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WORKSHOP ON NONLINEAR CONTROL AND CONTROL OF CHAOS

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"Control of Nonlinear Dynamics in Engineering Systems"

presented by:

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Control of Nonlinear Dynamics in Engineering Systems

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Which control problems do we consider?

Those for which:

1. Linear models & tools do not suffice
2. Nonlinear dynamical phenomena are present
3. The control objectives can be met by harnessing the nonlinear phenomena

Linearized models & linear tools are not always appropriate. E.g., in designing

large &/or rapid maneuvers

of

aircraft, spacecraft, robotic mechanisms, etc.

→ Nonlinear control is needed

But what about nonlinear dynamics?

Nonlinear dynamics (bifurcation & chaos, etc.) is a well-developed area. (~ 100 years old.)

Control of nonlinear dynamics is a developing area (~ 10 years old)

Families of systems & stressed operation

controls

$$\dot{x} = f_\mu(x, u)$$

state vector

parameters
(angle-of-attack setting, voltage reference, etc.)

The operating envelope is associated w/ the range of values of μ & x in which the system is operable.

tressed operation:

1. A reduced level of linearized system controllability
2. A reduced linear stability margin.

teps in control design for a family of systems:

1. Feedforward control
2. Feedback control

Selection of $u_1(x,\mu)$ in

Selection of $u_0 = u_0(\mu)$

Let the feedforward part be subsumed in the system equations (i.e., let $u = u_1(x, \mu)$). Then we have to design a feedback for

$$\dot{x} = f^{\mu}(x, u)$$

The feedback should ensure good properties of the nominal operating condition $x^{\circ}(\mu)$ (equilibrium point or limit cycle)

simplicity, for μ that $x^{\circ}(\mu)$ exists,

$$x^{\circ}(\mu) \equiv 0$$

(the origin)

earization (also a
nily of systems):

$$\dot{x} = A^{\mu} x + b^{\mu} u$$

e

$$A^{\mu} := \frac{\partial f^{\mu}}{\partial x}(0,0)$$

$$b^{\mu} := \frac{\partial f^{\mu}}{\partial u}(0,0)$$

(Linear) Stabilizability by feedback is implied by controllability:

$(b^\mu, A^\mu b^\mu, A^{\mu^2} b^\mu, \ldots, A^{\mu^{n-1}} b^\mu)$ is full rank

For such values of μ, can find a family of linear controllers

$$u = -k^\mu x$$

giving

$$\dot{x} = f(x, -k^\mu x)$$

Most existing results linking nonlinear dynamics to control are of the

NONLINEAR DYNAMICS OF CONTROL SYSTEMS

type.

To proceed, we need some knowledge of bifurcation phenomena and chaos. Only two bifurcations will be discussed in any detail: Andronov-Hopf and pitchfork.

Nonlinear dynamics entails:

- Ordinary behavior (stability) of nonlinear systems
- Bifurcation
- Multiple steady states
- Periodic orbits
- Hysteretic phenomena
- Exotic behavior
- Chaos
- Quasiperiodicity

ELEMENTARY BIFURCATIONS

Systems which function at or near the limits of their operating envelopes tend to experience *bifurcations* under moderate disturbances. These bifurcations often result in:

- A jump to a new low-performance operating point
- Oscillatory behavior
- Chaotic behavior

1. Dangerous
2. Safe

Discontinuous
Hard

Continuous
Soft

Consider a one-parameter family of nonlinear systems

$$\dot{x} = f_\mu(x) \qquad (1)$$

where $x \in \mathbb{R}^n$, and $\mu \in \mathbb{R}$ is the distinguished *bifurcation parameter*. Let $x_0(\mu)$ be an equilibrium point of (1) for a parameter range of interest. Bifurcations from $x_0(\mu)$ are termed *primary bifurcations*.

Two basic bifurcations from $x_0(\mu)$:

Hopf bifurcation: Emergence of a periodic solution from $x_0(\mu)$ at a *critical value* $\mu = \mu_c$.

Static bifurcation: A sudden change in the number of equilibrium points near $x_0(\mu)$ for μ near a *critical value* μ_c.

What is a bifurcation point for a system

$$\dot{x} = f_\mu(x) \; ?$$

Basically it is a member of this family

$$\dot{x} = f_{\mu_c}(x)$$

which is <u>not</u> <u>structurally stable</u> w.r.t. variations in μ.

μ = bifurcation parameter

μ_c = <u>critical value</u>

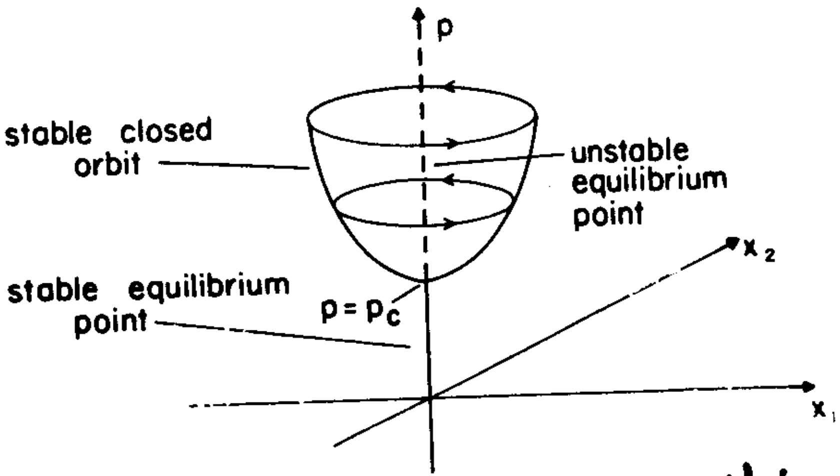
Bifurcations from equilibria are also called local bifurcations. These include

- pitchfork
- Hopf
- saddle-node
- transcritical
- other bifurcations which are nongeneric for one-parameter families ("degenerate bifurcations")

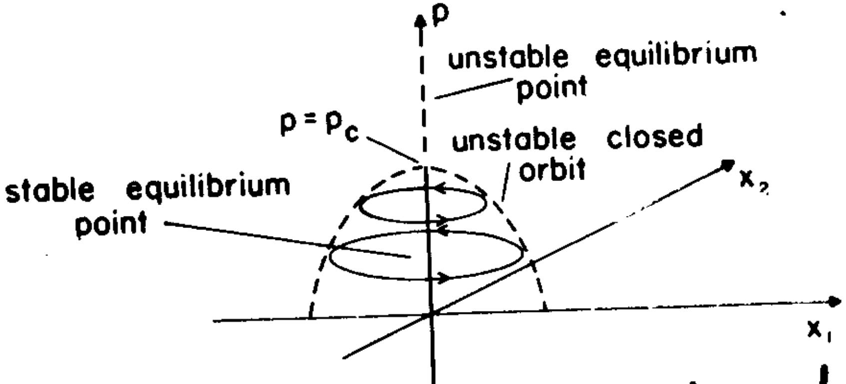
- The *occurrence* of these (primary) bifurcations is *determined from the linearization* of Eq. (1), without any assumption on the nonlinear terms in $f_\mu(x)$. However, conclusions regarding stability and "direction" of the bifurcated (i.e., new) solutions depend heavily on the nonlinear terms.

Hopf Bifurcation Theorem. A family of periodic solutions emerges from the equilibrium $x_0(\mu)$ at any critical parameter value $\mu = \mu_c$ such that, among the eigenvalues of the linearization of (1):

- A simple complex conjugate pair $\lambda(\mu)$, $\bar{\lambda}(\mu)$ satisfies $Re(\lambda(\mu_c)) = 0$, $Im(\lambda(\mu_c)) \neq 0$, $Re'(\lambda'(\mu_c)) \neq 0$, and
- All other eigenvalues have negative real parts for $\mu = \mu_c$.



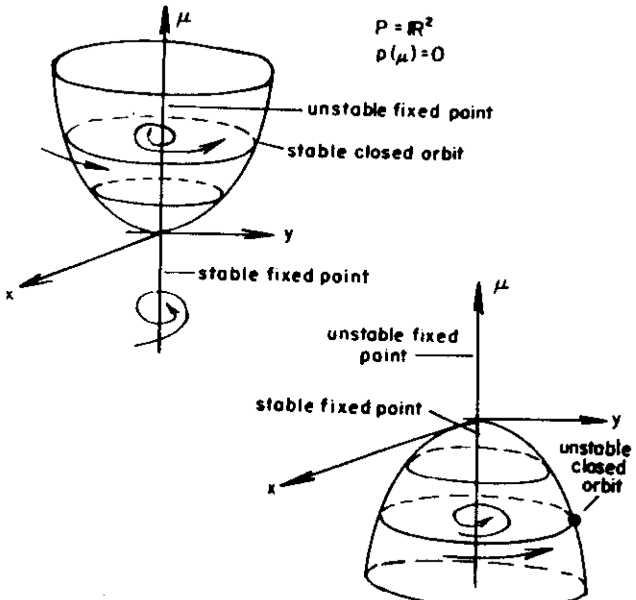
(a) Supercritical case



(b) Subcritical case

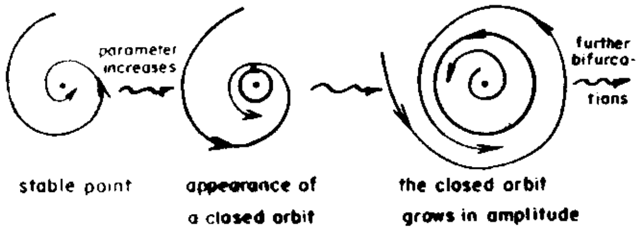
Andronov- Hopf Bifurcation
("Hopf Bifurcation")

The Hopf Bifurcation

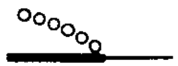


(a) Supercritical Bifurcation (Stable Closed Orbits)

(b) Subcritical Bifurcation (Unstable Closed Orbits)

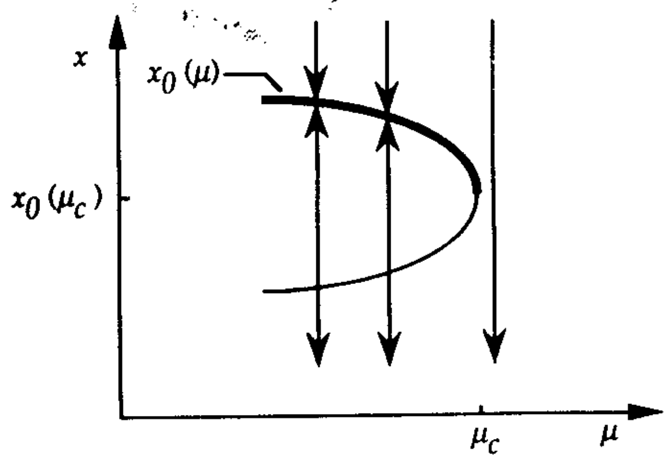


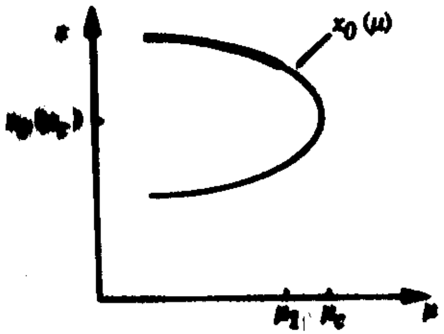
(a) Supercritical case



(b) Subcritical case

ANDRONOV-HOPF BIFURCATION





LOSS OF STABILITY BEFORE SADDLE NODE BIFURCATION

Stability Criteria

Write Eq. (1) as

$$\begin{aligned}\dot{x} &= L(\mu)x + Q_\mu(x,x) + C_\mu(x,x,x) + \cdots \\ &= L_0x + \mu L_1x + \mu^2 L_2 x + \cdots \\ &\quad + Q_0(x,x) + \mu Q_1(x,x) + \cdots \\ &\quad + C_0(x,x,x) + \cdots \end{aligned} \tag{2}$$

Let l (resp. r) = the left (resp. right) eigenvector of L_0 corresponding to the critical eigenvalue.

Stability coefficients:

- Hopf case: Require $\beta_2 < 0$ where

$$\beta(\epsilon) = \beta_2\epsilon^2 + \beta_4\epsilon^4 + \cdots$$

= Floquet exponent

- Static case: Require $\beta_1 = 0$ and $\beta_2 < 0$ where

$$\beta(\epsilon) = \beta_1\epsilon + \beta_2\epsilon^2 + \cdots$$

= Eigenvalue of bifurcated equilibrium

Many procedures for calculating these coefficients exist, most of which utilize the Center Manifold Theorem.

The projection method applied to the Taylor expansion (2) yields the following simplified technique.

Hopf case: Floquet coefficient β_2

Let a, b solve

$$-L_0 a = \frac{1}{2} Q(r, \bar{r}) \quad (3)$$

$$(2i\omega_c I - L_0) b = \frac{1}{2} Q(r, r) \quad (4)$$

for a and b.

Then,

$$\beta_2 = 2\mathrm{Re}\,\{\, 2lQ(r, a) + lQ(\bar{r}, b) + \frac{3}{4} lC(r, r, \bar{r}) \,\} \quad .(5)$$

Static case: Eigenvalue coefficients β_1, β_2

$$\beta_1 = lQ(r, r) \quad (6)$$

and, if $\beta_1 = 0$,

$$\beta_2 = 2l(2Q(r, x_2) + C(r, r, r)), \quad (7)$$

where

$$x_2 = -[\begin{pmatrix} L_0^T & l^T \end{pmatrix} \begin{pmatrix} L_0 \\ l \end{pmatrix}]^{-1} \begin{pmatrix} L_0^T & l^T \end{pmatrix} \begin{pmatrix} Q_0(r, r) \\ 0 \end{pmatrix} \quad (8)$$

Representative Results - Case S

Hypothesis (S) leads to a *stationary bifurcation.*

Consider the system

$$\dot{x} = F_\mu(x) \quad (2)$$

having an equilibrium point $x_0(\mu)$ at which hypothesis (S) holds.

Denote the new equilibrium as $(x(\epsilon), \mu(\epsilon))$. Write

$$\mu(\epsilon) = \mu_1 \epsilon + \mu_2 \epsilon^2 + \cdots \quad (3)$$

$$x(\epsilon) = x_1 \epsilon + x_2 \epsilon^2 + \cdots \quad (4)$$

The second equilibrium point also has an eigenvalue β which vanishes at $\mu = 0$, with a series expansion

$$\beta(\epsilon) = \beta_1 \epsilon + \beta_2 \epsilon^2 + \cdots \quad (5).$$

Moreover, the exchange of stability formula

$$\beta_1 = -\mu_1 \lambda_l'(0) \quad (6)$$

holds. If $\mu_1 = 0$ and $\mu_2 \neq 0$, the appropriate exchange of stability formula is

$$\beta_2 = -2\mu_2 \lambda_l'(0). \quad (7)$$

Suppose $\lambda_l'(0) > 0$. Then these facts imply that supercritical solution branches are stable while subcritical branches are unstable.

Rewrite (2) as

$$\dot{x} = L(\mu)x + Q_\mu(x, x) + C_\mu(x, x, x) + \cdots$$

$$= L_0 x + \mu L_1 x + \mu^2 L_2 x + \cdots$$

$$+ Q_0(x,x) + \mu Q_1(x,x) + \cdots$$

$$+ C_0(x,x,x) + \cdots \tag{8}$$

If x is any real (unknown) solution of $F_\mu(x) = 0$, define the parameter ϵ by

$$\epsilon := lx, \tag{9}$$

and attempt a series expansion of the form

$$\begin{pmatrix} x(\epsilon) \\ \mu(\epsilon) \end{pmatrix} = \sum_{k=1}^{\infty} \epsilon^k \begin{pmatrix} x_k \\ \mu_k \end{pmatrix}. \tag{10}$$

Substituting the expansion (10) in the equation obtained by equating the right side of (8) to 0, and equating coefficients of like powers of ϵ yields:

$$x_1 = r$$

$$x_2 = (R^T R)^{-1} R^T \begin{pmatrix} -\mu_1 L_1 r - Q_0(r,r) \\ 0 \end{pmatrix}.$$

here R (full rank) is

$$R := \begin{pmatrix} L_0 \\ l \end{pmatrix}$$

With x_2 now available, one applies the Fredholm Alternative to solve for the coefficient μ_2.

$$\mu_2 = -\frac{1}{\lambda_1'(0)} \{ \mu_1 l L_1 x_2 + \mu_1^2 l L_2 r + 2 l Q_0(r, x_2)$$

$$+ \mu_1 l Q_1(r,r) + l C_0(r,r,r) \}.$$

Also,

$$\beta_1 = l Q_0(r,r).$$

If $\mu_1 = 0$ (implying also $\beta_1 = 0$), then the exchange of stability formula (7) is valid. In that case, one finds that β_2 is

PROBLEM:

Find $u(x)$ to render the bifurcation supercritical or nearly supercritical, in the controlled system

$$\dot{x} = f_\mu(x, u(x)) \tag{3}$$

- **This is closely related to the feedback stabilization problem for the *critical system***

$$\dot{x} = f_0(x, u(x))$$

- **The stated problem is local. An interesting global problem, studied by Mehra (1977) (see also Casti, "Connectivity, Complexity, and Catastrophe in Large Scale Systems,". Wiley, 1979) is the following:**

- **Global Bifurcation Control Problem (Mehra 1977):**

Find $u(x)$ to remove all bifurcations from equilibrium points of system (3).

Theorem (Mehra, 1977). System (3) will exhibit no stationary bifurcations if $u(x)$ is such that

(i) $\det \frac{\partial f_\mu}{\partial x} \neq 0$ for all $\mu \times R$, $x \in R^n$, and

(ii) $\|f_\mu(x, u(x))\| \to \infty$ as $\|x\| \to \infty$.

No analogous result available for Hopf bifurcations.

Feedback Stabilization

Set-up:

Consider the parametrized nonlinear *control system*

$$\dot{x} = f_\mu(x,u) \tag{10}$$

where $f_\mu(0,0) \equiv 0$. Expand the RHS of (10) to get

$$\begin{aligned}\dot{x} = L_0 x \; + \mu L_1 x \; + u\bar{L}_1 x \; + \gamma u \; + Q_0(x,x)\\ + \mu^2 L_2 x \; + \mu Q_1(x,x) \; + u\bar{Q}_1(x,x)\\ + C_0(x,x,x) \; + \cdots\end{aligned} \tag{11}$$

Apply the purely nonlinear feedback control

$$u(x) = x^T Q_u x \; + C_u(x,x,x) \tag{12}$$

to stabilize the bifurcation assumed to occur in (10) for $u = 0$.

For cases in which stabilizing feedbacks of the form (12) exist, parametrizations of all such feedback controls have been constructed.

Stabilization Theorems

- Hopf bifurcation case

Theorem 1. *If $l\gamma \neq 0$, that is, the critical imaginary eigenvalues are controllable for the linearized system, then Eq. (10) is stabilizable with a feedback of the form (12). This can be accomplished using only cubic terms in x in the feedback.*

Theorem 2. *If $l\gamma = 0$, that is, the critical imaginary eigenvalues are not controllable for the linearized system, then generically Eq. (10) is stabilizable with a feedback of the form (12).*

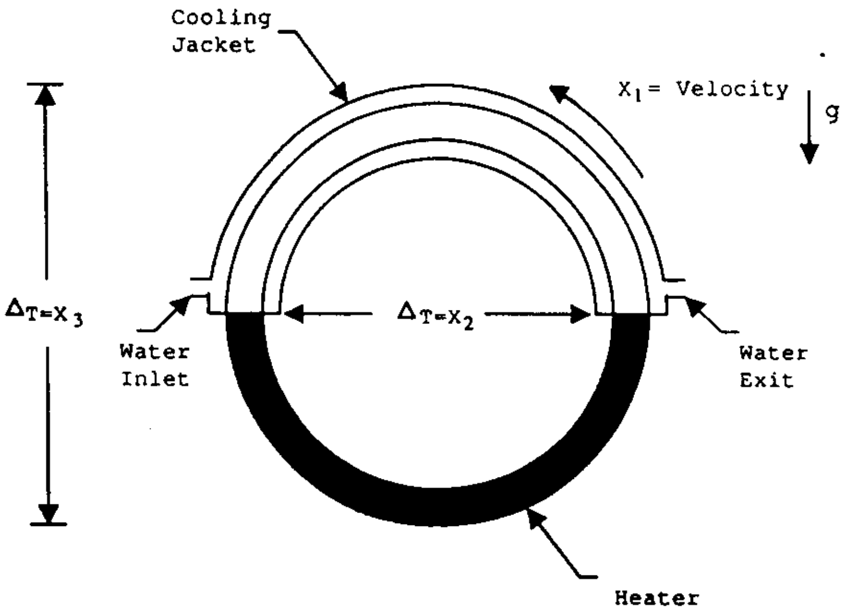
- Static bifurcation case

Theorem 3 *If $l\gamma \neq 0$, that is, the critical zero eigenvalue is controllable for the linearized version of Eq. (10), then (10) is stabilizable with a feedback of the form (12). Two-stage designs exist in which Q_u is chosen to ensure that $\beta_1 = 0$ and C_u is then chosen so that $\beta_2 < 0$.*

Theorem 4. *If $l\gamma = 0$, that is, the critical zero eigenvalue is uncontrollable for the linearized version of (10), then* generically *no feedback of the form (12) can stabilize the static bifurcation in (10).* (but O.K. if $\beta_1 = 0$)

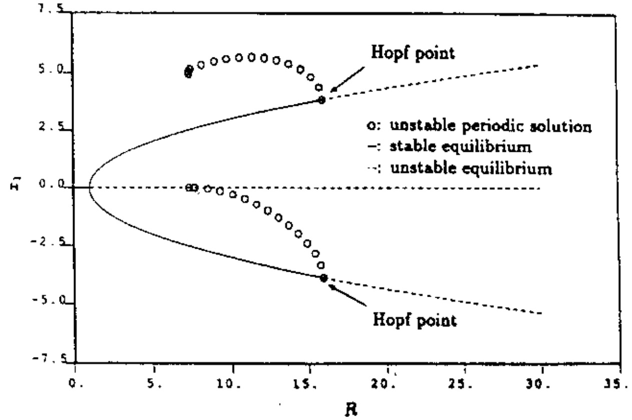
Robust Bifurcation Control of a Chaotic System

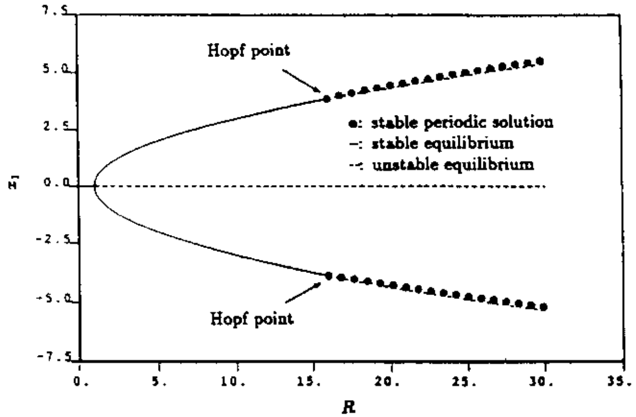
Problem: "Robustly" suppress chaotic and transient chaotic flow in a thermal convection loop (see experiment by J. Singer, Y-Z. Wang and H.H. Bau, *Physical Review Letters*, 1991).



A Thermal Convection Loop

Control Strategy: A washout filter-aided purely nonlinear feedback stabilizes bifurcated periodic solutions over a large parameter range while exactly preserving all system equilibria.



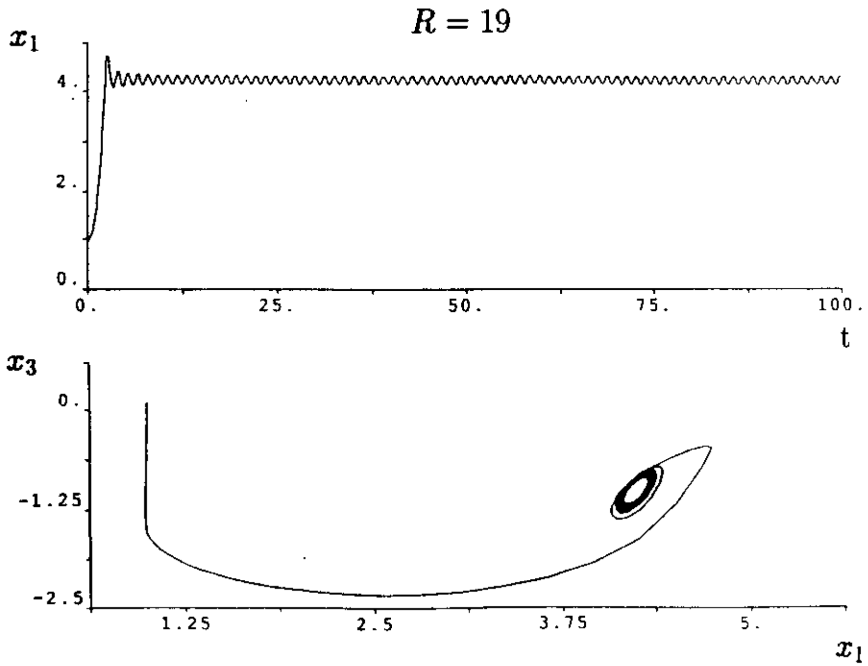


- Simulations show suppression of both chaos and transient chaos.

Controlled System Dynamics:

$$
\begin{aligned}
\dot{x}_1 &= -px_1 + px_2 \\
\dot{x}_2 &= -x_1x_3 - x_2 \\
\dot{x}_3 &= x_1x_2 - x_3 - R + u \\
\longrightarrow\ \dot{x}_4 &= x_3 - cx_4 \\
\longrightarrow\ y &= x_3 - cx_4 \\
\longrightarrow\ u &= -ky^3
\end{aligned}
$$

- x_4 is the washout filter state



Voltage Collapse Control:

$$
\begin{aligned}
\dot{\delta}_m &= \omega \\
M\dot{\omega} &= -d_m\omega + P_m + E_mVY_m\sin(\delta - \delta_m - \theta_m) \\
&\quad + E_m^2Y_m\sin\theta_m \\
K_{qw}\dot{\delta} &= -K_{qv2}V^2 - K_{qv}V + Q(\delta_m, \delta, V) - Q_0 - Q_1 \\
TK_{qw}K_{pv}\dot{V} &= K_{pw}K_{qv2}V^2 + (K_{pw}K_{qv} - K_{qw}K_{pv})V \\
&\quad + K_{qw}(P(\delta_m, \delta, V) - P_0 - P_1) \\
&\quad - K_{pw}(Q(\delta_m, \delta, V) - Q_0 - Q_1) + u(\omega)
\end{aligned}
$$

Nonlinear control:

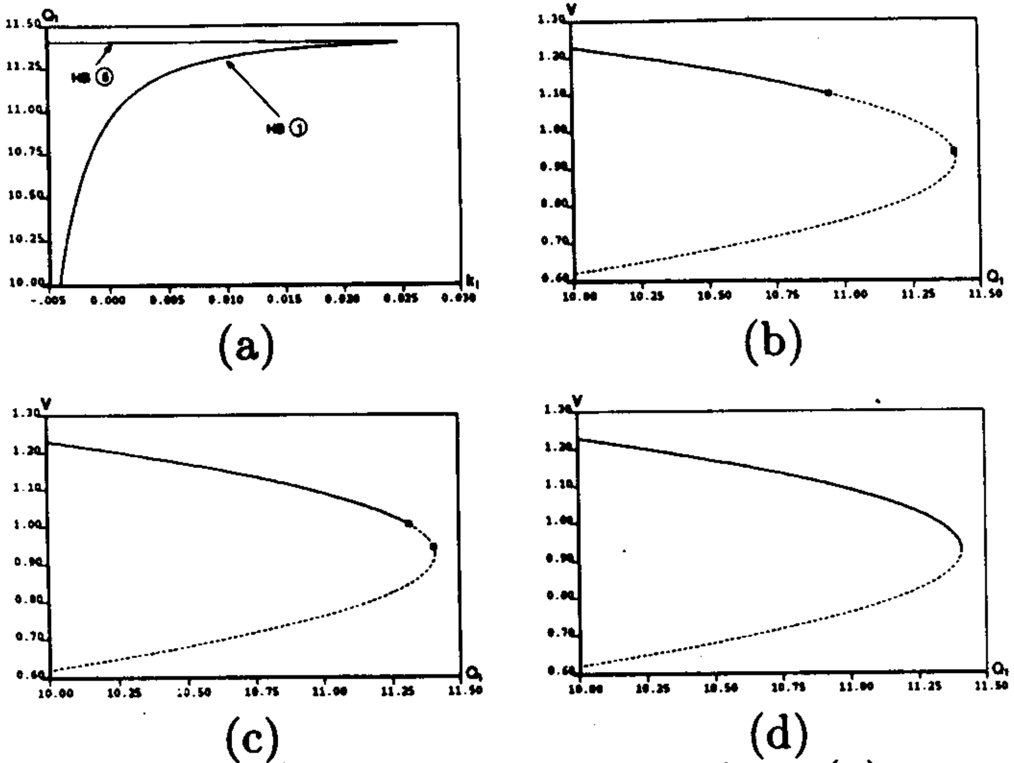
$$u = k_n\omega^3$$

Linear control:

$$u = k_l\omega$$

Linear control

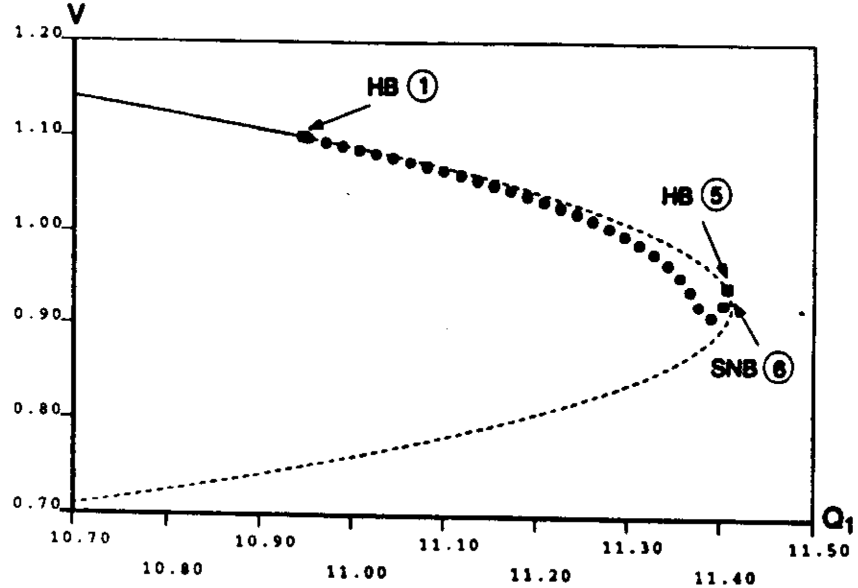
$$u = k_l \omega$$



With linear bifurcation control: (a) two-parameter curves of the Hopf bifurcation points; bifurcation diagrams with (b) $k_l = 0$ (no control), (c) $k_l = 0.01$, (d) $k_l = 0.025$.

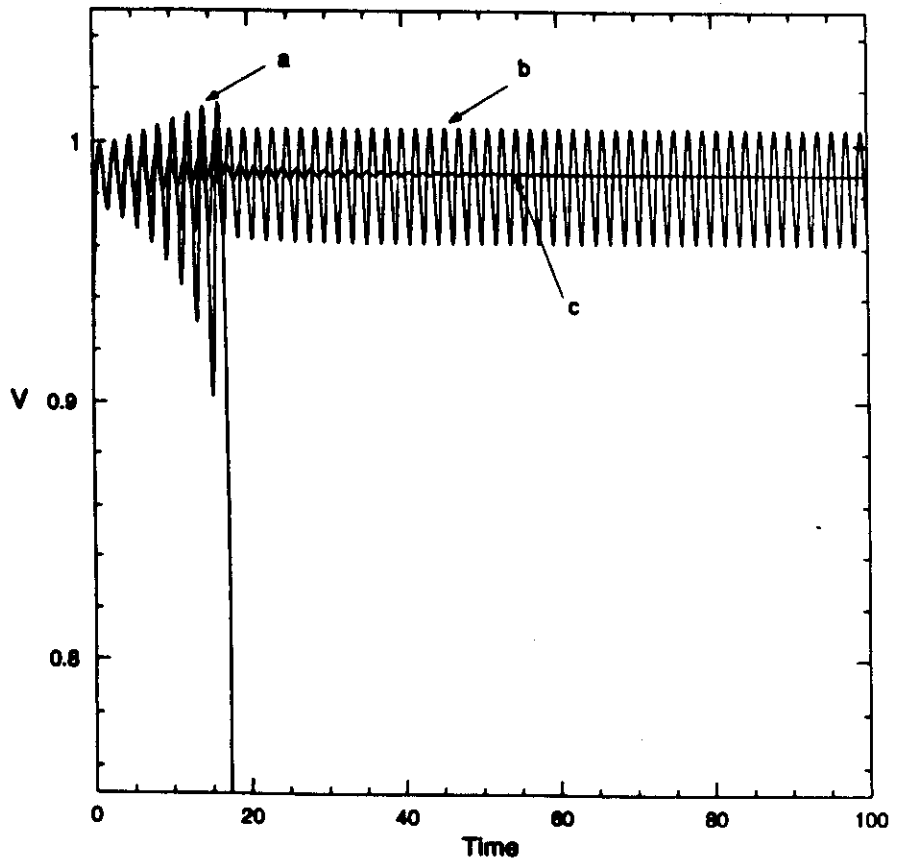
Nonlinear control

$$u = k_n \omega^3$$



Bifurcation diagram of closed loop system with cubic control ($k_n = 0.5$)

Sample simulation results

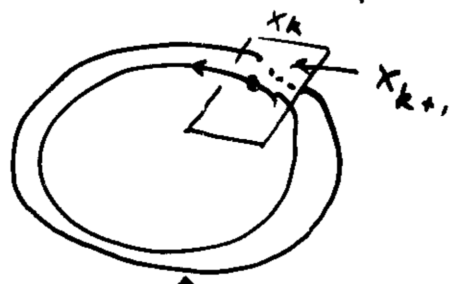


Dynamic responses of the system at $Q_1 = 11.35$: **a.** without control; **b.** with nonlinear control $k_n = 0.5$; **c.** with linear control $k_l = 0.025$.

PDB

$x_{k+1} = F_\mu(x_k)$

1^{st} return map



$\dot{x} = f_\mu(x)$

THm p. 59.

PD)

$F_\mu(0) \equiv 0$

$\lambda_1(\mu) \in \sigma(DF_\mu(0))$ s.t.

$\lambda_1(\mu_c) = -1$

$\lambda_1'(\mu_c) \neq 0$

$\lambda_2, \ldots, \lambda_n(\mu_c)$

have mag. < 1.

Pef:

$x_{k+2} = G_\mu(x_k)$

THm 6.1.3. Under (PD), a PDB occurs, stability if $\beta_2 < 0$ / instability if $\beta_2 > 0$.

Period-doubling bifurcation

Consider

$$x_{k+1} = f_\mu(x_k)$$

Let

$$\lambda_1(0) = -1, \text{and} \quad |\lambda_i(0)| < 1 \quad \text{for} \quad i = 2, \ldots, n.$$

$$A(0)r = -r, \quad \ell A(0) = -\ell$$

Expanding

$$x_{k+1} = A(\mu)x_k + Q(x_k, x_k) + C(x_k, x_k, x_k) + \ldots$$

For the sped-up sytem

$$0 = \tilde{A}(\mu)x + \tilde{Q}(x,x) + \tilde{C}(x,x,x) + \ldots$$

where

$$\begin{aligned}
\tilde{A}(\mu) &:= A^2(\mu) - I \\
\tilde{Q}(x,x) &:= A(0)Q(x,x) + Q(A(0)x, A(0)x) \\
\tilde{C}(x,x,x) &:= A(0)C(x,x,x) + 2Q(A(0)x,\ Q(x,x)) \\
&\quad + C(A(0)x, A(0)x, A(0)x)
\end{aligned}$$

Apply formulas (Abed-Fu 1986)for stationary bifurcation: $\beta(\sigma) = \beta_1 \sigma + \beta_2 \sigma^2 + \ldots$

$$\beta_1 = \ell \tilde{Q}(r,r) = 0$$

$$\beta_2 = 2\ell[\tilde{C}(r,r,r) - 2\tilde{Q}(r, \tilde{A}^{-}\tilde{Q}(r,r))]$$

(P) The vector field f is sufficiently smooth and has a fixed point at $x = 0$ for all μ. The linearization of (1) possess an eigenvalue $\lambda_1(\mu)$ with $\lambda_1(0) = -1$ and $\lambda_1'(0) \neq 0$. All remaining eigenvalues have magnitude less than unity.

Theorem (Period Doubling Bifurcation Theorem) If (P) holds, then a period doubled orbit bifurcates from the origin at $\mu = 0$. If $\beta_2 \neq 0$, then the period doubled orbit is supercritical and stable if $\beta_2 < 0$ but is subcritical and unstable if $\beta_2 > 0$.

PDB Control

$$x_{k+1} = f_\mu(x_k, u)$$

$$u = x_k^T Q_u x_k + C_u(x_k, x_k, x_k)$$

Henon example

$$\begin{cases} x_{k+1} = \rho - x_k^2 + p y_k \boxed{+u_k} \\ y_{k+1} = x_k \end{cases}$$

ike z_i $\longrightarrow$ $w_{k+1} = x_k + (1-d) w_k$

ike y_i $\longrightarrow$ $z_k = x_k - d w_k$

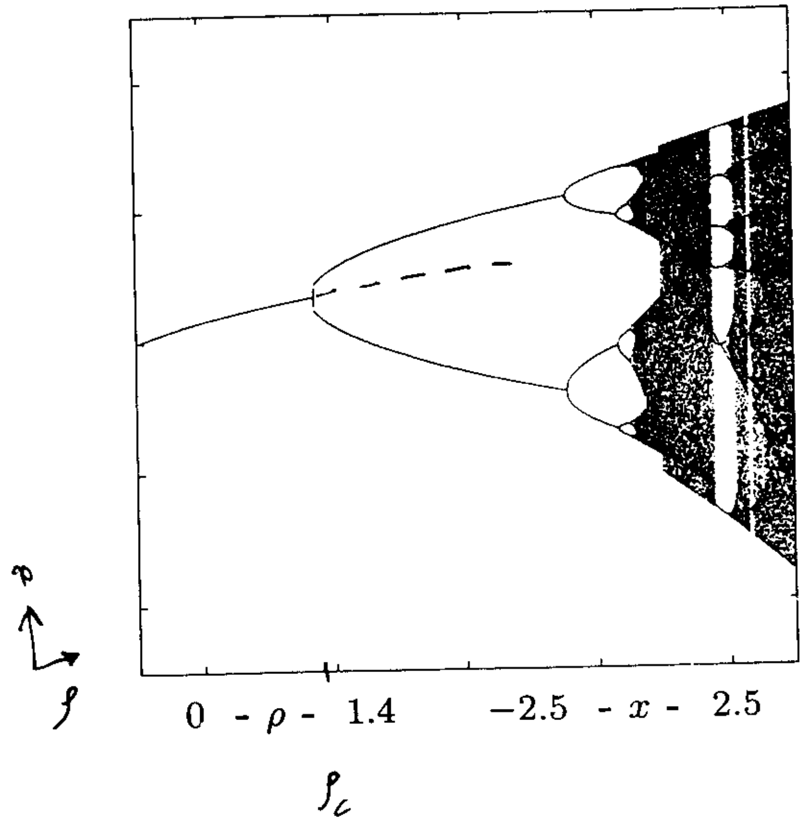
$$u_k = K_1 z_k^3$$

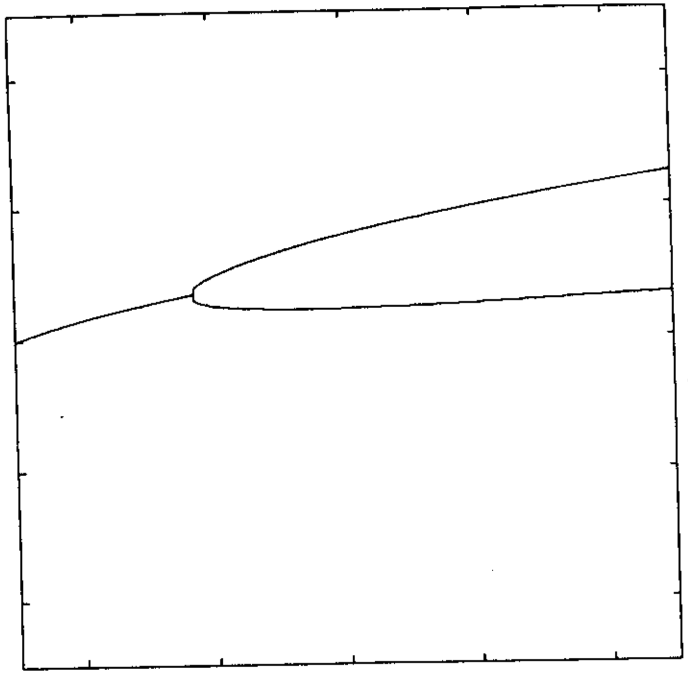
At equilibrium:

$$w_{k+1} = w_k$$

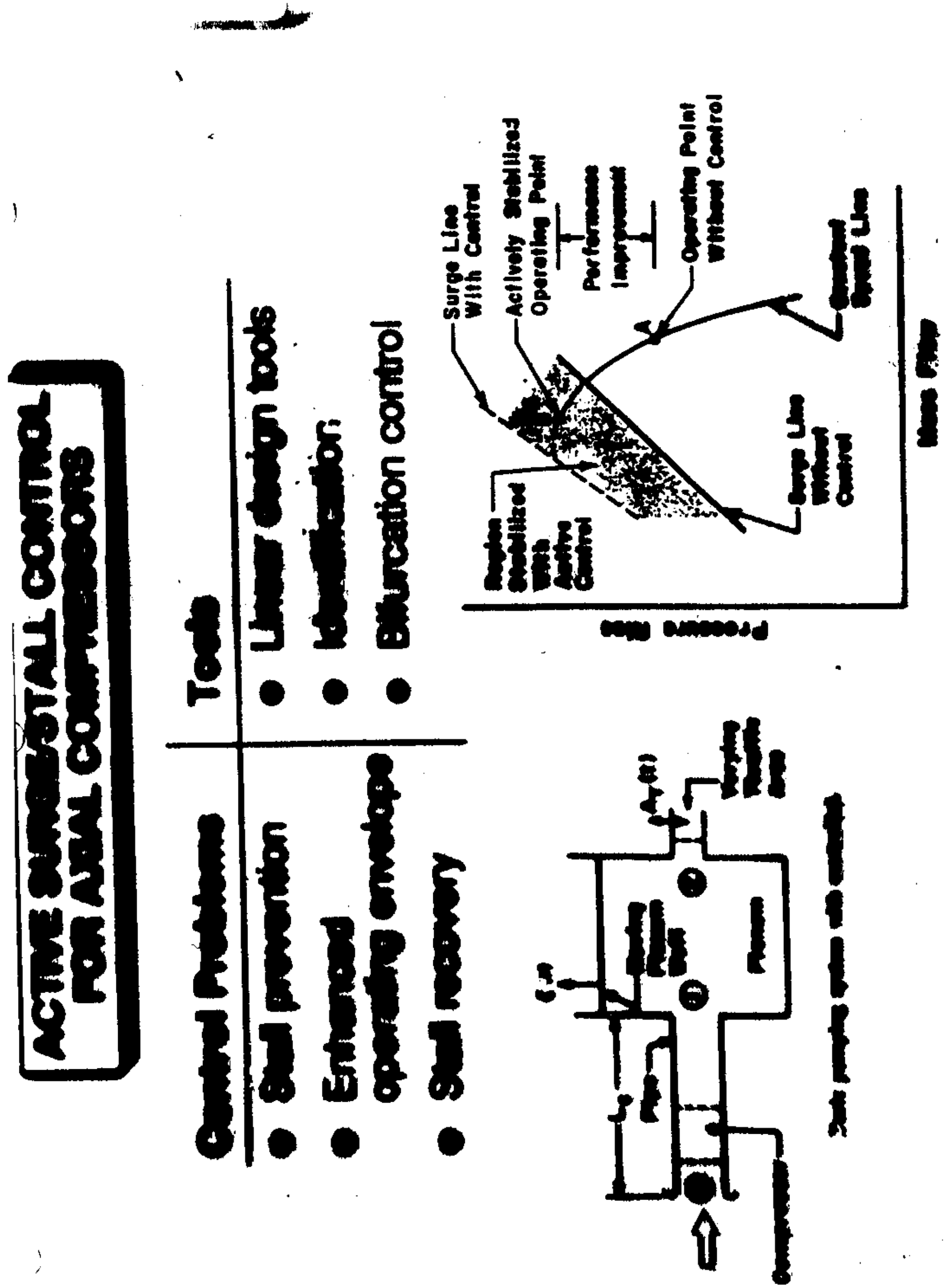
$$\Rightarrow x_k - d w_k = 0$$

$$\Rightarrow u_k = 0 \quad \text{AT EQUIL.}$$





0 - ρ - 1.4 $\qquad$ -2.5 - x - 2.5



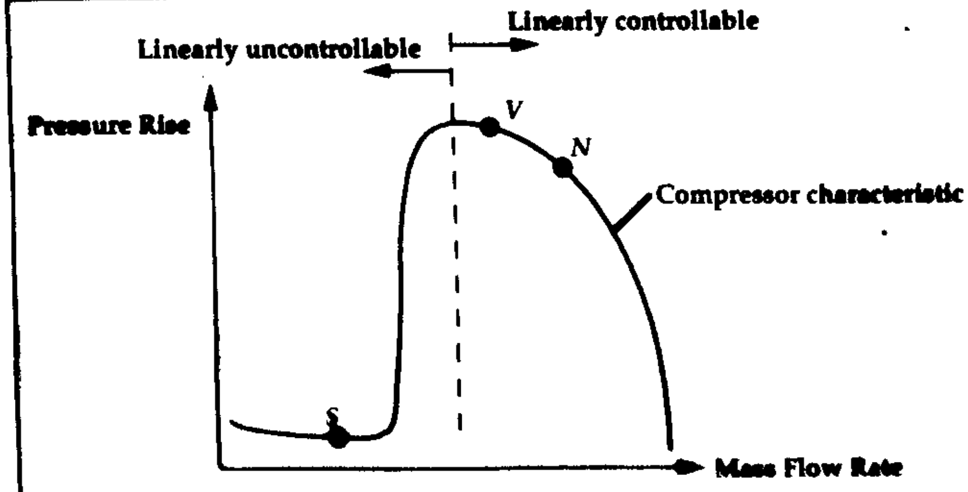
BIFURCATION CONTROL OF COMPRESSOR STALL INCEPTION

PROBLEM:

- **DESIGN AND IMPLEMENT AN ACTIVE CONTROL LAW TO ALLOW OPERATION NEAR MAXIMUM PRESSURE RISE OF AXIAL FLOW COMPRESSORS.**

MAIN DIFFICULTIES:

- **LINEAR CONTROL DESIGNS ARE INEFFECTIVE NEAR MAXIMUM PRESSURE RISE.**
- **NO ACCURATE SYSTEM MODEL FOR OFF-DESIGN OPERATION.**
- **SENSING AND ACTUATION.**



N : **Normal Operating State**

V : **Vulnerable, High Performance Operating State**

S : **Stalled Operating State**

2.1. Local Stability Analysis

A third-order lumped parameter model, in terms of nondimensional variables, for an axial flow compressor is introduced by Moore and Greitzer as follows:

$$\frac{d\dot{m}_C}{dt} = -\Delta P + \frac{1}{2\pi}\int_0^{2\pi} C_{ss}(\dot{m}_C + WA\sin\theta)d\theta \qquad (2.1a)$$

$$\frac{d\Delta P}{dt} = \frac{1}{4B^2}\{\dot{m}_C - F^{-1}(\Delta P)\} \qquad (2.1b)$$

$$\frac{dA}{dt} = \frac{\alpha}{\pi W}\int_0^{2\pi} C_{ss}(\dot{m}_C + WA\sin\theta)\sin\theta d\theta, \qquad (2.1c)$$

where $W, \alpha > 0$ are two constants and A denotes the amplitude of the first mode of rotating stall.

It is observed that $A = 0$ always makes $dA/dt = 0$ in Eq. (2.1c). So, we have $(\dot{m}_C^0, \Delta P^0, 0)^T$ as an equilibrium point for (2.1), where $\dot{m}_C^0 = F^{-1}(\Delta P^0)$ and $\Delta P^0 = C_{ss}(\dot{m}_C^0)$. However, $A = 0$ may not be the necessary condition for the existence of equilibrium point for (2.1).

Denote $x^0 = (0, \dot{m}_C^0, \Delta P^0)^T$ an equilibrium point for (2.1). The linearization of (2.1) at x^0 gives

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{pmatrix} = \begin{pmatrix} \alpha C'_{ss}(\dot{m}_C^0) & 0 & 0 \\ 0 & C'_{ss}(\dot{m}_C^0) & -1 \\ 0 & \frac{1}{4B^2} & -\frac{1}{4B^2} D_{\Delta P} F(\gamma^0, \Delta P^0) \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

where $x_1 := A$, $x_2 := \dot{m}_C - \dot{m}_C^0$ and $x_3 := \Delta P - \Delta P^0$.

Suppose $B > 0$ and F is a strictly increasing function, i.e., $F'(\dot{m}_C^0) > 0$ for all $\dot{m}_C^0$, then we have following stability results at the equilibrium point x^0:

i) asymptotically stable: if $C'_{ss}(\dot{m}_C^0) < 0$.

ii) unstable: if $C'_{ss}(\dot{m}_C^0) > 0$.

iii) the linearized model has one zero eigenvalue while $C'_{ss}(\dot{m}_C^0) = 0$.

- Possibilities of stationary (or static) bifurcation

Let x^0 be the equilibrium point at which $C'_{ss}(\dot{m}_C^0) = 0$ for some $\gamma = \gamma^0$. Taking the Taylor series expansion of (2.1) at the point (x^0, γ^0), we have

$$\frac{dX}{dt} = L_0 X + Q_0(X, X) + C_0(X, X, X) + (\gamma - \gamma^0) L_1 X + \cdots, \quad (2.4)$$

where

$$L_0 = \begin{pmatrix} \alpha C'_{ss}(\dot{m}_C^0) & 0 & 0 \\ 0 & C'_{ss}(\dot{m}_C^0) & -1 \\ 0 & \frac{1}{4B^2} & -\frac{1}{4B^2} D_{\Delta P} F(\gamma^0, \Delta P^0) \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

$$Q_0(X, X) = \begin{pmatrix} \alpha C''_{ss}(\dot{m}_C^0) x_1 x_2 \\ \frac{1}{4} C''_{ss}(\dot{m}_C^0)(W^2 x_1^2 + 2x_2^2) \\ -\frac{1}{4B^2} D_{(\Delta P)^2} F(\gamma^0, \Delta P^0) x_3^2 \end{pmatrix}$$

$$C_0(X, X, X) = \begin{pmatrix} \frac{\alpha}{8} C'''_{ss}(\dot{m}_C^0)(W^2 x_1^3 + 4x_1 x_2^2) \\ C'''_{ss}(\dot{m}_C^0)(\frac{1}{4} W^2 x_1^2 x_2 + \frac{1}{6} x_2^3) \\ -\frac{1}{4B^2} D_{(\Delta P)^3} F(\gamma^0, \Delta P^0) x_3^3 \end{pmatrix}$$

$$L_1 = \begin{pmatrix} \alpha H C''_{ss}(\dot{m}_C^0) & 0 & 0 \\ 0 & H C''_{ss}(\dot{m}_C^0) & 0 \\ 0 & 0 & -\frac{H}{4B^2} \end{pmatrix}$$

with $\quad H = D_{\Delta P \gamma} F(\gamma^0, \Delta P^0)$.

2.3. Stationary Bifurcation Control

• **Including Dynamics of A**

We consider a throttle control compressor system as given follows:

$$\frac{d\dot{m}_C}{dt} = -\Delta P + \frac{1}{2\pi}\int_0^{2\pi} C_{ss}(\dot{m}_C + WA\sin\theta)d\theta \tag{2.7a}$$

$$\frac{d\Delta P}{dt} = \frac{1}{4B^2}\{\dot{m}_C - f(\Delta P, u)\} \tag{2.7b}$$

$$\frac{dA}{dt} = \frac{\alpha}{\pi W}\int_0^{2\pi} C_{ss}(\dot{m}_C + WA\sin\theta)\sin\theta d\theta, \tag{2.7c}$$

where f is a smooth function with $f(\Delta P, 0) = F^{-1}(\Delta P)$ for all u.

Denote $x^0 = (\dot{m}_C^0, \Delta P^0, 0)^T$ is the equilibrium point of (2.7), where $C'_{ss}(\dot{m}_C^0) = 0$ and $u = 0$. It is observed that $A = 0$ is an invariant submanifold of (2.7) disregard the value of control u, which implies that system (2.7) is uncontrollable.

Let the control function u be in the form of $u = q_A A^2$. By calculating the bifurcation coefficients of the controlled system at x^0, we have $\beta_1^* = \beta_1 \equiv 0$ and $\beta_2^* = \beta_2 + 2\alpha q_A D_2 f(\Delta P^0, 0)C''_{ss}(\dot{m}_C^0)$, where β_1 and β_2 are the bifurcation coefficients of the uncontrolled version of (2.7).

We have following result.

Lemma. The stationary bifurcation of (2.7) can be guaranteed to be a supercritical pitchfork bifurcation by a purely quadratic feedback control if $C''_{ss}(\dot{m}_C^0) \neq 0$. The feedback control is in the form of $u = q_A A^2$.

Choose $l = (1, 0, 0)$ and $r = l^T$ as the left and right eigenvectors corresponding to the zero eigenvalue of L_0.

We have the stability coefficients $\beta_1 \equiv 0$ and

$$\beta_2 = \frac{\alpha W^2}{4}\{2D_{\Delta P}F(\gamma^0, \Delta P^0)(C''_{ss}(\dot{m}_C^0))^2 + C'''_{ss}(\dot{m}_C^0)\}. \tag{2.5}$$

By checking Eqs. (2.1a)-(2.1c), we have

i) the right handside of Eq. (2.1a) is an odd function of A.

ii) the right handsides of Eqs. (2.1b) and (2.1c) are even functions of A.

The next result follows readily from Theorem 1 and the discussions above.

Theorem 2. Suppose $HC''_{ss}(\dot{m}_C^0) \neq 0$ and $2D_{\Delta P}F(\gamma^0, \Delta P^0)(C''_{ss}(\dot{m}_C^0))^2 + C'''_{ss}(\dot{m}_C^0) \neq 0$ for $\gamma = \gamma^0$. Then, with respect to the small variation of γ at $\gamma = \gamma^0$, we have the following results:

i) system (2.1) has a pitchfork stationary bifurcation near the point $(0, \dot{m}_C^0, \Delta P^0)^T$ as well as the transformed model of (2.1) (with A being replaced by $J := A^2$ as one of system states) has a transcritical stationary bifurcation near the point $(0, \dot{m}_C^0, \Delta P^0)$.

ii) the stability of the bifurcated solutions for positive A and positive J of two different models (i.e., system (2.1) and its transformed model) near 0 are equivalent.

A stability criterion for the system equilibrium of (2.1) is given as follows.

Theorem 3. Suppose $HC''_{ss}(\dot{m}_C^0) \neq 0$ and the stability coefficient β_2 given in (2.5) is nonzero. Then system (2.1) exhibits a pitchfork bifurcation with respect to the small variation of γ at the point (x^0, γ^0), where $C'_{ss}(\dot{m}_C^0) = 0$. Moreover, if $\beta_2 < 0$ (resp. $\beta_2 > 0$) the local bifurcated solutions near x^0 will be asymptotically stable (resp. unstable).

Stationary Bifurcation Control

We consider a throttle control compressor system as given follows:

$$\frac{dA}{dt} = \frac{\alpha}{\pi W} \int_0^{2\pi} C_{ss}(\dot{m}_C + WA\sin\theta)\sin\theta d\theta \tag{2.7a}$$

$$\frac{d\dot{m}_C}{dt} = -\Delta P + \frac{1}{2\pi}\int_0^{2\pi} C_{ss}(\dot{m}_C + WA\sin\theta)d\theta \tag{2.7b}$$

$$\frac{d\Delta P}{dt} = \frac{1}{4B^2}\{\dot{m}_C - f(u, \Delta P)\}. \tag{2.7c}$$

where f is a smooth function with $f(u, \Delta P) = F(\gamma - \gamma^0, \Delta P)$.

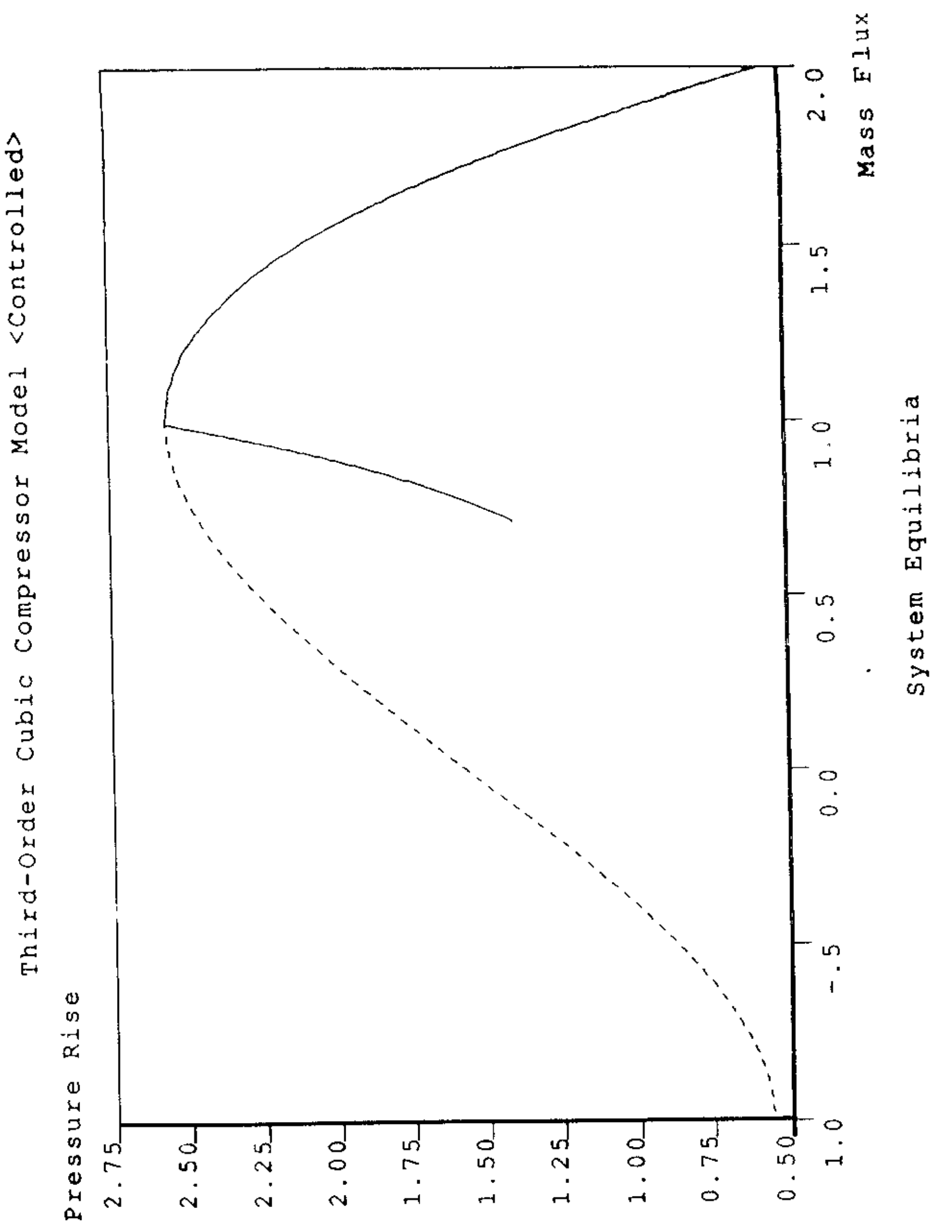
Denote $x^0 = (\dot{m}_C^0, \Delta P^0, 0)^T$ the equilibrium point of (2.7) at $\gamma = \gamma^0$, $C'_{ss}(\dot{m}_C^0) = 0$ and $u = 0$. It is observed that $A = 0$ is an invariant submanifold of (2.7) disregard the value of control u. which implies that system (2.7) is uncontrollable.

Let the control function u be in the form of $u = q_A A^2$. By calculating the bifurcation coefficients of the controlled system at x^0. we have $\beta_1^* = \beta_1 \equiv 0$ and $\beta_2^* = \beta_2 + 2\alpha q_A D_u f(0, \Delta P^0) C''_{ss}(\dot{m}_C^0)$. where β_1 and β_2 are the bifurcation coefficients of the uncontrolled version of (2.7).

We have the following result.

Lemma. The stationary bifurcation of (2.7) can be guaranteed to be a supercritical pitchfork bifurcation by a purely quadratic feedback control if $C''_{ss}(\dot{m}_C^0) \neq 0$. The feedback control is in the form of $u = q_A A^2$.

Note: According to Theorem 3, the same result as the one given in Theorem 4 can be obtained via the transformed model in terms of $J (:= A^2)$. However, the feedback control, in this case, is a linear state feedback in the form of $u = kJ$.



Goal of This Talk:

Discuss importance of and tools for the

Determination of features of measured signals that provide a warning of incipient instability

Sub-goals:

- Real-time precursor-based monitoring system
- Real-time precursor-based diagnostic system
- These precursor-based systems would address slow drift toward instability, not sudden jumps

Such features, called precursors, should:

- Be computable in real time
- Be mainly nonparametric, i.e., not model-based
- Account for noise of different statistics and coloration
- Allow design of optimal test inputs (noisy, deterministic, or chaotic) for system monitoring

Past Work:

Theory:

- K. Wiesenfeld, Noisy precursors of nonlinear instabilities, J. Stat. Phys., 1985
- K. Wiesenfeld, Observation of noisy precursors of dynamical instabilities, Phys. Rev. A, 1985
- K. Wiesenfeld and B. McNamara, Small-signal amplification in bifurcating dynamical systems, Phys. Rev. A, 1986

Application:

- V.H. Garnier, A.H. Epstein and E.M. Greitzer, Rotating waves as a stall inception indication in axial compressors, J. Turbomachinery, 1991
- I.J. Day, Stall inception indication in axial flow compressors, ASME Gas Turbine Conf., 1991
- T. Poinsot, F. Bourienne, S. Candel and E. Esposito, Suppression of combustion instabilities by active control, J. Propulsion, 1987

Related Application Papers:

- K. Wiesenfeld, Parametric amplification in semiconductor lasers, Phys. Rev. A, 1986
- D.R. Ostojic and G.T. Heydt, Transient stability assessment by pattern recognition in the frequency domain, IEEE Trans. Power Systems, 1991

(These don't aim to obtain warning signals for instability per se.)

Consider a system

$$\dot{x}(t) = f_\mu(x(t)) + g_\mu(x(t))\, w(t)$$

driving term ("noise")

Let

$$x^0(\mu)$$

be a nominal ("desired") operating condition — <u>stable</u> for

$$\underline{\mu < \mu_c}$$

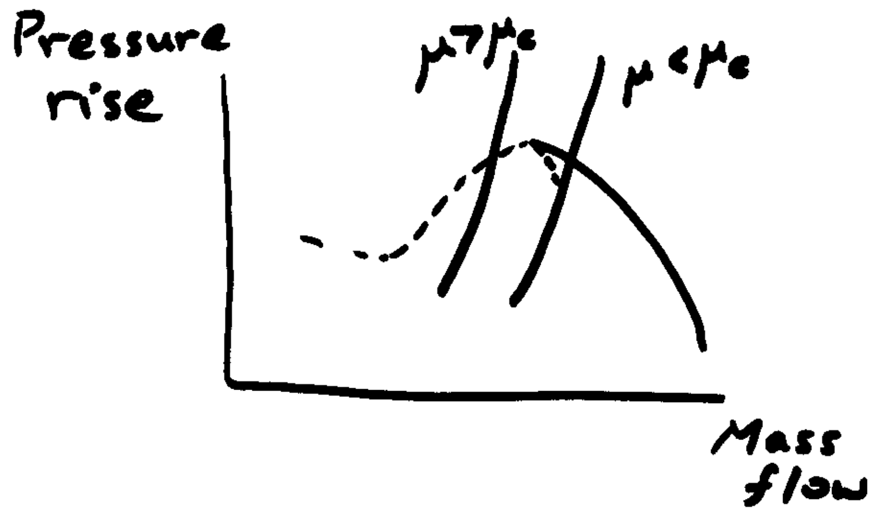
and <u>unstable</u> for

$$\underline{\mu > \mu_c}$$

The driving "noise" $w(t)$ is what allows us to detect instability prior to its onset (i.e., for $\mu < \mu_c$ with $|\mu - \mu_c|$ small).

We don't have to wait till instability is initiated to detect nearness to it.

It is important to study precursors in detail & consider control to enhance precursor effectiveness.

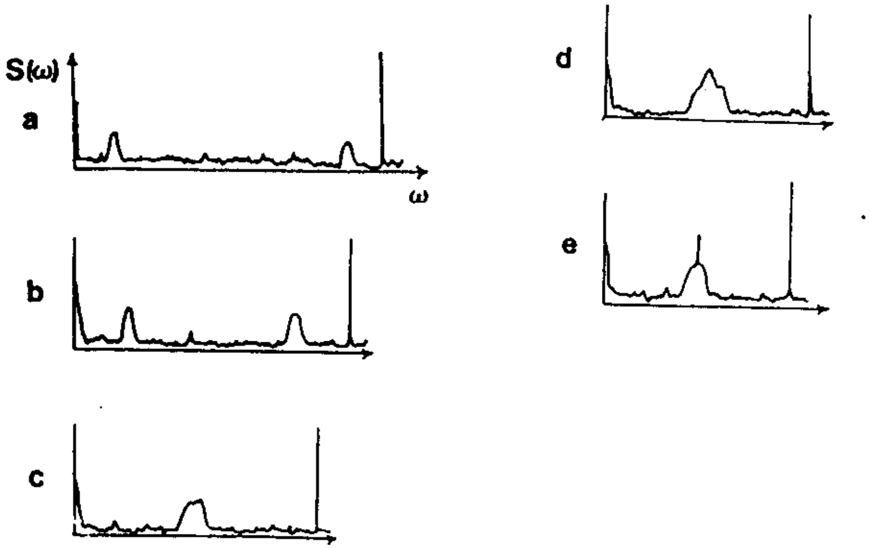


Uncontrolled bifurcation diagram for axial compressor

- We can detect nearness to stall for $\mu < \mu_c$

For a stabilized bifurcation, we don't want a false alarm
($\rightarrow$ nonlinear precursors needed

rom: K. Wiesenfeld
Noisy precursors of
nonlinear instabilities
J. Stat. Phys. 1985



g. 1. Qualitative sketch of a typical sequence of power spectra for the nonlinear circuit studied in Ref. 1, an external parameter is varied, the onset of period doubling occurs somewhere between Fig. 1c and Fig.

Motivation:

Determination of the compressor stage where an instability initiates is important for 'localized' control using, e.g., dithering of stator blade row angles.

(see Hosny et al., ASME Gas Turbines Conf., 1991)

Analytical Tool:

SMA (Selective Modal Analysis), and, specifically, participation factors.

(introduced by Pérez-Arriaga, Verghese, Schweppe – MIT, 1981)

Applies to LTI systems

$$\dot{x} = Ax$$

states : $x_1, x_2, \ldots, x_n$
modes : $\lambda_1, \lambda_2, \ldots, \lambda_n$
part. factors : p_{ki}

$$x(t) = e^{At} x^0$$
$$= \sum_{i=1}^{n} (l^i x^0) e^{\lambda_i t} r^i$$
$$\Rightarrow x_k(t) = \sum_{i=1}^{n} (l^i x^0) e^{\lambda_i t} r_k^i$$
$$= x^{0T} (r_k^i l^{iT})$$
$$\Rightarrow p_{ki} = l_k^i r_k^i$$

Justification of participation factors

(or participation factors revisited)

(w/ David Lindsay)

For initial condition x^0 we can show that the i^{th} mode contributes

$$(l^i x^0) r_k^i$$

at $t=0$ to the k^{th} state variable $x_k(t)$. Relative to the total value of x_k, this is

$$\frac{(l^i x^0) r_k^i}{\sum_{j=1}^{n} (l^j x^0) r_k^j}$$

Since the initial state x^0 can be viewed as an 'arbitrary' perturbation of $x=0$, we can define the participation as a (local) average of the above:

$$\text{"}p_{ki}\text{"} := \operatorname*{Avg}_{\|x^0\| \le 1} \frac{(\ell^i x^0)\, r_k^i}{\sum_{j=1}^{n} (\ell^j x^0)\, r_k^j}$$

This integral can be evaluated and the formula of Perez-Arriaga et al. is retrieved.

p =

Columns 1 through 7

		x_3	x_4			
0.0101	0.0107	0.1391	0.1378	0.1308	0.1427	0.2825
0.0101	0.0107	0.1391	0.1378	0.1308	0.1427	0.2825
0.0884	0.0904	0.9167	0.9173	0.5569	0.6149	0.2318
0.0884	0.0904	0.9167	0.9173	0.5569	0.6149	0.2318
0.1875	0.2053	1.0000	1.0000	0.0002	0.0000	0.5415
0.1875	0.2053	1.0000	1.0000	0.0002	0.0000	0.5415
0.4322	0.4420	0.4748	0.5303	0.7578	0.7998	0.2830
0.4322	0.4420	0.4748	0.5303	0.7578	0.7998	0.2830
0.6638	0.7205	0.0335	0.0395	1.0000	1.0000	1.0000
0.6638	0.7205	0.0335	0.0395	1.0000	1.0000	1.0000
1.0000	1.0000	0.8793	0.6502	0.0088	0.0008	0.2901
1.0000	1.0000	0.8793	0.6502	0.0088	0.0008	0.2901
0.4558	0.4486	0.8717	0.7034	0.8172	0.3528	0.3028
0.4558	0.4486	0.8717	0.7034	0.8172	0.3528	0.3028
0.0276	0.0514	0.0531	0.0885	0.0497	0.0540	0.0223
0.0276	0.0514	0.0531	0.0885	0.0497	0.0540	0.0223

highest participation (stage 2)

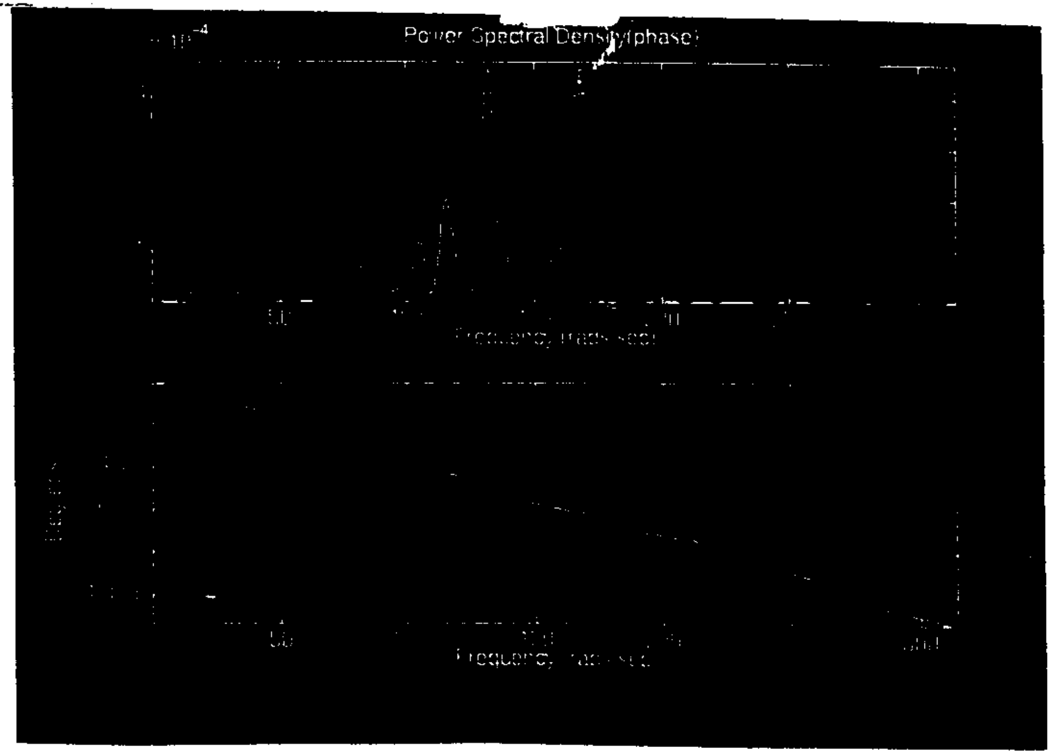
Columns 8 through 14

0.2997	0.5726	0.5027	0.4892	0.4715	0.2288	0.2402
0.2997	0.5726	0.5027	0.4892	0.4715	0.2288	0.2402
0.2544	0.1412	0.1232	0.1641	0.1561	0.1093	0.1138
0.2544	0.1412	0.1232	0.1641	0.1561	0.1093	0.1138
0.5908	0.2277	0.2037	0.2271	0.2037	0.7048	0.7372
0.5908	0.2277	0.2037	0.2271	0.2037	0.7048	0.7372
0.3120	0.7604	0.6331	0.0608	0.0709	0.8639	0.8859
0.3120	0.7604	0.6331	0.0608	0.0709	0.8639	0.8859
1.0000	0.0995	0.0786	0.6684	0.7028	1.0000	1.0000
1.0000	0.0995	0.0786	0.6684	0.7028	1.0000	1.0000
0.4479	1.0000	1.0000	1.0000	1.0000	0.5519	0.5210
0.4479	1.0000	1.0000	1.0000	1.0000	0.5519	0.5210
0.2414	0.2330	0.1357	0.1199	0.0669	0.0564	0.0248
0.2414	0.2330	0.1357	0.1199	0.0669	0.0564	0.0248
0.0475	0.0232	0.0402	0.0174	0.0386	0.0135	0.0501
0.0475	0.0232	0.0402	0.0174	0.0386	0.0135	0.0501

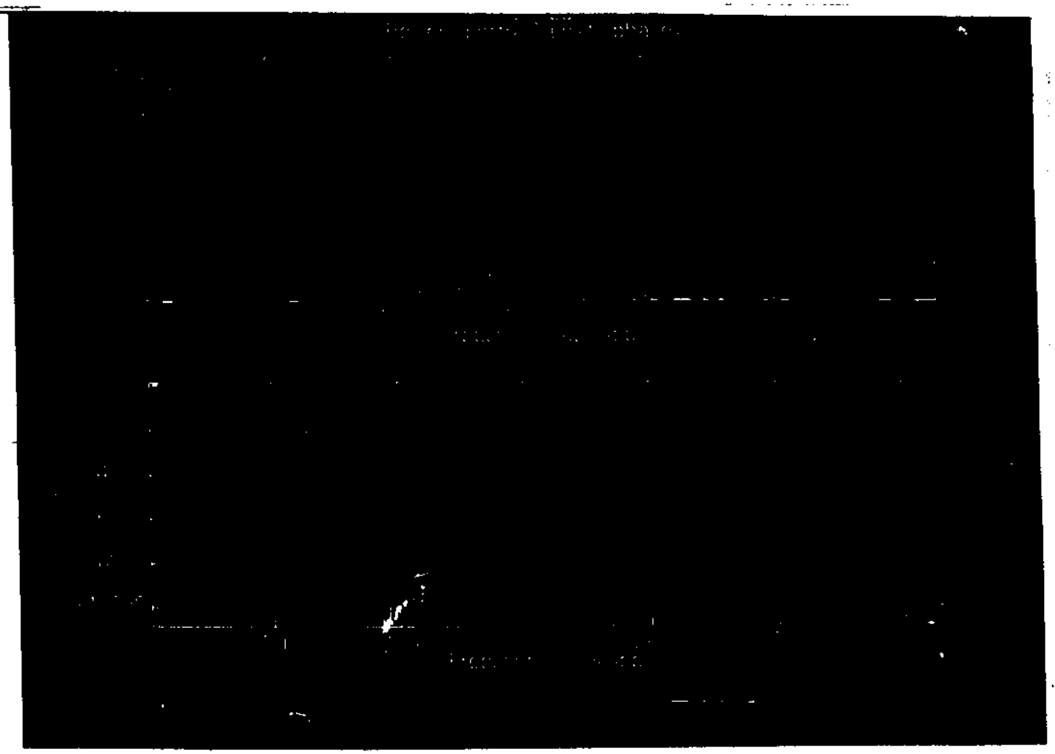
Columns 15 through 16

0.0000	0.0000
0.0000	0.0000
0.0000	0.0000
0.0000	0.0000
0.0000	0.0000
0.0000	0.0000
0.0000	0.0000
0.0000	0.0000
0.0001	0.0001
0.0001	0.0001
0.0009	0.0009
0.0009	0.0009
0.0113	0.0113
0.0113	0.0113

Power spectral density for 10th state (participation factor = 0.1232)



Power spectral density for 3rd state (participation factor = 0.91..)



The Control Handbook, W. S. Levine, Ed., CRC Press, Boca Raton, FL, 1996.

Ideal sliding mode: Motion of a system's state trajectory along a switching surface when switching in the control law is infinitely fast.

Matching condition: The condition requiring the plant's uncertainties to lie in the image of the input matrix, that is, the uncertainties can affect the plant dynamics only through the same channels as the plant's input.

Region of attraction: A set of initial states in the state space from which sliding is achievable.

Regular form: A particular form of the state-space description of a dynamic system obtained by a suitable transformation of the system's state variables.

Sliding surface: See equilibrium manifold.

Switching surface: See equilibrium manifold.

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57.6 Control of Bifurcations and Chaos

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57.6.1 Introduction

This chapter deals with the control of bifurcations and chaos in nonlinear dynamical systems. This is a young subject area that is currently in a state of active development. Investigations of control system issues related to bifurcations and chaos began relatively recently, with most currently available results having been published within the past decade. Given this state of affairs, a unifying and comprehensive picture of control of bifurcations and chaos does not yet exist. Therefore, the chapter has a modest but, it is hoped, useful goal: to summarize some of the motivation, techniques, and results achieved to date on control of bifurcations and chaos. Background material on nonlinear dynamical behavior is also given, to make the chapter somewhat self-contained. However, interested readers unfamiliar with nonlinear dynamics will find it helpful to consult nonlinear dynamics texts to reinforce their understanding.

Despite its youth, the literature on control of bifurcations and chaos contains a large variety of approaches as well as interesting applications. Only a small number of approaches and applications can be touched upon here, and these reflect the background and interests of the authors. The Further Reading section provides references for those who would like to learn about alternative approaches or to learn more about the approaches discussed here.

Control system design is an enabling technology for a variety of application problems in which nonlinear dynamical behavior arises. The ability to manage this behavior can result in significant practical benefits. This might entail facilitating system operability in regimes where linear control methods break down; taking advantage of chaotic behavior to capture a desired oscillatory behavior without expending much control energy; or purposely introducing a chaotic signal in a communication system to mask a transmitted signal from an adversary while allowing perfect reception by the intended party.

The control problems addressed in this chapter are characterized by two main features:

1. nonlinear dynamical phenomena impact system behavior
2. control objectives can be met by altering nonlinear phenomena

Nonlinear dynamics concepts are clearly important in *understanding* the behavior of such systems (with or without control). Traditional linear control methods are, however, often effective in the *design* of control strategies for these systems. In other cases, such as for systems of the type discussed in Section 57.6.5, nonlinear methods are needed both for control design and performance assessment.

The chapter proceeds as follows. In section two, some basic nonlinear system terminology is recalled. This section also includes a new term, namely "candidate operating condition", which facilitates subsequent discussions on control goals and strategies. Section three contains a brief summary of basic bifurcation and chaos concepts that will be needed. Section four provides application examples for which bifurcations and/or chaotic behavior occur. Remarks on the control aspects of these applications are also given. Section five is largely a review of some basic concepts related to control of parameterized families of (nonlinear) systems. Section five also includes a discussion of what might be called "stressed operation" of a system. Section six is devoted to control problems for systems exhibiting bifurcation behavior. The subject of section seven is control of chaos. Conclusions are given in section eight. The final section gives some suggestions for further reading.

57.6.2 Operating Conditions of Nonlinear Systems

In linear system analysis and control, a blanket assumption is made that the operating condition of interest is a particular equilibrium point, which is then taken as the origin in the state space. The topic addressed here relates to applying control to alter the dynamical behavior of a system possessing multiple possible operating conditions. The control might alter these conditions in terms of their location and amplitude, and/or stabilize certain possible operating conditions, permitting them to take the place of an undesirable open-loop behavior. For the purposes of this chapter, therefore, it is important to consider a variety of possible operating conditions, in addition to the single equilibrium point focused on in linear system theory. In this section, some basic terminology regarding operating conditions for nonlinear systems is established.

Consider a finite-dimensional continuous-time system

$$\dot{x}(t) = F(x(t)) \tag{57.180}$$

where $x \in \mathbb{R}^n$ is the system state and F is smooth in x. (The terminology recalled next extends straightforwardly to discrete-time systems $x^{k+1} = F(x^k)$.) An ***equilibrium point*** or ***fixed point*** of the system (57.180) is a constant steady-state solution, i.e., a vector x^0 for which $F(x^0) = 0$. A ***periodic solutio*** . of the system is a trajectory $x(t)$ for which there is a minimum $T > 0$ such that $x(t+T) = x(t)$ for all t. An ***invariant set*** is a set for which any trajectory starting from an initial condition within the set remains in the set for all future and past times. An ***isolated invariant set*** is a bounded invariant set a neighborhood of which contains no other invariant set. Equilibrium points and periodic solutions are examples of invariant sets. A periodic solution is called a *limit cycle* if it is isolated. An ***attractor*** is a bounded invariant set to which trajectories starting from all sufficiently nearby initial conditions converge as $t \to \infty$. The *positive limit set* of (57.180) is the ensemble of points that some system trajectory either approaches as $t \to \infty$ or makes passes nearer and nearer to as $t \to \infty$. For example, if a system is such that all trajectories converge either to an equilibrium point or a limit cycle, then the positive limit set would be the union of points on the limit cycle and the equilibrium point. The *negative limit set* is the positive limit set of the system run with time t replaced by $-t$. Thus, the positive limit set is the set where the system ends up at $t = +\infty$, while the negative limit sèt is the set where the system begins at $t = -\infty$ [30]. The *limit set* of the system is the union of its positive and negative limit sets.

It is now possible to introduce a term that will facilitate the discussions on control of bifurcations and chaos. A ***candidate operating condition*** of a dynamical system is an equilibrium point, periodic solution or other invariant subset of its limit set. Thus, a candidate operating condition is any possible steady-state solution of the system, without regard to its stability properties. This term, though not standard, is useful since it permits discussion of bifurcations, bifurcated solutions, and chaotic motion in general terms without having to specify a particular nonlinear phenomenon. The idea behind this term is that such a solution, if stable or stabilizable using available controls, might qualify as an operating condition of the system. Whether or not it would be *acceptable* as an actual operating condition would depend on the system and on characteristics of the candidate operating condition. As an extreme example, a steady spin of an airplane *is* a candidate operating condition, but certainly *is not* acceptable as an actual operating condition!

57.6.3 Bifurcations and Chaos

This section summarizes background material on bifurcations and chaos that is needed in the remainder of the chapter.

Bifurcations

A *bifurcation* is a change in the number of candidate operating conditions of a nonlinear system that occurs as a parameter is quasistatically varied. The parameter being varied is referred to as the bifurcation parameter. A value of the bifurcation parameter at which a bifurcation occurs is called a critical value of the bifurcation parameter. Bifurcations from a nominal operating condition can only occur at parameter values for which the condition (say, an equilibrium point or limit cycle) either loses stability or ceases to exist.

To fix ideas, consider a general one-parameter family of ordinary differential equation systems

$$\dot{x} = F^{\mu}(x) \tag{57.181}$$

where $x \in \mathbb{R}^n$ is the system state, $\mu \in \mathbb{R}$ denotes the bifurcation parameter, and F is smooth in x and μ. Equation 57.181 can be viewed as resulting from a particular choice of control law in a family of nonlinear control systems (in particular, as Equation 57.188 with the control u set to a fixed feedback function $u(x, \mu)$). For any value of μ, the equilibrium points of Equation 57.181 are given by the solutions for x of the algebraic equations $F^{\mu}(x) = 0$.

Local bifurcations are those that occur in the vicinity of an equilibrium point. For example, a small-amplitude limit cycle can emerge (bifurcate) from an equilibrium as the bifurcation parameter is varied. The bifurcation is said to occur regardless of the stability or instability of the "bifurcated" limit cycle. In another local bifurcation, a pair of equilibrium points can emerge from a nominal equilibrium point. In either case, the *bifurcated solutions* are close to the original equilibrium point — hence the name local bifurcation. *Global bifurcations* are bifurcations that are not local, i.e., those that involve a domain in phase space. Thus, if a limit cycle loses stability releasing a new limit cycle, a global bifurcation is said to take place.[3]

The nominal operating condition of Equation 57.181 can be an equilibrium point or a limit cycle. In fact, depending on the coordinates used, it is possible that a limit cycle in one mathematical model corresponds to an equilibrium point in another. This is the case, for example, when a truncated Fourier series is used to approximate a limit cycle, and the amplitudes of the harmonic terms are used as state variables in the approximate model. The original limit cycle is then represented as an equilibrium in the amplitude coordinates.

If the nominal operating condition of Equation 57.181 is an equilibrium point, then bifurcations from this condition can occur only when the linearized system loses stability. Suppose, for example, that the origin is the nominal operating condition for some range of parameter values. That is, let $F^{\mu}(0) = 0$ for all values of μ for which the nominal equilibrium exists. Denote the Jacobian matrix of (57.181) evaluated at the origin by

$$A(\mu) := \frac{\partial F^{\mu}}{\partial x}(0).$$

Local bifurcations from the origin can only occur at parameter values μ where $A(\mu)$ loses stability.

The scalar differential equation

$$\dot{x} = \mu x - x^3 \tag{57.182}$$

provides a simple example of a bifurcation. The origin $x^0 = 0$ is an equilibrium point for all values of the real parameter μ. The Jacobian matrix is $A(\mu) = \mu$ (a scalar). It is easy to see that the origin loses stability as μ increases through $\mu = 0$. Indeed, a bifurcation from the origin takes place at $\mu = 0$. For $\mu \leq 0$, the only equilibrium point of Equation 57.182 is the origin. For $\mu > 0$, however, there are two additional equilibrium points at $x = \pm\sqrt{\mu}$. This pair of equilibrium points is said to *bifurcate* from the origin at the *critical parameter value* $\mu_c = 0$. This is an example of a pitchfork bifurcation, which will be discussed later.

Subcritical vs. supercritical bifurcations In a very real sense, the fact that bifurcations occur when stability is lost is helpful from the perspective of control system design. To explain this, suppose that a system operating condition (the "nominal" operating condition) is not stabilizable beyond a critical parameter value. Suppose a bifurcation occurs at the critical parameter value. That is, suppose a new candidate operating condition emerges from the nominal one at the critical parameter value. Then it may be that the new operating condition is stable and occurs beyond the critical parameter value, providing an alternative operating condition near the nominal one. This is referred to as a *supercritical bifurcation*. In contrast, it may happen that the new operating condition is unstable and occurs prior to the critical parameter value. In this situation (called a *subcritical bifurcation*), the system state must leave the vicinity of the nominal operating condition for parameter values beyond the critical value. However, feedback offers the possibility of rendering such a bifurcation supercritical. This is true even if the nominal operating condition is not stabilizable. If such a feedback control can be found, then the system behavior beyond the stability boundary can remain close to its behavior at the nominal operating condition.

The foregoing discussion of bifurcations and their implications for system behavior can be gainfully viewed using graphical sketches called *bifurcation diagrams*. These are depictions of the equilibrium points and limit cycles of a system plotted against the bifurcation parameter. A bifurcation diagram is a schematic representation in which only a measure of the amplitude (or norm) of an equilibrium point or limit cycle need be plotted. In the bifurcation diagrams given in this chapter, a solid line indicates a stable solution, while a dashed line indicates an unstable solution.

Several bifurcation diagrams will now be used to further explain the meanings of supercritical and subcritical bifurcation, and to introduce some common bifurcations. It should be noted that not all bifurcations are supercritical or subcritical. For example, bifurcation can also be transcritical. In such a bifurcation, bifurcated operating conditions occur both prior to and beyond the critical parameter value. Identifying a bifurcation as supercritical, subcritical, transcritical, or otherwise is the problem of determining the *direction of the bifurcation*. A book on nonlinear dynamics should be consulted for a more extensive treatment

[3]This use of the terms local bifurcation and global bifurcation is common. However, in some books, a bifurcation from a limit cycle is also referred to as a local bifurcation.

(e.g., [10, 18, 28, 30, 32, 43, 46]). In this chapter only the basic elements of bifurcation analysis can be touched upon.

Suppose that the origin of Equation 57.181 loses stability as μ *increases* through the critical parameter value $\mu = \mu_c$. Under mild assumptions, it can be shown that a bifurcation occurs at μ_c.

Figure 57.13 serves two purposes: it depicts a subcritical bifurcation from the origin, and it shows a common consequence of subcritical bifurcation, namely hysteresis. A subcritical bifurcation occurs from the origin at the point labeled A in the figure. It leads to the unstable candidate operating condition corresponding to points on the dashed curve connecting points A and B. As the parameter μ is decreased to its value at point B, the bifurcated solution merges with another (stable) candidate operating condition and disappears. A *saddle-node bifurcation* is said to occur at point B. This is because the unstable candidate operating condition (the "saddle" lying on the dashed curve) merges with a stable candidate operating condition (the "node" lying on the solid curve). These candidate operating conditions can be equilibrium points or limit cycles — both situations are captured in the figure. Indeed, the origin can also be reinterpreted as a limit cycle and the diagram would still be meaningful. Another common name for a saddle-node bifurcation point is a *turning point*.

The physical scenario implied by Figure 57.13 can be summarized as follows. Starting from operation at the origin for small μ, increasing μ until point A is reached does not alter system behavior. If μ is increased past point A, however, the origin loses stability. The system then transitions ("jumps") to the available stable operating condition on the upper solid curve. This large transition can be intolerable in many applications. As μ is then decreased, another transition back to the origin occurs but at a lower parameter value, namely that corresponding to point B. Thus, the combination of the subcritical bifurcation at A and the saddle-node bifurcation at B can lead to a hysteresis effect.

Figure 57.14 depicts a *supercritical bifurcation* from the origin. This bifurcation is distinguished by the fact that the solution bifurcating from the origin at point A is stable, and occurs locally for parameter values μ beyond the critical value (i.e., those for which the nominal equilibrium point is unstable). In marked difference with the situation depicted in Figure 57.13, here as the critical parameter value is crossed a smooth change is observed in the system operating condition. No sudden jump occurs.

Suppose closeness of the system's operation to the nominal equilibrium point (the origin, say) is a measure of the system's performance. Then supercritical bifurcations ensure close operation to the nominal equilibrium, while subcritical bifurcations may lead to large excursions away from the nominal equilibrium point. For this reason, a supercritical bifurcation is commonly also said to be *safe* or *soft*, while a subcritical bifurcation is said to be *dangerous* or *hard* [49, 32, 52].

Given full information on the nominal equilibrium point, the occurrence of bifurcation is a consequence of the behavior of the linearized system at the equilibrium point. The manner in which the equilibrium point loses stability as the bifurcation parameter is varied determines the type of bifurcation that arises.

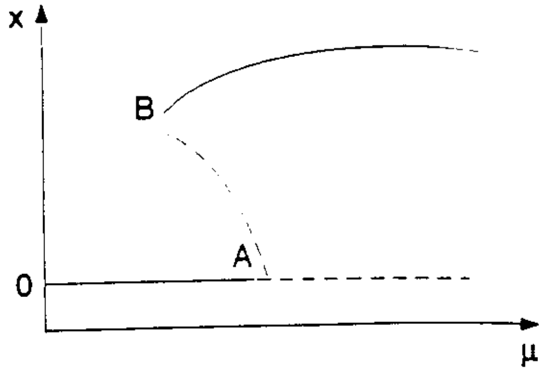

Figure 57.13 Subcritical bifurcation with hysteresis.

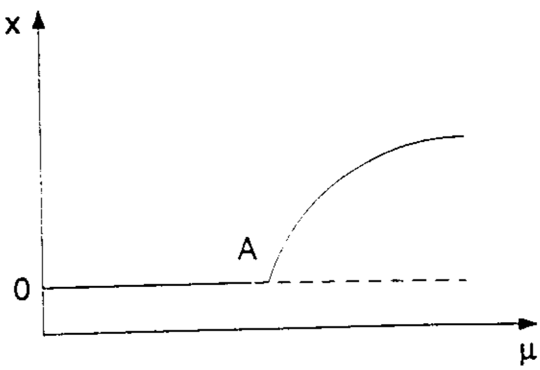

Figure 57.14 Supercritical bifurcation.

Three types of local bifurcation and a global bifurcation are discussed next. These are, respectively, the stationary bifurcation, the saddle-node bifurcation, the Andronov-Hopf bifurcation, and the period doubling bifurcation. All of these except the saddle-node bifurcation can be safe or dangerous. However, the saddle-node bifurcation is always dangerous.

There are analytical techniques for determining whether a stationary, Andronov-Hopf, or period doubling bifurcation is safe or dangerous. These techniques are not difficult to understand but involve calculations that are too lengthy to repeat here. The calculations yield formulas for so-called "bifurcation stability coefficients" [18], the meaning of which is addressed below. The references [1, 2, 4, 18, 28, 32] can be consulted for details.

What is termed here as "stationary bifurcation" is actually a special case of the usual meaning of the term. In the bifurcation theory literature [10], stationary bifurcation is any bifurcation of one or more equilibrium points from a nominal equilibrium point. When the nominal equilibrium point exists both before and after the critical parameter value, a stationary bifurcation "from a known solution" is said to occur. If the nominal solution disappears beyond the critical parameter value, a stationary bifurcation "from an unknown solution" is said to occur. To simplify the terminology, here the former type of bifurcation is called a *stationary bifurcation*. The saddle-node bifurcation is the most common example of the latter type of bifurcation.

Andronov-Hopf bifurcation also goes by other names. "Hopf bifurcation" is the traditional name in the West, but this name neglects the fundamental early contributions of Andronov and his co-workers (see, e.g., [6]). The essence of this phenomenon was also known to Poincaré, who did not develop a detailed theory but used the concept in his study of lunar orbital dynamics [37, Secs. 51–52]. The same phenomenon is sometimes called flutter bifurcation in the engineering literature. This bifurcation of a limit cycle from an equilibrium point occurs when a complex conjugate pair of eigenvalues crosses the imaginary axis into the right half of the complex plane at $\mu = \mu_c$. A small-amplitude limit cycle then emerges from the nominal equilibrium point at μ_c.

Saddle-node bifurcation and stationary bifurcation Saddle-node bifurcation occurs when the linearized system has a zero eigenvalue at $\mu = \mu_c$ but the origin doesn't persist as an equilibrium point beyond the critical parameter value. Saddle-node bifurcation was discussed briefly before, and will not be discussed in any detail in the following. Several basic remarks are, however, in order.

1. Saddle-node bifurcation of a nominal, stable equilibrium point entails the disappearance of the equilibrium upon its merging with an unstable equilibrium point at a critical parameter value.
2. The bifurcation occurring at point B in Figure 57.13 is representative of a saddle-node bifurcation.
3. The nominal equilibrium point possesses a zero eigenvalue at a saddle-node bifurcation.
4. An important feature of the saddle-node bifurcation is the *disappearance, locally, of any stable bounded solution of the system* (57.181).

Stationary bifurcation (according to the agreed upon terminology above) is guaranteed to occur when a single real eigenvalue goes from being negative to being positive as μ passes through the value μ_c. More precisely, the origin of Equation 57.181 undergoes a stationary bifurcation at the critical parameter value $\mu = 0$ if hypotheses (S1) and (S2) hold.

(S1) F of system (57.181) is sufficiently smooth in x, μ, and $F^\mu(0) = 0$ for all μ in a neighborhood of 0. The Jacobian $A(\mu) := \frac{\partial F^\mu}{\partial x}(0)$ possesses a simple real eigenvalue $\lambda(\mu)$ such that $\lambda(0) = 0$ and $\lambda'(0) \neq 0$.

(S2) All eigenvalues of the critical Jacobian $\frac{\partial F^\mu}{\partial x}(0)$ besides 0 have negative real parts.

Under (S1) and (S2), two new equilibrium points of Equation 57.181 emerge from the origin at $\mu = 0$. Bifurcation stability coefficients are quantities that determine the direction of bifurcation, and in particular the stability of the bifurcated solutions. Next, a brief discussion of the origin and meaning of these quantities is given.

Locally, the new equilibrium points occur for parameter values given by a smooth function of an auxiliary small parameter ϵ (ϵ can be positive or negative):

$$\mu(\epsilon) = \mu_1\epsilon + \mu_2\epsilon^2 + +\mu_3\epsilon^3 + \cdots \tag{57.183}$$

where the ellipsis denotes higher order terms. One of the new equilibrium points occurs for $\epsilon > 0$ and the other for $\epsilon < 0$. Also, the stability of the new equilibrium points is determined by the sign of an eigenvalue $\beta(\epsilon)$ of the system linearization at the new equilibrium points. This eigenvalue is near 0 and is also given by a smooth function of the parameter ϵ:

$$\beta(\epsilon) = \beta_1\epsilon + \beta_2\epsilon^2 + \beta_3\epsilon^3 + \cdots \tag{57.184}$$

Stability of the bifurcated equilibrium points is determined by the sign of $\beta(\epsilon)$. If $\beta(\epsilon) < 0$ the corresponding equilibrium point is stable, while if $\beta(\epsilon) > 0$ the equilibrium point is unstable. The coefficients β_i, $i = 1, 2, \ldots$ in the expansion above are the bifurcation stability coefficients mentioned earlier, for the case of stationary bifurcation. The values of these coefficients determine the local nature of the bifurcation.

Since ϵ can be positive or negative, it follows that if $\beta_1 \neq 0$ the bifurcation is neither subcritical nor supercritical. (This is equivalent to the condition $\mu_1 \neq 0$.) The bifurcation is therefore generically transcritical. In applications, however, special structures of system dynamics and inherent symmetries often result in stationary bifurcations that are not transcritical. Also, it is sometimes possible to render a stationary bifurcation supercritical using feedback control. For these reasons, a brief discussion of subcritical and supercritical pitchfork bifurcations is given next.

If $\beta_1 = 0$ and $\beta_2 \neq 0$, a stationary bifurcation is known as a *pitchfork bifurcation*. The pitchfork bifurcation is subcritical if $\beta_2 > 0$; it is supercritical if $\beta_2 < 0$. The bifurcation diagram of a subcritical pitchfork bifurcation is depicted in Figure 57.15, and that of a supercritical pitchfork bifurcation is depicted in Figure 57.16. The bifurcation discussed previously for the example system (57.182) is a supercritical pitchfork bifurcation.

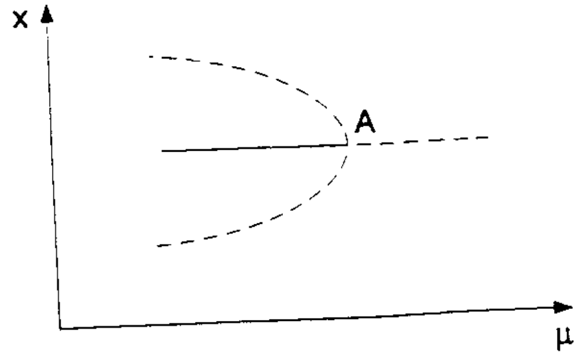

Figure 57.15 Subcritical pitchfork bifurcation.

Andronov-Hopf bifurcation Suppose that the orig of Equation 57.181 loses stability as the result of a complex co jugate pair of eigenvalues of $A(\mu)$ crossing the imaginary ax All other eigenvalues are assumed to remain stable, i.e., their r parts are negative for all values of μ. Under this simple cc dition on the linearization of a nonlinear system, the nonlin system typically undergoes a bifurcation. The word "typical is used because there is one more condition to satisfy, but it

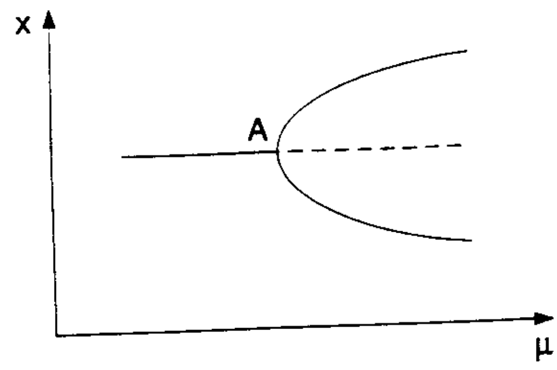

Figure 57.16 Supercritical pitchfork bifurcation.

mild condition. The type of bifurcation that occurs under these circumstances involves the emergence of a limit cycle from the origin as μ is varied through μ_c. This is the Andronov-Hopf bifurcation, a more precise description of which is given next. The following hypotheses are invoked in the Andronov-Hopf Bifurcation Theorem. The critical parameter value is taken to be $\mu_c = 0$ without loss of generality.

(AH1) F of system (57.181) is sufficiently smooth in x, μ, and $F^\mu(0) = 0$ for all μ in a neighborhood of 0. The Jacobian $\frac{\partial F^\mu}{\partial x}(0)$ possesses a complex-conjugate pair of (algebraically) simple eigenvalues $\lambda(\mu) = \alpha(\mu) + i\omega(\mu), \overline{\lambda(\mu)}$, such that $\alpha(0) = 0, \alpha'(0) \neq 0$ and $\omega_c := \omega(0) > 0$.

(AH2) All eigenvalues of the critical Jacobian $\frac{\partial F^\mu}{\partial x}(0)$ besides $\pm i\omega_c$ have negative real parts.

The Andronov-Hopf Bifurcation Theorem asserts that, under (AH1) and (AH2), a small-amplitude nonconstant limit cycle (i.e., periodic solution) of Equation 57.181 emerges from the origin at $\mu = 0$. Locally, the limit cycles occur for parameter values given by a smooth and even function of the amplitude ϵ of the limit cycles:

$$\mu(\epsilon) = \mu_2\epsilon^2 + \mu_4\epsilon^4 + \cdots \tag{57.185}$$

where the ellipsis denotes higher order terms.

Stability of an equilibrium point of the system (57.181) can be studied using eigenvalues of the system linearization evaluated at the equilibrium point. The analogous quantities for consideration of limit cycle stability for Equation 57.181 are the *characteristic multipliers* of the limit cycle. (For a definition, see for example [10, 18, 28, 30, 32, 43, 46, 48].) A limit cycle is stable (precisely: orbitally asymptotically stable) if its characteristic multipliers all have magnitude less than unity. This is analogous to the widely known fact that an equilibrium point is stable if the system eigenvalues evaluated there have negative real parts. The stability condition is sometimes stated in terms of the characteristic exponents of the limit cycle, quantities which are easily obtained from the characteristic multipliers. If the characteristic exponents have negative real parts, then the limit cycle is stable. Although it isn't possible to discuss the basic theory of limit cycle stability in any detail here, the reader is referred to almost any text on differential equations, dynamical systems, or bifurcation theory for a detailed discussion (e.g. [10, 18, 28, 30, 32, 43, 46, 48]).

The stability of the limit cycle resulting from an Andronov-Hopf bifurcation is determined by the sign of a particular characteristic exponent $\beta(\epsilon)$. This characteristic exponent is real and vanishes in the limit as the bifurcation point is approached. It is given by a smooth and even function of the amplitude ϵ of the limit cycles:

$$\beta(\epsilon) = \beta_2\epsilon^2 + \beta_4\epsilon^4 + \cdots \tag{57.186}$$

The coefficients μ_2 and β_2 in the expansions above are related by the exchange of stability formula

$$\beta_2 = -2\alpha'(0)\mu_2. \tag{57.187}$$

Generically, these coefficients do not vanish. Their signs determine the direction of bifurcation. The coefficients $\beta_2, \beta_4, \ldots$ in the expansion (57.186) are the bifurcation stability coefficients for the case of Andronov-Hopf bifurcation.

If $\beta_2 > 0$, then locally the bifurcated limit cycle is unstable and the bifurcation is subcritical. This case is depicted in Figure 57.17. If $\beta_2 < 0$, then locally the bifurcated limit cycle is stable (more precisely, one says that it is orbitally asymptotically stable [10]). This is the case of supercritical Andronov-Hopf bifurcation, depicted in Figure 57.18. If it happens that β_2 vanishes, then stability is determined by the first nonvanishing bifurcation stability coefficient (if one exists).

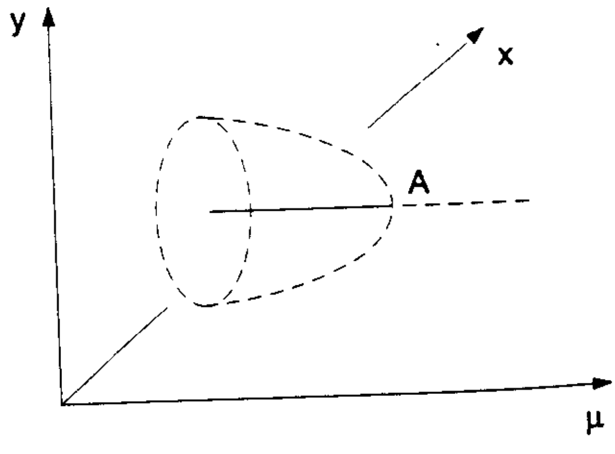

Figure 57.17 Subcritical Andronov-Hopf bifurcation.

Period doubling bifurcation The bifurcations considered above are all local bifurcations, i.e., bifurcations from an equilibrium point of the system (57.181). Solutions emerging at these bifurcation points can themselves undergo further bifurcations. A particularly important scenario involves a global bifurcation known as the *period doubling bifurcation*.

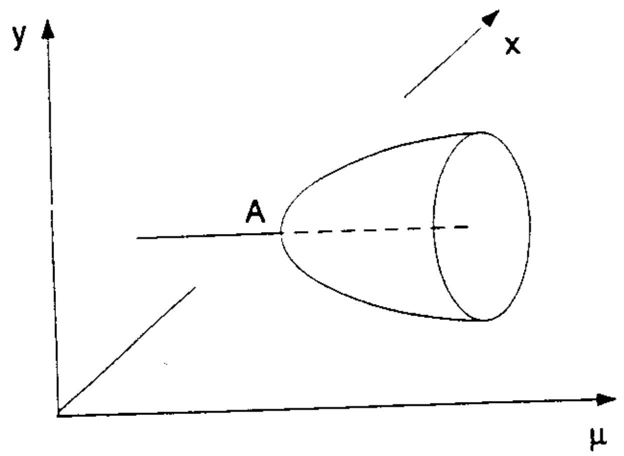

Figure 57.18 Supercritical Andronov-Hopf bifurcation.

To describe the period doubling bifurcation, consider the one-parameter family of nonlinear systems (57.181). Suppose that Equation 57.181 has a limit cycle γ^{μ} for a range of values of the real parameter μ. Moreover, suppose that for all values of μ to one side (say, less than) a critical value μ_c, all the characteristic multipliers of γ^{μ} have magnitude less than unity. If exactly one characteristic multiplier exits the unit circle at $\mu = \mu_c$, and does so at the point $(-1, 0)$, and if this crossing occurs with a nonzero rate with respect to μ, then one can show that a period doubling bifurcation from γ^{μ} occurs at $\mu = \mu_c$. (See, e.g., [4] and references therein.)

This means that another limit cycle, initially of twice the period of γ^{μ_c}, emerges from γ^{μ} at $\mu = \mu_c$. Typically, the bifurcation is either *supercritical* or *subcritical*. In the supercritical case, the period doubled limit cycle is stable and occurs for parameter values on the side of μ_c for which the limit cycle γ^{μ} is unstable. In the subcritical case, the period doubled limit cycle is unstable and occurs on the side of μ_c for which the limit cycle γ^{μ} is stable. In either case, an *exchange of stability* is said to have occurred between the nominal limit cycle γ^{μ} and the bifurcated limit cycle. Figure 57.19 depicts a supercritical period doubling bifurcation. In this figure, a solid curve represents a stable limit cycle, while a dashed curve represents an unstable limit cycle. The figure assumes that the nominal limit cycle loses stability as μ increases through μ_c.

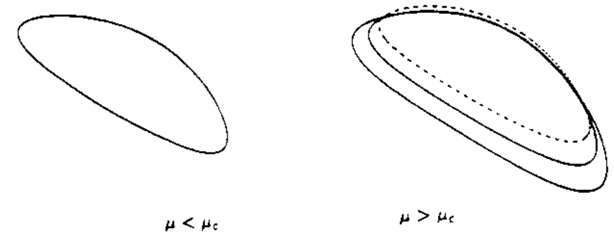

Figure 57.19 Period doubling bifurcation (supercritical case).

The direction of a period doubling bifurcation can easily be determined in discrete-time, using formulas that have been derived in the literature (see, e.g., [4]). Recently, an approximate test that applies in continuous-time has been derived using the harmonic balance approach [47].

Chaos

Bifurcations of equilibrium points and limit cycles are well understood and there is little room for alternative definitions of the main concepts. Although the notation, style, and emphasis may differ among various presentations, the main concepts and results stay the same. Unfortunately, the situation is not quite as tidy in regard to discussions of chaos. There are several distinct definitions of chaotic behavior of dynamical systems. There are also some aspects of chaotic motion that have been found to be true for many systems but have not been proved in general. The aim of this subsection is to summarize in a nonrigorous fashion some important aspects of chaos that are widely agreed upon.

The following working definition of chaos will suffice for the purposes of this chapter. The definition is motivated by [46, p. 323] and [11, p. 50]. It uses the notion of "attractor" defined in section two, and includes the definition of an additional notion, namely that of "strange attractor."

> A solution trajectory of a deterministic system (such as Equation 57.180) is *chaotic* if it converges to a strange attractor. A *strange attractor* is a bounded attractor that: (1) exhibits sensitive dependence on initial conditions, and (2) cannot be decomposed into two invariant subsets contained in disjoint open sets.

A few remarks on this working definition are in order. An aperiodic motion is one that is not periodic. Long-term behavior refers to steady-state behavior, i.e., system behavior that persists after the transient decays. Sensitive dependence on initial conditions means that for almost any initial condition lying on the strange attractor, there exists another initial condition as close as desired to the given one such that the solution trajectories starting from these two initial conditions separate by at least some prespecified amount after some time. The requirement of nondecomposability simply ensures that strange attractors are considered as being distinct if they are not connected by any system trajectories.

Often, sensitive dependence on initial conditions is defined in terms of the presence of at least one positive "Liapunov exponent" (e.g., [34, 35]). Further discussion of this viewpoint would entail technicalities that are not needed in the sequel.

From a practical point of view, a chaotic motion can be defined as a bounded invariant motion of a deterministic system that is not an equilibrium solution or a periodic solution or a quasiperiodic solution [32, p. 277]. (A quasiperiodic function is one that is composed of finitely many periodic functions with incommensurate frequencies. See [32, p. 231].)

There are two more important aspects of strange attractors and chaos that should be noted, since they play an important role in a technique for control of chaos discussed in section 57.6.7. These are:

1. A strange attractor generally has embedded within itself infinitely many unstable limit cycles. For example, Figure 57.20(a) depicts a strange attractor, and Figure 57.20(b) depicts an unstable limit cycle that is embedded in the strange attractor. Note that the shape of the limit cycle resembles that of the strange attractor. This is to be expected in general. The limit cycle chosen in the plot happens to be one of low period.
2. The trajectory starting from any point on a strange attractor will, after sufficient time, pass as close as desired to any other point of interest on the strange attractor. This follows from the indecomposability of strange attractors noted previously.

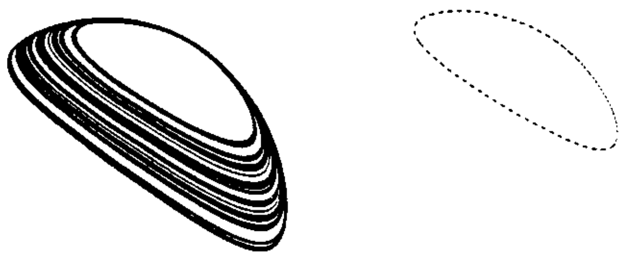

Figure 57.20 Strange attractor with embedded unstable limit cycle.

An important way in which chaotic behavior arises is through sequences of bifurcations. A well-known such mechanism is the *period doubling route to chaos*, which involves the following sequence of events:

1. A stable limit cycle loses stability, and a new stable limit cycle of double the period emerges. (The original stable limit cycle might have emerged from an equilibrium point via a supercritical Andronov-Hopf bifurcation.)
2. The new stable limit cycle loses stability, releasing another stable limit cycle of twice its period.
3. There is a cascade of such events, with the parameter separation between each two successive events decreasing geometrically.[4] This cascade culminates in a sustained chaotic motion (a strange attractor).

57.6.4 Bifurcations and Chaos in Physical Systems

In this brief section, a list of representative physical systems that exhibit bifurcations and/or chaotic behavior is given. The purpose is to provide practical motivation for study of these phenomena and their control.

Examples of physical systems exhibiting bifurcations and/or chaotic behavior include the following.

- An aircraft stalls for flight under a critical speed or above a critical angle-of-attack (e.g., [8, 13, 39, 51]).
- Aspects of laser operation can be viewed in terms of bifurcations. The simplest such observation is that a laser can only generate significant output if the pump energy exceeds a certain threshold (e.g., [19, 46]). More interestingly, as the pump energy increases, the laser operating condition can exhibit bifurcations leading to chaos (e.g., [17]).
- The dynamics of ships at sea can exhibit bifurcations for wave frequencies close to a natural frequency of the ship. This can lead to large-amplitude oscillations, chaotic motion, and ship capsize (e.g., [24, 40]).
- Lightweight, flexible, aircraft wings tend to experience flutter (structural oscillations) (e.g., [39]) (along with loss of control surface effectiveness (e.g., [15])).
- At peaks in customer demand for electric power (such as during extremes in weather), the stability margin of an electric power network may become negligible, and nonlinear oscillations or divergence ("voltage collapse") can occur (e.g., [12]).
- Operation of aeroengine compressors at maximum pressure rise implies reduced weight requirements but also increases the risk for compressor stall (e.g., [16, 20, 25]).
- A simple model for the weather consists of fluid in a container (the atmosphere) heated from below (the sun's rays reflected from the ground) and cooled from above (outer space). A mathematical description of this model used by Lorenz [27] exhibits bifurcations of convective and chaotic solutions. This has implications also for boiling channels relevant to heat-exchangers, refrigeration systems, and boiling water reactors [14].
- Bifurcations and chaos have been observed and studied in a variety of chemically reacting systems (e.g., [42]).
- Population models useful in the study and formulation of harvesting strategies exhibit bifurcations and chaos (e.g., [19, 31, 46]).

57.6.5 Control of Parametrized Families of Systems

Tracking of a desired trajectory (referred to as regulation when the trajectory is an equilibrium), is a standard goal in control system design [26]. In applying linear control system design to this standard problem, an evolving (nonstationary) system is modeled by a *parametrized family* of time-invariant (stationary)

[4] This is true exactly in an asymptotic sense. The ratio in the geometric sequence is a universal constant discovered by Feigenbaum. See, e.g., [11, 32, 34, 46, 43, 48].

systems. This approach is at the heart of the gain scheduling method, for example [26].

In this section, some basic concepts in control of parameterized families of systems are reviewed, and a notion of stressed operation is introduced. Control laws for nonlinear systems usually consist of a feedforward control plus a feedback control. The feedforward part of the control is selected first, followed by the feedback part. This decomposition of control laws is discussed in the next subsection, and will be useful in the discussions of control of bifurcations and chaos. In the second subsection, a notion of "stressed operation" of a system is introduced. Stressed systems are not the only ones for which control of bifurcations and chaos are relevant, but they are an important class for which such control issues need to be evaluated.

Feedforward/Feedback Structure of Control Laws

Control designs for nonlinear system can usually be viewed as proceeding in two main steps [26, Chapters 2, 14], [45, Chapter 3]:

1. feedforward control
2. feedback control

In Step 1, the feedforward part of the control input is selected. Its purpose is to achieve a desired candidate operating condition for the system. If the system is considered as depending on one or more parameters, then the feedforward control will also depend on the parameters. The desired operating condition can often be viewed as an equilibrium point of the nonlinear system that varies as the system parameters are varied. There are many situations when this operating condition is better viewed as a limit cycle that varies with the parameters. In Step 2, an additional part of the control input is designed to achieve desired qualities of the transient response in a neighborhood of the nominal operating condition. Typically this second part of the control is selected in feedback form.

In the following, the feedforward part of the control input is taken to be designed already and reflected in the system dynamical equations. The discussion centers on design of the feedback part of the control input. Because of this, it is convenient to denote the feedback part of the control simply by u and to view the feedforward part as being determined *a priori*. It is also convenient to take u to be small (close to zero) near the nominal operating condition. (Any offset in u is considered part of the feedforward control.)

It is convenient to view the system as depending on a single exogeneous parameter, denoted by μ. For instance, μ can represent the set-point of an aircraft's angle-of-attack, or the power demanded of an electric utility by a customer. In the former example, the control u might denote actuation of the aircraft's elevator angles about the nominal settings. In the latter example, u can represent a control signal in the voltage regulator of a power generator.

Consider, then, a nonlinear system depending on a single parameter μ

$$\dot{x} = f^{\mu}(x, u). \qquad (57.188)$$

Here $x \in \mathbf{R}^n$ is the n-dimensional state vector, u is the feedback part of the control input, and μ is a parameter. Both u and μ are taken to be scalar-valued for simplicity. The dependence of the system equations on x, u, and μ is assumed smooth; i.e., f is jointly continuous and several times differentiable in these variables. This system is thus actually a one-parameter *family* of nonlinear control systems. The parameter μ is viewed as being allowed to change so slowly that its variation can be taken as quasistatic.

Suppose that the nominal operating condition of the system is an equilibrium point $x^0(\mu)$ that depends on the parameter μ. For simplicity of notation, suppose that the state x has been chosen so that this nominal equilibrium point is the origin for the range of values of μ for which it exists ($x^0(\mu) \equiv 0$). Recall that the nominal equilibrium is achieved using feedforward control. Although the process of choosing a feedforward control is not elaborated on here, it is important to emphasize that in general this process aims at securing a particular desired candidate operating condition. The feedforward control thus is expected to result in an acceptable form for the nominal operating condition, but there is no reason to expect that other operating conditions will also behave in a desirable fashion. As the parameter varies, the nominal operating condition may interact with other candidate operating conditions in bifurcations. The design of the feedback part of the control should take into account the other candidate operating conditions in addition to the nominal one.

For simplicity, suppose the control u isn't allowed to introduce additional dynamics. That is, suppose u is required to be a direct state feedback (or static feedback), and denote it by $u = u(x, \mu)$. Note that in general u can depend on the parameter μ. That is, it can be *scheduled*. Since in the discussion above it was assumed that u is small near the nominal operating condition, it is assumed that $u(0, \mu) \equiv 0$ (for the parameter range in which the origin is an equilibrium).

Denote the linearization of Equation 57.188 at $x^0 = 0$, $u = 0$ by

$$\dot{x} = A(\mu)x + b(\mu)u. \qquad (57.189)$$

Here,

$$A(\mu) := \frac{\partial f^{\mu}}{\partial x}(0, 0)$$

and

$$b(\mu) := \frac{\partial f^{\mu}}{\partial u}(0, 0).$$

Consider how control design for the linearized system depends on the parameter μ. Recall the terminology from linear system theory that the pair $(A(\mu), b(\mu))$ is controllable if the system (57.189) is controllable. Recall also that there are several simple tests for controllability, one of which is that the controllability matrix

$$(b(\mu), A(\mu)b(\mu), (A(\mu))^2 b(\mu), \ldots, (A(\mu))^{(n-1)} b(\mu))$$

is of full rank. (In this case this is equivalent to the matrix being nonsingular, since here the controllability matrix is square.)

If μ is such that the pair $(A(\mu), b(\mu))$ is controllable, then a standard linear systems result asserts the existence of a linear feedback $u(x, \mu) = -k(\mu)x$ stabilizing the system. (The associated closed-loop system would be $\dot{x} = (A(\mu) - b(\mu)k(\mu))x$.) Stabilizability tests not requiring controllability also exist, and these are more relevant to the problem at hand. Even more interesting from a practical perspective is the issue of output feedback stabilizability, since not all state variables are accessible for real-time measurement in many systems. As μ is varied over the desired regime of operability, the system (57.189) may lose stability and stabilizability.

Stressed Operation and Break-Down of Linear Techniques

A main motivation for the study of control of bifurcations is the need in some situations to operate a system in a condition for which the stability margin is small and linear (state or output) feedback is ineffective as a means for increasing the stability margin. Such a system is sometimes referred to as being "pushed to its limits", or "heavily loaded". In such situations, the ability of the system to function in a state of increased loading is a measure of system performance. Thus, the link between increased loading and reduced stability margin can be viewed as a performance vs. stability trade-off. Systems operating with a reduced achievable margin of stability may be viewed as being "stressed". This trade-off is not a general fact that can be proved in a rigorous fashion, but has been found to occur in a variety of applications. Examples of this trade-off are given at the end of this subsection.

Consider a system that is weakly damped and for which the available controls cannot compensate with sufficient additional damping. Such a situation may arise for a system for some ranges of parameter values and not for others. Let the *operating envelope* of a system be the possible combinations of system parameters for which system operability is being considered. Linear control system methods lose their effectiveness on that part of the operating envelope for which the operating condition of interest of system (57.189) is either:

1. not stabilizable with linear feedback using available sensors and actuators
2. linearly stabilizable using available sensors and actuators but only with unacceptably high feedback gains
3. vulnerable in the sense that small parameter changes can destroy the operating condition completely (as in a saddle-node bifurcation)

Operation in this part of the desired operating envelope can be referred to by terms such as "stressed operation".

An example of the trade-off noted previously is provided by an electric power system under conditions of heavy loading. At peaks in customer demand for electric power (such as during extremes in weather), the stability margin may become negligible, and nonlinear dynamical behaviors or divergence may arise (e.g., [12]). The divergence, known as voltage collapse, can lead to system blackout. Another example arises in operation of an aeroengine compressor at its peak pressure rise. The increased pressure rise comes at the price of nearness to instability. The unstable modes that can arise are strongly related to flow asymmetry modes that are unstabilizable by linear feedback to the compression system throttle. However, bifurcation control techniques have yielded valuable nonlinear throttle actuation techniques that facilitate operation in these circumstances with reduced risk of stall [25, 16].

57.6.6 Control of Systems Exhibiting Bifurcation Behavior

Most engineering systems are designed to operate with a comfortable margin of stability. This means that disturbances or moderate changes in system parameters are unlikely to result in loss of stability. For example, a jet airplane in straight level flight under autopilot control is designed to have a large stability margin. However, engineering systems with a usually comfortable stability margin may at times be operated at a reduced stability margin. A jet airplane being maneuvered at high angle-of-attack to gain an advantage over an enemy aircraft, for instance, may have a significantly reduced stability margin. If a system operating condition actually loses stability as a parameter (like angle-of-attack) is slowly varied, then generally it is the case that a bifurcation occurs. This means that at least one new candidate operating condition emerges from the nominal one at the point of loss of stability. The purpose of this section is to summarize some results on control of bifurcations, with an emphasis placed on control of local bifurcations. Control of a particular global bifurcation, the period doubling bifurcation, is considered in the next section on control of chaos. This is because control of a period doubling bifurcation can result in quenching of the period doubling route to chaos summarized at the end of section three.

Bifurcation control involves designing a control input for a system to result in a desired modification to the system's bifurcation behavior. In section 57.6.5, the division of control into a feedforward component and a feedback component was discussed. Both components of a control law can be viewed in terms of bifurcation control. The feedforward part of the control sets the equilibrium points of the system, and may influence the stability margin as well. The feedback part of the control has many functions, one of which is to ensure adequate stability of the desired operating condition over the desired operating envelope. Linear feedback is used to ensure an adequate margin of stability over a desired parameter range. Use of linear feedback to "delay" the onset of instability to parameter ranges outside the desired operating range is a common practice in control system design. An example is the gain scheduling technique [26]. Delaying instability modifies the bifurcation diagram of a system. Often, the available control authority does not allow stabilization of the nominal operating condition beyond some critical parameter value. At this value, instability leads to bifurcations of new candidate operating conditions. For simplicity, suppose that a single candidate operating condition is born at the bifurcation point. Another important goal in bifurcation control is to ensure that the bifurcation is *supercritical* (i.e., *safe*) and that the resulting candidate operating condition remains stable and close to the original operating condition for a range of parameter values beyond the critical value. The need for control laws that soften (stabilize) a hard (unstable) bifurcation has been discussed earlier in this chapter. This need is greatest in stressed systems, since in such systems delay of the bifurcation by linear feedback is not viable. A soft bifurcation provides the possibility of an alternative operating condition beyond the regime of operability at the nominal condition.

Both of these goals (delaying and stabilization) basically involve local considerations and can be approached analytically (if a good system model is available). Another goal might entail a reduction in amplitude of any bifurcated solutions over some prespecified parameter range. This goal is generally impossible to work with on a completely analytical basis. It requires extensive numerical study in addition to local analysis near the bifurcation(s).

In the most severe local bifurcations (saddle-node bifurcations), neither the nominal equilibrium point nor any bifurcated solution exists past the bifurcation. Even in such cases, an understanding of bifurcations provides some insight into control design for safe operation. For example, it may be possible to use this understanding to determine (or introduce via added control) a warning signal that becomes more pronounced as the severe bifurcation is approached. This signal would alert the high-level control system (possibly a human operator) that action is necessary to avoid catastrophe.

In this section, generally it is assumed that the feedforward component of the control has been pre-determined, and the goal is to design the feedback component. An exception is the following brief discussion of a real-world example of the use of feedforward control to modify a system's operating condition and its stability margin in the face of large parameter variations. In short, this is an example where feedforward controls are used to successfully *avoid* the possibility of bifurcation. During takeoff and landing of a commercial jet aircraft, one can observe the deployment of movable surfaces on the leading- and trailing-edges of the wings. These movable surfaces, called flaps and slats, or camber changers [13], result in a nonlinear change in the aerodynamics, and, in turn, in an increased lift coefficient [51, 13]. This is needed to allow takeoff and landing at reduced speeds. Use of these surfaces has the drawback of reducing the critical angle-of-attack for stall, resulting in a reduced stability margin. A common method to alleviate this effect is to incorporate other devices, called vortex generators. These are small airfoil-shaped vanes, protruding upward from the wings [13]. The incorporation of vortex generators results in a further modification to the aeodynamics, moving the stall angle-of-attack to a higher value. Use of the camber changers and the vortex generators are examples of feedforward control to modify the operating condition within a part of the aircraft's operating envelope. References [13] and [51] provide further details, as well as diagrams showing how the lift coefficient curve is affected by these devices.

Local Direct State Feedback

To give a flavor of the analytical results available in the design of the feedback component in bifurcation control, consider the nonlinear control system (57.188), repeated here for convenience:

$$\dot{x} = f^{\mu}(x, u). \tag{57.190}$$

Here, u represents the feedback part of the control law; the feedforward part is assumed to be incorporated into the function f. The technique and results of [1, 2] form the basis for the following discussion. Details are not provided since they would require introduction of considerable notation related to multivariate Taylor series. However, an illustrative example is given based on formulas available in [1, 2].

Suppose for simplicity that Equation 57.190 with $u \equiv 0$ possesses an equilibrium at the origin for a parameter range of interest (including the value $\mu = 0$). Moreover, suppose that the origin of Equation 57.190 with the control set to zero undergoes either a subcritical stationary bifurcation or a subcritical Andronov-Hopf bifurcation at the critical parameter value $\mu = 0$. Feedback control laws of the form $u = u(x)$ ("static state feedbacks") are derived in [1, 2] that render the bifurcation supercritical.

For the Andronov-Hopf bifurcation, this is achieved using a formula for the coefficient β_2 in the expansion (57.186) of the characteristic exponent for the bifurcated limit cycle. Smooth nonlinear controls rendering $\beta_2 < 0$ are derived. For the stationary bifurcation, the controlled system is desired to display a supercritical pitchfork bifurcation. This is achieved using formulas for the coefficients β_1 and β_2 in the expansion (57.184) for the eigenvalue of the bifurcated equilibrium determining stability. Supercriticality is insured by determining conditions on $u(x)$ such that $\beta_1 = 0$ and $\beta_2 < 0$.

The following example is meant to illustrate the technique of [2]. The calculations involve use of formulas from [2] for the bifurcation stability coefficients β_1 and β_2 in the analysis of stationary bifurcations. The general formulas are not repeated here.

Consider the one-parameter family of nonlinear control systems

$$\dot{x}_1 = \mu x_1 + x_2 + x_1 x_2^2 + x_1^3, \tag{57.191}$$

$$\dot{x}_2 = -x_2 - x_1 x_2^2 + \mu u + x_1 u. \tag{57.192}$$

Here x_1, x_2 are scalar state variables, and μ is a real-valued parameter. This is meant to represent a system after application of a feedforward control, so that u is to be designed in feedback form. The nominal operating condition is taken to be the origin $(x_1, x_2) = (0, 0)$, which is an equilibrium of (57.191),(57.192) when the control $u = 0$ for all values of the parameter μ.

Local stability analysis at the origin proceeds in the standard manner. The Jacobian matrix $A(\mu)$ of the right side of (57.191),(57.192) is given by

$$A(\mu) = \begin{pmatrix} \mu & 1 \\ 0 & -1 \end{pmatrix}. \tag{57.193}$$

The system eigenvalues are μ and -1. Thus, the origin is stable for $\mu < 0$ but is unstable for $\mu > 0$. The critical value of the bifurcation parameter is therefore $\mu_c = 0$. Since the origin persists as an equilibrium point past the bifurcation, and since the critical eigenvalue is 0 (not an imaginary pair), it is expected that a stationary bifurcation occurs. The stationary bifurcation that occurs for the open-loop system can be studied by solving the pair of algebraic equations

$$0 = \mu x_1 + x_2 + x_1 x_2^2 + x_1^3, \tag{57.194}$$

$$0 = -x_2 - x_1 x_2^2 \tag{57.195}$$

for a nontrivial (i.e., nonzero) equilibrium (x^1, x^2) near the origin for μ near 0. Adding Equation 57.194 to Equation 57.195 gives

$$0 = \mu x_1 + x_1^3. \tag{57.196}$$

Disregarding the origin, this gives two new equilibrium points that exist for $\mu < 0$, namely

$$x(\mu) = \begin{pmatrix} \pm\sqrt{-\mu} \\ 0 \end{pmatrix}. \tag{57.197}$$

Since these bifurcated equilibria occur for parameter values ($\mu < 0$) for which the nominal operating condition is unstable, the bifurcation is a *subcritical* pitchfork bifurcation.

The first issue addressed in the control design is the possibility of using linear feedback to delay the bifurcation to some positive value of μ. This would require stabilization of the origin at $\mu = 0$. Because of the way in which the control enters the system dynamics, however, the system eigenvalues are not affected by linear feedback at the parameter value $\mu = 0$. To see this, simply note that in Equation 57.192, the term μu vanishes when $\mu = 0$, and the remaining impact of the control is through the term $x_1 u$. This latter term would result only in the addition of *quadratic* terms to the right side of Equations 57.191 and 57.192 for any linear feedback u. Hence, the system is an example of a stressed nonlinear system for μ near 0.

Since the pitchfork bifurcation cannot be delayed by linear feedback, next consider the possibility of rendering the pitchfork bifurcation supercritical using nonlinear feedback. This rests on determining how feedback affects the bifurcation stability coefficients β_1 and β_2 for this stationary bifurcation. Once this is known, it is straightforward to seek a feedback that renders $\beta_1 = 0$ and $\beta_2 < 0$. The formulas for β_1 and β_2 derived in [2] simplify for systems with no quadratic terms in the state variables. For such systems, the coefficient β_1 always vanishes, and the calculation of β_2 also simplifies. Since the dynamical equations (57.191) and (57.192) in the example contain no quadratic terms in x, it follows that $\beta_1 = 0$ in the absence of control. Moreover, if the control contains no linear terms, then it will not introduce quadratic terms into the dynamics. Hence, for any smooth feedback $u(x)$ containing no linear terms in x, the bifurcation stability coefficient $\beta_1 = 0$.

As for the bifurcation stability coefficient β_2, the pertinent formula in [2] applied to the open-loop system yields $\beta_2 = 2$. Thus, a subcritical pitchfork bifurcation is predicted for the open-loop system, a fact that was established above using simple algebra. Now let the control consist of a quadratic function of the state and determine conditions under which $\beta_2 < 0$ for the closed-loop system.[5] Thus, consider u to be of the form

$$u(x_1, x_2) = -k_1 x_1^2 - k_2 x_1 x_2 - k_3 x_2^2. \tag{57.198}$$

The formula in [2] yields that β_2 for the closed-loop system is then given by

$$\beta_2 = 2(1 - k_1). \tag{57.199}$$

Thus, to render the pitchfork bifurcation in (57.191),(57.192) supercritical, it suffices to take u to be the simple quadratic function

$$u(x_1, x_2) = -k_1 x_1^2, \tag{57.200}$$

with any gain $k_1 > 1$. In fact, other quadratic terms, as well as any other cubic or higher order terms, can be included along with this term without changing the local result that the bifurcation is rendered supercritical. Additional terms can be useful in improving system behavior as the parameter leaves a local neighborhood of its critical value.

Local Dynamic State Feedback

Use of a static state feedback control law $u = u(x)$ has potential disadvantages in nonlinear control of systems exhibiting bifurcation behavior. To explain this, consider the case of an equilibrium $x^0(\mu)$ as the nominal operating condition. The equilibrium is not translated to the origin to illustrate how it is affected by feedback. In general, a static state feedback

$$u = u(x - x^0(\mu)) \tag{57.201}$$

designed with reference to the nominal equilibrium path $x^0(\mu)$ of Equation 57.190 will affect not only the stability of this equilibrium but also the location and stability of other equilibria. Now suppose that Equation 57.190 is only an approximate model for the physical system of interest. Then the nominal equilibrium branch will also be altered by the feedback. A main disadvantage of such an effect is the wasted control energy that is associated with the forced alteration of the system equilibrium structure. Other disadvantages are that system performance is often degraded by operating at an equilibrium that differs from the one at which the system is designed to operate. Moreover, by influencing the locations of system equilibria, the feedback control is competing with the feedforward part of the control.

For these reasons, dynamic state feedback-type bifurcation control laws have been developed that do not affect the locations

[5] Cubic terms are not included because they would result in quartic terms on the right side of Equations 57.191 and 57.192, while the formula for β_2 in [2] involves only terms up to cubic order in the state.

of system equilibria [22, 23, 50]. The method involves incorporation of filters called "washout filters" into the controller architecture. A washout filter-aided control law preserves all system equilibria, and does so without the need for an accurate system model.

A washout filter is a stable high pass filter with zero static gain [15, p. 474]. The typical transfer function for a washout filter is

$$G(s) \quad = \quad \frac{y_i(s)}{x_i(s)} = \frac{s}{s+d}. \qquad (57.202)$$

where x_i is the input variable to the filter and y_i is the output of the filter. A washout filter produces a nonzero output only during the transient period. Thus, such a filter "washes out" sensed signals that have settled to constant steady-state values. Washout filters occur in control systems for power systems [5, p. 277], [41, Chapter 9] and aircraft [7, 15, 39, 45].

Washout filters are positioned in a control system so that a sensed signal being fed back to an actuator first passes through the washout filter. If, due to parameter drift or intentional parameter variation, the sensed signal has a steady-state value that deviates from the assumed value, the washout filter will give a zero output and the deviation will not propagate. If a direct state feedback were used instead, the steady-state deviation in the sensed signal would result in the control modifying the steady-state values of the other controlled variables. As an example, washout filters are used in aircraft yaw damping control systems to prevent these dampers from "fighting" the pilot in steady turns [39, p. 947].

From a nonlinear systems perspective, the property of washout filters described above translates to achieving equilibrium preservation, i.e., zero steady-state tracking error, in the presence of system uncertainties.

Washout filters can be incorporated into bifurcation control laws for Equation 57.190. This should be done only after the feedforward control has been designed and incorporated *and* part of the feedback control ensuring satisfactory equilibrium point structure has also been designed and incorporated into the dynamics. Otherwise, since washout filters preserve equilibrium points, there will be no possibility of modifying the equilibria. It is assumed below that these two parts of the control have been chosen and implemented.

For each system state variable x_i, $i = 1, \ldots, n$, in Equation 57.190, introduce a washout filter governed by the dynamic equation

$$\dot{z}_i \quad = \quad x_i - d_i z_i \qquad (57.203)$$

along with output equation

$$y_i = x_i - d_i z_i. \qquad (57.204)$$

Here, the d_i are positive parameters (this corresponds to using stable washout filters). Finally, require the control u to depend only on the measured variables y, and that $u(y)$ satisfy $u(0) = 0$.

In this formulation, n washout filters, one for each system state, are present. In fact, the actual number of washout filters needed, and hence also the resulting increase in system order, can usually be taken less than n.

It is straightforward to see that washout filters result in equilibrium preservation and automatic equilibrium (operating point) following. Indeed, since $u(0) = 0$, it is clear that y vanishes at steady-state. Hence, the x subvector of a closed-loop equilibrium point (x, z) agrees exactly with the open-loop equilibrium value of x. Also, since y_i may be re-expressed as

$$\begin{aligned} y_i &= x_i - d_i z_i \\ &= (x_i - x_i^0(\mu)) - d_i(z_i - z_i^0(\mu)), \end{aligned} \qquad (57.205)$$

the control function $u = u(y)$ is guaranteed to center at the correct operating point.

57.6.7 Control of Chaos

Chaotic behavior of a physical system can either be desirable or undesirable, depending on the application. It can be beneficial for many reasons, such as enhanced mixing of chemical reactants, or, as proposed recently [36], as a replacement for random noise as a masking signal in a communication system. Chaos can, on the other hand, entail large amplitude motions and oscillations that might lead to system failure. The control techniques discussed in this section have as their goal the replacement of chaotic behavior by a nonchaotic steady-state behavior. The first technique discussed is that proposed by Ott, Grebogi, and Yorke [33]. The Ott-Grebogi-Yorke (OGY) method sparked significant interest and activity in control of chaos. The second technique discussed is the use of bifurcation control to delay or extinguish the appearance of chaos in a family of systems.

Exploitation of Chaos for Control

Ott, Grebogi, and Yorke [33] proposed an approach to control of chaotic systems that involves use of small controls and exploitation of chaotic behavior. To explain this method, recall from section three that a strange attractor has embedded within itself a "dense" set (infinitely many) of unstable limit cycles. The strange attractor is a candidate operating condition, according to the definition in section two. In the absence of control, it is the actual system operating condition. Suppose that system performance would be significantly improved by operation at one of the unstable limit cycles embedded in the strange attractor. The OGY method replaces the originally chaotic system operation with operation along the selected unstable limit cycle.

Figure 57.20 depicts a strange attractor along with a particular unstable limit cycle embedded in it. It is helpful to keep such a figure in mind in contemplating the OGY method. The goal is to reach a limit cycle such as the one shown in Figure 57.20, or another of some other period or amplitude. Imagine a trajectory that lies on the strange attractor in Figure 57.20, and suppose the desire is to use control to force the trajectory to reach the unstable limit cycle depicted in the figure and to remain there for all subsequent time.

Control design to result in operation at the desired limit cycle is achieved by the following reasoning. First, note that the desired unstable limit cycle is also a candidate operating condition. Next, recall from section three that a trajectory on the strange attractor will, after sufficient time, pass as close as desired to any other point of interest on the attractor. Thus, the trajectory will eventually come close (indeed, arbitrarily close) to the desired limit cycle. Thus, no control effort whatsoever is needed in order for the trajectory to reach the desired limit cycle — chaos guarantees that it will. To maintain the system state on the desired limit cycle, a small stabilizing control signal is applied once the trajectory enters a small neighborhood of the limit cycle. Since the limit cycle is rendered stable by this control, the trajectory will converge to the limit cycle. If noise drives the trajectory out of the neighborhood where control is applied, the trajectory will wander through the strange attractor again until it once again enters the neighborhood and remains there by virtue of the control.

Note that the control obtained by this method is an example of a variable structure control, since it is "on" in one region in state space and "off" in the rest of the space. Also, the particular locally stabilizing control used in the neighborhood of the desired limit cycle has not been discussed in the foregoing. This is because several approaches are possible, among them pole placement [38]. See [34, 35] for further discussion.

Two particularly significant strengths of the OGY technique are:

1. it requires only small controls
2. it can be applied to experimental systems for which no mathematical model is available

The first of these strengths is due to the assumption that operation at an unstable limit cycle embedded in the strange attractor is desirable. If none of the embedded limit cycles provides adequate performance, then a large control could possibly be used to introduce a new desirable candidate operating condition within the strange attractor. This could be followed by a small control possibly designed within the OGY framework. Note that the large control would be a feedforward control, in the terminology used previously in this chapter. For a discussion of the second main strength of the OGY control method mentioned above, see, e.g., [17, 33, 34]. It suffices to note here that a construction known as experimental delay coordinate embedding is one means to implement this control method without *a priori* knowledge of a reliable mathematical model.

Several interesting applications of the OGY control method have been performed, including control of cardiac chaos (see [35]). In [17], a multimode laser was controlled well into its usually unstable regime.

Bifurcation Control of Routes to Chaos

The bifurcation control techniques discussed in section six have direct relevance for issues of control of chaotic behavior of dynamical systems. This is because chaotic behavior often arises as a result of bifurcations, such as through the period doubling route to chaos. The bifurcation control technique discussed earlier in this chapter is model-based. Thus, the control of chaos applications of the technique also require availability of a reliable mathematical model.

Only a few comments are given here on bifurcation control of routes to chaos, since the main tool has already been discussed in section six. The cited references may be consulted for details and examples.

In [50], a thermal convection loop model is considered. The model, which is equivalent to the Lorenz equations mentioned in section four, displays a series of bifurcations leading to chaos. In [50], an Andronov-Hopf bifurcation that occurs in the model is rendered supercritical using *local* dynamic state feedback (of the type discussed in section six). The feedback is designed using local calculations at the equilibrium point of interest. It is found that this simple control law results in elimination of the chaotic behavior in the system. From a practical perspective, this allows operation of the convection loop in a *steady* convective state with a desired velocity and temperature profile.

Feedback control to render supercritical a previously subcritical period doubling bifurcation was studied in [4, 47]. In [4], a discrete-time model is assumed, whereas a continuous-time model is assumed in [47]. The discrete-time model used in [4] takes the form (k is an integer)

$$x(k+1) \quad = \quad f^{\mu}(x(k), u(k)) \qquad (57.206)$$

where $x(k) \in \mathbb{R}^n$ is the state, $u(k)$ is a scalar control input, $\mu \in \mathbb{R}$ is the bifurcation parameter, and the mapping f^{μ} is sufficiently smooth in x, u, and μ. The continuous-time model used in [47] is identical to Equation 57.188.

For discrete-time systems, a limit cycle must have an integer period. A period-1 limit cycle sheds a period-2 limit cycle upon period doubling bifurcation. The simplicity of the discrete-time setting results in explicit formulas for bifurcation stability coefficients and feedback controls [4]. By improving the stability characteristics of a bifurcated period-2 limit cycle, an existing period doubling route to chaos can be extinguished. Moreover, the period-2 limit cycle will then remain close to the period-1 limit cycle for an increased range of parameters.

For continuous-time systems, limit cycles cannot in general be obtained analytically. Thus, [47] employs an approximate analysis technique known as *harmonic balance*. Approximate bifurcation stability coefficients are obtained, and control to delay the onset of period doubling bifurcation or stabilize such a bifurcation is discussed.

The washout filter concept discussed in section six is extended in [4] to discrete-time systems. In [47], an extension of the washout filter concept is used that allows approximate preservation of limit cycles of a certain frequency.

57.6.8 Concluding Remarks

Control of bifurcations and chaos is a developing area with many interesting avenues for research and for application. Some of the tools and ideas that have been used in this area were discussed. Connections among these concepts, and relationships to traditional control ideas, have been emphasized.

57.6.9 Acknowledgment

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Further Reading

Detailed discussions of bifurcation and chaos are available in many excellent books (e.g., [6, 10, 11, 18, 19, 28, 30, 32, 34, 35, 43, 46, 48]). These books also discuss a variety of interesting applications. Many examples of bifurcations in mechanical systems are given in [52]. There are also several journals devoted to bifurcations and chaos. Of particular relevance to engineers are *Nonlinear Dynamics*, the *International Journal of Bifurcation and Chaos*, and the journal *Chaos, Solitons and Fractals*.

The book [45] discusses feedforward control in the context of "trimming" an aircraft using its nonlinear equations of motion and the available controls. Bifurcation and chaos in flight dynamics are discussed in [8]. Lucid explanations on specific uses of washout filters in aircraft control systems are given in [15, pp. 474–475], [7, pp. 144–146], [39, pp. 946–948 and pp. 1087–1095], and [45, pp. 243–246 and p. 276]. The book [31] discussed applications of nonlinear dynamics in biology and population dynamics.

Computational issues related to bifurcation analysis are addressed in [43]. Classification of bifurcations as safe or dangerous is discussed in [32, 43, 48, 49].

The edited book [14] contains interesting articles on research needs in applications of bifurcations and chaos.

The article [3] contains a large number of references on bifurcation control, related work on stabilization, and applications of these techniques. The review papers [9, 44] address control of chaos methods. In particular, [44] includes a discussion of use of sensitive dependence on initial conditions to direct trajectories to targets. The book [35] includes articles on control of chaos, detection of chaos in time series, chaotic data analysis, and potential applications of chaos in communication systems. The book [32] also contains discussions of control of bifurcations and chaos, and of analysis of chaotic data.

asily determined (8) from the sequence of oints 164 through 167 in Fig. 2A. Lastly, is the sensitivity of points near the fixed oint to an advance in the timing δp, hich we approximate with the change in ne fixed point with respect to δp:

$$g \approx \frac{\delta \xi_F}{\delta p} \quad (3)$$

he constant C is inversely proportional to ne projection of this change onto the nstable manifold.

We must stress that Eqs. 1 to 3 are lentical to the OGY control equations except that the systemwide parameter p used in ne OGY method is replaced in our method y the variable that controls the system erturbation. Thus our system perturbation p (which replaces the change, δp, in the OGY systemwide parameter) represents the mount of time we must shorten an anticiated natural beat (through the introduction f a stimulus) to force the state point onto ne stable manifold.

Our proportional perturbation feedback ontrol consisted of two parts: a learning hase and an intervention phase. In the earning phase the computer monitored the nterbeat intervals until it determined the pproximate locations of the unstable fixed oint and the stable and unstable manifolds 9, 10). The application of Eqs. 1 and 2 uring the intervention phase was then traightforward once the geometry of the ocal map around the fixed point had been etermined and the quantity g had been ound. The interbeat interval was directly nanipulated by shortening it with an electrically stimulated beat. The advance in the iming of the next interbeat interval is the erturbation δp. The sensitivity of the state oint to changes in p was determined experimentally by noting the change in the nterbeat interval in response to a single lectrical stimulus when the state point was iear the fixed point (Eq. 3).

Chaos control with this approach was omplicated by the fact that the intervenion was, of necessity, unidirectional; that s, by delivering an electrical stimulus beore the next spontaneous beat, the interbeat interval could be shortened but it ould not directly be lengthened. This is ecause a stimulus that elicits a beat from he heart must, by definition, shorten the nterbeat interval between the previous pontaneous beat and the beat elicited by he stimulus. Although a stimulus that does not elicit a beat may prolong the interbeat nterval by electrotonic effects, in the inact heart this is highly dependent both on he precise location of the pacing site in elation to the site of origin of the arrhythmia and on the history of preceding interbeat intervals. Thus such a stimulus is highly unstable and unpredictable during an aperiodic rhythm and therefore is unsuitable as a cardiac pacing strategy.

The learning phase typically lasted from 5 to 60 seconds, after which the computer waited for the system to make a close approach to the unstable fixed point (indi-

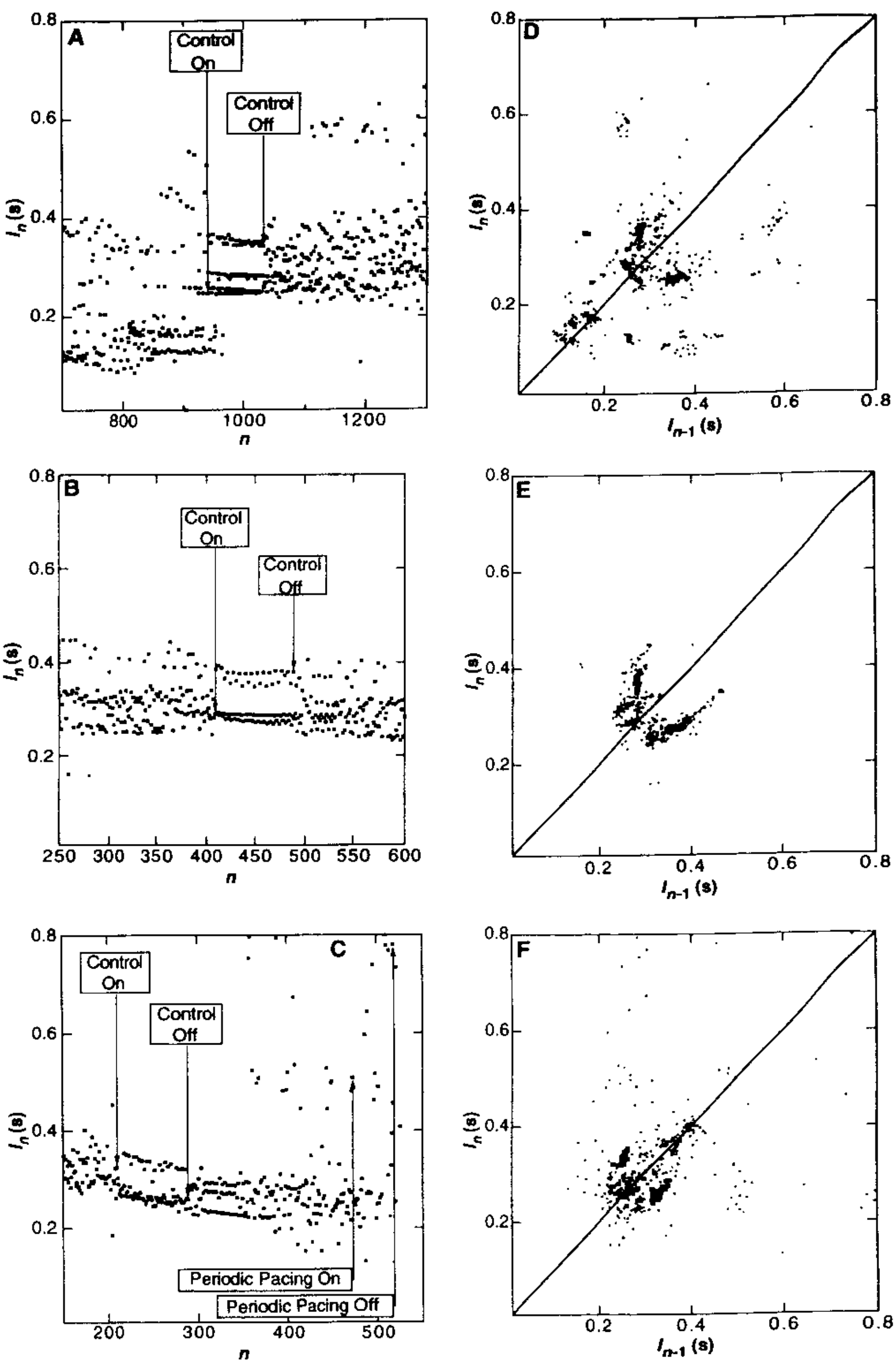

Fig. 3. (**A**, **B**, and **C**) Interbeat interval I_n versus beat number n during the chaotic phase of the ouabain-epinephrine-induced arrhythmia in three typical hearts. The region of chaos control is indicated. In (C) a region during which periodic pacing was applied is also indicated. Note that periodic pacing failed to control the arrhythmia. (**D**, **E**, and **F**) The corresponding Poincaré maps for (A), (B), and (C), respectively. The small points represent interbeat intervals during the uncontrolled arrhythmia, while the larger points represent interbeat intervals during the chaos control. In the (A) and (D) data set, there were several losses of control that occurred early in the control sequence (indicated by the lower points in the early part of the control region). These were always immediately followed by reacquisition of the chaos control.

cated by the point α within the circle in Fig. 2B.) The next point would normally fall further out along the unstable manifold (as well as on the opposite side of the stable manifold) as indicated by point β. However at this point the computer intervened by injecting an electrical stimulus early enough so that this point actually occurred at β', lying directly below β and, by construction, near the stable manifold. Because the system now lay close to the stable manifold, ideally the subsequent beat would tend to move closer to the fixed point along the stable manifold, as indicated by the point γ. Thus the state point would be confined to the region near the unstable fixed point, thereby regularizing the arrhythmia. However, in actual practice this degree of accuracy was not typically obtained. Figure 2C illustrates the usual result. When β' did not fall precisely on the stable manifold, γ often was not extremely close to the fixed point (fell outside the circle) but still lay fairly close to the stable manifold (since β' was close to the stable manifold). Yet γ lay closer to the stable manifold than point α did. Thus the next point fell within the circle in the vicinity of α and restarted the cycle (as a new point α). As these patterns repeated, a period 3 beating resulted. In this manner the chaotic (arrhythmic) beating was made periodic by only intermittent stimuli. The subsequent stabilized period 3 motion represents the combined dynamics of the cardiac tissue, sensor, and computer control. We emphasize that this approach did not require any theoretical model of the heart; all of the quantities needed were calculated in real time from the data. It should also be noted that a control stimulus has only a transient effect on the cardiac preparation (which may influence the timing of several subsequent beats) but otherwise does not affect the preparation in any permanent fashion.

Results of chaos control. Using PPF, control of the chaotic phase of the ouabain-epinephrine–induced arrhythmia was attempted in 11 separate experimental runs and was successful in 8. Figure 3 shows the results of three successful cases. When the arrhythmia became chaotic, the chaos control program was activated. Our criteria for chaos were that the Poincaré map exhibit stable and unstable manifolds with a flip saddle along the linear part of the unstable manifold (*18*). Specifically we looked for the state point to walk toward the unstable fixed point along one direction (thereby determining the stable manifold) and then to walk out from the unstable fixed point along a clearly different direction (the unstable manifold). The program then chose and delivered electrical stimuli as described above. In order to prove that we had achieved and maintained control of the chaos [defined as a clear conversion of a chaotic sequence to a periodic one with a low-integer (2 to 3) period], we turned off the chaos control program and consistently saw a return to chaotic behavior, sometimes preceded by transient complex periodicities, as in Fig. 3C.

Several observations should be made about the pattern of the stimuli delivered by the chaos control program. First, these stimuli did not simply overdrive the heart. Stimuli were delivered sporadically, not on every beat and never more than once in every three beats on average. The pattern of stimuli was initially erratic and aperiodic but soon became approximately periodic as the arrhythmia was controlled into a nearly periodic rhythm. For example, in Fig. 3, the chaos control program rapidly converted the aperiodic behavior of the arrhythmia to a period 3 rhythm. In contrast, periodic pacing, in which stimuli were delivered at a fixed rate, was never effective at restoring a periodic rhythm and often made the aperiodicity more marked. This is illustrated in Fig. 3C, in which the chaos control program converted chaotic behavior to an approximate period 3, whereas periodic pacing at a rate of 75 beats per minute (nearly identical to the time-averaged rate at which stimuli were delivered during chaos control) had no effect. Irregular pacing was similarly ineffective at converting chaotic to periodic behavior, as illustrated in Fig. 3A. In this case chaotic behavior was converted to a period 3 by the chaos control program, but the rhythm quickly became aperiodic again when the parameters of the algorithm were modified by arbitrarily changing C (Eqs. 1 and 2) to eliminate effective chaos control (at the arrow labeled "control off").

In two cases, one of which is shown in Fig. 3A, chaos control had the additional effect of eliminating the shortest interbeat intervals, hence reducing the average rate of the tachycardia. Without an understanding of the chaotic nature of the system, it would seem paradoxical that an intervention that only shortened the interbeat intervals lengthened the average interval. However, because very long interbeat intervals tend to be followed by very short interbeat intervals (a consequence of the properties of a flip saddle), elimination of the very long intervals also tends to eliminate very short intervals. In cases in which very short intervals predominate during the arrhythmia, their elimination during chaos control will tend to lengthen the average interbeat interval.

Future possibilities for chaos control. In the cases where chaos was successfully controlled, the chaotic pattern of the arrhythmia was converted to a low-order periodic pattern. However, as discussed previously for Fig. 2, B and C, we did n observe any period 1 patterns during cha control. There is no a priori reason why period 1 pattern cannot be achieved; on period 3 chaos control is established, it possible that a refinement of the contr parameters could "walk" the points alor the stable manifold, in essence spiraling i toward the fixed point. This could be in plemented in the future either as a secon learning phase in which the details of th Poincaré map are learned to higher accura cy than during the first attempts at chac control or as an adaptive algorithm tha responds to the state of the heart after eacl beat. An adaptive algorithm might als allow chaos control to adapt to changin physiological conditions such as variation in autonomic tone or other extracardia factors. It may even be possible to use chao control to walk the fixed point upwarc along the diagonal (*19*), thereby reducin the rate of the tachycardia.

The relevance of the ouabain-inducec arrhythmia model used in our study tc clinically important arrhythmias in humans is not established. Although toxicity from cardiac glycosides is a common cause of human arrhythmias, only in lethal doses would it be likely to produce the severe aperiodic arrhythmias observed in our study. However many clinically important rapid cardiac arrhythmias are aperiodic, such as atrial and ventricular fibrillation, polymorphic ventricular tachycardia, and multifocal atrial tachycardia. In cases where aperiodic arrhythmias are examples of deterministic chaos, it is conceivable that a chaos control strategy, perhaps implemented by a "smart" pacemaker, could be used to restore the cardiac rhythm to normal. In this context it is encouraging to note that in several instances in which chaos control was achieved in the ouabain-induced arrhythmia model, the average heart rate decreased as a result of the elimination of very short interbeat intervals (Fig. 3). It remains a challenge to determine whether this chaos control strategy in in vitro cardiac ventricle can be successfully applied to the in vivo heart.

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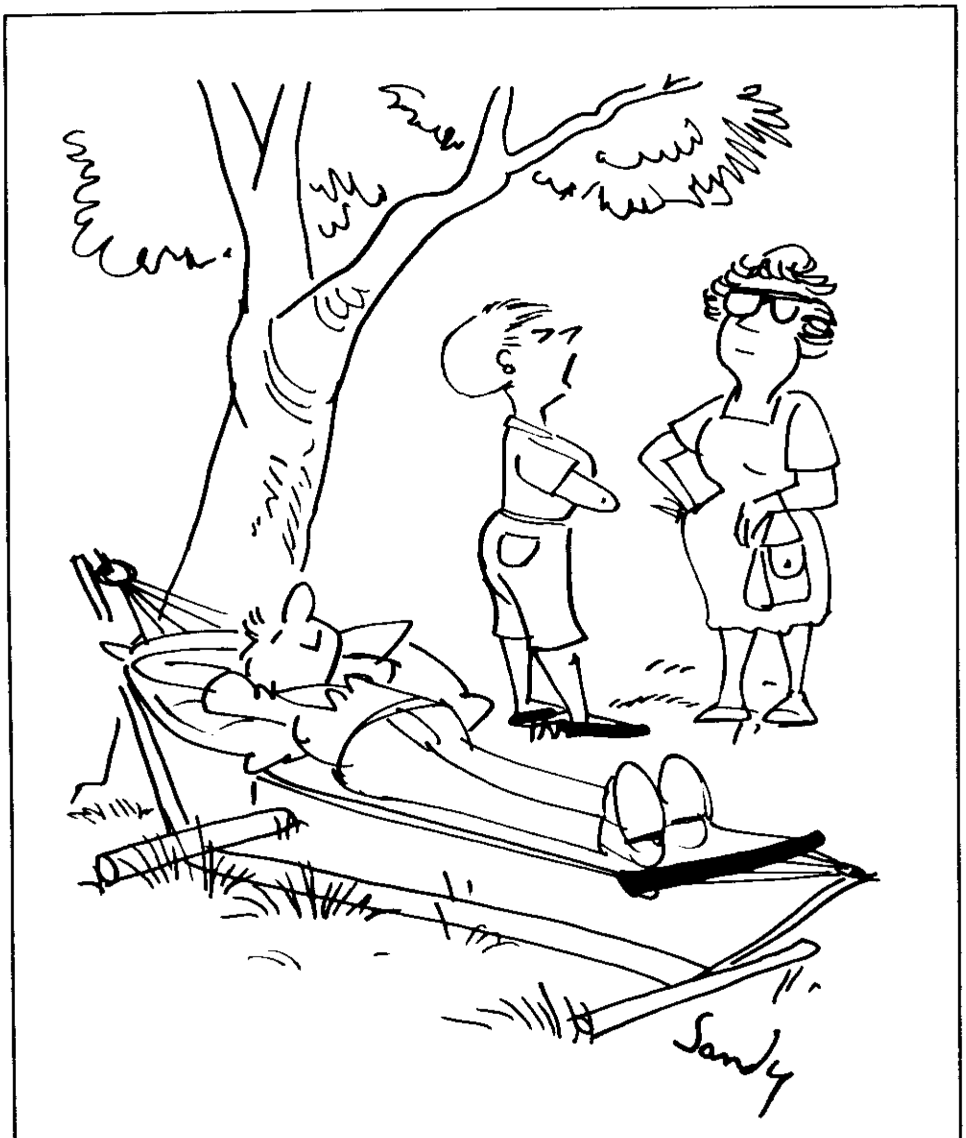


"Thom wants to prove *Newton's law— 'A body at rest, tends to remain at rest'."*

Mastering Chaos

It is now possible to control some systems that behave chaotically. Engineers can use chaos to stabilize lasers, electronic circuits and even the hearts of animals

by William L. Ditto and Louis M. Pecora

What good is chaos? Some would say it is unreliable, uncontrollable and therefore unusable. Indeed, no one can ever predict exactly how a chaotic system will behave over long periods. For that reason, engineers have typically dealt with chaos in just one way: they have avoided it. We find that strategy somewhat shortsighted. Within the past few years we and our colleagues have demonstrated that chaos is manageable, exploitable and even invaluable.

Chaos has already been applied to increase the power of lasers, synchronize the output of electronic circuits, control oscillations in chemical reactions, stabilize the erratic beat of unhealthy animal hearts and encode electronic messages for secure communications. We anticipate that in the near future engineers will no longer shun chaos but will embrace it.

There are at least two reasons why chaos is so useful. First, the behavior of a chaotic system is a collection of many orderly behaviors, none of which dominates under ordinary circumstances. In recent years, investigators have shown that by perturbing a chaotic system in the right way, they can encourage the system to follow one of its many regular behaviors. Chaotic systems are unusually flexible because they can rapidly switch among many different behaviors.

Second, although chaos is unpredictable, it is deterministic. If two nearly identical chaotic systems of the appropriate type are impelled, or driven, by the same signal, they will produce the same output, even though no one can say what that output might be [see "The Amateur Scientist," page 120]. This phenomenon has already made possible a variety of interesting technologies for communications.

For more than a century, chaos has been studied almost exclusively by a few theoreticians, and they must be credited for developing some of the concepts on which all applications are based. Most natural systems are nonlinear: a change in behavior is not a simple function of a change in conditions. Chaos is one type of nonlinear behavior. The distinguishing feature of chaotic systems is that they exhibit a sensitivity to initial conditions. To be more specific, if two chaotic systems that are nearly identical are in two slightly different states, they will rapidly evolve toward very different states.

To the casual observer, chaotic systems appear to behave in a random fashion. Yet close examination shows that they have an underlying order. To visualize dynamics in any system, the Irish-born physicist William Hamilton and the German mathematician Karl Jacobi and their contemporaries devised, more than 150 years ago, one of the fundamental concepts necessary for understanding nonlinear dynamics: the notion of state space.

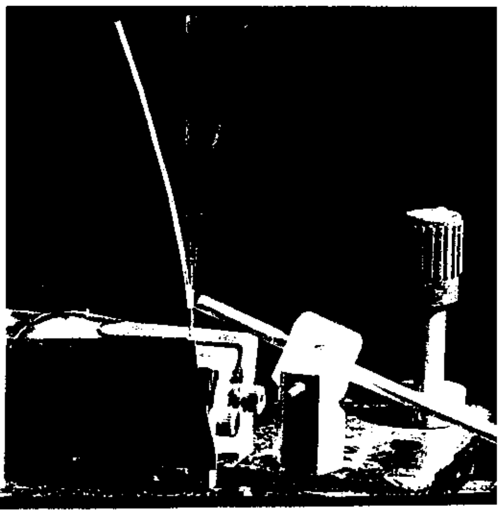
Any chaotic system that can be described by a mathematical equation includes two kinds of variables: dynamic and static. Dynamic variables are the fundamental quantities that are changing all the time. For a chaotic mechanism, the dynamic variables might be the position of a moving part and its velocity. Static variables, which might also be called parameters, are set at some point but then are never changed. The static variable of a chaotic mechanism might be the length of some part or the speed of a motor.

State space is essentially a graph in which each axis is associated with one dynamic variable. A point in state space represents the state of the system at a given time. As the system changes, it moves from point to point in state space, defining a trajectory, or curve. This trajectory represents the history of the dynamic system.

Chaotic systems have complicated trajectories in state space. In contrast, linear systems have simple trajectories, such as loops [*see box on page 80*]. Yet

WILLIAM L. DITTO and LOUIS M. PECORA have pioneered methods for controlling the chaotic behavior of mechanical, electrical and biological systems. Ditto is assistant professor of physics at the Georgia Institute of Technology. He won the Office of Naval Research Young Investigator Award. Ditto and his collaborators were the first to control chaos in an experimental system. Since 1977 Pecora has been a research physicist at the U.S. Naval Research Laboratory and now heads a program in nonlinear dynamics in solid-state systems there. Pecora is in the process of patenting four devices that exploit chaos.

METALLIC RIBBON sways chaotically in a magnetic field whose strength fluctuates periodically. Recent advances make it possible to control such chaotic motions. The ribbon is made of a material whose stiffness depends on the strength of the field. The magnetic field changes at a rate of around one cycle per second. A strobe light flashes at the same frequency so that long-exposure photography captures the irregularities in the motion of the ribbon. A computer system analyzes the movement of the ribbon using a scheme developed by theorists at the University of Maryland. By making slight changes to the field, the computer system can transform the chaotic motions (*right*) to periodic oscillations of almost any frequency (*below*).

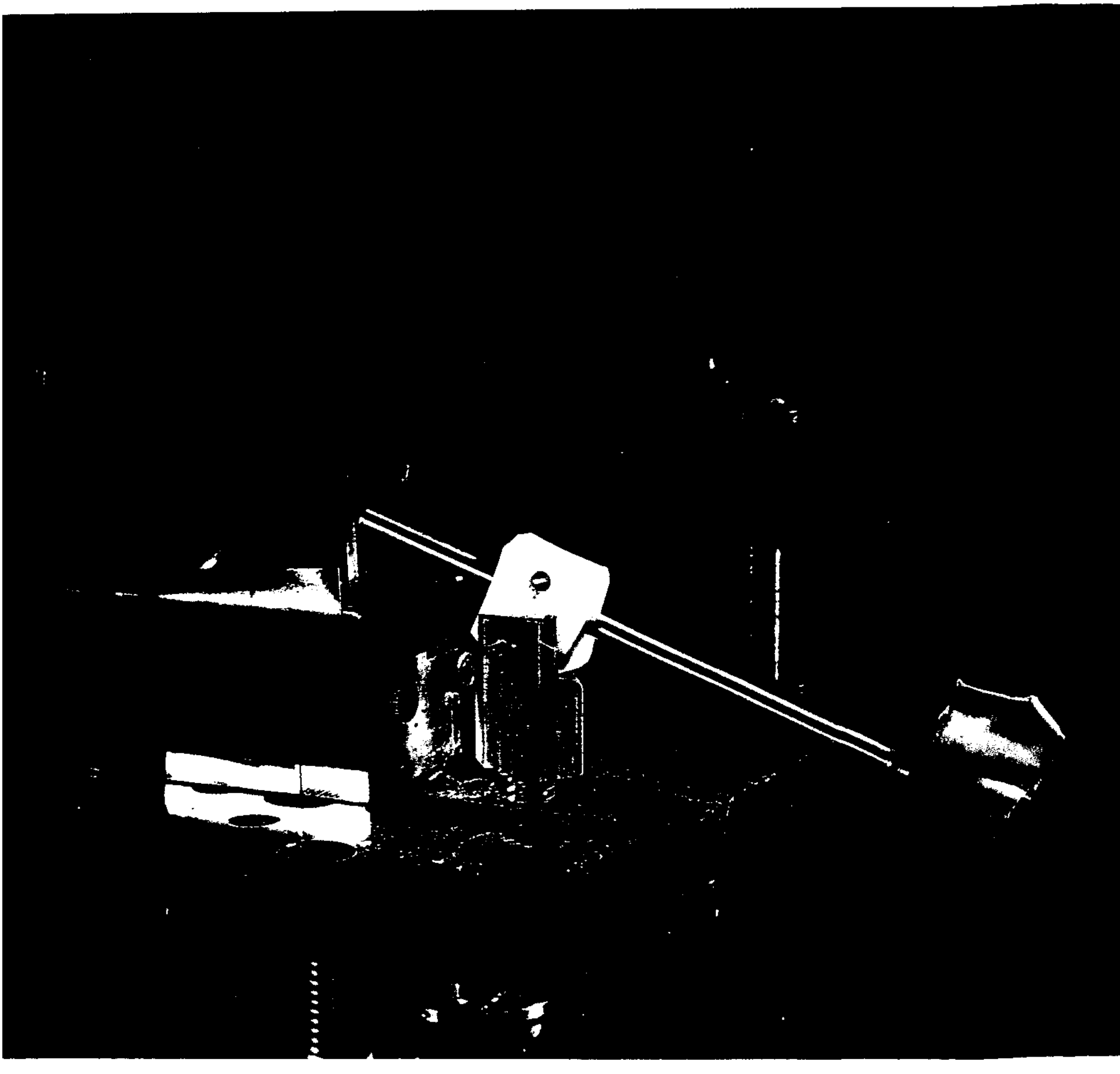


the trajectory of a chaotic system is not random; it passes through certain regions of state space while avoiding others. The trajectory is drawn toward a so-called chaotic attractor, which in some sense is the very essence of a chaotic system. The chaotic attractor is the manifestation of the fixed parameters and equations that determine the values of the dynamic variables.

So if one measures the trajectory of a chaotic system, one cannot predict where it will be on the attractor at some point in the distant future. The chaotic attractor, on the other hand, remains the same no matter when one measures it. Once researchers have obtained information about the chaotic attractor of a system, they can begin to use chaos to their advantage.

In 1989 one of us (Pecora) discovered that a chaotic system could be built in such a way that its parts would act in perfect synchrony. As is well known, isolated chaotic systems cannot synchronize. If one could construct two chaotic systems that were virtually identical but separate, they would quickly fall out of step with each other because any slight difference between the two systems would be magnified. In some cases, however, the parts of a system can be arranged so that they exhibit identical chaotic behavior. In fact, these parts can be located at great distances from one another, making it possible to use synchronized chaos for communications.

To build a chaotic system whose parts are synchronous, one needs to understand the concept of stability. A system is stable if, when perturbed somewhat, its trajectory through state space changes only a little from what it might have been otherwise. The Russian mathematician Aleksandr M. Lyapunov realized that a single number could be used to represent the change caused by a perturbation. He divided the size of the perturbation at one instant in time by its size a moment before. He then performed the same computation at various intervals and averaged the results.



This quantity, now known as the Lyapunov multiplier, describes how much, on average, a perturbation will change. If the Lyapunov multiplier is less than one, the perturbations die out, and the system is stable. If the multiplier is greater than one, however, the disturbances grow, and the system is unstable. All chaotic systems have a Lyapunov multiplier greater than one and so are always unstable. This is the root of the unpredictability of chaos.

The parts of a chaotic system must be stable if they are to be synchronized. This does not mean that the entire system cannot be chaotic. If two similar, stable parts are driven by the same chaotic signal, they will both seem to exhibit chaotic behavior, but they will suppress rather than magnify any differences between them, thereby creating an opportunity for synchronized chaos.

One design developed by Pecora and Thomas L. Carroll of the U.S. Naval Research Laboratory uses a subsystem of an ordinary chaotic system as the synchronizing mechanism. They first distinguished the synchronizing subsystem from other supporting parts and then duplicated the synchronizing subsystems [*see illustrations on opposite page*]. The supporting subsystem then supplied a driving signal for the original synchronizing subsystem and its duplicate.

If the synchronizing subsystems have Lyapunov multipliers that are less than one, that is, if these subsystems are stable, they will behave chaotically but will be in complete synchrony. The stability of the subsystems guarantees that any small perturbations will be damped out, and therefore the synchronizing subsystems will react to the signals from the supporting subsystem in practically the same way, no matter how complex the signals are.

To demonstrate synchronized chaos initially, Pecora created a computer simulation based on the chaotic Lorenz system, which is named after American meteorologist Edward N. Lorenz, who in 1963 discovered chaotic behavior in a computer study of the weather. The Lorenz system has three dynamic variables, and consequently the state-space picture of such systems is three-dimensional. Plotting the trajectory of the Lorenz system in state space reveals what is known as the Lorenz chaotic attractor [see "Chaos," by James P. Crutchfield, J. Doyne Farmer, Norman H. Packard and Robert S. Shaw; SCIENTIFIC AMERICAN, December 1986].

For the computer simulation, Pecora started with the three dynamic variables of the Lorenz system and chose one of them as the driving signal. The subsystem consisting of the two remaining dynamic variables was then duplicated. Although the subsystem and its duplicate were initially set up to produce different outputs, they quickly converged, generating chaotic signals in synchrony. In state space the two subsystems began at different points, caught up to each other and waltzed about on their chaotic attractors in a synchronized ballet.

Until this discovery, scientists had no reason to believe that the stability of a subsystem could be independent of the stability of the rest of the system. Nor

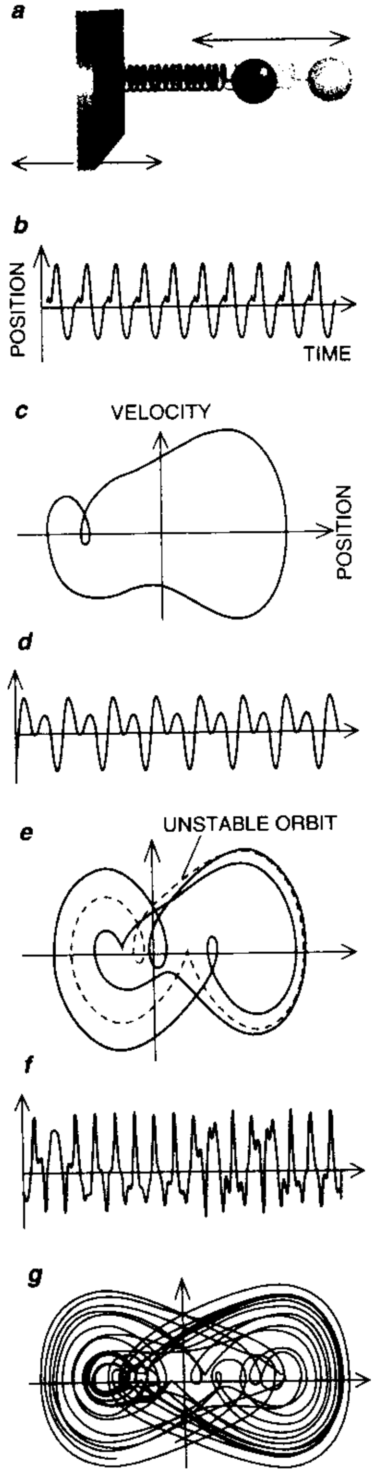
Representing Chaos

Conventionally, the motion of an object is represented as a "time series," a graph showing the change in position over time. Chaotic motions, however, can often be more conveniently visualized in "state space," a plot of the history of the changing variables, which are typically the position and velocity of the object. Consider, for example, the motion of the ball in the mechanism shown at the right (*a*). The ball is attached to a spring that gets stiffer as it is stretched or compressed. (In technical terms, the spring delivers a nonlinear restoring force.) As the board moves cyclically back and forth, the spring is pushed and pulled and causes the ball to move. Hence, the movement of the ball ultimately depends on the force with which the spring is pushed.

If the force is weak, the ball moves in a simple trajectory, which repeats with each cycle of the force from the board (*b*). The state-space path (*c*) shows all the information about the motion of the ball at each instant of time. Because the motion is periodic, the path will retrace itself with each board cycle. The state space reveals a so-called period-one attractor. The same kind of behavior would be observed if the ball were attached to an ordinary (linear) spring.

The interesting behavior occurs as the driving force on the spring is increased. At a certain point, the ball can be made to move back and forth in a more complicated way for each oscillation of the board. Hence, the time series and state-space path change (*d*, *e*). Specifically, the previous period-one orbit becomes unstable, and the system produces a period-two attractor, that is, it takes two board cycles before the path retraces itself.

If the force on the spring increases past some threshold, the ball moves chaotically. No pattern is apparent in the time series (*f*). The state space, on the other hand, reveals a chaotic attractor (*g*): the state-space path never retraces itself, yet it occupies only certain regions of state space. Indeed, a chaotic attractor can be considered a combination of unstable periodic orbits.



had anyone thought that a nonlinear system could be stable when driven with a chaotic signal.

Stability depends not only on the properties of the subsystem itself but on the driving signal. A subsystem may be stable when driven by one type of chaotic signal but not when driven by another. The trick is to find those subsystems that react to a chaotic signal in a stable way. In some cases, the stability of a subsystem can be estimated using a mathematical model, but in general such predictions are difficult.

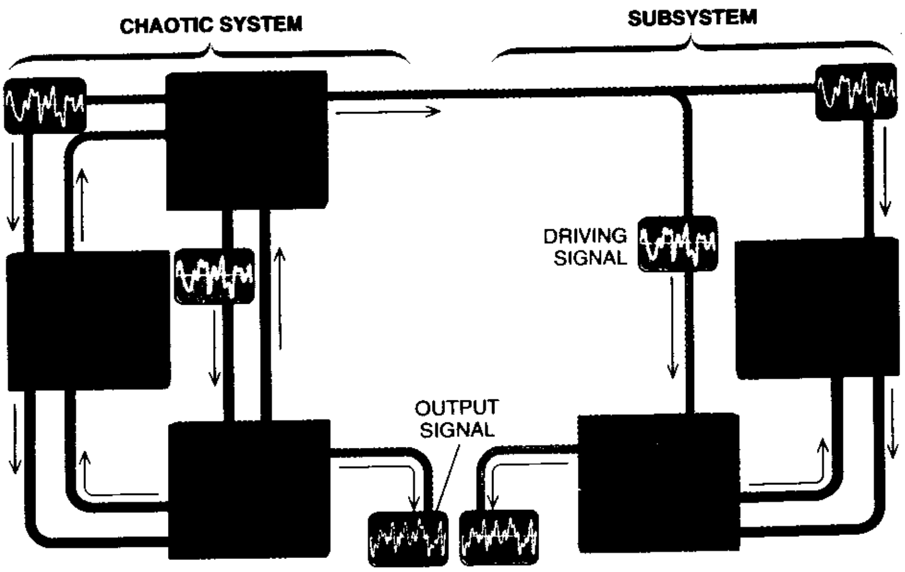
Investigators have only begun to experiment with synchronized chaos. In 1989 Carroll built the first synchronized chaos circuit. By duplicating part of a circuit that exhibited chaotic behavior, he created a circuit in which the subsystem and its duplicate were driven by the same chaotic signal. These two subsystems generated voltages that fluctuated in a truly chaotic fashion but were always in step with each other.

Carroll realized that his synchronized chaos circuit might be used for a private communications system. For example, imagine that Bill wants to send a secret message to Al. Bill has a device that generates a chaotic drive signal, and he has a subsystem that reacts to the signal in a stable way. Al has a copy of the subsystem. Bill translates his secret message into an electronic signal and combines it with the chaotic output of his subsystem. Bill then transmits the encoded message as well as the drive signal.

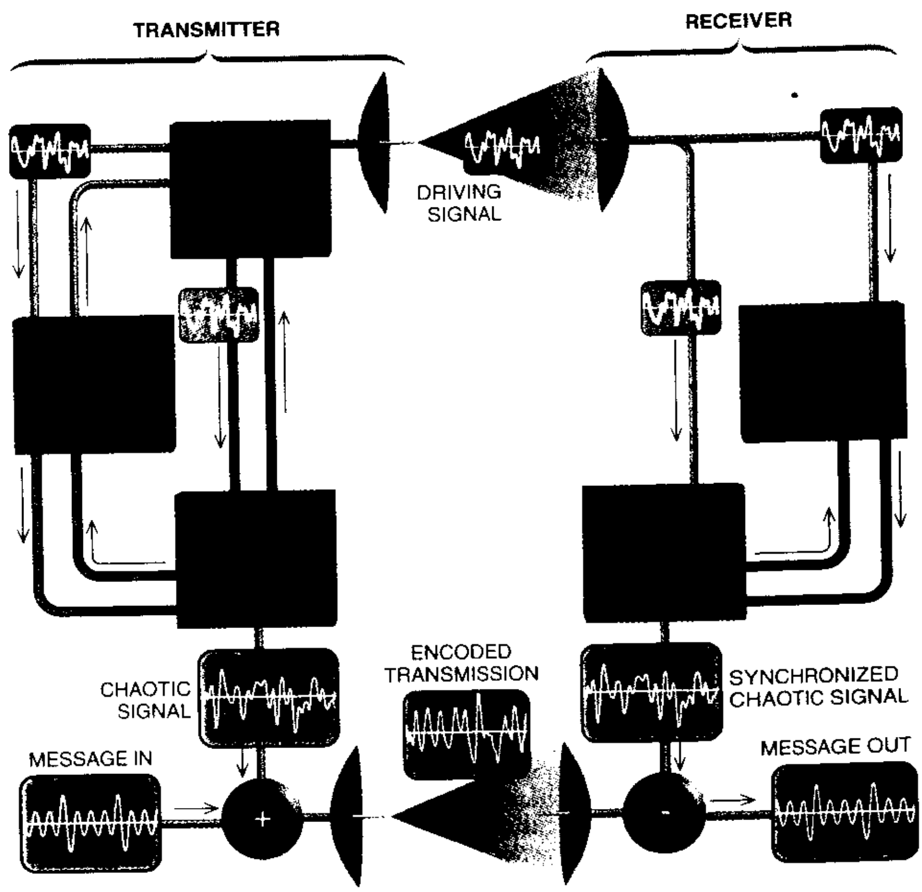
Anyone who intercepts these signals will detect only chaotic noise and will be unable to extract any information (unless, of course, he or she manages to get a copy of Bill's or Al's subsystem). When Al receives the drive signal, he sends it through his subsystem, which reproduces the chaotic output of Bill's subsystem. Al can then subtract this output from the encoded information signal to recover the secret message.

It is rather easy, as Carroll has demonstrated, to recover signals buried in chaos. To build a practical, secure communications system, however, engineers will probably need to develop sophisticated schemes in which the encoding process involves more than just adding chaos to message signals.

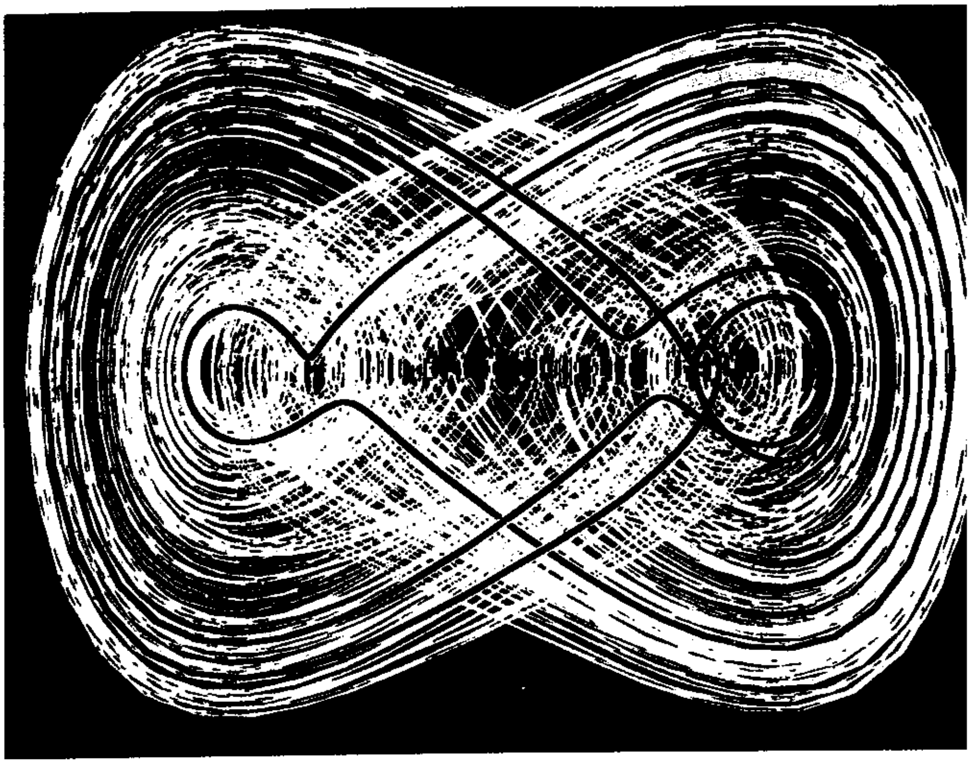
Researchers at the Massachusetts Institute of Technology, Washington State University and the University of California at Berkeley are building new combinations of chaotic subsystems for signal processing and communications. We have also continued our work and found a chaotic system that performs the same operations as a phase-locked



CHAOTIC SIGNALS produced by a system and a subsystem can be generated in complete synchrony. The important feature of the system is that the blue and green components are stable when driven by the chaotic signals from the red part. In other words, small changes in the initial settings of the blue and green parts have little or no effect on the behavior they eventually settle into. The subsystem consists of duplicates of the blue and green components of the system. Although the red, blue and green parts of the system send signals to one another, there is no feedback between the red component and the subsystem.



PRIVATE COMMUNICATIONS can be achieved using the scheme for synchronized chaos [*see upper illustration on this page*]. The transmitter adds a chaotic signal to a message signal; it then broadcasts the encoded signal and a drive signal. The drive signal is sent to two components in the receiver, causing them to generate a chaotic signal in synchrony with the signal produced in the transmitter. The chaotic signal is then subtracted from the encoded signal to yield the original information.



CHAOTIC ATTRACTOR consists of many periodic orbits—for example, the period-one orbit (*red*) and the period-two orbit (*blue*). This attractor represents a system whose velocity and position change along a single direction. One axis represents position, and the other is velocity. Attractors may be multidimensional because systems can have many different state-space variables, that is, positions and velocities that vary in three dimensions.

loop—a device that, among other things, allows an FM radio receiver to track the changes in the transmitted signal.

Carroll and Pecora had such success with using chaotic signals to drive stable parts of chaotic systems that they suspected there might be an advantage to employing chaotic signals to drive stable periodic systems. Their intuition led them to another series of promising applications. Imagine two systems that are driven by the same periodic signal and have the same nonchaotic period-two attractor. The term "period two" means that in two periods of the driving signal, the system travels around the attractor once. If two such systems are driven with the same signal but are initially at opposite ends of the attractor, they will cycle around the attractor and never catch up to each other. In other words, if the two systems start out of phase, they will remain out of phase forever.

By changing the periodic driving signal to a certain type of chaotic signal, workers have recently discovered that two systems can be coaxed to operate in phase. Yet not all types of chaotic signals will perform this task. Some such signals may cause the system themselves to behave chaotically, thereby eliminating any chance of getting them in phase. Engineers must therefore figure out what type of chaotic signal to use for each kind of system.

A good first choice is the signal produced by a system originally invented by Otto E. Rössler. The Rössler signal is nearly periodic and resembles a sine wave whose amplitude and wavelength have been randomized somewhat from one cycle to the next. Such signals, which have come to be known as pseudoperiodic, can be made, in general, simply by taking a periodic signal and adding chaos to it. If several identical systems whose attractors are all period two or greater are driven by the appropriate pseudoperiodic signal, they will all get in step.

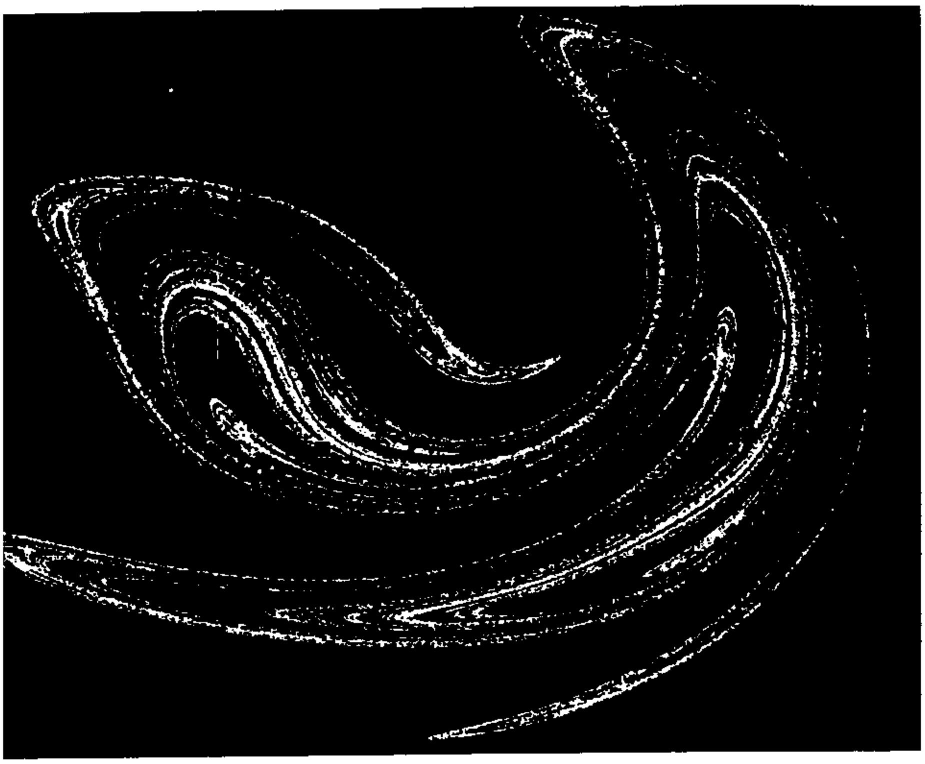
This application was first demonstrated by Carroll in 1990. He built a set of electronic circuits that had period-two attractors and drove them with one of several pseudoperiodic signals, including the Rössler type. In each case, he found he was able to solve the out-of-step problem. Yet the circuits did get out of phase for short periods. That occurred because it is impossible to build several circuits that are exactly the same. Nevertheless, it is quite feasible to keep the circuits in phase 90 percent of the time.

Whereas some investigators have searched for uses of chaos, others have sought to control it. One key to this effort lies in the realization that a chaotic attractor is an infinite collection of unstable periodic behaviors. The easiest way to illustrate this point is to consider a system consisting of a weight, a "nonlinear" spring and a motor. One end of the spring hangs from the motor; the other is attached to the weight. The motor lifts the spring up and down with a force that can be adjusted. The dynamic variables of the system are the position of the weight and its velocity, and these define the state space. If the driving force of the motor is weak, the weight will move up and down once for each cycle of the motor. At the same time, the weight speeds up and slows down, and therefore the trajectory in state space is a single loop, or what is technically called a period-one orbit.

If the driving force of the motor is increased somewhat, the period-one orbit becomes an unstable behavior, and the weight will move up and down once for every two cycles of the motor. In state space the trajectory is now a double loop, or period-two orbit. If the driving force is increased again, the period-two orbit becomes unstable, and a period-four orbit will emerge. Indeed, if the driving force is sufficiently strong, all periodic orbits become unstable, and a chaotic attractor will appear. In a rigorous sense, the chaotic attractor is an ensemble of unstable periodic orbits.

Four years ago Edward Ott, Celso Grebogi and James A. Yorke of the University of Maryland developed a scheme in which a chaotic system can be encouraged to follow one particular unstable orbit through state space. The scheme can therefore take advantage of the vast array of possible behaviors that make up the chaotic system.

The method devised by Ott, Grebogi and Yorke (referred to as OGY) is conceptually straightforward. To start, one obtains information about the chaotic system by analyzing a slice of the chaotic attractor. After the information about this so-called Poincaré section has been gathered, one allows the system to run and waits until it comes near a desired periodic orbit in the section. Next the system is encouraged to remain on that orbit by perturbing the appropriate parameter. One strength of this method is that it does not require a detailed model of the chaotic system but only some information about the Poincaré section. It is for this reason that the method has been so successful in controlling a wide variety of chaotic systems.



POINCARÉ SECTION is, more or less, a perpendicular slice through a chaotic attractor, in this case, a three-dimensional version of the attractor shown on the opposite page. The points of the Poincaré section represent different unstable periodic orbits. The period-one orbit is a single point (*red*), and the period-two orbit is two points (*blue*). The determination of the Poincaré section is a key step in controlling chaos.

The practical difficulties of the OGY method involve getting the Poincaré section and then calculating the control perturbations. One of the simplest ways to obtain a Poincaré section is to measure, at some regular interval, the position of the state-space trajectory of the system. For example, the measurements could be made at the end of every cycle of the driving signal. This technique produces a map in which a period-one orbit appears as one point, a period-two orbit appears as two points and so on.

To calculate the control perturbations, one analyzes the Poincaré section to determine how the system approaches the desired orbit, or fixed point. This analysis requires three steps. First, one determines the directions along which the system converges toward or diverges away from the fixed point. Second, one must obtain the rate of convergence or divergence—a quantity that is related to the Lyapunov multiplier. Last, one must figure out how much the orbit shifts in phase space if the control parameter is changed in one way or another. This three-step calculation provides the information necessary to determine the perturbations that will nudge the system toward the desired periodic orbit.

When the control parameter is actually changed, the chaotic attractor is shifted and distorted somewhat. If all goes according to plan, the new attractor encourages the system to continue on the desired trajectory. If the system starts to go astray, the control parameter is changed again, producing yet another attractor with the desired properties.

The process is similar to balancing a marble on a saddle. If the marble is initially placed in the center of the saddle, it tends to roll off one side or the other but is unlikely to fall off the front or back. To keep the marble from rolling off, one needs to move the saddle quickly from side to side. Likewise, one needs to shift the attractor to compensate for the system's tendency to fly off the desired trajectory in one direction or another. And just as the marble reacts to small movements of the saddle, the trajectory is very sensitive to changes in the attractor.

What is truly extraordinary about the work of Ott, Grebogi and Yorke is that it showed that the presence of chaos could be an advantage in controlling dynamic behavior. Because chaotic systems are extremely sensitive to initial conditions, they also react very rapidly to implemented controls.

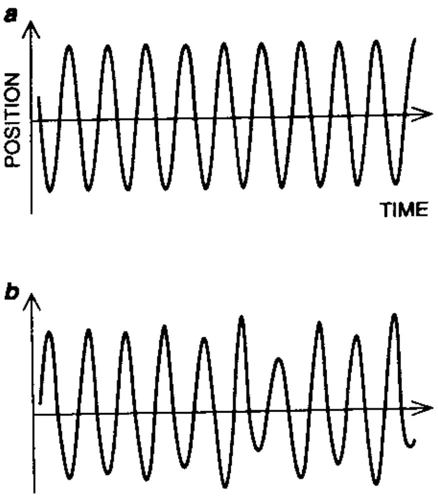
In 1990 one of us (Ditto), along with Mark L. Spano and Steven N. Rauseo of the Naval Surface Warfare Center, set out to test the ideas of Ott, Grebogi and Yorke. We did not expect things to work right away. Few theories ever survive initial contact with experiment. But the work of Ott, Grebogi and Yorke proved to be an exception. Within a couple of months we achieved control over chaos in a rather simple experimental system.

The experiment requires a metallic ribbon whose stiffness can be changed by applying a magnetic field. The bottom end of the ribbon is clamped to a base; the top flops over either to the left or right. When the ribbon is exposed to a field whose strength is varied periodically at a rate around one cycle per second, the ribbon buckles chaotically. A second magnetic field served as the control parameter.

Our goal was to change the chaotic motion of the ribbon to a periodic one by applying the OGY method. We first needed to obtain the Poincaré map of the system's chaotic attractor. The map was created by recording the position of the ribbon once per drive cycle, that is, the frequency with which the magnetic field was modulated. A computer stored and analyzed the map; it then calculated how the control parameter should be altered so that the system would follow a period-one orbit through state space. We found that even in the presence of noise and imprecise measurements, the ribbon could easily be controlled about an unstable period-one orbit in the chaotic region. We were genuinely surprised at how easy it was to implement and exploit the chaos.

The ribbon remained under control without a failure for three days, after which we got bored and wanted to attempt something else. We tried to make the system fail by adding external noise, altering the parameters and demonstrating the system to visitors (a sure way to make any experiment fail). In addition, we succeeded in forcing the system to follow period-one, period-two and period-four orbits, and we could switch between the behaviors at will.

As often happens in science, discoveries lead in unexpected directions. Earle R. Hunt of Ohio University took notice of our experiments and decided to attempt to control chaos in an electronic circuit. Hunt's circuit was made out of readily available components. He was able to coax the system into orbits with periods as high as 23 and at drive frequencies as high as 50,000 cycles per second. Not only did he prove that chaos could be controlled in electronic circuits, but he also showed that



NONCHAOTIC SYSTEMS that are out of phase can sometimes be made to operate in phase if they are driven with chaotic signals. For example, if two period-two systems are driven with the same periodic signal (*a*) and start out of phase, they will remain out of phase. If the same two systems are driven with a Rössler signal (*b*), which is mildly chaotic, they can be coaxed to operate in phase.

scientific advances can be made on a low budget.

Hunt's chaos controller—which employs a variation of the OGY method—has the advantage that it eliminates the need to obtain a Poincaré section; information for the controller can be obtained directly from measurements of the chaotic behavior. Hunt's circuit enables the manipulation of chaos in rapidly changing systems. Armed with these results, Hunt attended, in October 1991, the First Experimental Chaos Conference, having solved a problem he did not know existed.

Hunt had unknowingly overcome a technological obstacle that had stymied Rajarshi Roy and his colleagues at the Georgia Institute of Technology. Roy had been studying the effects of a "doubling crystal" on laser light. The crystal doubles the frequency of the incident light or, equivalently, halves the wavelength. The laser used by Roy generated infrared light at a wavelength of 1,064 nanometers, and the crystal converted the light to green at a wavelength of 532 nanometers. The crystal did not work perfectly, however. If the doubling crystal was oriented in certain ways, the intensity of green light would fluctuate chaotically.

Roy was eager to control the chaos in the laser system, but he knew the OGY method would not work, because it was difficult and impractical to obtain the Poincaré section for the laser system. Furthermore, Roy was skeptical that a computer system could perform calculations fast enough to implement the OGY method. Fortunately, he had learned about Hunt's work at the conference. Less than two weeks later Roy constructed a prototype controller for their laser, and to their astonishment, it worked on the first try. Within days they were controlling periods as high as 23 and at driving frequencies of 150,000 cycles per second.

Roy demonstrated that he could control chaotic fluctuations in the laser intensity and could stabilize unstable high-period oscillations. The energy of the laser could therefore be dumped into desired frequencies instead of being distributed across a broad band of frequencies. (Roy has also employed this technique to eliminate almost completely intensity fluctuations in his laser system.) Consequently, investigators can design laser systems with more flexibility and stability than ever before.

The field of laser research has also benefited from other advances in chaos control. For example, the team of Ira B. Schwartz, Ioana A. Triandaf, Carroll and Pecora at the Naval Research Laboratory developed a method known as tracking to extend the range over which the control of chaos can be maintained. Tracking compensates for parameters that change as the system ages or that slowly drift for one reason or another.

In recent months, tracking has been applied to both chaotic circuits and lasers with astounding results. For example, the laser remains stable only for a very limited power range if the parameters of the control mechanism do not adapt. By using tracking, researchers can maintain control over a much wider power range, and amazingly, they find they can increase the output power by a factor of 15.

One problem with the OGY method is that an unacceptable amount of time may elapse as one waits for the system naturally to approach the desired orbit in the chaotic attractor. Troy Shinbrot and his colleagues at the University of Maryland and the Naval Surface Warfare Center have demonstrated a technique that rapidly moves the chaotic system from an arbitrary initial state to a desired orbit in the attractor. When Shinbrot and his co-workers implemented their method in the magnetic-ribbon experiment, they were able to reduce the time needed to acquire unstable orbits by factors as high as 25.

Another collaboration that emerged from the First Experimental Chaos Conference led to the first technique for controlling chaos in a biological system. The team—which included Ditto, Spano, Alan Garfinkel and James N. Weiss of the School of Medicine, University of California at Los Angeles—studied an isolated part of a rabbit heart. We were able to induce fast, irregular contractions of the heart muscle by injecting a drug called ouabain into the coronary arteries. Once this arrhythmia started, we stimulated the heart with electric signals that were generated according to a scheme we adapted from the OGY method. These seemingly random signals were sufficient to establish a regular beat and sometimes reduced the heart rate to normal levels. On the other hand, signals that were random or periodic did not stop the arrhythmia and often made it worse.

Investigators have already begun to test whether variations of the OGY method could be used to control arrhythmias in human hearts. They suspect the technique might work for atrial or ventricular fibrillation, in which upper or lower chambers of the heart contract irregularly and cease to pump blood effectively. In the near future, it may be possible to develop pacemakers and defibrillators that take advantage of techniques for controlling chaos.

Scientists and engineers have just begun to appreciate the advantages of designing devices to exploit, rather than avoid, nonlinearity and chaos. Whereas linear systems tend to do only one thing well, nonlinear devices may be capable of handling several tasks. Nonlinear applications promise more flexibility, faster response and unusual behaviors. As we continue to investigate the nonlinearity inherent in natural and physical systems, we may learn not just to live with chaos, not just to understand it, but to master it.

FURTHER READING

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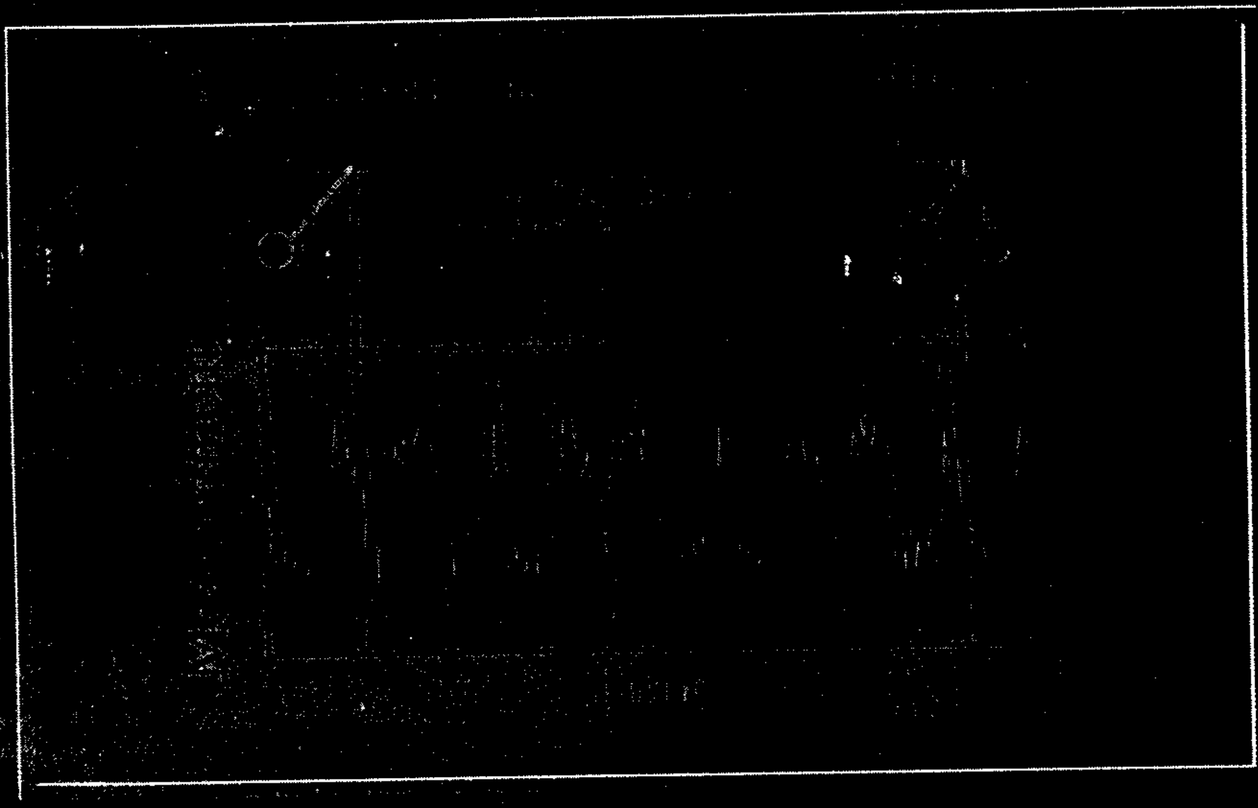
Applied Chaos: Oxymoron to Reality

becomes impossible



Features of Chaos

Regular Behavior	*Irregular Behavior*

REGULAR　　　　RANDOM

CONTROLLABLE　　UNCONTROLLABLE

From Measurement to Understanding

Measurement

1.333

2.643

-1.46

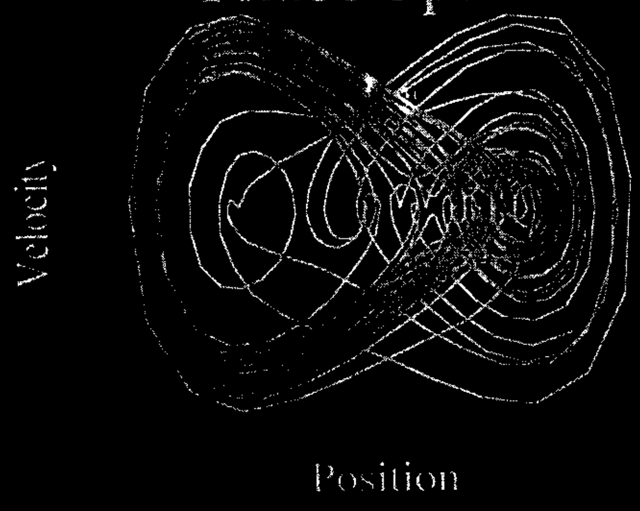
-2.34

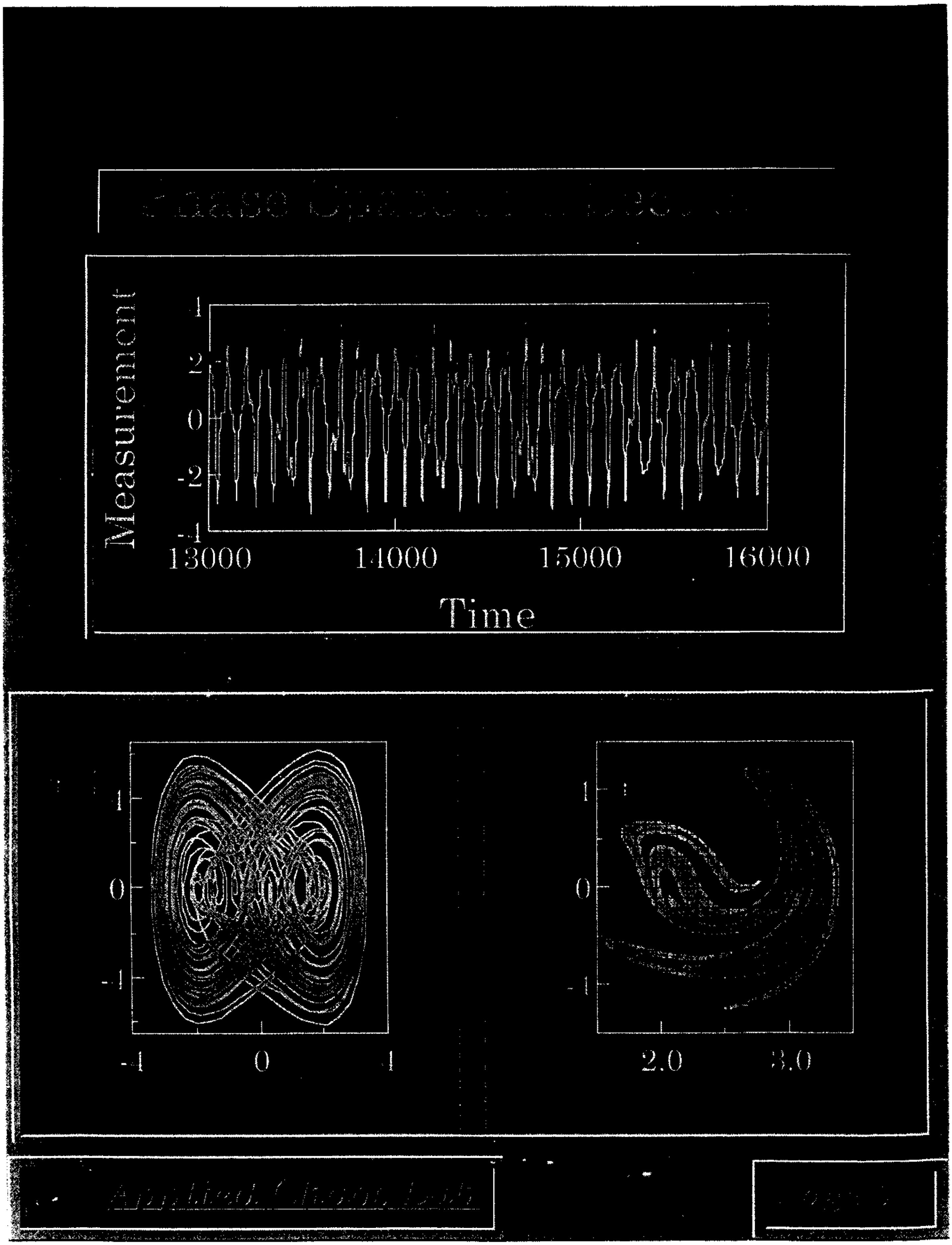
0.034

...

Time Series

Geometry of Phase Space

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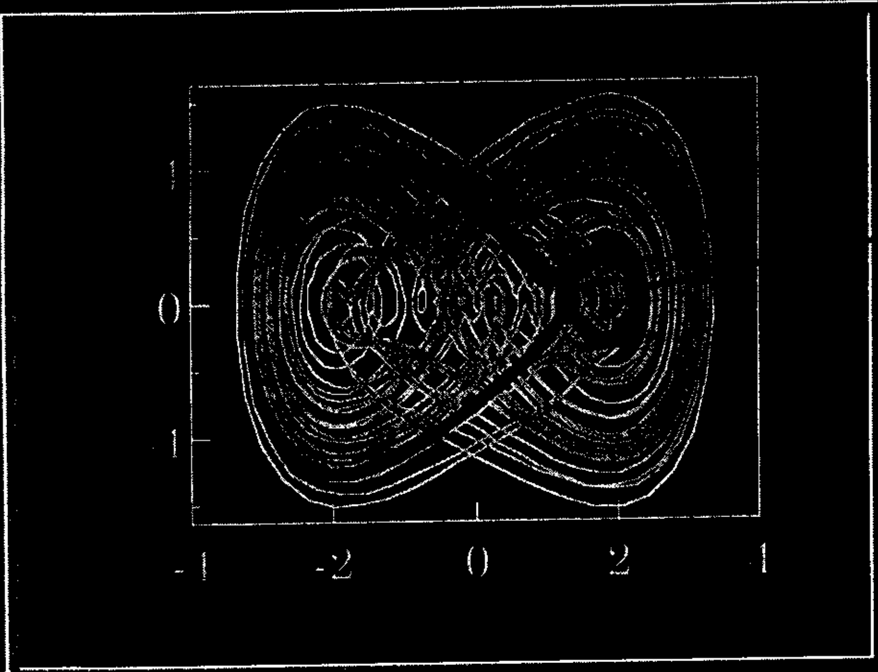
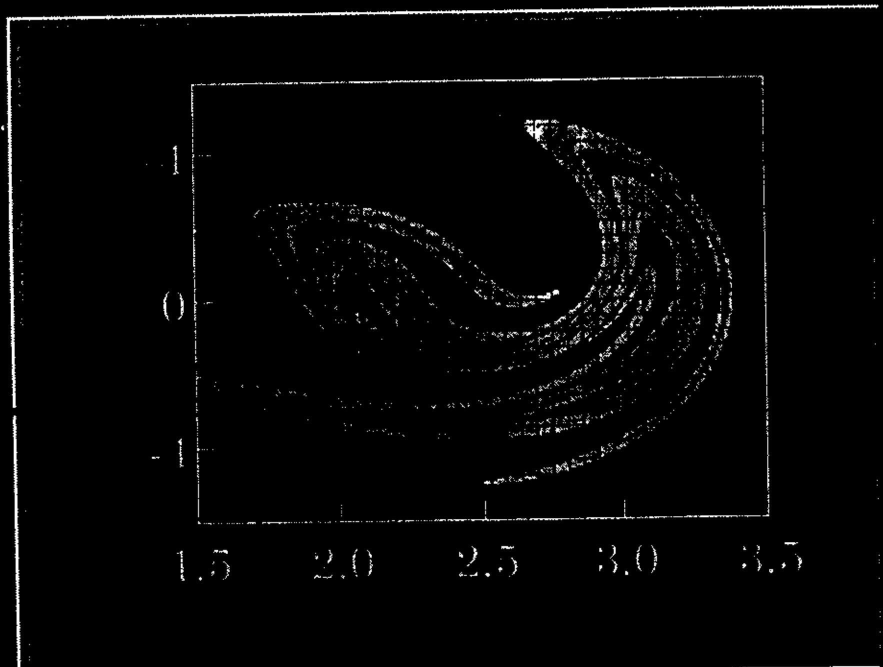
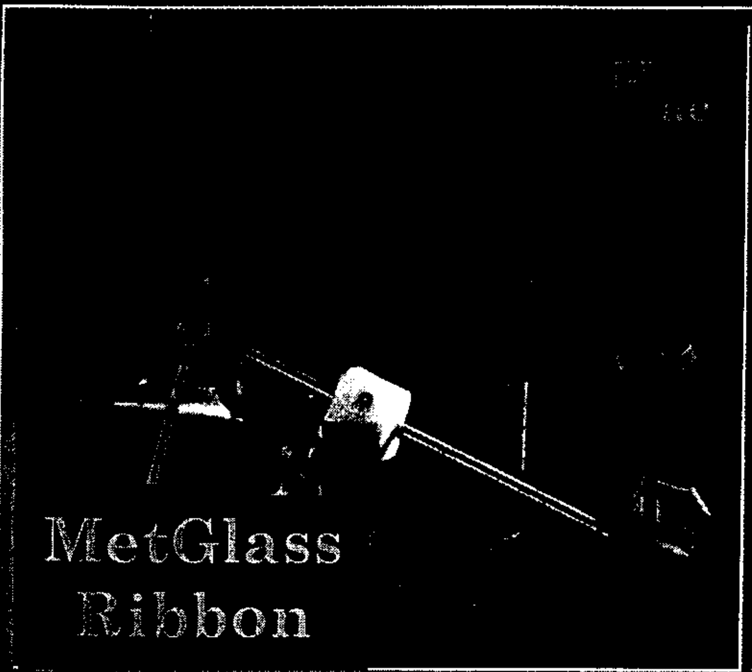
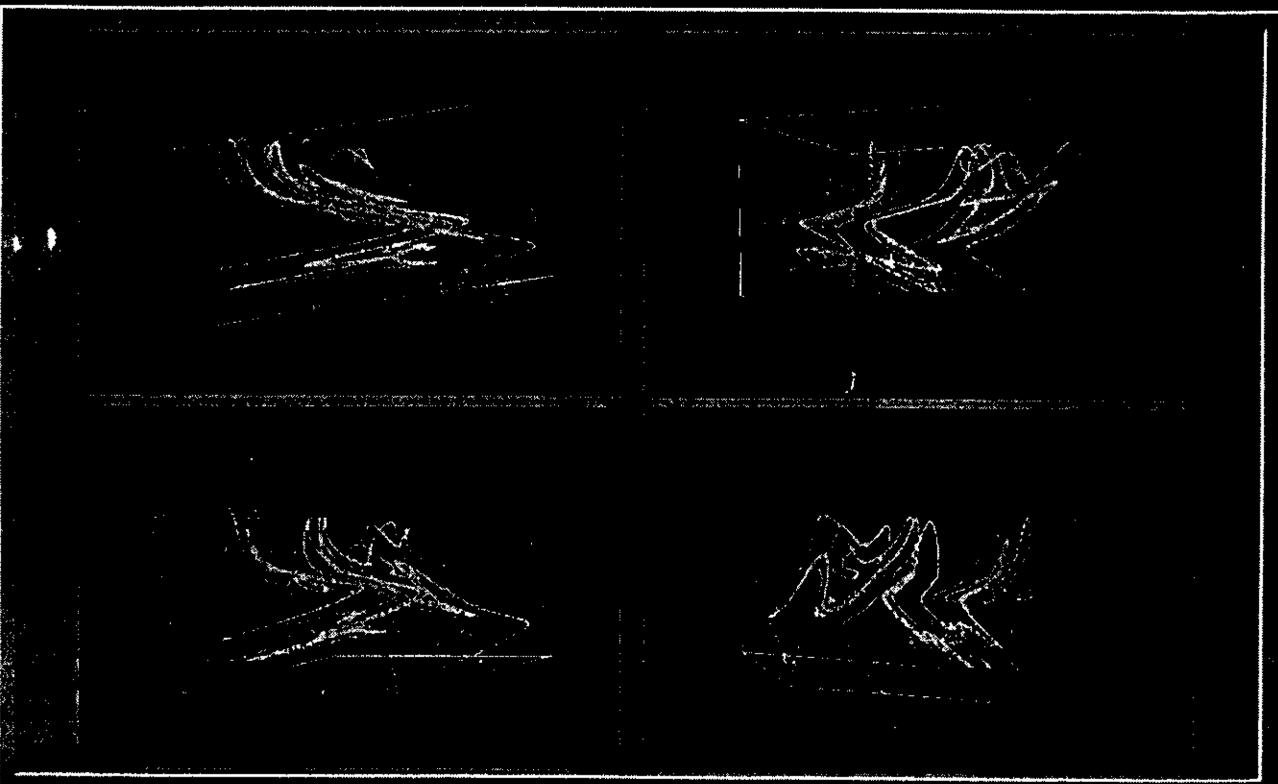
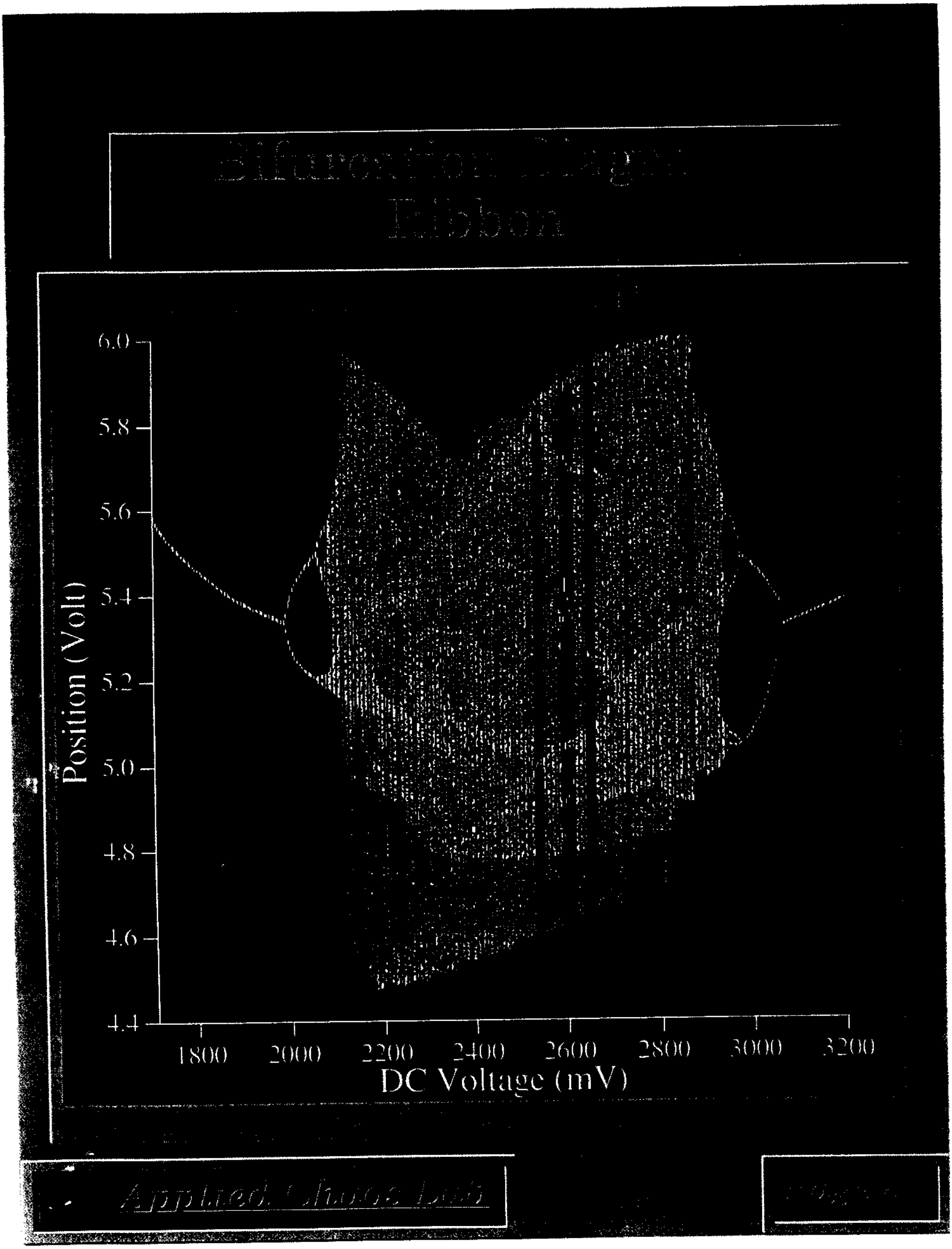




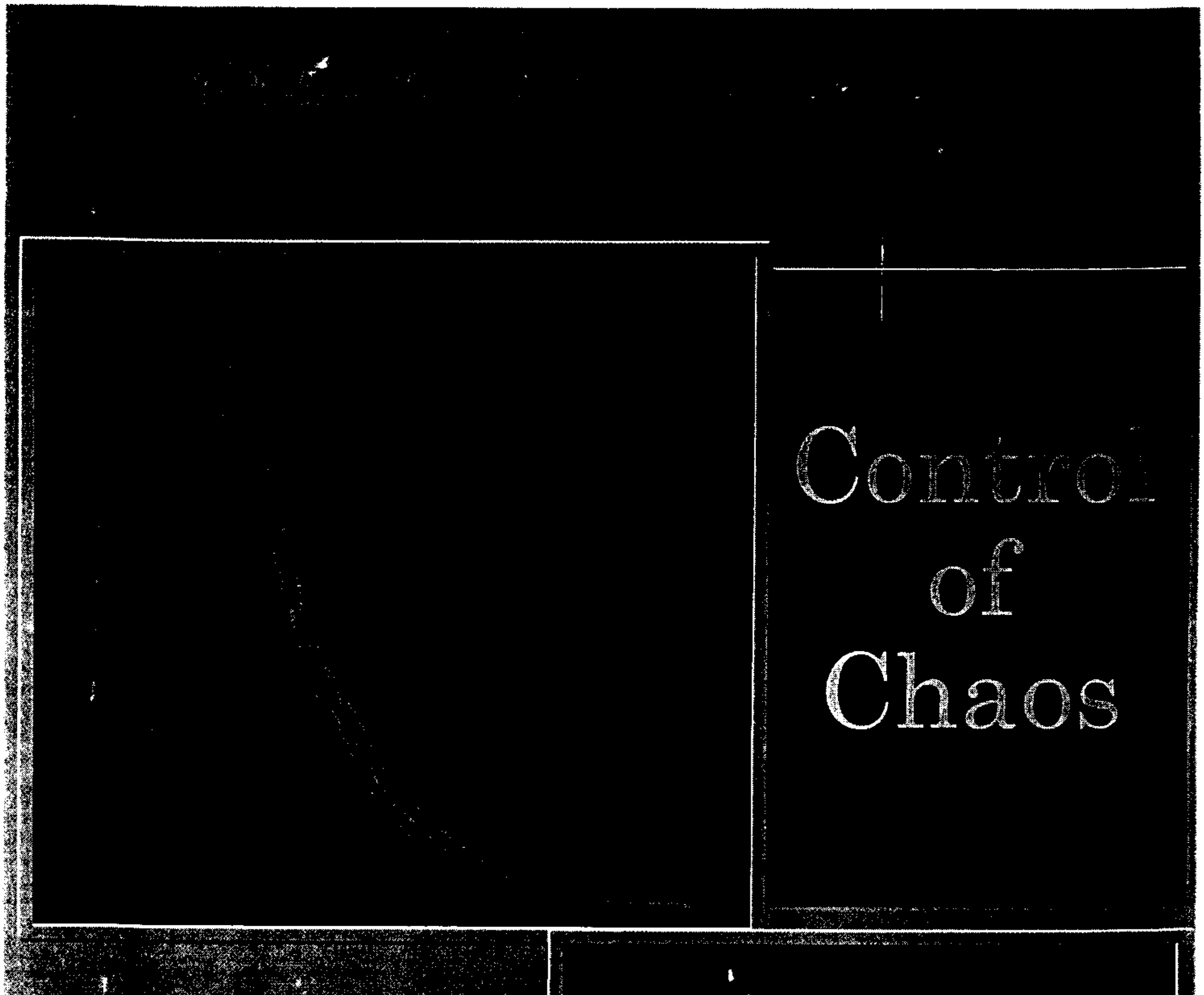
Chaos in Magnetoelastic Vibrations







Ribbon
6.0
5.8
5.6
5.4
5.2
5.0
4.8
4.6
4.4
Position (Volt)
1800
2000
2200
2400
2600
2800
3000
3200
DC Voltage (mV)
Applied Chaos Lab


Control
of
Chaos
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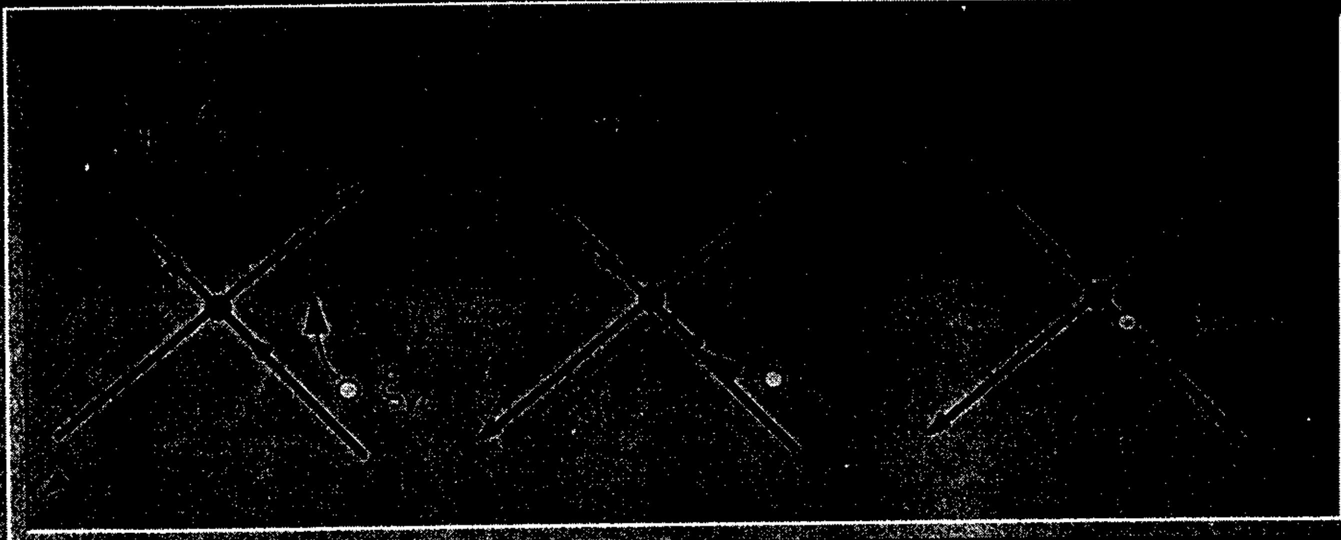
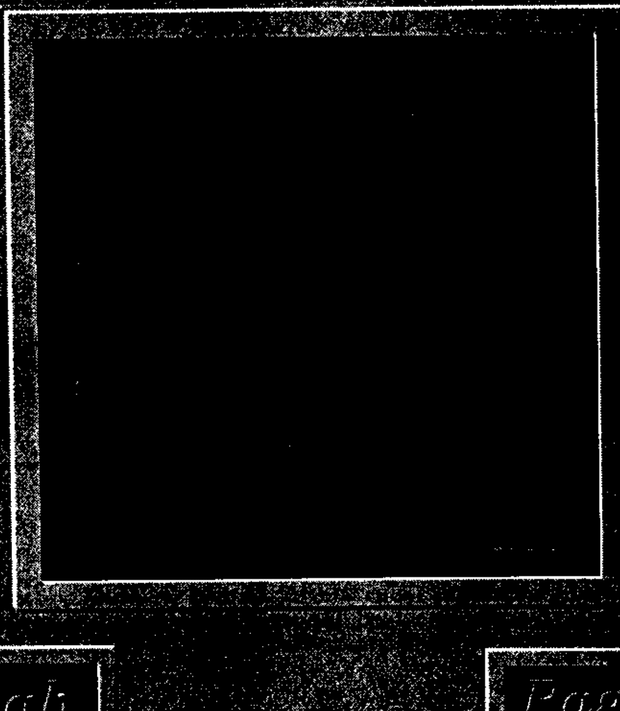
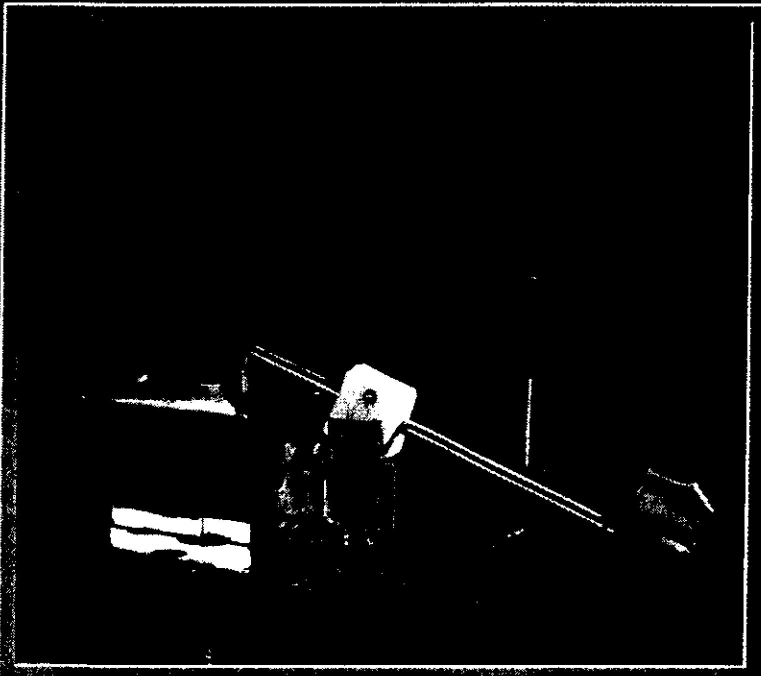
Chaos Contro



Change a parameter p_n to force the state point onto the stable direction. Key: Let chaotic dynamics do most of the work!

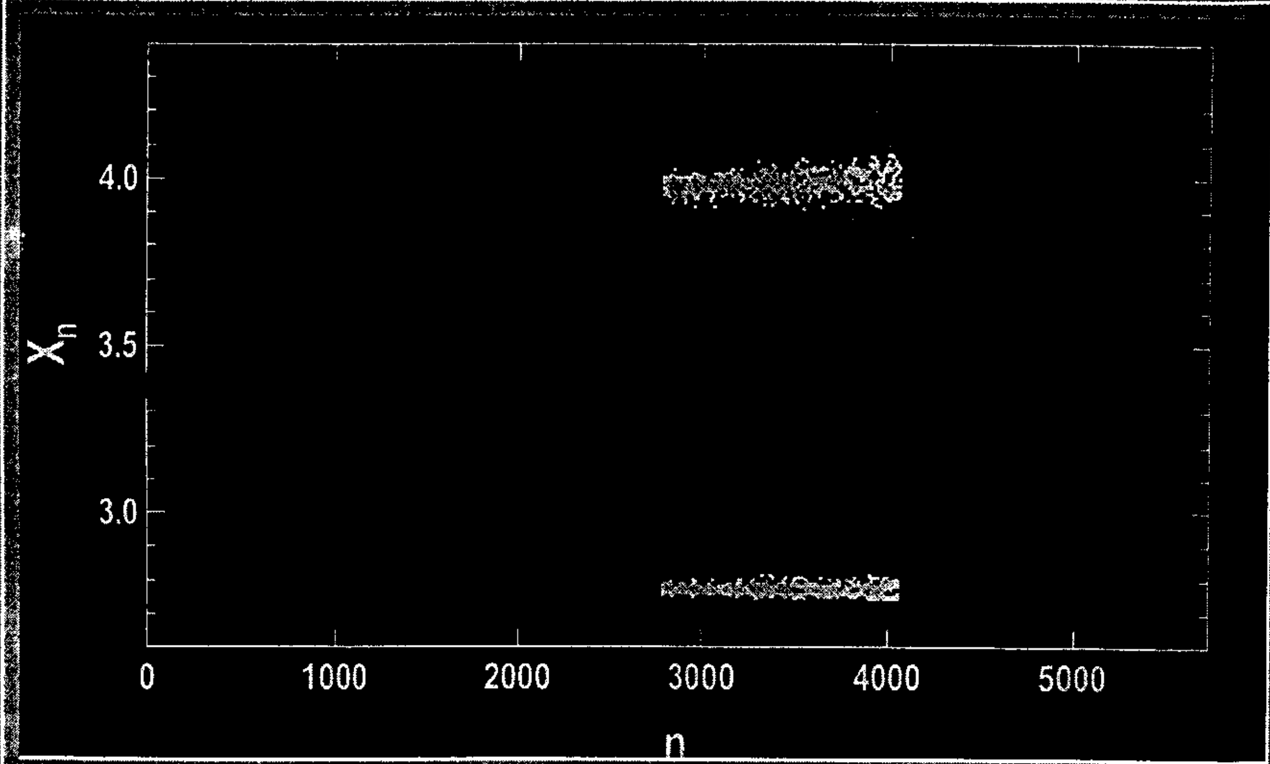
Like Balancing a Marble on a Saddle!



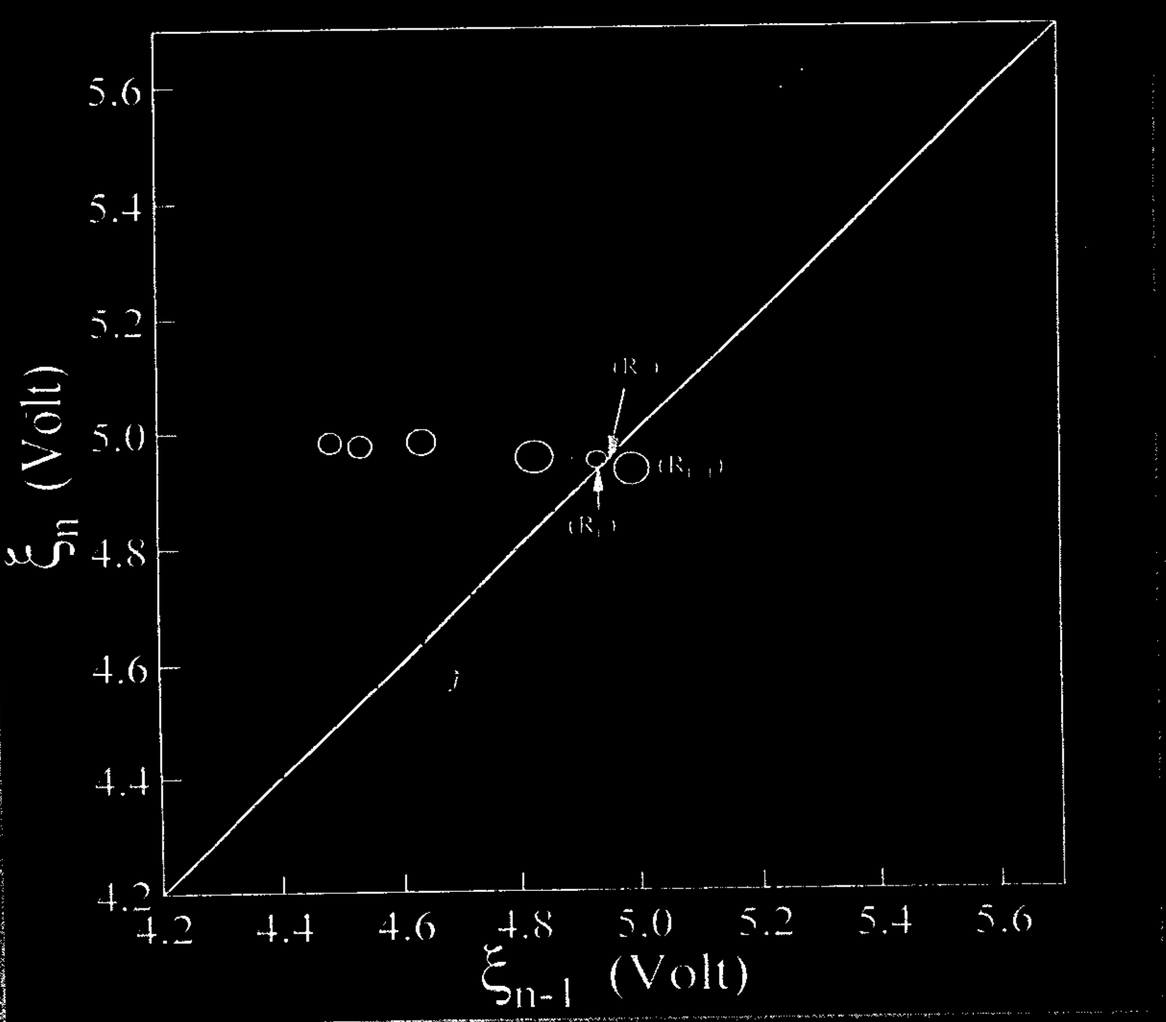


Controlling Chaos in Vibrating Ribbons

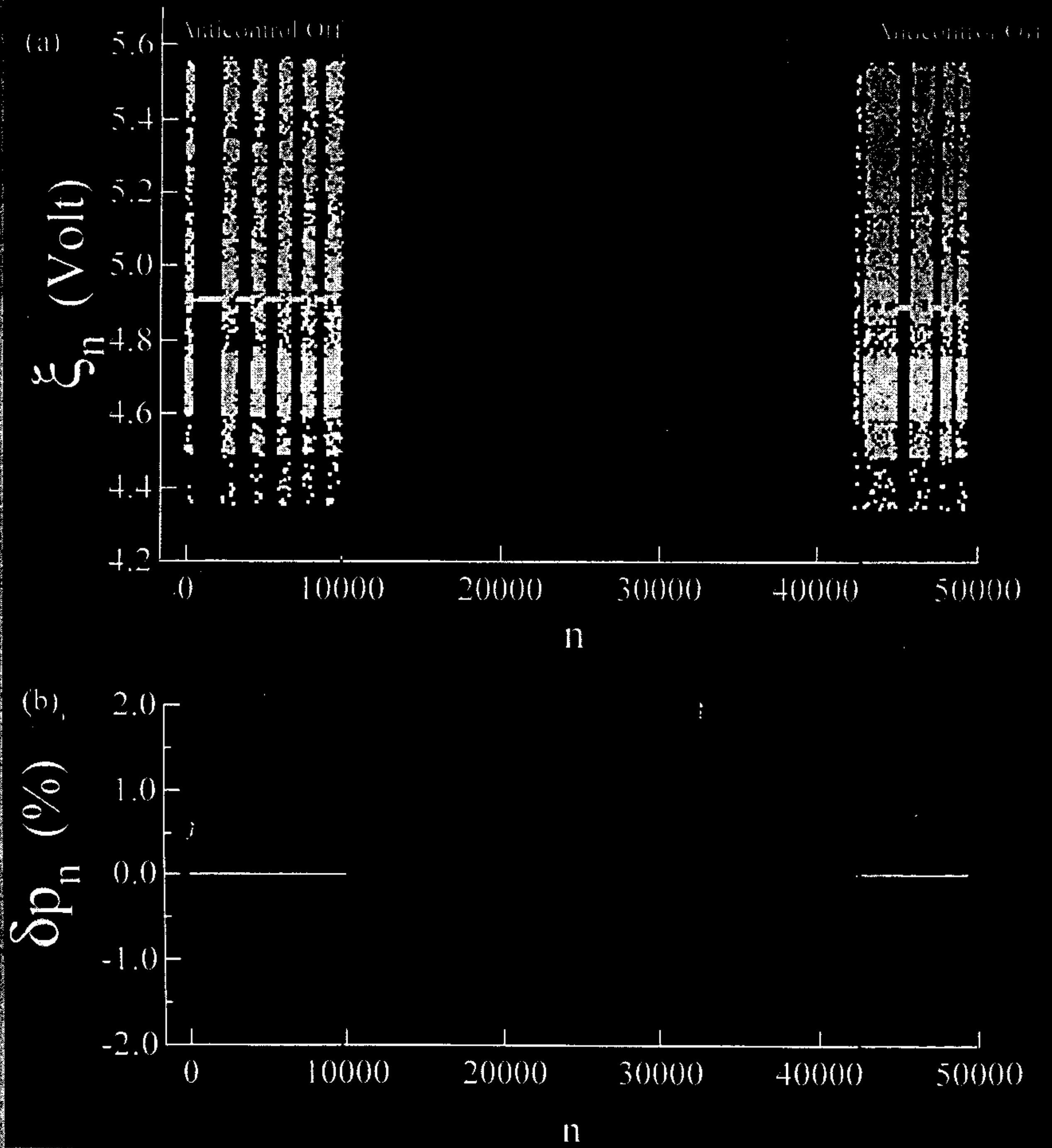
Chaos Control allows *small* perturbations ~ 0.01 % of DC Field



Anti-Chaos Control



Anti Chaos Control

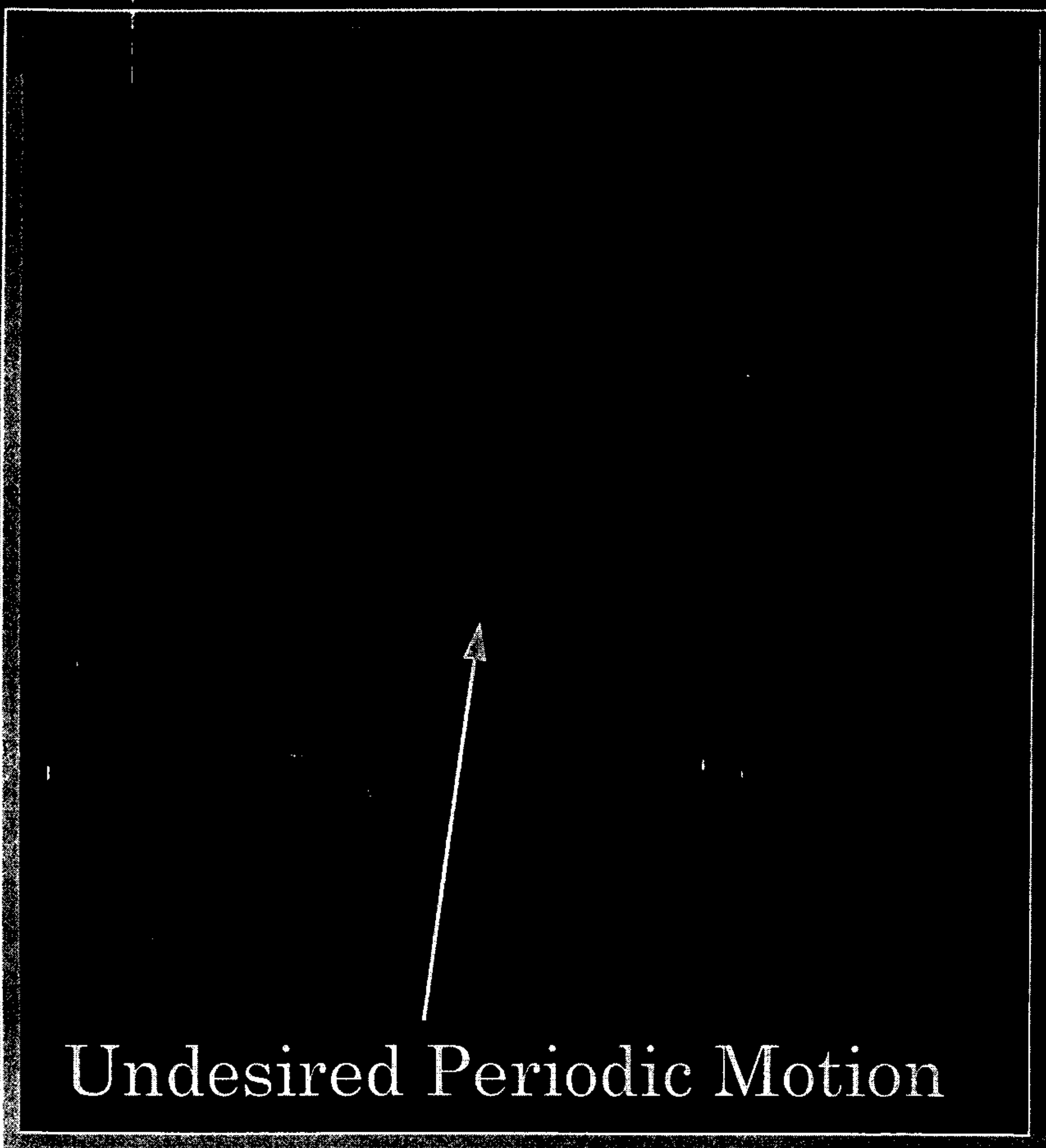


Anti-Chaos Control

In. Mahan, Ditto and Spano, *Phys. Rev. Lett.* 74, 1120

Undesired Periodic Motion

Anti-Chaos Control C

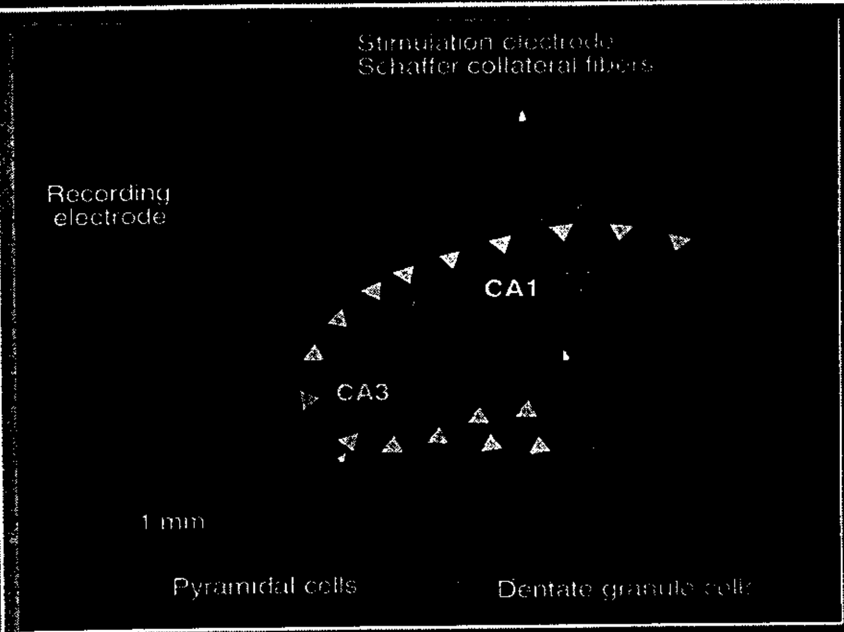


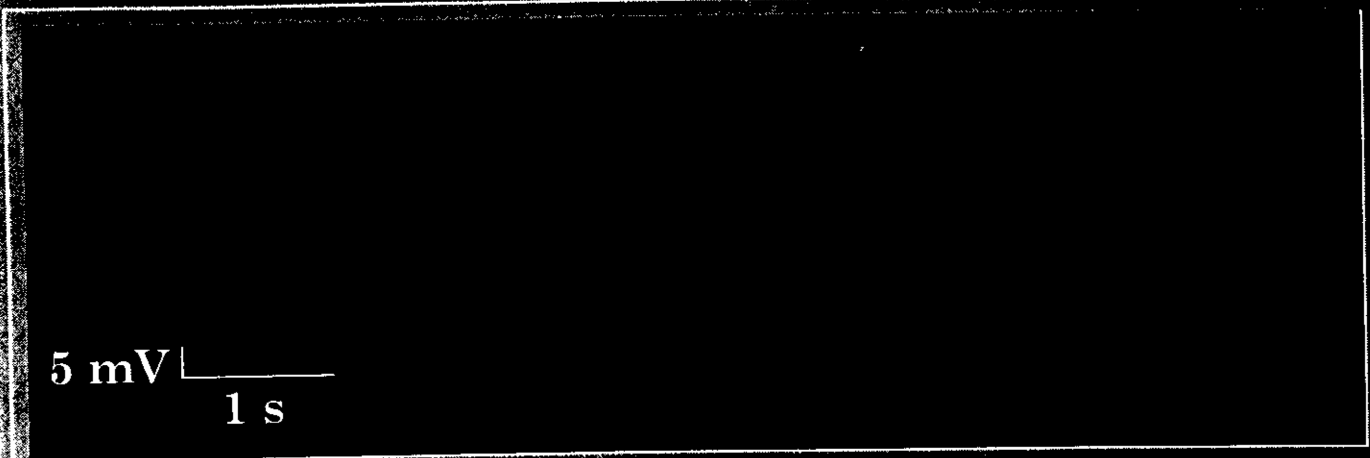
Anti-Control On only $\approx 0.12\%$ of Time

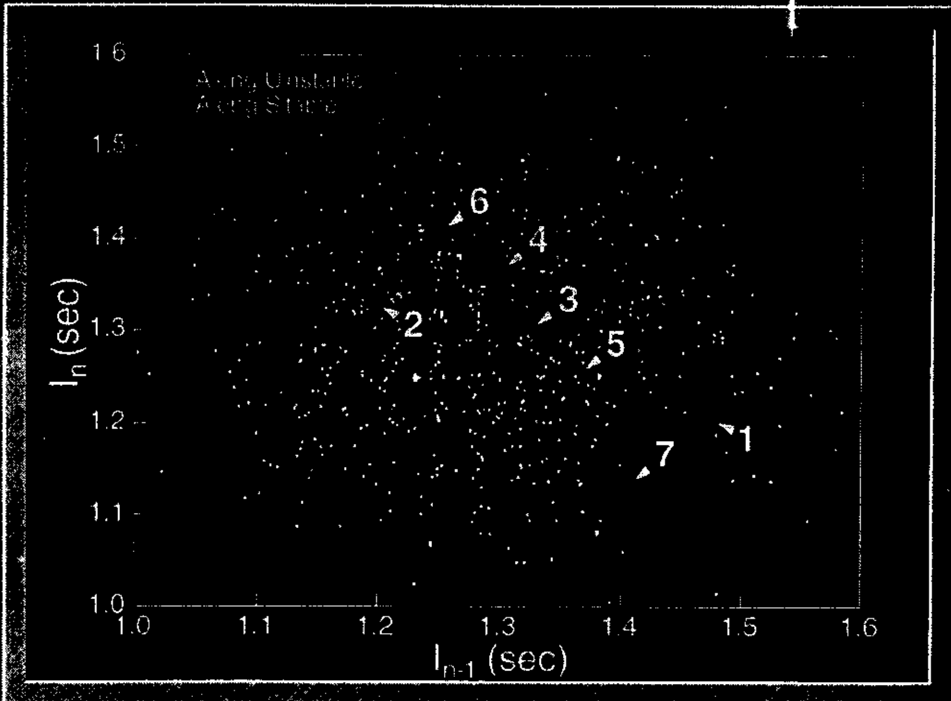
Controlling Chaos in the Brain

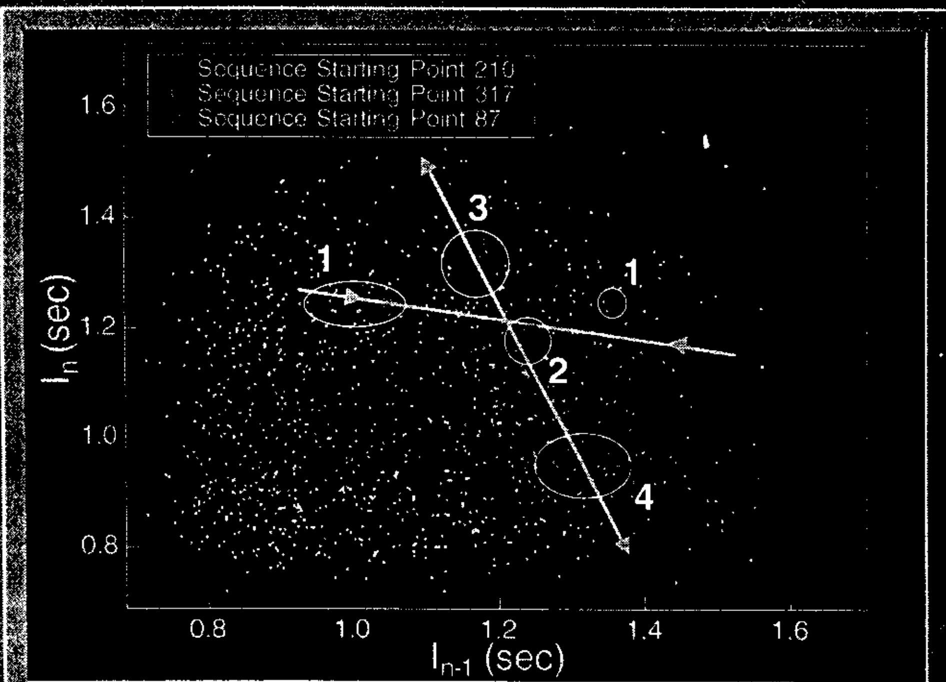
Nature, 370, 25 Aug 1994

Steve Schiff
Kristin Jerger
Duc Duong
Taeun Chang
Childrens National
&
Mark Spano
Navy
&
Bill Ditto
Applied Chaos Lab
Georgia Tech






Along Unstable
Along Stable
1.0
1.1
1.2
1.3
1.4
1.5
1.6
I_n (sec)
I_{n-1} (sec)
1
2
3
4
5
6
7


Sequence Starting Point 210
Sequence Starting Point 317
Sequence Starting Point 87
0.8
1.0
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1.4
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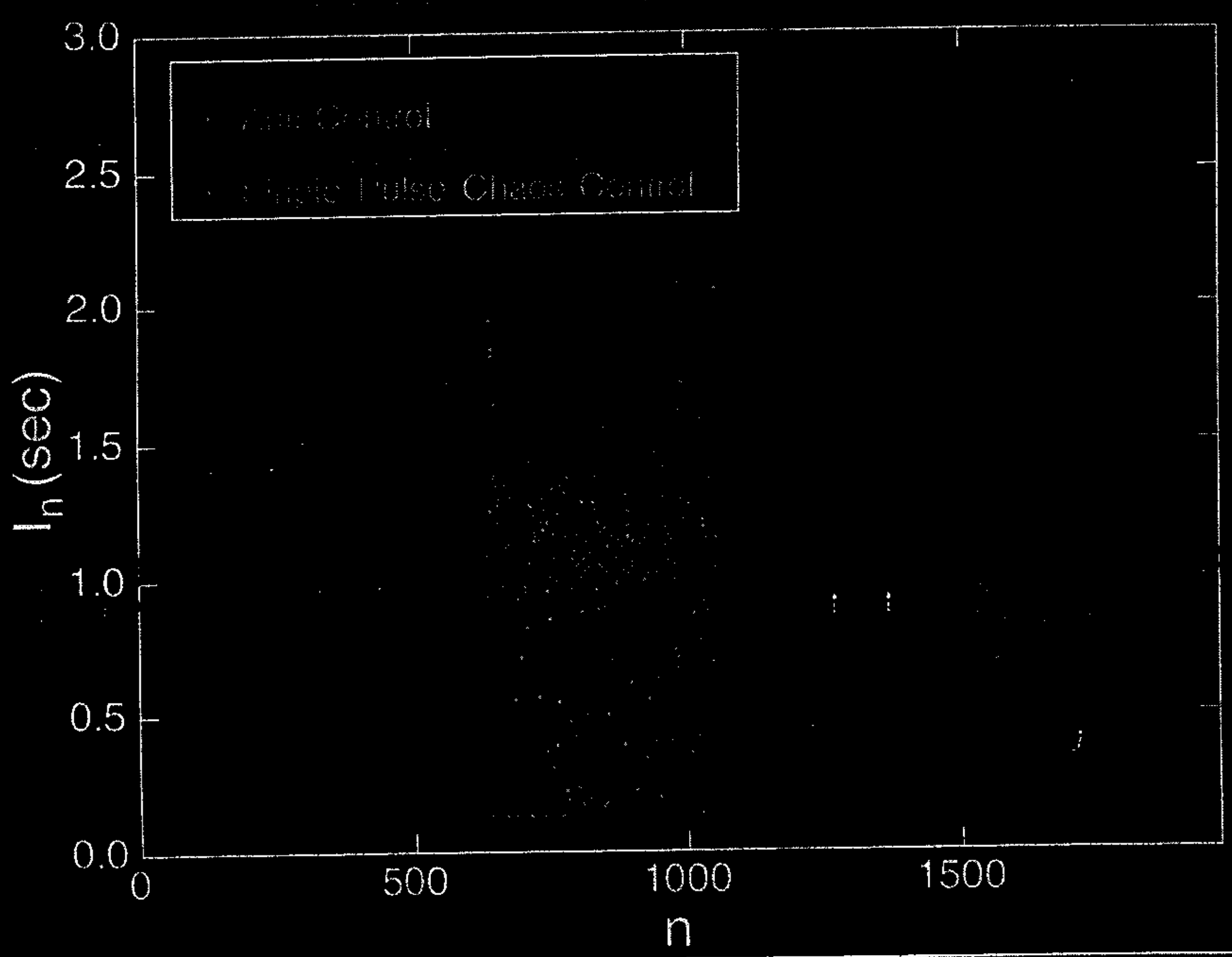
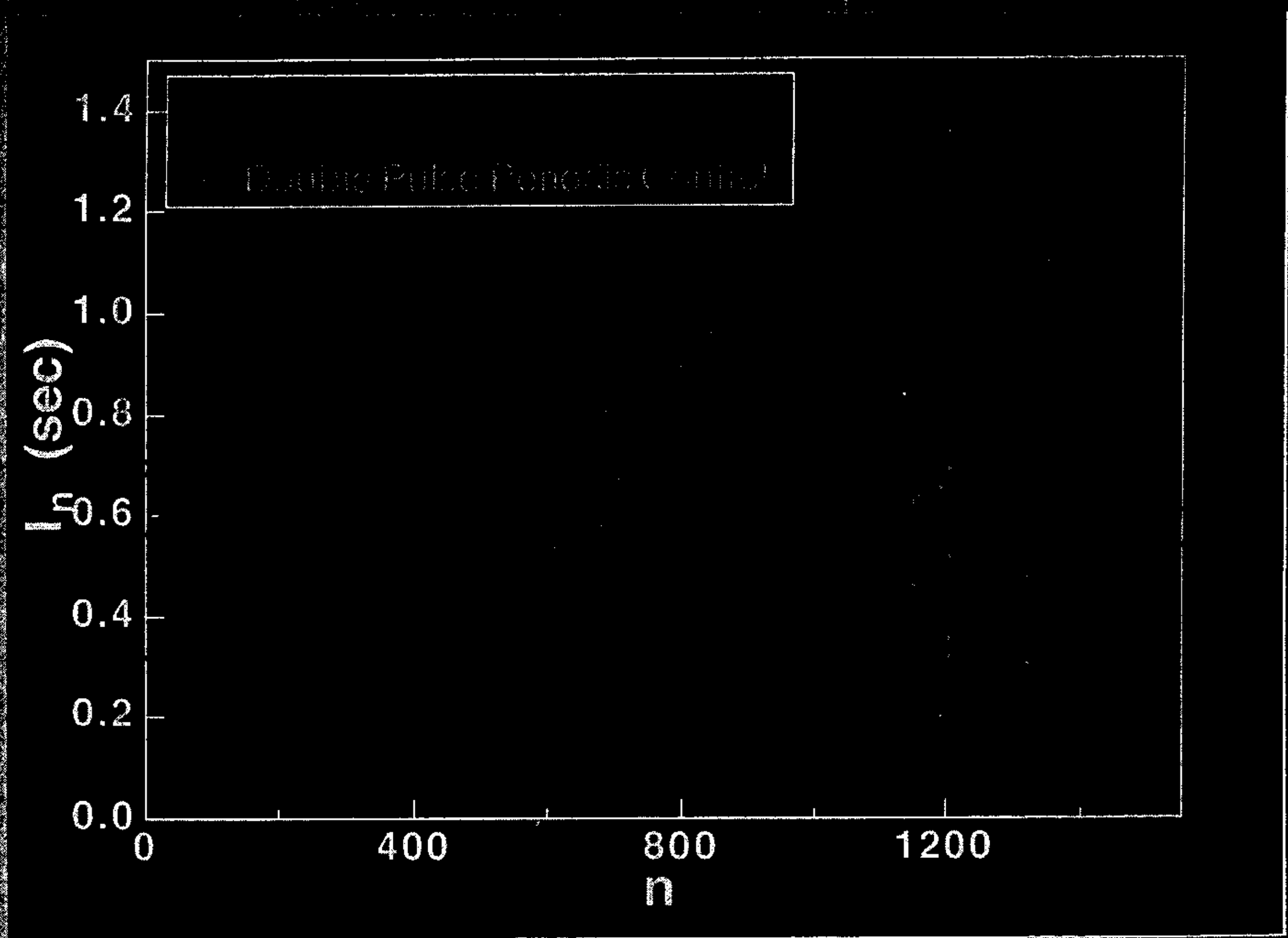
I_n (sec)
I_{n-1} (sec)
1
1
2
3
4

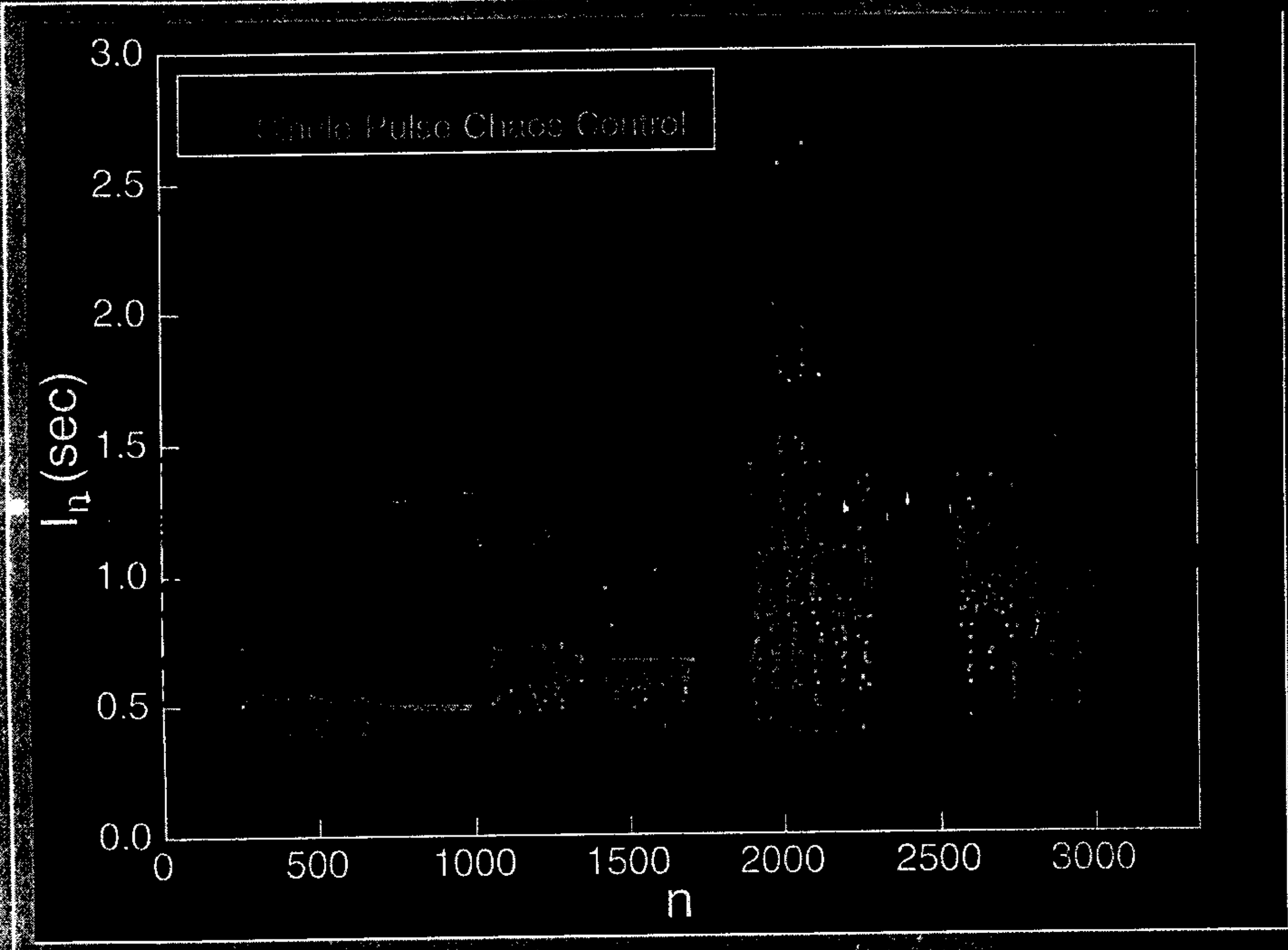
Controlling Brain Chaos



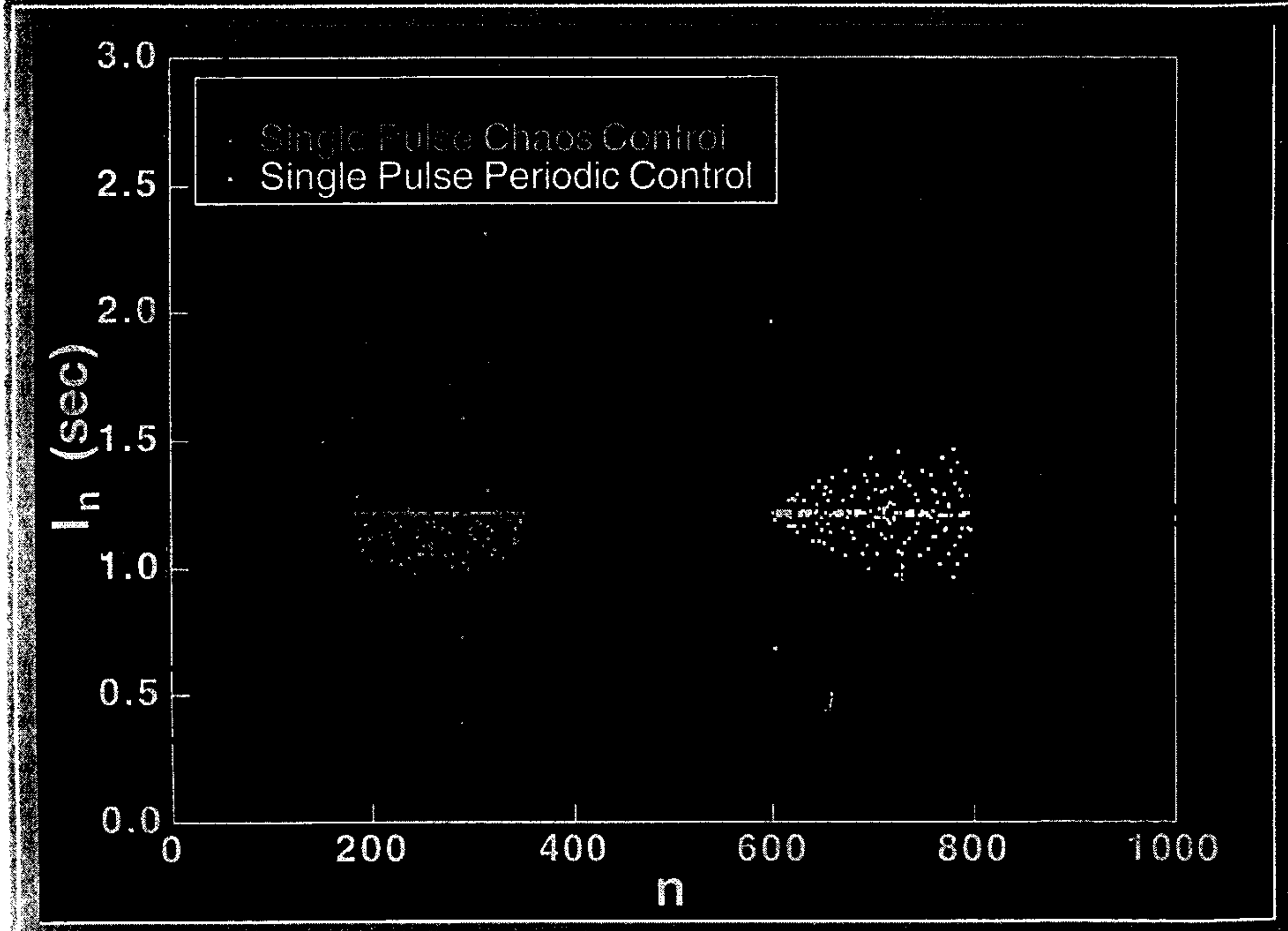
Controlling Brain Chaos



Controlling Brain Chaos



Controlling Brain Chaos



Ventricular Fibrillation

"[illegible] [illegible] [illegible] such [illegible] the ventricles is very violent at first. It lasts for 3 or 4 minutes and rapidly transforms itself into a fibrillary trembling or shuddering which persists also for a few minutes. Then all movement of the ventricles ceases. Throughout the duration of these irregular contractions one does not witness a single collective systole being produced by those parts of the heart."

— A. Vulpian 1874
(Arch Physiol 1:975, 1874)

Controlling Cardiac Chaos

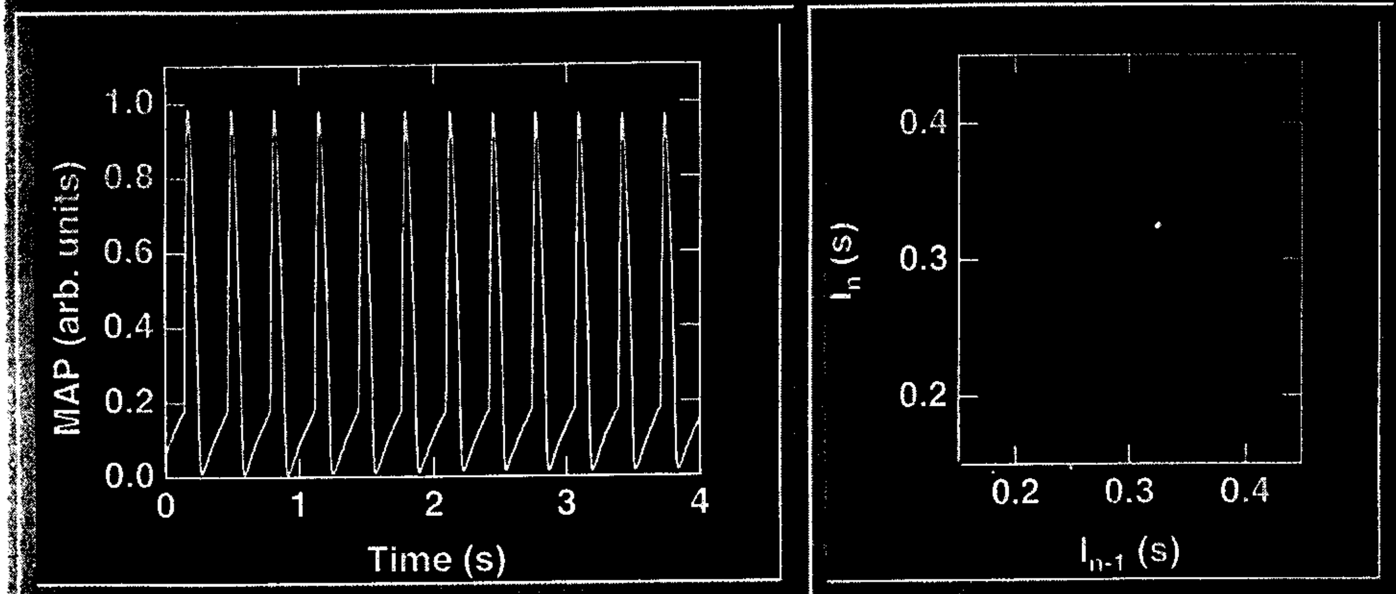
Science, 257, pp. 1230-1235 (1992)

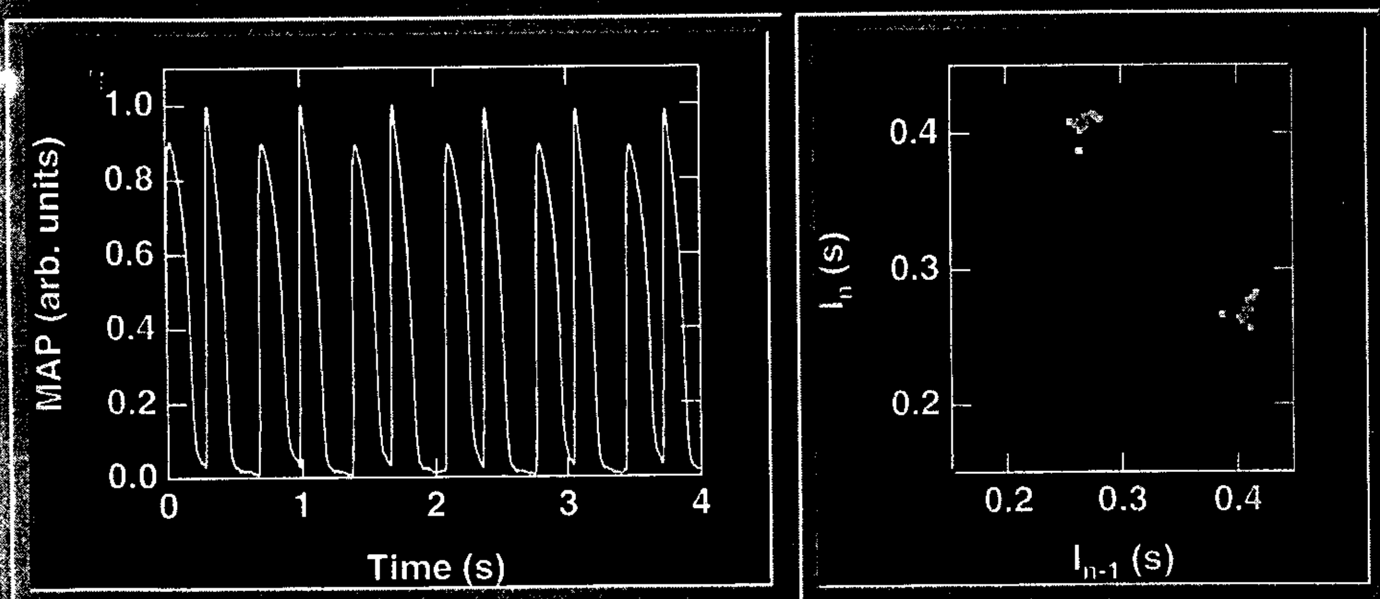
Alan Garfinkel • UCLA
James N. Weiss • UCLA

Mark L. Spano • Dept. of Navy

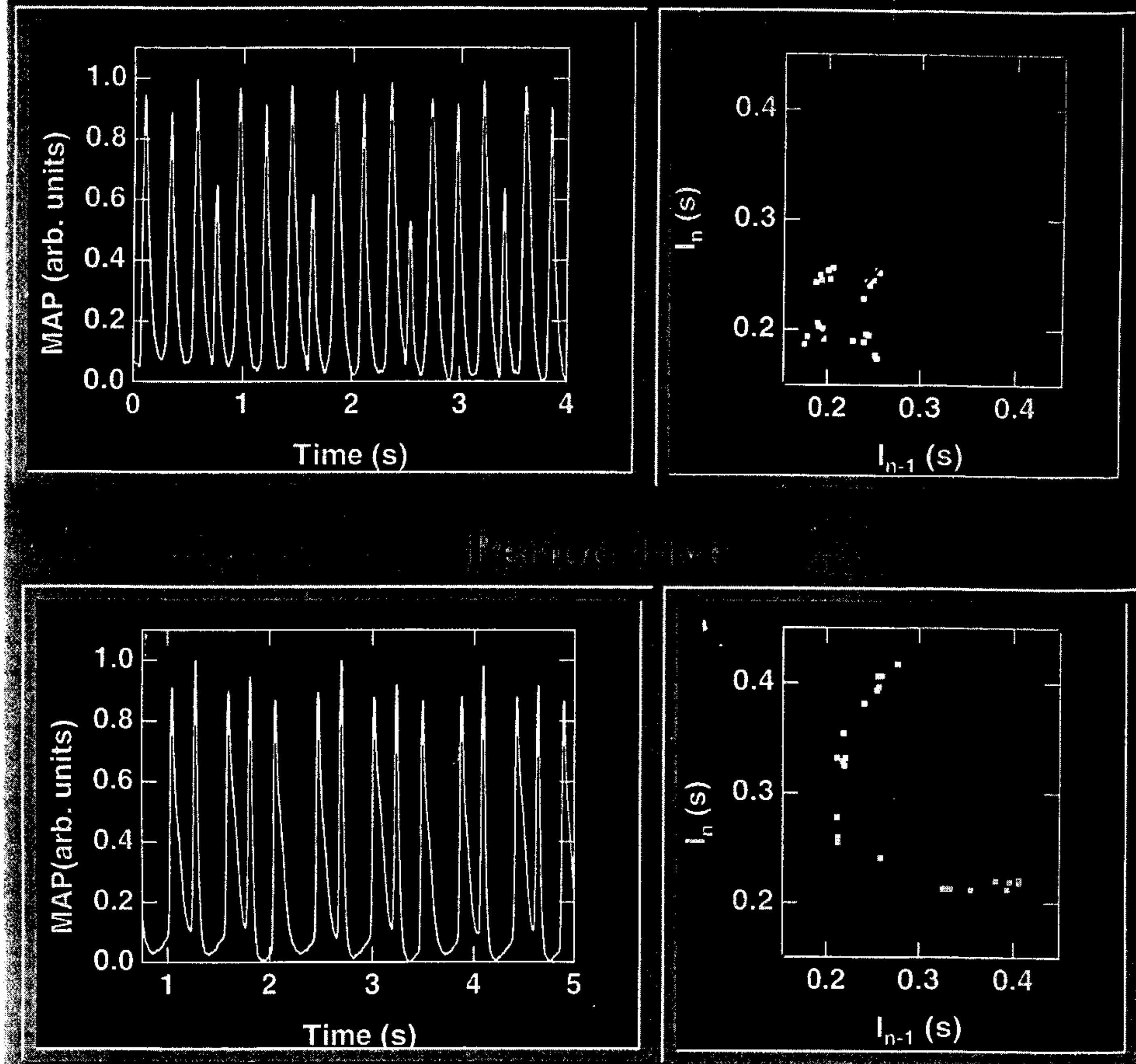
W. L. Ditto • Georgia Tech

- Interventricular septum from rabbit
  - arterially perfused with Krebs Solution
  - 2 to 5 μM Ouabain
  - 2 to 10 μM Epinephrine
- Silver Chloride Monitoring Electrodes
  - on surface of tissue.
- *Control Stimulus*:
  - 3 msec constant voltage pulse
  - 10 - 30 V pulse (twice threshold)
  - platinum electrodes
  - Grass Stimulator driven by PC

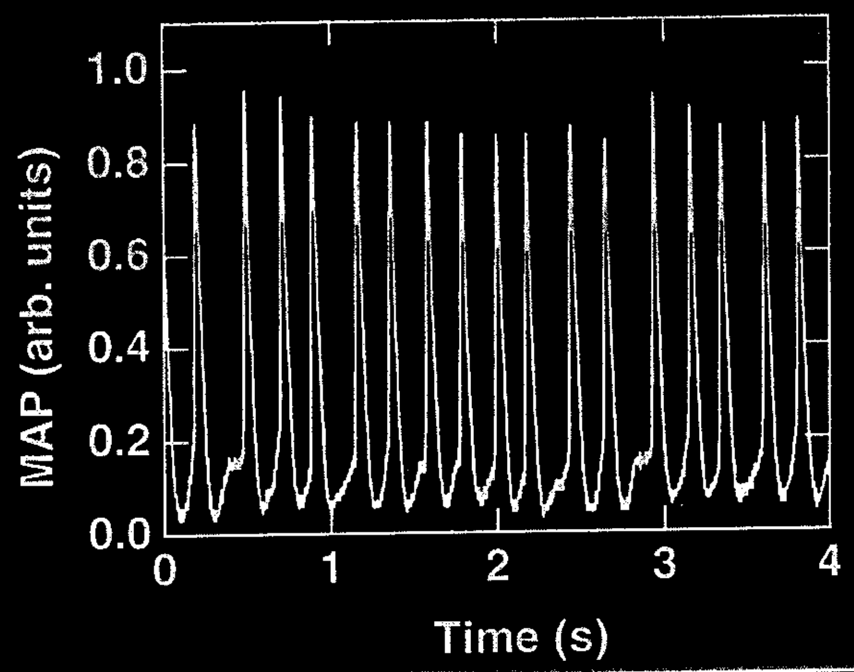

MAP (arb. units)
1.0
0.8
0.6
0.4
0.2
0.0
0
1
2
3
4
Time (s)
0.4
0.3
0.2
I_n (s)
0.2
0.3
0.4
I_{n-1} (s)

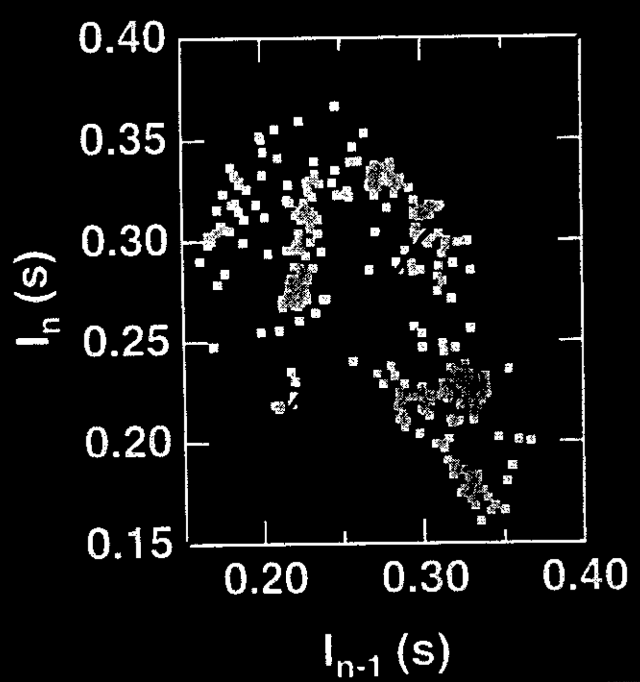

MAP (arb. units)
1.0
0.8
0.6
0.4
0.2
0.0
0
1
2
3
4
Time (s)
0.4
0.3
0.2
I_n (s)
0.2
0.3
0.4
I_{n-1} (s)

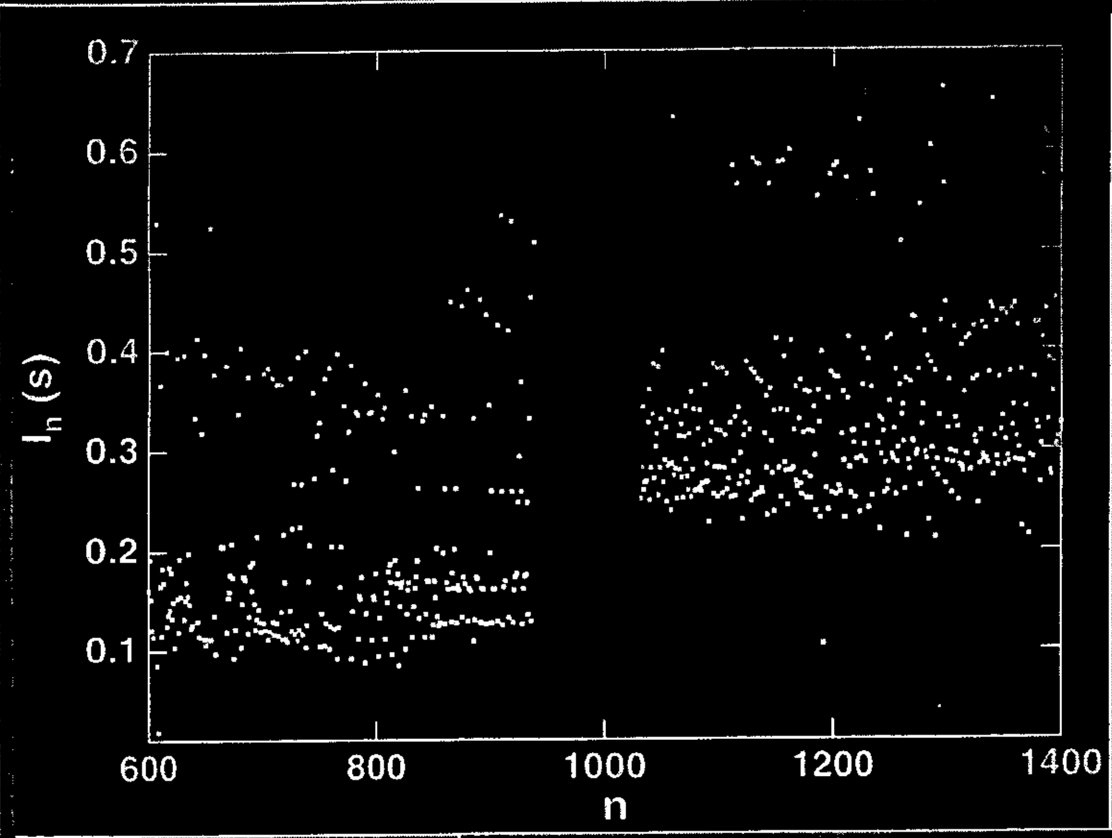
Rabbit Septum Expt



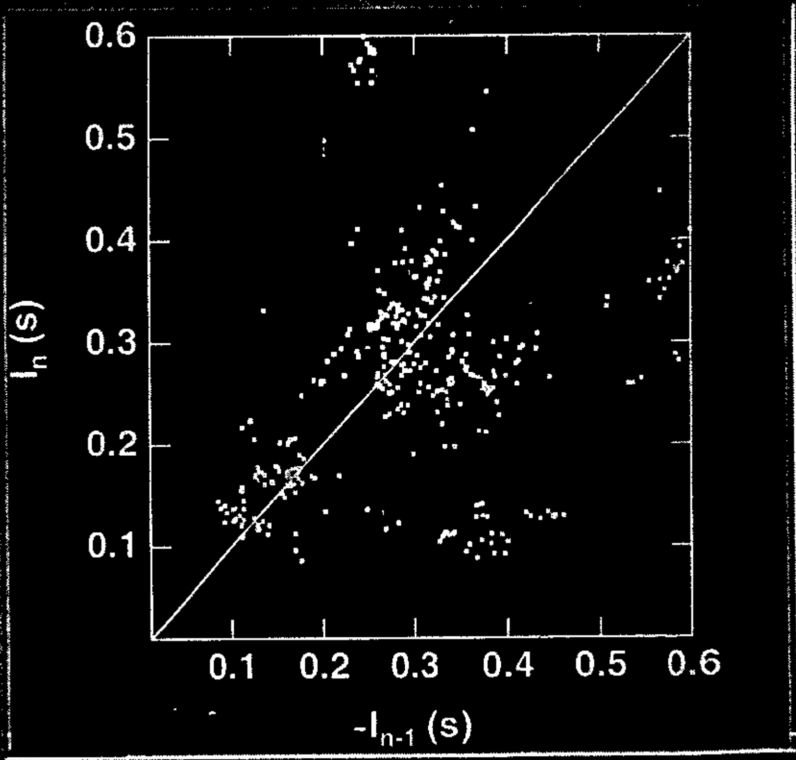
Cardiac Chaos







Controlling Cardiac Chaos



Control Successes

- Flame Fronts (?)
- Lasers *(Ga Tech)*
- Pendulum *(Univ. of Denver)*
- Diode Resonator Circuit *(Ohio U)*
- Fast Electronic Circuits *(Duke)*
- Coupled Diode Resonators *(Ohio U)*
- Coupled Circuits *(Duke)*
- MagnetoElastic Ribbons *(Ga Tech, NSWC)*
- Heart Tissue *(Ga Tech, UCLA, NSWC)*
- Brain Tissue *(Ga Tech, Children's, NSWC)*

Exploitation Applications

- Synchronized Chaos (NRL, MIT, Ga
- Mixing with Chaos (Northwestern)
- Encoding Information in Chaos (ARL)
- Anti-Chaos Control (Ga Tech, NSWC, UMD)
- Tracking of Chaos (NRL, Ga Tech, WVA)
- Targeting of Chaos (Ga Tech, NSWC, UMD)

The Future

- Spatio-Temporal Chaos Control
- Chaos Aware De-Fibrillators
- Information Processing Using Chaos
- Control/Improvement of Combustion
- Control of Power Grids/Turbines
- Control of Epilepsy/Seizures
- Encoding/Processing of Information in Neural Nets and Brain Tissue using Chaos Control

Controlling Chaos: Overview

I will put Chaos into fourteen lines
And keep him there; and let him thence escape
If he be lucky; let him twist and ape
Flood, fire, and demon - his adroit designs
Will strain to nothing in the strict confines
Of this sweet Order, where, in pious rape,
I hold his essence and amorphous shape,
Till he with Order mingles and combines.
Past are the hours, the years, of our duress,
His arrogance, our awful servitude:
I have him. He is nothing more nor less
Than something simple not yet understood;
I shall not even force him to confess;
Or answer. I will only make him good.

--Edna St. Vincent Millay

Stabilizing Unstable Periodic Orbits

OGY Control Strategy

The OGY technique of chaos control begins by reducing the phase space flow to a m: on a Poincaré section. The flow

$$\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x}, p), \quad \mathbf{x} \in \Re^N,$$

becomes the map

$$\mathbf{x}_{n+1} = \mathbf{F}_M(\mathbf{x}_n, p), \quad \mathbf{x}_n \in \Re^{N-1}$$

A UPO is represented in the section as a periodic point of the map, which we take f simplicity to be a fixed point, namely

$$\mathbf{x}_f = \mathbf{F}(\mathbf{x}_f, p_0),$$

where the nominal parameter value is denoted p_0. Nearby the fixed point, the map c: be linearized as

$$\delta\mathbf{x}_{n+1} \sim \mathbf{M}\delta\mathbf{x}_n + \mathbf{s}\delta p_n,$$

where the shift vector and map matrix of partial derivatives

$$\mathbf{s} = \frac{\partial \mathbf{F}}{\partial p}, \quad \mathbf{M} = \frac{\partial \mathbf{F}}{\partial \mathbf{x}},$$

are evaluated at the fixed point, and where the relative displacement $\delta\mathbf{x}_n \equiv \mathbf{x}_n - \mathbf{x}_f$ a: the parameter perturbation $\delta p_n \equiv p_n - p_0$. Assume that the relative perturbation proportional to the relative distance from the fixed point and try a linear control law,

$$\delta p_n \sim -\mathbf{K}^T \delta\mathbf{x}_n,$$

where $\mathbf{K}$ is the proportionality vector. Thus, near the fixed point the linearized m becomes

$$\delta\mathbf{x}_{n+1} \sim \mathbf{C}\delta\mathbf{x}_n,$$

where the control matrix

$$\mathbf{C} \equiv \mathbf{M} - \mathbf{s}\mathbf{K}^T .$$

If the proportionality vector $\mathbf{K}$ is chosen such that the moduli of the eigenvalues of the control matrix $\mathbf{C}$ are all less than unity, iterates of the map will contract toward the fixed point,

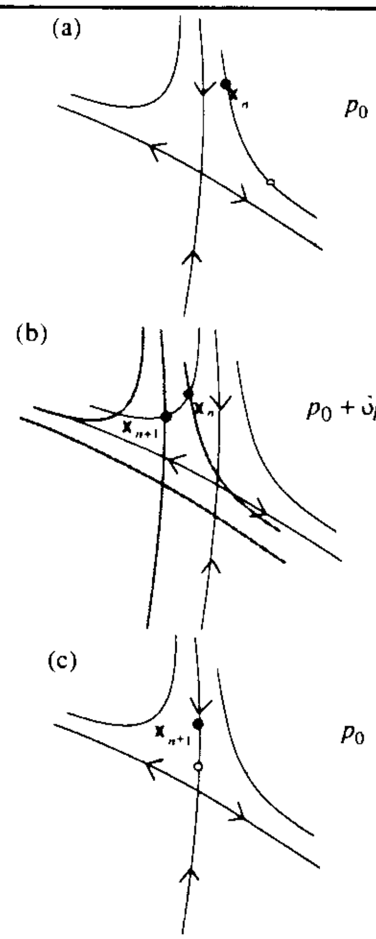
$$\lim_{n \to \infty} \delta \mathbf{x}_n = 0 .$$

and the UPO will be stabilized.

In practice, the map $\mathbf{M}$ and shift $\mathbf{s}$ are found by doing a linear fit to close returns in the section The perturbations are turned on only when the iterates ergodically wander near the fixed point. Thus the system is trapped when the required perturbations are small,

$$\left| K^T \delta p_n \right| < \delta p_{MAX} .$$

An excellent review of the OGY technique is (Shinbrot, Grebogi, Ott, Yorke: 1993).



Nearby iterate (a) will miss fixed point, but perturbation shifts the fixed point and its manifolds (b) so that next iterate falls on stable manifold (c).

Elementary Derivation of the OGY Control Formula

Here we present a simple and pedagogical derivation of the original OGY contr formula. Assume that the section is two-dimensional and that the fixed point is a sadd with one stable and one unstable manifold. Stabilizing the orbit is akin to balancing marble on a horse's saddle: the saddle must be jiggled back and forth in the direction the horse's ribs (rather than in the direction of its backbone).
The strategy is illustrated in the previous figure. It may be summarized as

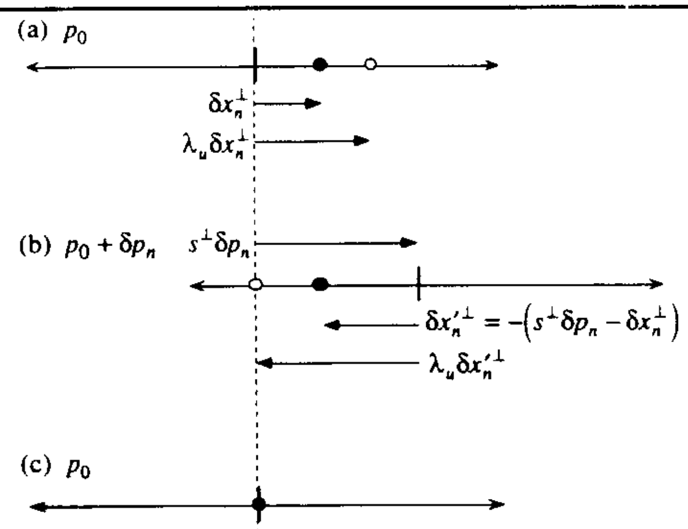
$$\mathbf{x}_{n+1} = \mathbf{F}_M\left(\mathbf{x}_n, p_0 + \delta p_n\right).$$

where the perturbation is

$$\delta p_n = \begin{cases} -k\delta x_n^{\perp}, & \delta p_n \le \delta p_{MAX} \\ 0, & \delta p_n > \delta p_{MAX} \end{cases}.$$

Motion of the map iterates in the vicinity of the fixed point is governed by the stab $|\lambda_s| < 1$ and unstable $|\lambda_u| > 1$ eigenvalues along the stable and unstable eigenvectors of tl linearized map matrix M. The following figure projects the iterates onto the directic perpendicular to the stable direction. OGY demands that the *next* iterate fall on tl stable manifold, so that its distance perpendicular to the stable manifold vanished:

$$\delta x_{n+1}^{\perp} = 0.$$



Previous figure collapsed onto the direction perpendicular to the stable manifold. Projected point, represented by the black dot, would iterate away from fixed point in (a), except perturbation applied in (b) causes it to move onto fixed point in (c).

To accomplish this we must shift the fixed point by

$$s^{\perp}\delta p_n = -\lambda_u \delta x_n'^{\perp},$$

where

$$\delta x_n'^{\perp} = \delta x_n^{\perp} - s^{\perp}\delta p_n.$$

Thus, the necessary perturbation is

$$\delta p_n = \frac{\lambda_u}{\lambda_u - 1}\frac{\delta x_n^{\perp}}{s^{\perp}}.$$

Note that we require $\lambda_u \neq 1$ and $s^{\perp} \neq 0$. The proportionality vector **K** has been chosen so that the eigenvalues of the control matrix **C** vanish in the direction perpendicular to the stable direction and are the same as the eigenvalues of the map matrix **M** in the stable direction.
OGY verified this control strategy in numerical experiments involving the two-dimensional Hénon map. The length of the chaotic transient (or time-to-control) was found to decrease as the maximum allowed parameter perturbation increased. Control in the presence of noise and modeling errors were found to necessitate larger perturbations and sometimes lead to chaotic outbursts as the orbit escaped.

OGY remarked that control of higher-period orbits required merely a simple extension, because a period-*n* orbit of a map is a fixed point (period-1) of a new map which is composited of *n* of the original maps. For example, if x_1 is a period-2 point of a map f,

$$\left.\begin{aligned} x_2 &= f(x_1) \\ x_1 &= f(x_2) \end{aligned}\right\},$$

then x_1 is also a period-1 (fixed) point of the composite map f_2, where

$$x_1 = f(x_2) = f(f(x_1)) \equiv f_2(x_1).$$

(Ditto, Rauseo, Spano: 1990a) quickly confirmed the OGY strategy in a physical experiment by controlling a chaotic system around UPOs of period-1 and period-2, and by switching between them at will. The apparatus consisted of a vertical magnetoelast ribbon clamped at its base. The stiffness of the ribbon was a very nonlinear function an applied magnetic field. Small changes in the magnetic field could soften the ribbc so that it would buckle under gravity. A sinusoidal field would alternately stiffen ar soften it so that, at high enough frequencies (~1Hz), chaotic motion would ensue. Tl position of the ribbon was measured once a forcing period by a light sensor Analysis the resulting "stroboscopic" time series allowed identification of low-order UPOs ar their local dynamics (shift vector and map matrix). Applications of OGY perturbation to a DC magnetic field easily controlled the UPOs. Furthermore, control was found be insensitive to errors in modeling the local dynamics.

(Romeiras, Ott, Grebogi, Ott, Dayawansa: 1992) extended OGY in several direction Control theory provided a more general choice of feedback matrix. The problems implementation in higher dimensions and the stabilization of periodic points with moi than one unstable eigenvalues were addressed. OGY control was illustrated in numerical experiment on a model of the kicked double rotor, whose equations of motic were reduced to a four-dimensional dissipative map.

Stabilizing Unstable Periodic Orbits
OGY with Delay Coordinates

(Dressler, Nitsche: 1992) realized that the original OGY control formula must be modified if the attractor is reconstructed using delay coordinates. In this case, because the coordinates of the delay vector are obtained at different times when different parameters apply, the section map depends on the current parameter *and* on (at least) the previous parameter:

$$\mathbf{x}_{n+1} = \mathbf{F}_M\left(\mathbf{x}_n, p_n, p_{n+1}\right).$$

Further, to avoid the possible instability of a growing perturbation, the OGY control criterion was modified so that the next perturbation be zero *and* the next but one iterate stabilize:

$$\delta p_{n+2} = 0 \quad \& \quad \delta x_{n+2}^{\perp} = 0.$$

As a result, the current perturbation depends recursively on the previous perturbation,

$$\delta p_n = c_1 \delta x_n^{\perp} + c_2 \delta p_{n-1},$$

where the constants c_1 and c_2 enfold information about the local dynamics. This modified control equation was tested in numerical experiments on the Duffing attractor reconstructed from a time series by a three-dimensional delay coordinate embedding. It was found to be superior to the original OGY in that it succeeded in cases where the original failed.

Stabilizing Unstable Periodic Orbits
Highly Dissipative Systems (OPF)

In a highly dissipative system, the contraction in the phase space is so large that th attractor is squeezed down to low dimensions. In cases were the contraction is such tha the attractor can be well described by a one-dimensional map in a two-dimensional dela coordinate embedding (or "time-one return map"), then the OGY technique simplifies t a geometrical algorithm involving Occasional Proportional Feedback (OPF). The ma becomes one-dimensional

$$x_{n+1} = F_M(x_n, p),$$

and the control becomes

$$\delta p_n = \frac{\delta x_n}{s},$$

where $s = dx/dp$. Notice that the feedback is simply proportional to the distance to th periodic point. The OPF technique has proven to be especially useful in controllin high-frequency chaos in diodes and lasers.

Inspired by OGY, (Hunt: 1991) developed the OPF into an analog technique to contr chaos in an electrical circuit consisting of a rectifying diode in series with an inducto (and resistor), which were sinusoidally driven To control, the peak current was sample at the driving frequency and, if it was in a given window, the drive voltage wa modulated with a signal proportional to the current offset. The window's size and th center were adjusted manually in seeking stable orbits. The technique successfull controlled high-period orbits (period 91 and beyond!) on millisecond and faster (53 kH ~1/0.05 ms) time scales. Note, however, that these were generic reductions fro chaotic to periodic; there was no systematic way to stabilize a particular orbit.

(Roy, Murphy, Maier, Gills, Hunt: 1992) utilized the OPF analog variation of OGY t control optical chaos on microsecond time scales. The apparatus was a multimode las with an intracavity period-doubling crystal. A specific orientation of the crystal coul render the laser intensity output chaotic. The source of the chaos was the nonline global coupling of longitudinal modes of the system. In the absence of any periodi forcing, the intensity of the laser was sampled at natural period of relaxation of the las cavity. Using the OPF technique, the chaotic output of the laser was converted t periodic output (at ~100 kHz).

Working independently, (Peng, Petrov, Showalter: 1991) devised and demonstrated OPF in numerical experiments by controlling an open three-variable isothermal self-catalyzing chemical reaction Then (Petrov, Gaspar, Masere, Showalter: 1993) utilized OPF to control the chaotic regime of a famous oscillatory chemical system, the Belousov-Zhabotinsky reaction where, acidified by sulfuric acid, cerium catalyzes the oxidation and bromination of malonic acid. The flow rate of the reactants into and out of a continuously stirred tank determines whether the system exhibits steady, periodic, or chaotic behavior. The attractor was reconstructed from smoothed values of the potential of a bromide electrode. Feedback was applied to the system by varying the flow rates of cerium and bromate (but holding constant the flow rate of malonic acid). Oscillatory behavior of the chemical reaction could be converted back and forth among period-1, period-2, and chaos simply by adjusting the proportional feedback.

Stabilizing Unstable Periodic Orbits
Switching Parametric Feedback

Given a target flow $\mathbf{x}_g(t)$, one such strategy is to make small adjustments $\pm\varepsilon$ to parameter according to whether a variable $x^i(t)$ is too large or too small. Thus,

$$\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x}, p + \delta p(t))$$

with the controller

$$\delta p(t) = \varepsilon \operatorname{sgn}\left(x^i(t) - x_g{}^i(t) \right)$$

where ε is the stiffness of the control signal.

(Singer, Wang, Bau: 1991) used such a discontinuous on-off "switching" feedback controller to stabilize nonstable stationary flows in a physical experiment and its numerical model. The system was a convection loop under thermal and gravity gradients and it was modeled by the Lorenz equations. The apparatus consisted of a water pipe bent into a torus in the vertical plane. The lower half of the pipe was warmed by a resistance heater and the upper half was cooled by a coolant jacket. Vertical and horizontal temperature differences were measured. The system was capable of no motion (at low heating rates), steady clockwise or counter clockwise flow (at higher rates), or chaotic flow (at still higher rates). The chaotic flow was "laminarized" by incrementing or decrementing the heating rate by a preset amount depending upon whether or not the vertical temperature difference was above or below an average value. The control was robust with respect to small amplitude disturbances. However, (Chen, Bau: 1993) commented that switching-type functions can result in control "chattering" and in practice should be replaced with smoother functions.

Stabilizing Unstable Periodic Orbits
Using Recursion (RPF)

OPF can fail if the attractor changes when a control parameter is altered. (Rollins, Parmananda, Sherard: 1993) applied the insights of (Nitsche, Dressler: 1992) to the OPF technique to forge a Recursive Proportional Feedback (RPF) and overcome this problem. The return map becomes

$$x_{n+1} = F_M\left(x_n, p_n, p_{n+1}\right),$$

and the control is

$$\delta p_n = k_1 \delta x_n + k_2 \delta p_{n-1},$$

where the constants k_1 and k_2 enfold the local dynamics. Like the OPF, the RPF is useful in highly dissipative systems where dynamics exhibits a nearly one-dimensional return map, but it can succeed where the OPF will fail, as was demonstrated in numerical simulations of a three-dimensional electrochemical model and a two-dimensional bacterial model.

(Parmananda, Sherard, Rollins, Dewald: 1993) provided experimental verification of the RPF algorithm. The apparatus was an electrochemical cell with a (rotating and dissolving) copper anode and a platinum cathode in a sodium acetate-acetic acid buffer. Successive current minima between the anode and the cathode provided the time series. The potential between the anode and a reference electrode provided the control parameter. RPF was used to stabilize the chaotic dissolution of the anode as a period-1 orbit, corresponding to large amplitude oscillations only.

Stabilizing Unstable Periodic Orbits
with No Accesible Parameters (PPF)

Suppose there is no accessible system parameter. Then try to alter a system variab instead. Suppose the variables can be moved along a specific direction $\mathbf{u}_i$, then if tl map is

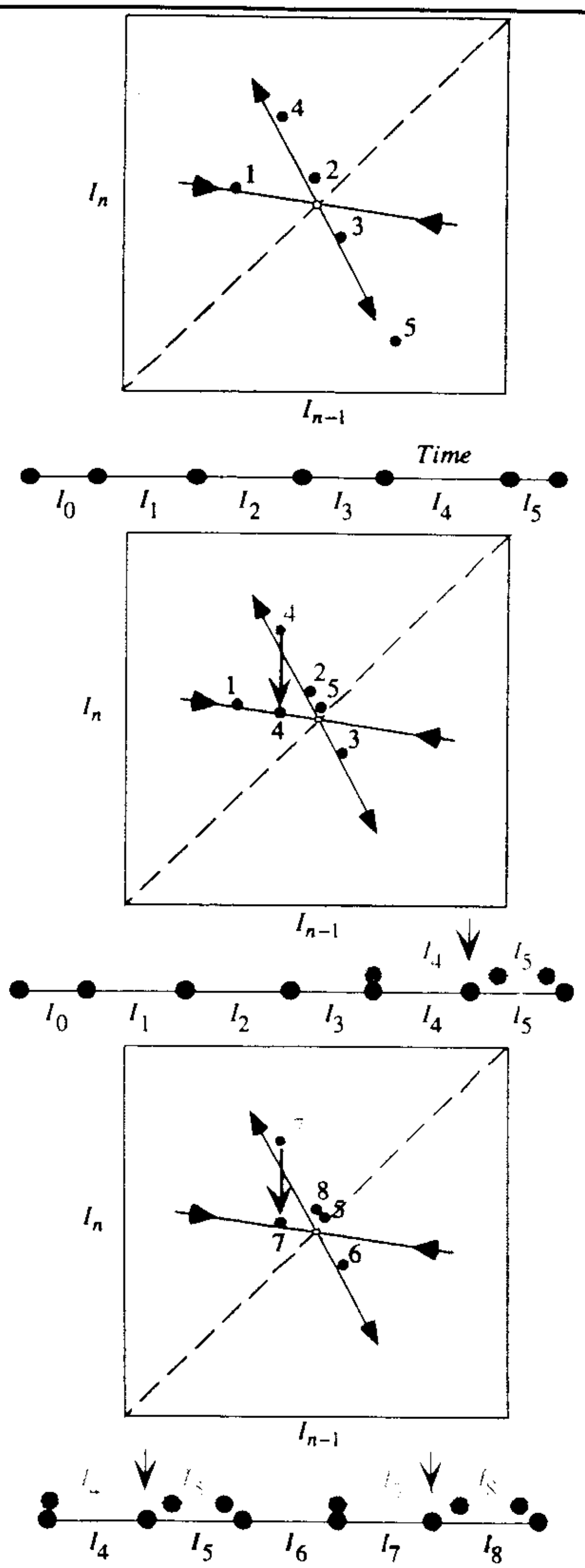
$$\mathbf{x}_{n+1} = \mathbf{F}_M(\mathbf{x}_n),$$

define a new map by

$$\mathbf{x}_{n+1} = \mathbf{F}'_M(\mathbf{x}_n, p) \equiv \mathbf{F}_M(\mathbf{x}_n + p\mathbf{u}_i),$$

and proceed with the OGY strategy, except rather than move the saddle instead move th system state point in the opposite direction.

(Garfinkel, Spano, Ditto, Weiss: 1992) invented this method of chaos control, calle Proportional Perturbation Feedback (PPF), of necessity, when first controlling cardia chaos. In this experiment, a section of a rabbit's heart was induced to beat chaoticall by the injection of the drug ouabain. No adjustable parameter could be modifie quickly enough to implement OGY control. Instead electrical stimuli were used t *shorten* (but not lengthen) interbeat intervals. The advance in the next beat wa calculated to place the state point directly onto the stable manifold of the fixed poin While it was not possible, in these early experiments, to achieve a "normal" period- heartbeat, the chaotic beating was replaced with a period-3 beat, which is much mor efficient at pumping blood than chaotic beating.
The following figure is a return plot of I_n vs. I_{n-1} typically constructed from (hea beat) interval data. Note the presence of an unstable fixed point along the 45 degre line. It demonstrates how to pull the interval down onto the stable direction an consequently onto the unstable fixed point. The intervals between beats and contr stimuli are shown below the return map. (The grayed are without control). The figu depicts the subsequent behavior of the system under control Note the application c stimuli (arrows) only intermittently.



PPF control as applied to cardiac chaos.

Stabilizing Unstable Periodic Orbits
Complex Eigenvalues

The original OGY formula is inappropriate when complex eigenvalues or multipl unstable eigenvalues occur. (Reyl, Flepp, Badii, Brun: 1993) introduced a Minima Expected Deviation (MED) variation of OGY to circumvent these problems. Instead o demanding that the next iterate fall on the stable manifold of the target fixed point, i was required that the next iterate come as close as possible. Minimizing

$$\left\|\delta\mathbf{x}_{n+1}\right\| = \left\|\mathbf{M}\delta\mathbf{x}_n + \mathbf{s}\delta p_n\right\|,$$

results in the new control

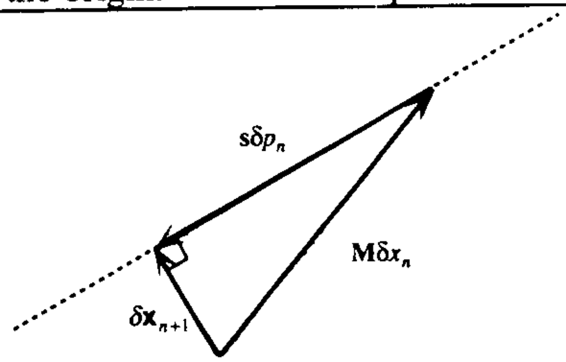
$$\mathbf{s}\delta p_n = -(\mathbf{s}/s)^T\mathbf{M}\delta\mathbf{x}_n$$

(see following figure) or

$$\delta p_n = -\frac{\mathbf{s}^T\mathbf{M}\delta\mathbf{x}_n}{s^2}.$$

Note how the map matrix $\mathbf{M}$ appears in this expression rather than the unstabl eigenvalue λ_u.

The MED successfully controlled a chaotic system whose attractor needed to b embedded in a six-dimensional space. The apparatus studied system was an NMR lase with a sinusoidally modulated quality factor. In this experiment, the MED techniqu proved more reliable than the original OGY technique.



The MED minimization

Stabilizing Unstable Periodic Orbits
Continuous Feedback: External

A slightly different strategy is to apply a *continuous* perturbation that stabilizes without altering a desired UPO. The OGY strategy is based on maps from sections, and hence its calculated small and rare perturbations are applied intermittently, allowing occasional bursts of lost control. In addition, the OGY method can stabilize only those UPOs whose expanding directions are sufficiently small; the time between perturbations must be small compared to the reciprocal of the maximal Lyapunov exponent. (Pyragas: 1992) introduced the strategy of continuous control to overcome these limitations.

Stabilizing Unstable Periodic Orbits

Continuous Feedback: Delayed

One can use negative feedback to modify a variable of the flow based upon the motio of an external oscillator. Let

$$\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x}, p) + \mathbf{u}_i C(t),$$

where the control signal is sensitive to the ith variable

$$C(t) = \varepsilon\left(x_g{}^i(t) - x^i(t) \right),$$

and where the external oscillator goal $\mathbf{x}_g(t) = \mathbf{x}_g(t+T)$ is periodic, and the stiffness $\varepsilon >$ is positive. The critical feature of this control is that it does not change the flow : $\mathbf{x}(t) = \mathbf{x}_g(t)$ is a UPO extracted from the chaotic attractor. Furthermore, whe stabilization obtains $\mathbf{x}(t) \sim \mathbf{x}_g(t)$ and the control perturbation becomes very small.

Because the control is continuous, it is also permanent, and no chaotic bursts resul Further, the control can be initiated at any time, even when the flow is far from th UPO. However, excessively large perturbations can drive some systems to alternativ solutions.

(Qu, Hu, Ma: 1993) numerically investigated controlling chaos by continuous feedbac in the Lorenz and Duffing models. The feedback rule was modified to include diagonal perturbation weight (or stiffness) matrix **M**:

$$C(t) = \mathbf{E}\left(\mathbf{x}_g(t) - \mathbf{x}(t) \right),$$

These numerical experiments suggested that higher-dimensional systems may be controlled by feeding back a single variable, but not always; sometimes it is necessary to feedback several variables.

Stabilizing Unstable Periodic Orbits
Continuous Delayed Feedback

The previous strategy can be simplified by replacing the external periodic oscillator by a delayed output signal from the system itself. Let

$$\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x}, p) + \mathbf{u}_i C(t),$$

where the control signal is now

$$C(t) = \varepsilon\left(x^i(t-\tau) - x^i(t)\right),$$

where τ is the delay time. Notice that if $\tau = T$ control does not change the flow if $\mathbf{x}(t) = \mathbf{x}_g(t)$. Again, the hope is that appropriate choice of stiffness can stabilize the UPO.

(Pyragas: 1992) achieved such stabilization for the Rössler system and the Duffing oscillator in numerical experiments. The advantage is that no computer or external perturbation is needed to perform the control. Unfortunately, for the Rössler system, the delayed perturbations can stabilize *multiple* orbits, not just the desired orbit, depending upon the initial conditions.

(Pyragas, Tamasevicius: 1993) succeeded in using continuous delayed feedback to control a physical system, namely an analog electrical circuit using a simple delay line. The experiment required no computer analysis of the system and was robust to noise.

Stabilizing Unstable Periodic Orbits
Continuous Feedback: Synchronized

(Pyragas: 1993) combined the above ideas of controlling chaos with the idea of synchronizing chaos suggested by (Pecora, Carroll: 1990) to synchronize the current behavior of a system with its past behavior. Here the continuous feedback perturbation is proportional to the difference between the current and past output signals of one of the system variables. In this way, *aperiodic* orbits were controlled in numerical experiments on the Duffing, Lorenz, and Rössler systems.

Entrainment of Chaos to Goal Dynamics

If an accurate model of the system is available, then one approach is to entrain the phase space flow, or system dynamics, into a goal dynamics $\mathbf{x}_g(t)$ by adding a controlling term $\mathbf{C}(t)$ such that

$$\lim_{t \to \infty} \left| \mathbf{x}(t) - \mathbf{x}_g(t) \right| = 0.$$

For example, if we take the phase space control to be additive

$$\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x}, p) + \mathbf{C}(t),$$

with a governing system

$$\mathbf{C}(t) = \dot{\mathbf{x}}_g - \mathbf{F}\left(\mathbf{x}_g, p\right),$$

then clearly $\mathbf{x}(t) = \mathbf{x}_g(t)$ is *a* solution. However, the challenge is to find such a solution that is stable and has a nonzero basin of entrainment.

(Jackson, Hübler: 1990) studied these issues by numerically entraining orbits of the logistics map to periodic sets. Both the measure (size) and connectivity of the basin of entrainment were found to depend on the mapping in a complex way.

Parametric Control Dynamics

Another approach is to apply an error signal to control a perturbed parametei (Huberman, Lumer: 1990) suggested coupling the flow

$$\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x}, p(t)),$$

to a parameter dynamics governed by the error signal,

$$\dot{p} = -\varepsilon\left(x_g{}^i - x^i\right),$$

where x^i is the ith variable and ε is the stiffness of the control. The error signa encourages the parameter of the system to readjust so as to reduce the error to zero.] *adaptively* controls the system. Note that the error signal is proportional to th difference between an actual and goal *variable*; alternately, the error could be taken t be proportional to the difference between an actual and goal *parameter*.

(Sinha, Ramaswamy, Rao: 1990) extended and generalized this strategy in a number of ways. Numerical experiments suggested that parametric control is robust with respect to noise and is indifferent to the exact form of the control function. Recovery times were consistently found to be inverse to the stiffness of the control.

Control with Perturbations

Variable Periodic perturbations

Periodic or stochastic (random) perturbations applied to either the variables or the parameters of a nonlinear system may remove or suppress chaotic behavior. This strategy is often easy to apply but is not goal-oriented; the final state is untargeted and unpredictable.

Periodic Perturbations of a Variable

Consider suppressing chaos by weak periodic external forcing. In this scheme, the phase space flow is modified by periodically perturbing (modulating) one of the components,

$$\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x}, p) + \mathbf{u}_i A \cos \omega t,$$

where $\mathbf{u}_i$ is a unit vector in the ith direction. This is a simple and often easy way to adjust a system. However, the adjustments may need to be large, and they are not goal-oriented.

(Braiman, Goldhirsch: 1991) used this technique to tame the chaos of a periodically driven damped pendulum in a numerical experiment, as measured by a decrease in the leading Lyapunov exponent. This study emphasized that a chaotic system's defining sensitivity is to weak time-dependent perturbations as well as to initial conditions.

Control with Perturbations

Parametric Periodic perturbations

It is well known that an inverted pendulum can be stabilized by oscillating its bas Consider then removing or suppressing chaos by resonant parametric perturbatior Imagine modifying the phase space flow by periodically perturbing one of i parameters,

$$\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x}, p + \delta p \cos \omega t).$$

(Lima, Pettini: 1990) first proposed this technique and succeeded in removing chaos a numerical experiment involving the Duffing oscillator. It was found that parametı perturbations could convert chaotic dynamics to regular dynamics provided tl perturbation amplitude was sufficiently large (~1% of the unperturbed parameter) *aı* provided the perturbation frequency was in resonance with the forcing frequency of tl oscillator.

(Fronzoni, Giocondo, Pettini: 1991) used periodic parametric perturbations to suppre chaos in a physical experiment that can be modeled as a Duffing oscillator. Tl apparatus was a physical realization of the Duffing differential equation. One end of ; elastic steel beam was clamped while the other end was free to move in the double-wı potential created by two magnets. When the apparatus was electromagnetically shake sustained chaotic motion of the beam was observed. The magnetic fields, and hence tl shape of the potential, were modulated sinusoidally by varying the current throu; copper coils wrapped around the magnets. Small modulations at particular resona frequencies were successful in regularizing the motion.

(Azevedo, Rezende: 1991) succeeded in suppressing chaos in a microwave-pumped spin-wave-instability experiment. The apparatus consisted of a small polished ferromagnetic sphere at the center of a microwave cavity placed between the poles of a biasing magnet. For certain parameters, the amplitude of the reflected microwave radiation developed chaotic oscillations due to nonlinear interactions between spin-wave modes. However, a small modulation of the biasing magnetic field, induced by current flowing in a small copper loop inserted into the cavity, was able to suppress this chaos. The reduction in chaos was measured by the decrease in the (information) dimension of the chaotic attractor reconstructed from the time series using delay coordinates, and by the change in the power spectra of the oscillations.

Control with Perturbations
Stochastic perturbations

Consider suppressing chaos by the simple device of adding external noise,

$$\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x}, p) + \mathrm{N}(t),$$

where $\mathrm{N}(t)$ is a stochastic process.

In the review (Rajasekar, Lakshmanan: 1993), the Bonhoeffer-Van der Pol oscillator is numerically integrated in a chaotic regime. Gaussian random numbers are added every time step. For sufficiently large spread (standard deviation) of the noise, the maximal Lyapunov exponent becomes negative and the system loses its sensitive dependence on initial conditions. Interestingly, although the added noise appears to smear out the attractor, its (correlation) dimension remains nonintegral; it becomes a *strange nonchaotic* attractor.This is normal

Improvements to Control of Chaos Techniques
Targeting

The exponential sensitivity of chaotic systems to tiny perturbations can be used to rapidly direct orbits from an arbitrary initial state to a desired state. Different trajectories arising from small changes in parameters or initial conditions may be patched together to come arbitrarily close to a desired target.
(Shinbrot, Ott, Grebogi, Yorke: 1990) first introduced this strategy and verified it in a numerical experiment involving the Hénon map. Targeting was shown to be effective if the noise and modeling errors were sufficiently small. However, to date it has been achieved experimentally in only one system. (Shinbrot, Ditto, Grebogi, Ott, Spano, Yorke: 1992) successfully targeted states in the magnetoelastic ribbon, the attractor of which was low-dimensional.

Improvements to Control of Chaos Techniques

Tracking

Several groups have experimentally demonstrated that UPOs can continue to be controlled while system parameters are continuously varied, even through bifurcations. Heretofore unobtainable phase space regions can be reached with this method. In addition, tracking may be critical to chaos control applications in the heart and brain, biological systems which are typically nonstationary.

(Carroll, Triandaf, Schwartz, Pecora: 1992) tracked a UPO of a Duffing-like electrical circuit as they slowly drove the system through a bifurcation. (Gills, Iwata, Roy, Schwartz, Triandaf: 1992) extended the range over which OPF stabilized a multimode laser by an order of magnitude by tracking an unstable steady state as the pump excitation was slowly varied. This boosted the power output 15-fold. Recently, (In, Ditto, Spano: 1994) achieved tracking in a mechanical system, a magnetolastic ribbon, using the full OGY technique.

Control of Higher Dimensional Chaos

The dynamics of some systems is not confined to a lower-dimensional attractor but roams over many dimensions. Control of such systems is more difficult due to their complicated nature. For example, OGY control may not be feasible due to the large amount of information (shift vector, map matrix) needed from the data. (Auerbach, Grebogi, Ott, Yorke: 1992) extended the OGY scheme in an attempt to overcome this problem. This still model-independent extension requires knowledge only of a single time series. It has succeeded in controlling higher-dimensional chaos in a numerical experiment involving the kicked double rotor. However, unless the contracting directions are very strong, significant information must still be extracted from the time series, and this technique has still not succeeded in controlling a physical system.