in the $(q_d, \dot{q}_d)$ -plane are given in Figs. 1 and 2. Additionally we visualize in these figures the error-trajectories of the closed-loop system (71), (72), (73), (74) with the above parameter selections (75), (76), (77).

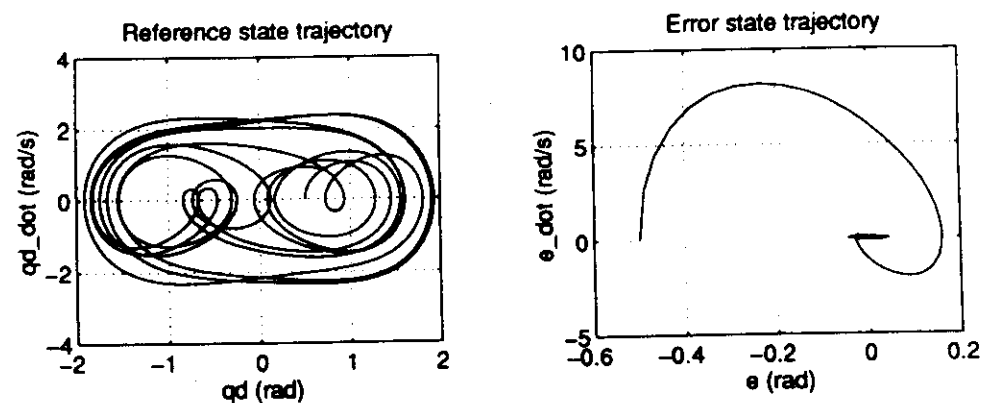


Fig. 2. Results for chaotic solution of Duffing equation (74).

The error-trajectories in closed loop were initialized at, respectively, $(e(t_0), \dot{e}(t_0)) = (-0.5909, 0)$ for the periodic orbit (Fig. 1) and $(e(t_0), \dot{e}(t_0)) = (-0.5, 0)$ for the chaotic trajectory (Fig. 2). Notice that in order to eliminate the transient effects we have initialized the error-trajectories at some time $t_0 > 0$. Inspection of both simulations clearly shows that the error trajectories readily converge to a neighbourhood of $(0, 0)$ despite the fact that our controller-observer combination (72), (73) does not use model information nor velocity measurements. By increasing the gains this neighborhood of stability can be made smaller, cf. Proposition 6.

5. Conclusions

We have shown in this paper how physically based controllers and controller-observer combinations can be designed for solving the tracking control problem for rigid robot manipulators. These so-called passivity-based techniques centre around the idea of reshaping the energy of the manipulator in such a way so as to fulfil the control objective, see (Takegaki and Arimoto, 1981). The modified energy is then used as a Lyapunov-function for the closed-loop system. We show that this type of Lyapunov control can also be used when no model information is used; in this case the closed-loop system is shown to be practically stable (instead of asymptotically stable). The Lyapunov control we use in robot manipulators can also be used in other physical control systems. In particular we illustrate this fact here for the controlled Duffing equation. The nice feature of our techniques is that, no matter how the initial dynamical system behaves (stable, periodic, chaotic), the controlled system can follow *any* desired trajectory with any degree of accuracy. In this way this work complements ongoing research on the control of chaotic dynamics, cf. (Chen and Dong, 1993a; 1993b).

Appendix

Function (22) can be written as

$$V_1(\cdot) = \frac{1}{2} y^T R(q) y \quad (\text{A1})$$

where

$$y^T = [s_1^T, (\lambda e)^T] \quad (\text{A2})$$

and

$$R(q) = \begin{bmatrix} M(q) & 0 \\ 0 & 2\lambda^{-1}K_d - M(q) \end{bmatrix} \quad (\text{A3})$$

According to (16), we have

$$K_{d,m} > \lambda M_M \quad (\text{A4})$$

Hence, we obtain

$$\frac{1}{2} R_m \|y\|^2 \leq V_1(\cdot) \leq \frac{1}{2} R_M \|y\|^2 \quad (\text{A5})$$

where

$$R_m = M_m, \quad R_M = 2\lambda^{-1} K_{d,M} \quad (\text{A6})$$

By definition,

$$y = T x_1 \quad (\text{A7})$$

with

$$T = \begin{bmatrix} \begin{bmatrix} I & I \\ 0 & I \end{bmatrix} & 0 \\ 0 & \begin{bmatrix} I & I \\ 0 & I \end{bmatrix} \end{bmatrix} \quad (\text{A8})$$

Note from (A7)

$$\frac{1}{3} \|x_1\|^2 \leq \|y\|^2 \leq 3 \|x_1\|^2 \quad (\text{A9})$$

Together with (A5) and (A6) this implies (23).

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replacing the nonlinear load by a $1\ \Omega$ resistor. The dominant impulse response poles and residues of the circuit are then calculated using AWE technique. For this circuit, two sets of poles and residues are required: from input to output and from a current source connected parallel to load to output. The first order response is then obtained using the first set of pole-residue by exciting the circuit with the actual input. Subsequently, the input is killed and the higher order responses are calculated by exciting the circuit with a current source connected parallel to load and using the second set of pole-residue. The order is increased until the algorithm converges. The output waveforms, obtained with the proposed method and HSPICE, are given in Fig. 2 for comparison. The first, third, fifth, and eleventh order outputs are shown in Fig. 3.

Example 2: The second example, which is shown in Fig. 4, has been taken from [16]. The nonlinear elements are defined as: $I_a = 0.001V_a^3$, $I_b = V_b/750 + 0.002V_b^3$, and $I_{out} = 0.001V_{out}^3$. The applied input voltage waveform for this circuit is 4.5 ns pulse with 1.5 ns rise and fall times. The amplitude of the input pulse is 5 volts. The output waveform obtained using the proposed method is compared with HSPICE result in Fig. 5.

IV. CONCLUSION

A new method has been proposed for the transient analysis of circuits with relatively few and mildly nonlinear terminations. In this approach, the method of Volterra-series analysis of the nonlinear elements is combined with AWE-based techniques for the linear part of the circuit. The method is noniterative and corresponds to recursive analysis of a linear circuit with different excitations. Therefore, it has no convergence problem. Since it is based on AWE technique, it uses a very small number of LU decompositions with respect to the traditional methods.

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On Lyapunov Control of the Duffing Equation

Henk Nijmeijer and Harry Berghuis

Abstract—In this brief, we develop feedback control strategies for a chaotic dynamic system such as the Duffing equation. Our controllers are of the so-called Lyapunov-type and are inspired by robot manipulator feedback controls. The different controllers we propose include observer-based controllers that even can cope with parametric uncertainties of the original system. Some simulation examples support the developed methods.

I. INTRODUCTION

Recently, an increasing interest has been developed in controlling chaotic nonlinear systems as arising in physics and engineering: from the various relevant references we mention [4]–[8], [12], [13], [15], and references therein. A very essential element in the control of chaos is that, in many cases, the ultimate goal of control is to decrease random effects and to stabilize the system at an equilibrium point, or more general, about a given reference trajectory. In such cases, one is in fact naturally led to reduce or even more completely annihilate the chaotic dynamics that an uncontrolled system may exhibit. Depending on the specific desired behavior of the system, several methods for controlling chaotic systems have been proposed, see, e.g., [6], [13]. Among the methods given there, a prominent role is played by the so-called *Lyapunov-type methods*. At the same time and earlier, various authors have investigated stabilizing control schemes for second-order mechanical systems, as in particular robot manipulators. Let us give a sample of relevant references [1], [10], [14], [17], [19], noting that also this field is strongly progressing at the moment. It should be noted that also in this context Lyapunov-type methods are very popular and useful.

The purpose of this paper is essentially to develop a controller-observer scheme for controlling a chaotic second-order system such as

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H. Nijmeijer is with the University of Twente, Department of Applied Mathematics, 7500 AE Enschede, The Netherlands.
H. Berghuis is with Hollandse Signaalapparaten B.V., Department RDT-R&S-MFE-SER, 7550 GD Hengelo, The Netherlands.
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the Duffing equation; see also [5] where a controller without observer has been derived. Our controller-observer analysis is inspired by the aforementioned papers on robot control. Recall that Duffing's equation describes a specific nonlinear circuit or a pendulum moving in a viscous medium and is in controlled form given by

$$\ddot{x} + p\dot{x} + p_1x + p_2x^3 = u(t) + q \cos(\omega t) \quad (1.1)$$

or, setting $x_1 \equiv x$, $x_2 \equiv \dot{x}$, as

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -p x_2 - p_1 x_1 - p_2 x_1^3 + u(t) + q \cos(\omega t) \end{cases} \quad (1.2)$$

where p , p_1 , p_2 , q , and ω are constants, and $u(\cdot)$ is the physical control input. Depending on the choice of these constants, it is known that solutions of (1.2) exhibit periodic, almost periodic, and chaotic behavior; see [5]. The general problem that we will study throughout this note is whether we are able to find a suitable output-feedback controller

$$u = o(x_1, \dot{x}_2, x_d, t) \quad (1.3)$$

such that for the closed loop (1.2), (1.3) the solution $x(t) = x_1(t)$ asymptotically converges to a desired trajectory $x_d(t)$, $t \geq 0$. Here, $\dot{x}_2$ is obtained via an observer for the "velocity" x_2 , and $x_d(t)$ may represent any smooth time-function, including fixed points or periodic orbits.

The manuscript is organized as follows: In Section II, we briefly recall some results on feedback control of second-order (mechanical) systems. Next, in the third section, we translate these results to the control of the Duffing equation (1.2). In Section IV, we discuss some related issues. Section V provides simulations that support our findings. Finally, Section VI contains the conclusions.

II. FEEDBACK CONTROL OF SECOND-ORDER SYSTEMS

Consider the second-order dynamics

$$J\ddot{q} + F\dot{q} = \tau + T(t) \quad (2.1)$$

where $J > 0$, $F \geq 0$, τ is the control, and $T(t)$ is a (known) disturbance. Note that this is exactly the dynamics of a one degree-of-freedom robot system [17], where in that case q represents the angular position. Our control objective is to let the system (2.1) follow an arbitrary smooth reference trajectory $q_d(t)$. For this purpose, we select the control input as

$$\tau = J\ddot{q}_d + F\dot{q}_d - T(t) - K_d\dot{e} - K_p e \quad (2.2)$$

where $e \equiv q - q_d$ represents the tracking error, and $K_d > 0$, $K_p > 0$. This controller consists of three components, namely

- 1) a position error feedback part $-K_p e$
- 2) a velocity error feedback part $-K_d \dot{e}$
- 3) a feedforward part $J\ddot{q}_d + F\dot{q}_d - T(t)$

The feedback terms are required to guarantee that the robot system converges towards $q_d(t)$. Once on this trajectory, the feedforward component keeps the robot moving along it.

The closed-loop consisting of (2.1) and (2.2) is described by the second-order dynamics

$$J\ddot{e} + (F + K_d)\dot{e} + K_p e = 0. \quad (2.3)$$

As can be seen from (2.3), the transient behavior of the error dynamics can be influenced by a suitable choice of the proportional and derivative gain K_p and K_d , respectively. For this system, we have the following result (cf. [19]):

Proposition 2.1: The feedback controller (2.2) guarantees that (2.1) asymptotically converges towards any smooth and bounded reference trajectory $q_d(t)$, i.e.,

$$\lim_{t \rightarrow \infty} e(t) = 0 \quad \text{and} \quad \lim_{t \rightarrow \infty} \dot{e}(t) = 0. \quad (2.4)$$

Proof: Consider the candidate Lyapunov function

$$V_1(e, \dot{e}) = \frac{1}{2}J(\dot{e} + \lambda e)^2 + \frac{1}{2}(K_p + \lambda(F + K_d) - \lambda^2 J)e^2 \quad (2.5)$$

with $\lambda > 0$ constant. A sufficient condition for $V_1(\cdot)$ to be positive definite in $(e, \dot{e})$ is

$$0 < \lambda < J^{-1}K_d. \quad (2.6)$$

Along the closed-loop error dynamics (2.3), the time-derivative of $V_1(e, \dot{e})$ becomes

$$\dot{V}_1(e, \dot{e}) = -(F + K_d - \lambda J)\dot{e}^2 - \lambda K_p e^2. \quad (2.7)$$

Because λ satisfies (2.6), we have that $\dot{V}_1(\cdot)$ is negative definite in the error state $(e, \dot{e})$. Consequently, the closed-loop system (2.3) is asymptotically stable (cf. [9]). $\square$

The Lyapunov function (2.5) is very familiar in robotics literature, e.g., [19]. This function originates from and closely resembles the natural energy contents of the open-loop (robot) dynamics (2.1); see, for instance, [10], [11], [18].

The controller (2.2) requires full-state information, i.e., "position" q and "velocity" $\dot{q}$ measurements are necessary in its actual implementation. However, in practice, velocity sensing equipment is generally not available. To overcome this velocity measurement problem, we modify the controller (2.2) as follows [1]:

$$\text{Controller} \begin{cases} \tau = J\ddot{q}_d + F\dot{q}_d - T(t) - K_d\dot{\hat{e}} - K_p\hat{e} \end{cases} \quad (2.8a)$$

$$\text{Observer} \begin{cases} \dot{\hat{e}} = \hat{u} + 2J^{-1}K_d(e - \hat{e}) - J^{-1}Fe \\ \dot{\hat{u}} = 2J^{-1}K_p(e - \hat{e}) \end{cases} \quad (2.8b)$$

This output-feedback controller (i.e., it only requires knowledge of the position q) consists of two parts: a linear observer part (2.8b) that generates an estimated error state $(\hat{e}, \hat{\dot{e}})$ from the position error e and a controller part (2.8a) that utilizes this estimated error state in the feedback loop. Let us assume that

$$K_p = \lambda K_d \quad (2.9)$$

where $\lambda > 0$ scalar. Then we can prove

Proposition 2.2: Let K_p satisfy (2.9). Under condition (2.6), the closed-loop dynamics (2.1, 8) is asymptotically stable.

Proof: The closed-loop system (2.1, 8) can be written as

$$J\ddot{e} + (F + K_d)\dot{e} + K_p e = K_d\dot{\hat{e}} + K_p\hat{e} \quad (2.10a)$$

$$J\ddot{\hat{e}} + K_d\dot{\hat{e}} + K_p\hat{e} = -K_d\dot{e} - K_p e \quad (2.10b)$$

where $\hat{e} \equiv e - \hat{e}$. Define the Lyapunov function as

$$V(e, \dot{e}, \hat{e}, \dot{\hat{e}}) = V_1(e, \dot{e}) + V_2(\hat{e}, \dot{\hat{e}})$$

with $V_1(e, \dot{e})$ as in (2.5) and in analogy with this

$$V_2(\hat{e}, \dot{\hat{e}}) = \frac{1}{2}J(\dot{\hat{e}} + \lambda\hat{e})^2 + \frac{1}{2}(K_p + \lambda K_d - \lambda^2 J)\hat{e}^2. \quad (2.11)$$

Condition (2.6) guarantees that $V(\cdot)$ is positive definite in $(e, \dot{e}, \hat{e}, \dot{\hat{e}})$. The time-derivative of $V(\cdot)$ along (2.10) equals

$$\dot{V}(x) = -x^T Q x \quad (2.12)$$

where $x^T = [\lambda\hat{e} \ \dot{\hat{e}} \ \lambda\hat{e} \ \dot{\hat{e}}]$ and

$$Q = \text{diag}(K_d, F + K_d - \lambda J, K_d, K_d - \lambda J). \quad (2.13)$$

Condition (2.6) is sufficient for $Q > 0$. This completes the proof. $\square$ We stress that Propositions 2.1 and 2.2 are valid for arbitrary bounded

reference trajectories $q_d(t)$. This is due to the presence of the feedforward term in the controller. In the next section, it is shown that the previous results can be employed in the context of controlling the Duffing equation.

III. FEEDBACK CONTROL OF DUFFING'S EQUATION

In [5], the (modified) Duffing equation in controlled form is introduced as

$$\ddot{x} + p\dot{x} + p_1x + x^3 = u + q \cos(\omega t) \quad (3.1)$$

where p , p_1 , q , and ω are constant. In contrast to [5], we do not assume beforehand that $p > 0$. The dynamics of this system is similar to that of (2.1) by putting $J \equiv 1$, $F \equiv p$, $\tau \equiv u$, and $T(t) \equiv q \cos(\omega t)$. The only differences are the linear and cubic term in x , but these terms do not cause any problems in both the control design and the stability analysis, as will be shown below.

As before, assume we want the system to follow any smooth desired trajectory $x_d(t)$. For this purpose, we define the control input as

$$u = \ddot{x}_d + p\dot{x}_d + p_1x_d + x_d^3 - q \cos(\omega t) - K_d\dot{e} - K_p e + \nu \quad (3.2a)$$

$$\nu = 3xx_d e \quad (3.2b)$$

with $e \equiv x - x_d$ and $K_d > 0$, $K_p > 0$. Essentially, one recognizes in (3.2a) the three parts as in (2.2), namely, proportional and derivative feedback and a feedforward term. Note that ν is introduced in order to deal with the cubic term in (3.1).

Proposition 3.1: The closed-loop system (3.1), (3.2) is asymptotically stable, so

$$\lim_{t \rightarrow \infty} e(t) = 0 \quad \text{and} \quad \lim_{t \rightarrow \infty} \dot{e}(t) = 0 \quad (3.3)$$

if

$$K_d > -p, \quad K_p > -p_1. \quad (3.4)$$

Proposition 3.1 demonstrates that the Duffing equation (3.1) can be forced asymptotically towards arbitrary smooth reference trajectories $x_d(t)$. This is achieved by incorporating in the controller a suitably selected feedforward compensation. A special situation that might be of general interest, see [5], [7], occurs when the desired trajectory $x_d(t)$ represents a (stable or unstable) equilibrium motion of the uncontrolled dynamics (3.1). Hence

$$\ddot{x}_d + p\dot{x}_d + p_1x_d + x_d^3 = q \cos(\omega t). \quad (3.5)$$

As a consequence, the feedforward component in (3.2a) reduces to zero. Then we have

Corollary 3.1: If (3.5) is satisfied, then the controller

$$u = -K_d\dot{e} - K_p e + 3xx_d e \quad (3.6)$$

guarantees that the Duffing equation asymptotically converges towards the equilibrium motion (3.5).

Corollary 3.1 relates our work to that of [5]. In [5], the problem of controlling the Duffing equation (3.1) with $p > 0$ to one of its periodic motions (3.5) is considered. For this purpose, the authors propose controllers of the form

$$u = -K_p e + 3xx_d e \quad (3.7)$$

and prove that these yield asymptotic convergence towards (3.5) when $K_p > -p_1$. If we assume $p > 0$, we may select $K_d = 0$; see (3.4), and consequently, (3.6) reduces to (3.7). However, this means that the error convergence of (3.1, 5, 6) stands or falls with the presence of *damping* (the physical meaning of $p > 0$) in the open-loop system

(3.1). More importantly, this open-loop damping plays a major role in the transient performance of the error dynamics, which can hardly be influenced in this way. Hence, it is attractive to inject additional damping in the system via velocity feedback, which motivates the velocity error component $-K_d\dot{e}$ in the controllers (3.2) and (3.6). Consequently, under condition (3.4a) on K_d asymptotic stability of the controlled Duffing equation is guaranteed even when $p \leq 0$. Furthermore, transient characteristics like overshoot and rise-time can naturally be selected by tuning K_d . In addition, via the inclusion of the feedforward term in (3.2), we do not necessarily require the desired motion to be a periodic solution of the Duffing dynamics, in contrast to [5].

Proof of Proposition 3.1 and Corollary 3.1: The closed-loop (3.1), (3.2) is described by

$$\ddot{e} + (p + K_d)\dot{e} + (p_1 + K_p)e + e^3 = 0. \quad (3.8)$$

In correspondence to (2.5), consider the candidate Lyapunov function

$$V_3(e, \dot{e}) = \frac{1}{2}(\dot{e} + \lambda e)^2 + \frac{1}{2}((p_1 + K_p) + \lambda(p + K_d) - \lambda^2)e^2 + \frac{1}{4}e^4 \quad (3.9)$$

with $\lambda > 0$ satisfying

$$0 < \lambda < p + K_d. \quad (3.10)$$

Together with (3.4), this implies that $V_3(\cdot)$ is positive definite. Along (3.8), $\dot{V}_3(e, \dot{e})$ becomes

$$\dot{V}_3(e, \dot{e}) = -(p + K_d - \lambda)\dot{e}^2 - \lambda(p_1 + K_p)e^2 - \lambda e^4 \quad (3.11)$$

which is negative definite if (3.4, 10) are satisfied. Then the proof can be completed along the lines of Section II. $\square$

In order to inject additional damping in the loop, controllers (3.2) and (3.6) require $\dot{x}$, as opposed to (3.7). The need for $\dot{x}$ can be eliminated by a simple observer, without affecting the stability properties of the closed loop. In particular, the output-feedback controller

$$\text{Controller} \begin{cases} u = \ddot{x}_d + p\dot{x}_d + p_1x_d + x_d^3 - q \cos(\omega t) \\ \quad - K_d\dot{\hat{e}} - K_p\hat{e} + 3xx_d e \end{cases} \quad (3.12a)$$

$$\text{Observer} \begin{cases} \dot{\hat{e}} = u + 2K_d(e - \hat{e}) - pe \\ \dot{\hat{e}} = 2K_p(e - \hat{e}) - p_1e - e^3 \end{cases} \quad (3.12b)$$

can be employed, where

$$K_p = \lambda K_d \quad (3.13)$$

and $\lambda > 0$ scalar. Since measuring $\dot{x}$ in the controlled Duffing equation (3.1) might be difficult and noise-sensitive, the controller-observer combination (3.12, 13) seems very attractive.

Proposition 3.2: The closed-loop system (3.1, 12, 13) is asymptotically stable under the conditions (3.4, 13) and

$$0 < \lambda < \min(K_d, p + K_d). \quad (3.14)$$

Corollary 3.2: Assume that (3.4, 13, 14) are satisfied, and that $x_d(t)$ satisfies (3.5). Then

$$u = -K_d\dot{\hat{e}} - K_p\hat{e} + 3xx_d e \quad (3.15)$$

where $(\hat{e}, \dot{\hat{e}})$ as in (3.12b), provides asymptotic error convergence.

Proof: Take the Lyapunov function candidate

$$V(e, \dot{e}, \ddot{e}) = V_3(e, \dot{e}) + V_4(\ddot{e}, \dot{\ddot{e}}) \quad (3.16)$$

where $V_3(e, \dot{e})$ as in (3.9) and

$$V_4(\ddot{e}, \dot{\ddot{e}}) = \frac{1}{2}(\dot{\ddot{e}} + \lambda \ddot{e})^2 + \frac{1}{2}(K_p + \lambda K_d - \lambda^2)\ddot{e}^2. \quad (3.17)$$

The function $V(\cdot)$ is positive definite under (3.14). This condition together with (3.4) are also sufficient for $\dot{V}(\cdot)$ to be negative definite along (3.1, 12, 13). This completes the proof. $\square$

Proposition 3.2 demonstrates that even in the absence of open-loop damping, i.e., $p \leq 0$, for the Duffing equation asymptotic convergence towards any desired trajectory $x_d(t)$ can be guaranteed with an output-feedback type of controller, i.e., by only using x .

IV. DISCUSSION

An important issue is that of robustness of the proposed controllers to parametric uncertainties and bounded disturbances. From Proposition 3.1, it follows that for asymptotic tracking of an arbitrary reference $x_d(t)$ exact knowledge of the system parameters p , p_1 , q , and ω is required, cf. (3.2). In practice, however, it may be difficult to determine exactly the parameters of a (chaotic) system (see also [7] about a related point on model uncertainties). So, it could be attractive to implement (3.6) instead of (3.2), even if $x_d(t)$ does not belong to the set of periodic motions (3.5). It is interesting to know that despite this simplification the boundedness of the tracking errors can still be guaranteed under some high-gain condition on K_d . In particular, the tracking errors can be shown to be ultimately uniformly bounded (UUB) or *practically stable*, which implies that the error state tends in final time towards a closed region around zero. The proof is a straightforward extension of the UUB-results for the robot dynamics as given in [16]. By analogy, also the system-parameter-independent output-feedback controller

$$\text{Controller} \begin{cases} u = -K_d \dot{e} - K_p e \end{cases} \quad (4.1a)$$

$$\text{Observer} \begin{cases} \dot{\hat{e}} = w + 2K_d(e - \hat{e}) \\ \dot{w} = 2K_p(e - \hat{e}) \end{cases} \quad (4.1b)$$

ields practical stability of the tracking errors for *arbitrary* smooth and bounded reference motions under a high-gain assumption; see [1]. Notice, however, that high-gain feedback may have practical limitations because of, for instance, noise amplification.

V. SIMULATIONS

To support our results, we simulated with MATLABTM Duffing's equation (3.1) under output-feedback control (3.12, 13). The Duffing parameters were selected as $p = 0.4$, $p_1 = -1.1$, $q = 2.100$, and $\omega = 3$, in which case the Duffing equation displays chaotic behavior [5]. To illustrate that feedback control enables us to completely annihilate chaotic dynamics and force the system towards an arbitrary desired trajectory, we define the reference motion as

$$x_d(t) = \sin(t), \quad t \geq 0. \quad (5.1)$$

To satisfy (3.4, 14), the controller parameters were chosen to be $K_p = 12.5$ and $\lambda = 4.0$. This choice yields a proportional feedback in $K_p = \lambda K_d = 50.0$, which corresponds to K_{21} in [5]. The resulting control performance is depicted in Fig. 1, where Fig. 1(a) and 1(b) shows the time-trajectories of $\dot{e}(t)$ and $\ddot{e}(t)$, respectively. Fig. 1(c) contains the state error trajectory $(e, \dot{e})$. To clearly show the effect of feedback control, the controller is only applied for $t \geq$

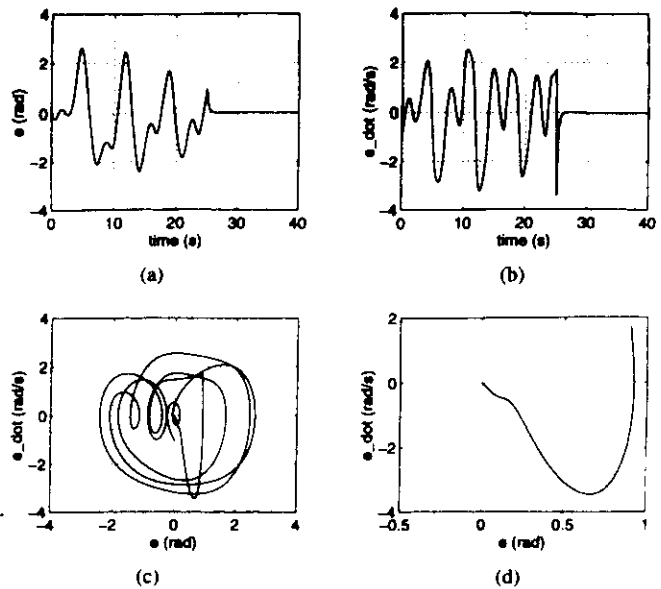


Fig. 1. Duffing's dynamics under output-feedback control (3.12, 13).

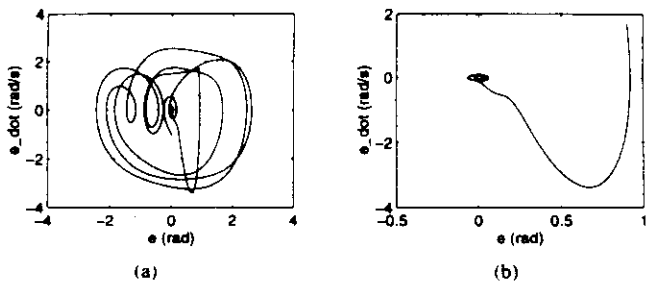


Fig. 2. Duffing's dynamics under robust output-feedback control (4.1).

25. After a short transient, the position tracking error e and velocity tracking error $\dot{e}$ converge to zero, and the control objective is attained. This can particularly be seen in Fig. 1(d), which contains the latter part of the state error trajectory.

As discussed in Section IV, when knowledge of p , p_1 , q , and ω is lacking, the system-parameter-independent controller (4.1) can be employed. The error-state performance of this controller is shown in Fig. 2, where all gains were selected as before. The results in Fig. 2 indicate that the error state does no longer converge to zero, but that it approaches a bounded region around zero as implied by UUB-stability; see Fig. 2(b). Note that the size of the ultimate error region is small relative to the amplitude of the reference motion (5.1). This implies that the state $(x, \dot{x})$ of the Duffing equation closely follows the reference trajectory $(x_d, \dot{x}_d)$. So, even in the absence of parameter knowledge, we can largely suppress chaotic dynamics by (output) feedback control!

In a third simulation, we select $p = -0.1$, so we have negative damping in the open-loop system. For this choice, the Duffing dynamics grows unstable without feedback control. To stabilize the dynamics (3.1), we need to inject (positive) damping in the system, which can be done with both the controllers (3.2) and (3.12). Fig. 3 gives the tracking error data obtained with the output-feedback controller (3.12), where the controller gain settings were as given above. Before control the phenomenon of instability can clearly be observed, whereas for $t \geq 25$, asymptotic error convergence is attained, as proved in Proposition 3.2. Note that controller (3.7) of [5] does not yield a stable closed-loop system in the absence of open-loop damping, i.e., when $p < 0$.

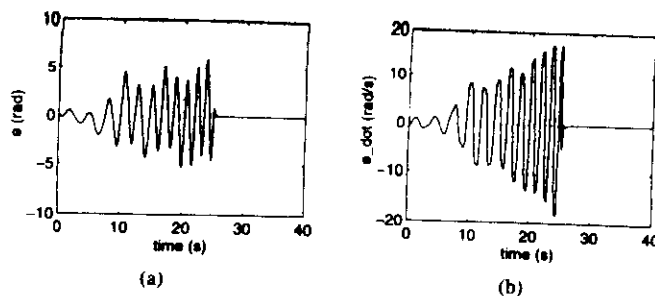


Fig. 3. Duffing's dynamics with negative damping under control (3.12, 13).

VI. CONCLUSION

In this paper, we have described how to design Lyapunov-type controllers to steer a chaotic dynamic system as the Duffing equation towards a given desired trajectory. Our methods are inspired by Lyapunov-type controllers that were recently developed for tracking control of rigid robots. The class of controllers that are discussed include observer-based controllers that may cope with parametric uncertainties and bounded disturbances in the to-be-controlled system. Some simulations illustrate the newly proposed feedback controllers.

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 Shift-Variant m -D Systems and Singularities on T^m : Implications for Robust Stability

Sandra A. Yost and Peter H. Bauer

Abstract—This brief addresses the robust asymptotic and BIBO (bounded-input bounded-output) stability of a class of linear shift-variant multidimensional systems. Using a shift-invariant comparison system, necessary and sufficient conditions for the stability of the entire family of systems are derived.

NOMENCLATURE

Λ_0^m	The first m -D hyperquadrant.
$\bar{U}^m$	The closed unit polydisk: $\{(\underline{z}) : z_i \leq 1, i = 1, \dots, m\}$.
U^m	The open unit polydisk: $\{(\underline{z}) : z_i < 1, i = 1, \dots, m\}$.
T^m	The distinguished boundary of unit polydisk: $\{(\underline{z}) : z_i = 1, i = 1, \dots, m\}$.
$\underline{n}, \underline{i}, \underline{j}$	The spatial vectors $(n_1, \dots, n_m)$, $(i_1, \dots, i_m)$, and $(j_1, \dots, j_m)$.
$y(\underline{n})$	The output of the m -D system.
$x(\underline{n})$	The input of the m -D system.
$a_i(\underline{n})$	The shift-varying coefficient of a shifted output in a m -D difference equation. (For example, in a 2-D difference equation, $a_{(3,2)}(n_1, n_2)$ is the coefficient of $y(n_1 - 3, n_2 - 2)$.)
$b_j(\underline{n})$	The shift-varying coefficient of a shifted input in a m -D difference equation.
N_j	The order of the m -D system in the n_j direction, $j = 1, \dots, m$.
$\mathcal{I}$	$\{(i_1, \dots, i_m) : 0 \leq i_k \leq N_k, k = 1, \dots, m, \text{ and } (i_1, \dots, i_m) \neq \underline{0}\}$.
$\mathcal{J}$	$\{(j_1, \dots, j_m) : 0 \leq j_k \leq N_k, k = 1, \dots, m, \text{ and } (j_1, \dots, j_m) \neq \underline{0}\}$.

I. INTRODUCTION

Results addressing the robust stability problem for 1-D discrete interval polynomials have generated interest in analogous results for the m -D case. Yet even in the work on shift-invariant m -D systems, only a few results address the $m > 2$ case [1], [2]. As for 1-D systems, conditions for the robust stability of shift-variant m -D systems are more restrictive than for the shift-invariant case. Some recent results concerning the robust stability of shift-variant m -D systems can be found in [3]–[5].

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The authors are with the Laboratory for Image and Signal Analysis, Department of Electrical Engineering, University of Notre Dame, Notre Dame, IN 46556 USA.

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Faculty of Applied Mathematics

University of Twente

University for Technical and Social Sciences

P.O. Box 217
7500 AE Enschede
The Netherlands
Phone +31-53-4893400
Fax +31-53-4892255
Email: c.h.g.hassing-
assink@math.utwente.nl

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An observer looks at synchronisation

H. NIJMEIJER AND I. MAREELS ¹

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¹Department of Engineering, Cooperative Research Centre for Robust and Adaptive Systems, Australian National University, Canberra ACT 0200, Australia

An Observer Looks At Synchronisation

H. Nijmeijer* † ‡

Department of Applied Mathematics
University of Twente
7500AE Enschede
The Netherlands
e-mail: twhenk@math.utwente.nl

and

I. Mareels†

Department of Engineering
Cooperative Research Centre for Robust and Adaptive Systems
Australian National University
Canberra ACT 0200
Australia
e-mail: iven.mareels@anu.edu.au

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Abstract

In the literature on dynamical systems analysis and the control of systems with complex behavior, the topic of synchronisation of the response of systems has received considerable attention. This concept is revisited in the light of the classical notion of observers from nonlinear control theory.

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‡Corresponding author.

1 Introduction

In recent years there has been considerable interest in the dynamics and control of systems exhibiting complex behavior. The number of papers related to this subject seems to grow at an almost exponential rate [1]. For an admittedly already 'dated' review of some of the prevailing research problems the reader may consult the seminal papers in [2].

The purpose of the present paper is to revisit the concept of *synchronisation* from a mathematical control theoretic perspective. More specifically we want to explore how the *observer* notion from nonlinear control theory links in with synchronisation. For an introduction to nonlinear control theory we refer to [4, 5, 6].

Synchronisation, as introduced by Pecora and Carroll [8, 7] has been studied from various angles, and it appears that the concept remains still somewhat fluid in that no formal definition seems to be coined. Often a master-slave formalism is taken eg [8, 7, 9, 10, 11] and [12]. Given a particular dynamical system, the master, together with an identical (sub)system, the aim is to synchronise to the master system the complete response of the slave system, by driving the latter with a (scalar) signal derived from the master system. In this context synchronisation is often considered to be a remarkable property when the master dynamics are chaotic and thus sensitive to initial condition variations. A promising application in secure communication suggested in [9] uses such a chaotic master dynamics to mask a message and a synchronised slave system to recover the message.

The above master-slave viewpoint leaves some ambiguity as to what the actual slave system should be, given the master system. A naive, but often realistic approach, would be to consider the master dynamics (transmitter) as transmitting a signal to the slave dynamics (receiver) and the receiver is requested to recover the full state trajectory of the transmitter. The problem is of course only interesting if the signal received is not equal to the full state. In this situation the receiver has in principle the freedom to build any dynamical system, it could be a copy of the master system, but it need not be, which driven by the received signal will synchronise to the transmitters' dynamics. In thus allowing the receiver the freedom of which dynamical system to implement, we enlarge the class of master/slave systems that allow synchronisation. Note that at this point we do not consider the actual physical realisation of the new receiver's dynamical system. In certain applications this may be crucial, but this aspect lies beyond the scope of the present paper.

The problem just described is closely related to the *observer* problem from control theory. For linear dynamical systems a complete solution to the problem is well known [13], [14]. For nonlinear systems a few partial results exist [15, 16, 26, 27, 28]. This observation is at the core of the paper. Another point that will transpire from the exposition is that the complexity of the dynamics involved is of little concern in the observer design or synchronisation problem.

Besides the master-slave perspective on synchronisation another viewpoint is expressed in [3] and references therein. Synchronisation is seen as the design

of a mechanism for the receiver, using the transmitted signal, so as to ensure that the controlled receiver synchronises with the transmitter. This approach to synchronisation is in essence a *control* problem, which we do not discuss in this paper.

A standard approach in solving the observer problem in control theory is to use as receiver a copy of the transmitter (of course with unknown initial state) modified with a term depending on the difference between the received signal and its prediction derived from the observer. The additional term aims at attenuating the difference between the state of the transmitter and the state of the observer system. This procedure may be shown to be successful in many instances, but certainly no global validity can be claimed. Moreover as stated the synchronisation problem requires one to establish global asymptotic stability for the error dynamics, the dynamics governing the difference between the transmitter state and observer state. Rigorous proofs often rely on Lyapunov arguments [22, 23] and [24]. Most of the existing results concerning synchronisation also rely on Lyapunov based arguments [9, 12] and [10].

That a solution to the above synchronisation problem, or observer problem, may be feasible under certain conditions may be deduced from the Takens embedding theorem [17], which is closely related to the observability property for nonlinear dynamical systems [18, 19]. In essence the observability property states that the history of the transmitted signal contains all the information to reconstruct a state variable for the master dynamics. However this falls short of implying the existence of an observer or receiver that synchronises. In the case of linear systems the link between observability (or better detectability) and the existence of an observer can be made explicitly. However in the nonlinear context the situation is not that clear, and apart from some local results, cited before, few results are available.

The observer complexity can be *reduced* by noting that with the given transmitter signal one only needs to reconstruct the complementary part of the transmitter state, that part which is not explicitly contained in the transmitted signal. This leads naturally to the concept of a reduced observer.

The presentation in the paper is kept as simple as possible. No attempt is made at stating the most general results available from (non)linear control theory. Appropriate references are provided. Also in order to avoid (sometimes nontrivial) complications, we assume throughout that all dynamical systems are defined on an open subset of an Euclidean space and have sufficient regularity such as to guarantee the existence of unique solutions. In this paper only dynamical systems described in continuous time are considered. A completely similar treatment for dynamical systems defined via maps is possible.

The observer perspective in approaching synchronisation is to the best of our knowledge new.

The paper is organised as follows. In the next section we define the problem of synchronisation in some detail and state clearly the standing assumptions. Sections 3 to 5 each revisit the observer problem for a specific subclass of systems. In section 6 we make some observations about the general problem.

2 Problem statement

We state two particular problems in the area of observer design, which are closely linked to synchronisation. First we introduce the full observer, next the reduced observer problem. This section is then concluded with a fairly general but negative result.

Let us consider dynamics governed by:

$$\dot{x}(t) = f(x(t), t) \quad x(0) = x_0 \quad t \geq 0 \quad (1)$$

We assume throughout that $x(t) \in \mathbb{R}^n$ and that the vectorfield f is such that the system (1) has a unique solution $x(t, x_0)$ passing through the initial state $x(0, x_0) = x_0$ defined on some interval $(-T_-, T_+)$, with $T_-, T_+ > 0$. We will consider systems for which $T_+ = +\infty$.

The state $x(t, x_0)$ is not directly available, only an output variable is measured, say:

$$y = h(x(t), t) \quad (2)$$

The signal y will be referred to as the *output*, typically $y \in \mathbb{R}^p$ with $p < n$.

2.1 Full observer problem

A natural question in the context of the system (1) with (2) is ‘When is it possible to reconstruct the state x from measurements of the output y ?’ This is the so called observability problem. Notice that this question is clearly aligned with the synchronisation problem discussed in the introduction ‘Given the signal $y(t)$ when can we synchronise the state of another dynamical system to $x(t)$, the state of (1)?’

A *full observer* for the system (1) with (2) is defined as:

$$\begin{aligned} \dot{\hat{x}}(t) &= \hat{f}(\hat{x}(t), y(t), t) \quad \hat{x}(0) = \hat{x}_0 \quad t \geq 0 \\ \hat{y}(t) &= h(\hat{x}(t), t) \end{aligned} \quad (3)$$

where $\hat{x} \in \mathbb{R}^n$ and $\hat{f}$ is a smooth vectorfield, parametrized by y and t , such that the error

$$e(t) = x(t) - \hat{x}(t) \quad (4)$$

asymptotically converges to zero as $t \rightarrow \infty$ for all initial conditions $\hat{x}_0$ and x_0 and moreover if $e(t_0) = 0$ then $e(t) \equiv 0$ for all $t \geq t_0$.

2.2 Reduced observer

In a sense if we reconstruct the state via equation (3), we are reconstructing more information than necessary. (In synchronisation terms, we synchronise more than is required, seeing we already have a copy of the output.) Indeed the

output y already contains some information about the state which we need not reconstruct. To discuss this point further let us specialise to the case where the output equation does not depend explicitly on time, $y(t) = h(x(t))$. The time dependence in the output equation (2) leads to some complications we prefer to avoid.

More precisely let us assume that there exists a diffeomorphism $\phi : \mathbb{R}^n \rightarrow \mathbb{R}^n$ such that:

$$\begin{aligned}\phi(x) &= \begin{pmatrix} h(x) \\ v(x) \end{pmatrix} = \begin{pmatrix} y \\ z \end{pmatrix} \\ x &= \phi^{-1}(y, z)\end{aligned}\tag{5}$$

Given the output measurements y it suffices to reconstruct z in order to know x . Now z is governed by the differential equation:

$$\dot{z}(t) = f_r(z(t), y(t), t) \quad z(0) = v(x_0) \quad t \geq 0\tag{6}$$

The vector field f_r is defined as $f_r(z, y, t) = Dv(\phi^{-1}(y, z))f(\phi^{-1}(y, z), t)$. This expression follows from (5) together with (1).

Let $\hat{z}$ be defined by:

$$\dot{\hat{z}}(t) = f_r(\hat{z}(t), y(t), t) \quad \hat{z}(0) = \hat{z}_0 \quad t \geq 0\tag{7}$$

If the diffeomorphism ϕ can be chosen such that the error $e_r(t) = z(t) - \hat{z}(t)$ converges to zero asymptotically as $t \rightarrow \infty$ and moreover if $e_r(t_0) = 0$ implies that $e_r(t) \equiv 0$ for all $t \geq t_0$ then we call the system (7) a *reduced observer* for the nonlinear system (1).

Of course once a reduced observer is found the full state is asymptotically recovered via $x(t) = \phi^{-1}(y(t), \hat{z}(t))$.

Remark 1 In general it appears quite hopeless to find conditions under which the existence/construction of either a full or reduced observer may be resolved. Some specific instances are discussed in the following sections.

Remark 2 In formulating the observer problem above, the links with synchronisation are obvious. We emphasise the importance of the state space transformations allowed for in the reduced observer problem and the freedom to choose the vector field $\hat{f}$ in the full observer problem. This additional freedom, which appears to be largely lacking in the discussions on synchronisation, enlarges considerably the class of systems for which the observer/synchronisation problem can be addressed successfully.

Remark 3 Notice also that the observer (synchronisation) problem, both for the full and reduced observer, is but well posed provided the solutions to (1) are defined on $(t_0, +\infty)$.

Remark 4 We observe that the input-to-state stability [29] for the system (6) with y viewed as its input and z as its state is at the heart of the observer property.

Remark 5 In the above only full or reduced observers are defined of dimension n respectively $n-p$. Obviously, there are further variants possible. In particular, the stable modes of the dynamics (1) need not be reconstructed at all. For linear time-invariant systems this will be described in the next section. In the nonlinear case, this observation is sometimes very relevant, and leads to further simplifications as in the linear case, but we will not elaborate this point.

2.3 Observability

Before presenting some positive results about the construction of observer systems, we present the notion of observability which can be used to decide on the non existence of observers.

Consider the time invariant system:

$$\begin{aligned}\dot{x}(t) &= f(x(t)) & x(0) &= x_0 & t &\geq 0 \\ y(t) &= h(x(t))\end{aligned}\tag{8}$$

Denote by $x(t, x_0)$ a solution to (8); ie. $x(0, x_0) = x_0$ and $\dot{x}(t, x_0) = f(x(t, x_0))$.

The system (8) is called *locally observable at x_0* if for all initial conditions x_1, x_2 in some neighborhood U of x_0 if $h(x(t, x_1)) = h(x(t, x_2))$ for all t such that $x(t, x_1), x(t, x_2) \in U$ implies that $x_1 = x_2$. The system (8) is called *locally observable* if it is locally observable at any $x_0 \in \mathbb{R}^n$.

The *observation space O* is the linear space over $\mathbb{R}$ of the functions $L_f^k h_i(x)$ * for $i = 1, \dots, p$ and $k = 0, 1, 2, \dots$.

A sufficient, and almost necessary, condition for local observability is that the codistribution

$$dO(x) = \text{span}\{DL_f^k h_i(x), i = 1, \dots, p, k = 0, 1, \dots\}\tag{9}$$

satisfies

$$\dim dO(x) = n\tag{10}$$

for all $x \in \mathbb{R}^n$. See [5] Chapter 3 for details. The observability codistribution is involutive as it is generated by exact one-forms.

In case $\dim dO(x) < n$, then the vectorfield leaves the kernel of the observability codistribution invariant (ie. $[f, \ker dO] \subset \ker dO$). [†](Notice that the

*Given a vectorfield $f(x)$ (see (8)) and a scalar valued function $h(x)$ (see (8)) define the (iterated) directional derivative of h in the direction of f as: $L_f^{k+1} h(x) = L_f(L_f^k h(x))$, with $L_f^0 h(x) = h(x)$ and $L_f h(x) = \sum_i (D_i h(x)) f_i(x)$.

[†] $[X, Y]$ denotes the Lie bracket of two vector fields, it is defined as $[X, Y](x) = (DY(x))X(x) - (DX(x))Y(x)$.

$\ker dO$ is an involutive distribution.) A necessary condition for the existence of a valid observer becomes that the dynamics of f restricted to $\ker(dO)$ are asymptotically stable. In this case we say that the system is *detectable*.

3 Linear time invariant systems

In the case of linear time invariant dynamics the problem of constructing a full or reduced observer is solved completely [14].

The relevant equations (1) and (2) or (8) now simplify to:

$$\begin{aligned}\dot{x}(t) &= Ax(t) \quad x(0) = x_0 \quad A \in \mathbb{R}^{n \times n} \\ y(t) &= Cx(t) \quad C \in \mathbb{R}^{p \times n}\end{aligned}\tag{11}$$

For any transition matrix A the solutions of (11) are defined on $(-\infty, +\infty)$, the observer problem is hence always well posed.

3.1 Full observer

For the linear system (11) an observer system (3) takes on the form

$$\begin{aligned}\dot{\hat{x}}(t) &= A\hat{x}(t) + K(\hat{y}(t) - y(t)) \quad \hat{x}(0) = \hat{x}_0 \\ \hat{y}(t) &= C\hat{x}(t)\end{aligned}\tag{12}$$

Here $K \in \mathbb{R}^{n \times p}$ is an output injection matrix.

The error $e(t) = x(t) - \hat{x}(t)$ dynamics are then governed by

$$\dot{e}(t) = (A + KC)e(t)\tag{13}$$

In order for these dynamics to represent a valid observer it is necessary that a gain matrix K can be found such that the matrix $A + KC$ has eigenvalues with negative real part. This control problem has been completely solved. Necessary and sufficient for the existence of a real gain matrix K such that $A + KC$ is asymptotically stable is that the matrix pair (A, C) be *detectable* which is equivalent to

$$\text{if } \text{rank} \begin{pmatrix} \lambda I - A \\ C \end{pmatrix} < n \text{ then } \text{Real}(\lambda) < 0\tag{14}$$

In case that the matrix

$$\begin{pmatrix} \lambda I - A \\ C \end{pmatrix}\tag{15}$$

has rank n for all complex λ we say that the system is observable. It can be verified that this condition for observability can equally be expressed by the more geometrically appealing condition:

$$\ker O := \ker \begin{pmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{pmatrix} = \{0\} \quad (16)$$

This condition (16) is nothing but the expression of the condition (10) as applied to a linear system (11).

Remark 6 The necessity of the condition (16) for observability is obvious from considering a point x_0 such that $Cx_0 = 0$ and $Cx(t, x_0) = 0$ for all t .

We refer the reader to [14] for further details.

3.2 Reduced observer

Clearly when measuring $y = Cx$, it appears that we only need to reconstruct $z = Hx$ where H is chosen such that $(C^T H^T)^T$ has full column rank. Let us assume that C has full row rank p . This amounts to stating that there are no redundant measurements in the output y . We can then find $H \in \mathbb{R}^{(n-p) \times n}$:

$$\begin{pmatrix} y \\ z \end{pmatrix} = \begin{pmatrix} C \\ H \end{pmatrix} x \quad \text{and} \quad x = \begin{pmatrix} S & T \end{pmatrix} \begin{pmatrix} y \\ z \end{pmatrix}$$

The partial state z satisfies the differential equation:

$$\dot{z}(t) = HATz(t) + HASy(t) \quad (17)$$

It can be shown that under the assumption that the matrix pair (A, C) be detectable that H can be chosen such that HAT is asymptotically stable. A *reduced order* observer is then given by:

$$\begin{aligned} \dot{\hat{z}}(t) &= HAT\hat{z}(t) + HASy(t) \\ \hat{x}(t) &= T\hat{z}(t) + Sy(t) \end{aligned} \quad (18)$$

Notice that if the matrix pair (A, C) is not observable, but detectable, it may be that a reduced observer of lower dimension than $n - p$ exists; eg. in the case that A is asymptotically stable, one could use as reduced observer $\hat{x}(t) \equiv 0$.

An example may serve to illustrate the ideas.

Example 1 Consider the two dimensional system:

$$\begin{aligned} \dot{x}(t) &= \begin{pmatrix} a & 1 \\ b & 0 \end{pmatrix} x(t) \\ y(t) &= (1 \ 0) x(t) \end{aligned} \quad (19)$$

The system (19) need not be stable, and indeed will not be stable if a or b are positive.

The above state space realization is often referred to as being in *observer canonical form*. The terminology stems from the fact that this state space structure makes observer design obvious.

It is easily verified that the matrix pair

$$\left(\begin{pmatrix} a & 1 \\ b & 0 \end{pmatrix}, (1 \ 0) \right) \quad (20)$$

is observable for all a, b . Hence both full and reduced observer problems are solvable, regardless of the stability of the system (19).

A full observer is eg. given by:

$$\dot{\hat{x}}(t) = \begin{pmatrix} a & 1 \\ b & 0 \end{pmatrix} \hat{x}(t) + \begin{pmatrix} -1 - a \\ -1 - b \end{pmatrix} (\hat{y}(t) - y(t)) \quad (21)$$

$$\hat{y}(t) = (1 \ 0) \hat{x}(t)$$

This leads to asymptotically stable error dynamics given by:

$$\dot{e}(t) = \begin{pmatrix} -1 & 1 \\ -1 & 0 \end{pmatrix} e(t) \quad (22)$$

A reduced observer can be obtained as follows. Let z be defined via:

$$\begin{pmatrix} y \\ z \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ k & 1 \end{pmatrix} x \quad (23)$$

Here k is a scalar which we need to choose judiciously. By construction for any scalar k the above (23) defines a diffeomorphism between the coordinates x and (y, z) .

Clearly, with the above definition (23), and using the system description (19) z obeys the differential equation:

$$\dot{z}(t) = kz(t) + (b + ka - k^2)y(t) \quad (24)$$

A reduced observer is thus obtained for any $k < 0$. In particular, the reduced observer takes the form:

$$\begin{aligned} \dot{\hat{z}}(t) &= k\hat{z}(t) + (b + ka - k^2)y(t) \\ \hat{x}(t) &= \begin{pmatrix} 1 & 0 \\ -k & 1 \end{pmatrix} \begin{pmatrix} y(t) \\ \hat{z}(t) \end{pmatrix} \end{aligned} \quad (25)$$

Let us finalise this section on observer design/ synchronisation for linear systems with the observation that the above ideas may be applied in the nonlinear context when we are interested in the neighborhood of an hyperbolic and stable fixed point to yield local results in observer/synchronisation theory. However this is of limited value. Stability is essential as otherwise we are not guaranteed that the solutions will remain in a small neighborhood of the fixed point. Hyperbolicity is needed in order that the linearization captures the local behavior. This leads to a trivial observer problem, as the system is trivially detectable, setting the state estimate equal to the fixed point is an allowable observer strategy! This situation is unsatisfactory, and hence the need to have a more global point of view. This will be considered in the following sections. In the case control can be applied the linear perspective may be sufficient, however control aspects will not be discussed in this paper.

4 Linear time varying systems

Apart from non trivial technical details, the situation for observer design in the case of linear time varying systems resembles strongly the theory for linear time invariant systems. We refer the interested reader to [20] and to [21] for an application of a time varying linear system observer in the context of controlling chaos. In particular we want to alert the reader to the fact that the definition of detectability in the context of time varying linear systems is normally defined via the existence of a full observer. We limit ourselves to discussing an example.

Example 2 Let us illustrate the concepts for the class of second order systems described by:

$$\begin{aligned}\dot{x}(t) &= \begin{pmatrix} -a(t) & 1 \\ -b(t) + \dot{a}(t) & 0 \end{pmatrix} x(t) \\ y(t) &= (1 \ 0) x(t)\end{aligned}\tag{26}$$

Equivalently described by the differential equation:

$$\frac{d^2}{dt^2}y(t) + a(t)\frac{d}{dt}y(t) + b(t)y(t) = 0\tag{27}$$

Throughout this example assume that $a : \mathbb{R} \rightarrow \mathbb{R}$ is a continuously differentiable function of time and that $b : \mathbb{R} \rightarrow \mathbb{R}$ is continuous. This ensures the existence and uniqueness of solutions on $\mathbb{R}$ and hence the well posedness of the observer problem.

Again the state space realization (26) for the differential equation (27) is referred to as being in *observer canonical form*.

Following Example 1 we find that a full order observer may be given by:

$$\begin{aligned}\dot{\hat{x}}(t) &= \begin{pmatrix} -a(t) & 1 \\ -b(t) + \dot{a}(t) & 0 \end{pmatrix} \hat{x}(t) + \\ &\quad \begin{pmatrix} -1 + a(t) \\ -1 + b(t) - \dot{a}(t) \end{pmatrix} (\hat{y}(t) - y(t)) \\ \hat{y}(t) &= (1 \ 0) \hat{x}(t)\end{aligned}\tag{28}$$

This leads to asymptotically stable error dynamics given by (22).

A reduced observer can be obtained as for the linear time invariant example. Let z be defined via (23). With the definition (23), and using the system description (26) z obeys the differential equation:

$$\dot{z}(t) = kz(t) + (-b(t) + \dot{a}(t) - ka(t) - k^2)y(t)\tag{29}$$

A reduced observer, with associated time invariant error dynamics, may thus be obtained for any constant $k < 0$:

$$\begin{aligned}\dot{\hat{z}}(t) &= k\hat{z}(t) + (-b(t) + \dot{a}(t) - ka(t) - k^2)y(t) \\ \hat{x}(t) &= \begin{pmatrix} 1 & 0 \\ -k & 1 \end{pmatrix} \begin{pmatrix} y(t) \\ \hat{z}(t) \end{pmatrix}\end{aligned}\tag{30}$$

Observe that the class of systems (27) considered in the Example 2 contains the so called Hill equations (set $a(t) \equiv 0$). The problem of deciding stability for the Hill equation is notoriously difficult, see eg. [22], yet the observer design as outlined above is straightforward and indeed independent of the stability properties of the Hill equation.

5 Systems with linearizable error dynamics

From the previous examples a straightforward, yet nontrivial extension towards nonlinear systems transpires. The idea is to consider systems that may give rise to linear error dynamics maybe via an appropriate change of coordinates and/or rescaling of the output variables. We first present an example, a direct extension of the observer forms considered so far. Then we introduce the general idea, which we illustrate with a number of typical examples and then present some fairly complete results which allow one to decide if a given system may give rise to linear error dynamics after suitable coordinate transformation. Again we treat the full observer case first, then the reduced observer.

5.1 Full observer

Example 3 Consider the n th order nonlinear, time dependent differential equation in the scalar output variable y :

$$\frac{d^n}{dt^n}y(t) + \frac{d^{n-1}}{dt^{n-1}}f_1(y(t),t) + \dots + \frac{d}{dt}f_{n-1}(y(t),t) + f_n(y(t),t) = 0 \quad (31)$$

Assume that the functions f_i have sufficient regularity to guarantee the existence of unique solutions on $\mathbb{R}$.

A particular state space realization for this differential equation (31) is obviously:

$$\begin{aligned} \dot{x}(t) &= Ex(t) + f(Cx(t), t) \\ y(t) &= Cx(t) \\ E &= \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & & & \ddots & \\ 0 & 0 & & & 1 \\ 0 & 0 & \dots & & 0 \end{pmatrix} \in \mathbb{R}^{n \times n} \\ C &= (1 \ 0 \ \dots \ 0) \in \mathbb{R}^{1 \times n} \end{aligned} \quad (32)$$

$$f(y, t) = (-f_1(y, t) \ -f_2(y, t) \ \dots \ -f_n(y, t))^T$$

Because the matrix pair (E, C) is observable, and due to the specific structure of the state equation (32), it is obvious how to construct a full observer yielding asymptotically stable linear error dynamics:

$$\begin{aligned} \dot{\hat{x}}(t) &= E\hat{x}(t) + f(y(t), t) + K(\hat{y}(t) - y(t)) \\ \hat{y}(t) &= C\hat{x}(t) \end{aligned} \quad (33)$$

It suffices to choose K such that $E + KC$ is asymptotically stable.

Starting from systems of the form

$$\begin{aligned} \dot{x}(t) &= f(x(t), t) \quad x(0) = x_0 \quad t \geq 0 \\ y(t) &= h(x(t)) \end{aligned} \quad (34)$$

the general idea is to find a coordinate transformation $\xi = \phi(x)$ and output transformation $\eta = \psi(y)$ such that in the new coordinates we have a system description of the form:

$$\begin{aligned}\dot{\xi}(t) &= A\xi(t) + g(\eta(t), t) \quad \xi(0) = \xi_0 \quad t \geq 0 \\ \eta(t) &= C\xi(t)\end{aligned}\tag{35}$$

Provided the matrix pair (A, C) is detectable we can construct an observer which gives rise to linear error dynamics in the usual way.

Notice that although we may obtain error dynamics which are seemingly defined on $\mathbb{R}$, this does not imply that the observer problem is well defined. The solutions to (35) have to be defined on $(0, +\infty)$ before it makes sense to discuss the error dynamics.

Some examples may serve to illustrate the basic idea.

Example 4 Consider the Duffing equation with periodic driving term.

$$\begin{aligned}x_1(t) &= x_2(t) \\ \dot{x}_2(t) &= -p_1 x_2(t) - p_2 x_1(t) - p_3 x_1^3(t) + q \cos(\omega t) \\ y(t) &= x_1(t)\end{aligned}\tag{36}$$

The solutions of (36) are well defined on $(0, +\infty)$ for any initial condition, provided $p_1, p_2, p_3 > 0$. (This can easily be verified by considering the derivative of the comparison function $V = x_2^2 + \frac{p_3}{2} x_1^4 + p_2 x_1^2$ along the solutions of (36).) It is well known that the solutions may exhibit chaos for particular parameter combinations.

This equation (36) is in the form of the equation (35) with $\xi = (x_1 \ x_2)^T$, $\eta = y = x_1 = \xi_1$ and

$$A = \begin{pmatrix} 0 & 1 \\ -p_2 & -p_1 \end{pmatrix} \quad C = (1 \ 0) \quad g(y, t) = \begin{pmatrix} 0 \\ -p_3 y^3 + q \cos(\omega t) \end{pmatrix}$$

Clearly the matrix pair (A, C) is observable, hence an observer may be constructed as follows:

$$\begin{aligned}\dot{\hat{x}}(t) &= A\hat{x}(t) + K(\hat{y}(t) - y(t)) + g(y(t), t) \\ \hat{y}(t) &= C\hat{x}(t)\end{aligned}\tag{37}$$

The ensuing error $e(t) = x(t) - \hat{x}(t)$ dynamics are hence linear and governed by:

$$\dot{e}(t) = \begin{pmatrix} k_1 & 1 \\ k_2 - p_2 & -p_1 \end{pmatrix} e(t)\tag{38}$$

By appropriate selection of the gain $K = (k_1 \ k_2)^T$ the error dynamics (38) can be made asymptotically stable.

This example illustrates that any simple second order mechanical system with position measurements is in fact observable.

Example 5 Chua's circuit [11] can be described by the following state equations:

$$\begin{aligned} \frac{d}{dt} \begin{pmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{pmatrix} &= \begin{pmatrix} -\frac{G}{C} & \frac{G}{C} & 0 \\ \frac{G}{Q} & -\frac{G}{Q} & \frac{1}{Q} \\ 0 & -\frac{1}{L} & 0 \end{pmatrix} \begin{pmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{pmatrix} \\ &\quad + \begin{pmatrix} -\frac{1}{C}f(x_1(t)) \\ 0 \\ 0 \end{pmatrix} \\ y(t) &= (1 \ 0 \ 0)x(t) \end{aligned} \tag{39}$$

The nonlinearity f in (39) is given by $f(y) = Q_1 y + 1/2(Q_2 - Q_1)(|y + E| - |y - E|)$ which is non smooth, but this does not affect the discussion. Because $|f(y)| \leq \gamma|y|$ for some $\gamma > 0$ it is easily shown that the solutions for all initial conditions are well defined on $\mathbb{R}$.

Clearly the linear part of the system equations (39) is observable. An observer may thus be constructed as before:

$$\begin{aligned} \frac{d}{dt} \begin{pmatrix} \hat{x}_1(t) \\ \hat{x}_2(t) \\ \hat{x}_3(t) \end{pmatrix} &= \begin{pmatrix} -\frac{G}{C} & \frac{G}{C} & 0 \\ \frac{G}{Q} & -\frac{G}{Q} & \frac{1}{Q} \\ 0 & -\frac{1}{L} & 0 \end{pmatrix} \begin{pmatrix} \hat{x}_1(t) \\ \hat{x}_2(t) \\ \hat{x}_3(t) \end{pmatrix} \\ &\quad + \begin{pmatrix} -\frac{1}{C}f(y(t)) \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} k_1 \\ k_2 \\ k_3 \end{pmatrix} (\hat{y}(t) - y(t)) \\ \hat{y}(t) &= (1 \ 0 \ 0)\hat{x}(t) \end{aligned} \tag{40}$$

In this case the choice $k_1 = k_2 = k_3 = 0$ will yield asymptotically stable linear error dynamics [11]. Using different gains allows us to select a faster error response.

Example 6 Let us consider the Rössler system:

$$\frac{d}{dt} \begin{pmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{pmatrix} = \begin{pmatrix} -x_2(t) - x_3(t) \\ x_1(t) + ax_2(t) \\ c + x_3(t)(x_1(t) - b) \end{pmatrix} \quad (41)$$

$$y = (0 \ 0 \ 1)x$$

In the above (41) the coefficients $a, b, c > 0$. Assume also that $x_3(0) > 0$, then $x_3(t) = y(t) > 0$ for all $t \geq 0$. Keeping this restriction in mind we may use the comparison function $V = x_1^2 + x_2^2 + x_3 > 0$. Taking the derivative along the solutions of (41) $\dot{V} = ax_2^2 + c - bx_3 \leq aV + c$ which implies that the solutions (with $x_3(0) > 0$) are well defined on $(0, +\infty)$. The observer problem is hence well posed.

We introduce now the following coordinates:

$$(\xi_1 \ \xi_2 \ \xi_3) = (x_1 \ x_2 \ \ln(x_3)) \quad \eta = \ln y \quad (42)$$

In the new coordinates the system equations are then given by:

$$\frac{d}{dt} \begin{pmatrix} \xi_1(t) \\ \xi_2(t) \\ \xi_3(t) \end{pmatrix} = \begin{pmatrix} 0 & -1 & 0 \\ 1 & a & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} \xi_1(t) \\ \xi_2(t) \\ \xi_3(t) \end{pmatrix} + \begin{pmatrix} -e^{\xi_3(t)} \\ 0 \\ -b + ce^{-\xi_3(t)} \end{pmatrix} \quad (43)$$

$$\eta(t) = (0 \ 0 \ 1)\xi(t)$$

The linear part of (43) is again observable and hence an observer with linear, asymptotically stable dynamics may be constructed as before.

We now present for time invariant systems of the form (8) with scalar output y conditions that allow one to transform via appropriate coordinate changes the system (8) into a system of the form (35). The more general case of multiple outputs has also been treated in [15, 14, 28].

To state the result we need some more notation. Given a vectorfield $f(x)$ (see (8)) and a scalar valued function $h(x)$ (see (8)) denote by $[X, Y]$ the Lie bracket of the vector fields X and Y . The Lie bracket is defined as $[X, Y](x) = (DY(x))X(x) - (DX(x))Y(x)$. Moreover the operation $ad_X^k Y$ is defined via $ad_X^0 Y = Y$, $ad_X Y = [X, Y]$ and $ad_X^{k+1} Y = [X, ad_X^k Y]$.

We consider the situation where the transformation from (8) to (35) only involves a state space coordinate change $\xi = \phi(x)$. This result is discussed in detail in [4]. Let n be the state dimension of (34). The conditions under which the transformation may be achieved are given by:

1. *Local observability at x* (see also (10))

$$\dim \left(\text{span} \{ Dh(x), DL_f h(x), \dots, DL_f^{n-1} h(x) \} \right) = n \quad \forall x \quad (44)$$

2. The mapping defined as

$$\xi := \phi(x) := \left(h(x) \ L_f h(x) \dots L_f^{n-1} h(x) \right)^T \quad (45)$$

is a global diffeomorphism on $\mathbb{R}^n$.

3. The unique vectorfield r on $\mathbb{R}^n$ that satisfies

$$\begin{aligned} L_r h(x) &= L_r L_f h(x) = \dots = L_r L_f^{n-2} h(x) = 0 \\ L_r L_f^{n-1} h(x) &= 1 \end{aligned} \quad (46)$$

is such that

$$[r, ad_f^k r] = 0 \quad \forall k = 1, 3, 5, \dots, 2n-1 \quad (47)$$

A necessary and sufficient result which involves both a state space transformation $\xi = \phi(x)$ as well as an output transformation $\eta = \psi(y)$ is discussed in [15]. See also [26, 27, 28].

Remark 7 Observe that condition (45) trivially implies (44).

5.2 Reduced observer

The above result(s) are clearly relevant for the full observer problem. If a full observer with linear error dynamics may be found, a reduced observer with linear error dynamics can also be constructed. The reverse may not be the case. As far as we are aware, no results are available that provide conditions, under which via a state transformation and output transformation, a reduced observer with linear error dynamics may be found.

The Examples 4, 5 and 6 readily allow for reduced observers to be found that yield linear error dynamics. After the coordinate transformation the methods valid for linear systems apply immediately.

Let us give an example for which the conditions (44), (45), (46) and (47) are not all satisfied, yet the system is such that we are still able to solve the reduced observer problem with error dynamics which are linear.

Example 7 Consider the van der Pol equation with driving term.

$$\begin{aligned} \dot{x}_1(t) &= x_2(t) \\ \dot{x}_2(t) &= \mu(1 - x_1^2(t))x_2(t) - x_1(t) + q \cos(\omega t) \\ y(t) &= x_1(t) \end{aligned} \quad (48)$$

The parameter $\mu > 0$. It is well known that the solutions of (48) are well defined on $\mathbb{R}^+$ and that they may exhibit chaotic behavior for certain parameter selections (μ, q and ω).

It is easily verified that (44) is satisfied, that the constant vectorfield $r(x) = (0 \ 1)^T$ satisfies (46) but fails to satisfy (47). Hence it is not possible to produce via a state coordinate transformation alone linear error dynamics for an observer.

However a reduced order observer with linear error dynamics can be found. Consider the new variable $z = x_2 + k_1 y + k_2 y^3$. It satisfies the differential equation:

$$\dot{z}(t) = (\mu + k_1)x_2(t) + (3k_2 - \mu)x_1^2(t)x_2(t) - x_1(t) + q \cos(\omega t) \quad (49)$$

Selecting $k_1 + \mu = -1$ and $3k_2 - \mu = 0$ we get

$$\dot{z}(t) = -z(t) + (-2 - \mu)y(t) + \frac{\mu}{3}y^3(t) + q \cos(\omega t) \quad (50)$$

This suggest the reduced observer:

$$\begin{aligned} \dot{\hat{z}}(t) &= -\hat{z}(t) + (-2 - \mu)y(t) + \frac{\mu}{3}y^3(t) + q \cos(\omega t) \\ \hat{x}_2(t) &= \hat{z}(t) + (1 + \mu)y(t) - \frac{\mu}{3}y^3(t) \\ \hat{x}_1(t) &= y(t) \end{aligned} \quad (51)$$

The corresponding error dynamics are $\dot{e}_r(t) = -e_r(t)$, where $e_r = z - \hat{z}$.

5.3 Linear timevarying error dynamics

In the above examples we were lead to time invariant error dynamics. This may not always be achievable, but it may be possible to attain linear time varying error dynamics. For a more complete discussion we refer to [30].

Example 8 Let us reconsider the van der Pol equation with driving term (48), but this time we want to obtain a full observer.

Design the observer to be of the form:

$$\begin{aligned} \dot{\hat{x}}_1(t) &= \hat{x}_2(t) + \alpha_1(y(t))e_1(t) \\ \dot{\hat{x}}_2(t) &= \mu(1 - \hat{x}_1^2(t))\hat{x}_2(t) - \hat{x}_1(t) + \alpha_2(y(t))e_1(t) + q \cos(\omega t) \\ \hat{y}(t) &= \hat{x}_1(t) \\ e(t) &= x(t) - \hat{x}(t) \end{aligned} \quad (52)$$

This gives rise to the following error dynamics

$$\begin{aligned} \dot{e}_1(t) &= e_2(t) + \alpha_1(y(t))e_1(t) \\ \dot{e}_2(t) &= -e_1(t) + \mu(1 - y^2(t))e_2(t) + \alpha_2(y(t))e_1(t) \end{aligned} \quad (53)$$

The error dynamics, considering $y(t)$ as given, are linear time varying. In order to select the functions α_1 and α_2 we proceed with a Lyapunov analysis. As candidate Lyapunov function we propose $V(e_1, e_2) = \frac{1}{2}(e_1 - \frac{1}{2\mu}e_2)^2 + \frac{1}{2}e_1^2$. For its derivative along the solutions of (53) we find:

$$\begin{aligned}\dot{V}(t) &= -\frac{1}{4\mu}(1 + y^2(t))e_2^2(t) \\ &\quad + (2\alpha_1(y(t)) - \frac{1}{2\mu}\alpha_2(y(t)))e_1^2(t) \\ &\quad + \left(\left(\frac{3}{2} + \frac{1}{2}y^2(t)\right) - \frac{1}{2\mu}\alpha_1(y(t)) + \frac{1}{4\mu^2}\alpha_2(y(t))\right)e_1(t)e_2(t)\end{aligned}$$

The following selection of α_1 and α_2 will make $\dot{V} < 0$, which implies asymptotic stability for the error dynamics.

$$\begin{aligned}\alpha_1(y) &= -\mu \left(3 + y^2 + \frac{1}{2\mu^2}\right) \\ \alpha_2(y) &= -1 - 4\mu^2(3 + y^2)\end{aligned}$$

The method followed in solving the above example appears misleadingly straightforward, in general it is extremely hard to establish stability properties for linear time varying systems. Finding suitable Lyapunov functions is difficult, nevertheless Lyapunov theory is a very useful tool.

Another example in the same vein is given by the Lorentz equations, as discussed eg. [9].

6 General systems

As stated before providing conditions under which the general observer problem as stated in Section 2 may be solved, is very difficult. However, under the reasonable restriction that the dynamics are constrained to a compact domain, which is in particular the case when the system dynamics evolve on some strange attractor, a positive result using a high gain observer can be derived.

6.1 High gain observer

If we limit ourselves to dynamics defined on some compact set, then the following result is available [16].

Consider a time invariant system of the form (8). Let the state $x(t)$ be an n -dimensional signal and the output $y(t) = h(x(t))$ be a scalar signal. Assume that $\Omega \subset \mathbb{R}^n$ is a compact and a positively invariant set for the dynamics (8). Assume that

$$\xi := \phi(x) := \left(h(x) \ L_f h(x) \dots \ L_f^{n-1} h(x)\right)^T$$

is a global diffeomorphism on Ω . Let the system equation (8) in the new coordinates ξ be given by:

$$\begin{aligned}\dot{\xi}(t) &= F(\xi(t)) \\ y(t) &= \xi_1(t)\end{aligned}\tag{54}$$

Consider also the system

$$\begin{aligned}\dot{\hat{\xi}}(t) &= F(\hat{\xi}(t)) + K_\theta(\hat{y}(t) - y(t)) \\ \hat{y}(t) &= \hat{\xi}_1(t)\end{aligned}\tag{55}$$

where the constant gain $K_\theta \in \mathbb{R}^n$ is defined via:

$$K_\theta = -S_\theta^{-1}C^T$$

where $S_\theta = S_\theta^T > 0$ solves

$$0 = \theta S_\theta + A^T S_\theta + S_\theta A - C^T C$$

where

$$C = (1 \ 0 \ \dots 0) \in \mathbb{R}^{1 \times n}$$

and

$$A = \begin{pmatrix} 0_{n-1 \times 1} & I_{n-1} \\ 0 & 0_{1 \times n-1} \end{pmatrix}$$

The system (55) is an observer for the system (54) for all sufficiently large $\theta > 0$, in that for all ξ_0 and all $\hat{\xi}_0$ in $\phi(\Omega)$, the error $e(t) = \xi(t) - \hat{\xi}(t)$ decreases exponentially.

The equation $0 = \theta S_\theta + A^T S_\theta + S_\theta A - C^T C$ to be solved for S_θ is known as an algebraic Riccati equation. The observability of (A, C) guarantees the existence of a positive definite solution with the property that $A + K_\theta C$ is an asymptotically stable matrix.

The error dynamics are in general nonlinear, but due to the *large* gain K_θ the error dynamics on $\phi(\Omega)$ are essentially dominated by the stability of $A + K_\theta C$.

Remark 8 Although a positive result, its usefulness in applications is limited. High gain observers are very sensitive to small measurement errors.

6.2 Nonlinear error dynamics

We conclude this section with one more example inspired by the Lorentz equations and which gives rise to a nonlinear observer with nonlinear error dynamics. We again exploit Lyapunov arguments [23], [24] to establish global asymptotic stability for the nonlinear error dynamics.

Example 9 Consider the nonlinear dynamical system:

$$\begin{aligned}
\dot{x}_1(t) &= \sigma(x_2(t) - x_1^3(t)) \\
\dot{x}_2(t) &= \rho x_1(t) - x_2^3(t) - x_1(t)x_2(t)x_3(t) \\
\dot{x}_3(t) &= x_1(t)x_2^2(t) - \beta x_3(t) \\
y(t) &= x_1(t)
\end{aligned} \tag{56}$$

This system is not observable, but detectable for $\beta > 0$ and $\sigma, \rho \neq 0$. Indeed computing the observability codistribution we obtain:

$$\begin{aligned}
dO(x) &= \text{span}\{(1, 0, 0); (*, \sigma, 0); (*, *, \sigma x_1 x_2); \\
&\quad (*, *, -\sigma^2 x_2^2 - \rho \sigma x_1^2 + \sigma x_1 x_2 g(x)); \dots\}
\end{aligned}$$

Clearly if either $x_1 \neq 0$ or $x_2 \neq 0$ we have that $\dim dO(x) = 3$. It can be seen that if $x_1 = 0$ and $x_2 = 0$ then $\dim dO(x) = 2$ and the $\ker dO(x) = \{(0, 0, x_3)\}$. Moreover this ker is invariant. Indeed $x_1(0) = 0$ and $x_2(0) = 0$ imply that $x_1(t, 0) = 0$, $x_2(t, 0) = 0$ for all t and also that $\dot{x}_3(t) = -\beta x_3(t)$. The system is thus detectable, but not observable.

Consider the situation where the parameters $\rho, \sigma, \beta > 0$. In this case the origin is a saddle point for the system (56). Using the comparison function $V = x_1^2 + x_2^2 + x_3^2$ it can be shown that the solutions are well defined on $\mathbb{R}^+$, and that the solutions are ultimately bounded. Indeed $\dot{V} = -\sigma x_1^4 - x_2^4 + (\rho + \sigma)x_1 x_2 - \beta x_3^2$. It follows that $\dot{V} \leq 0$ outside some compact domain Ω , let M^2 denote the maximum value that V attains on Ω . Ω is positively invariant.

We look for a reduced observer. Introduce the variable z :

$$z_1(t) = x_2(t) - \frac{k}{\sigma}y(t) \quad z_2(t) = x_3(t)$$

z is governed by the differential equation:

$$\begin{aligned}
\dot{z}_1(t) &= -k(z_1(t) + ky(t)) - (z_1(t) + ky(t))^3 \\
&\quad - y(t)(z_1(t) + ky(t))z_2(t) - ky^3(t) + \rho y(t) \\
\dot{z}_2(t) &= y(t)(z_1(t) + ky(t))^2 - \beta z_2(t)
\end{aligned} \tag{57}$$

Choosing $k > M^2 + \frac{M^4}{2\beta}$ leads to an exponentially stable observer, given by:

$$\begin{aligned}
\dot{\hat{z}}_1(t) &= -k(\hat{z}_1(t) + ky(t)) - (\hat{z}_1(t) + ky(t))^3 \\
&\quad - y(t)(\hat{z}_1(t) + ky(t))\hat{z}_2(t) - ky^3(t) + \rho y(t) \\
\dot{\hat{z}}_2(t) &= y(t)(\hat{z}_1(t) + ky(t))^2 - \beta \hat{z}_2(t)
\end{aligned} \tag{58}$$

The error dynamics with $e_1 = z_1 - \hat{z}_1$ and $e_2 = z_2 - \hat{z}_2$ are governed by the nonlinear and timevarying differential equation:

$$\begin{aligned}\dot{e}_1(t) &= -ke_1(t) - e_1(t)[3x_2^2(t) - 3x_2(t)e_1(t) + e_1^2(t)] \\ &\quad -y(t)x_2(t)e_2(t) - y(t)x_3(t)e_1(t) + y(t)e_1(t)e_2(t) \\ \dot{e}_2(t) &= -\beta e_2(t) - y(t)e_1^2(t) + 2y(t)x_2(t)e_1(t)\end{aligned}\quad (59)$$

Using the comparison function $W = e_1^2 + e_2^2$ we can now establish that the error dynamics (59) have a uniformly asymptotically stable trivial solution provided $k > M^2 + \frac{M^4}{2\beta}$.

7 Conclusion

We have drawn attention to the fact that the problem of synchronisation as introduced in the control of chaos literature can be viewed as a special case of the observer design problem, which is well known in the nonlinear control theory literature. The formalism offered via the observer theory allows us to provide a reasonable comprehensive framework for synchronisation issues. Some open research problems have been identified.

This paper was preoccupied with the existence question. In a companion paper we will focus attention on the control problem formulation of synchronisation and in particular consider issues relating to sensitivity with respect to measurement errors and/or errors in the representation of the system dynamics. These are of great importance when considering observer/synchronisation problems in a practical setting.

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