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WORKSHOP ON QUANTUM DISSIPATION AND APPLICATIONS

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"Quantum Dynamics of a Vortex in a Josephson Junction Array"

presented by:

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Quantum array of Josephson Junctions close to the Superconductor - Insulator transition

Introductory to :

Quantum dynamics of a vortex in a JJA

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Content :

a) vortices in classical JJA

b) Charging and quantum JJA

c) superconducting - insulating transition

d) effect of external charges on the transition

Next :

Quantum dynamics of a vortex in a JJA

a') Real time effective action for the vortex motion

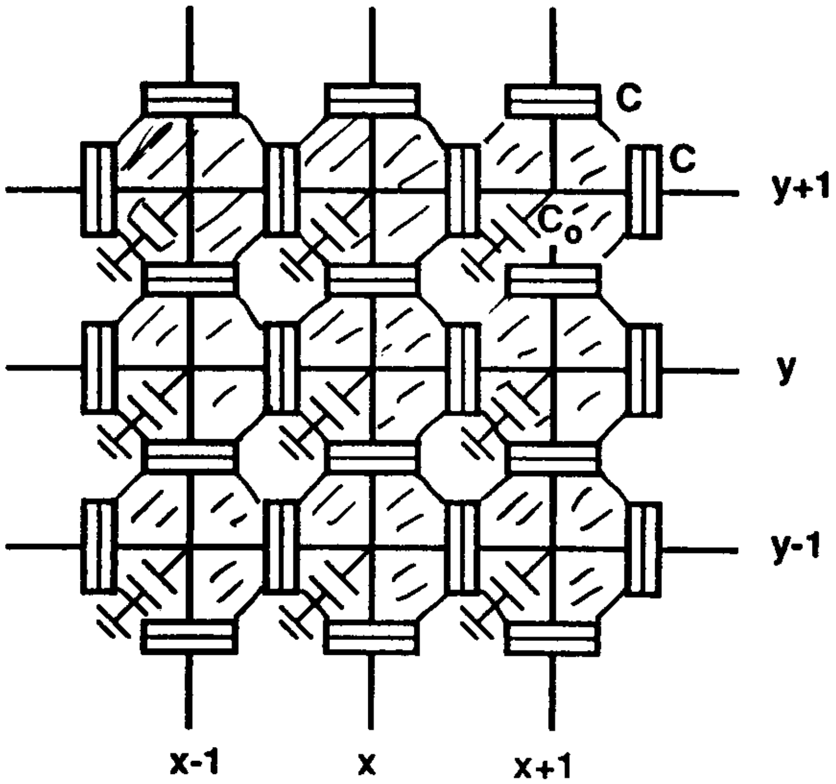
b') Diffusive motion and ballistic motion

c') Linear response to an external force

d') Radiating vortex

e') Dynamical mass

f') Vortex in a ring : Aharonov Casher effect



2D Josephson array

Classical fluctuations of the phase in a Josephson Array

(square lattice)

$$H = -J \sum_{(ij)} \cos(\theta_i - \theta_j)$$

High T:

$$< e^{i\theta_a} e^{-i\theta_b} > \sim e^{-\alpha d_{ab}} \qquad \beta \to 0$$

d_{ab} : minimal distance between a, b on the lattice

$\alpha = | \ln N_{ab} \frac{\beta J}{2} |$ $\qquad$ N_{ab} : number of path of minimal length connecting a, b

Very low T: spin wave approximation :

$$Z_{sw} = \int \prod_q d\theta_q \; \exp -\beta J \sum_q \theta_q \, G^{-1}(q) \, \theta_{-q}$$

$$G_q^{-1} = 4 - 2(\cos q_x + \cos q_y)$$

$$< e^{i\theta_a} e^{-i\theta_b} >_{sw} \sim \left(\frac{1}{d_{ab}} \right)^{1/2\pi\beta J} \qquad \beta \to \infty$$

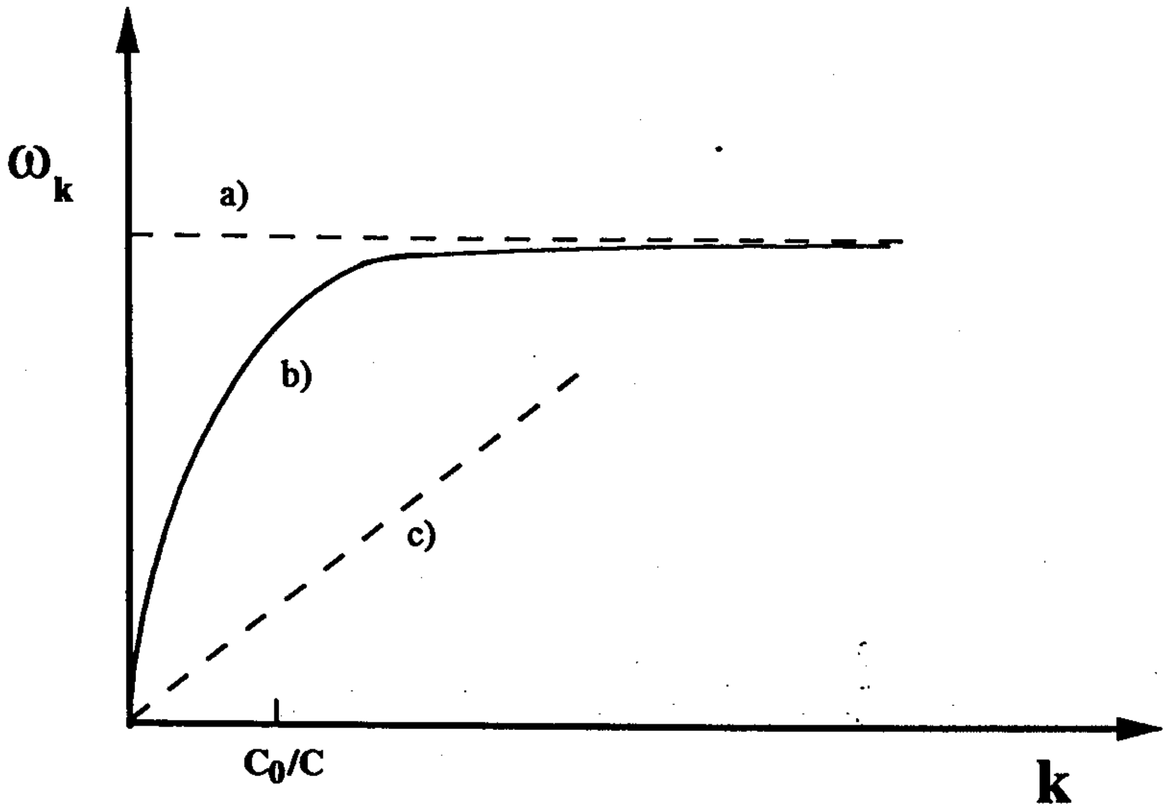


Fig. 6.

Intermediate temperatures

Continuum limit
$$S = \frac{1}{2}\beta J \int d^2r \, (\nabla\theta)^2$$

A solution of $\nabla^2\theta = 0$ (except at the origin) is
$$\theta_v = \operatorname{arctg} \frac{y - y_0}{x - x_0}$$

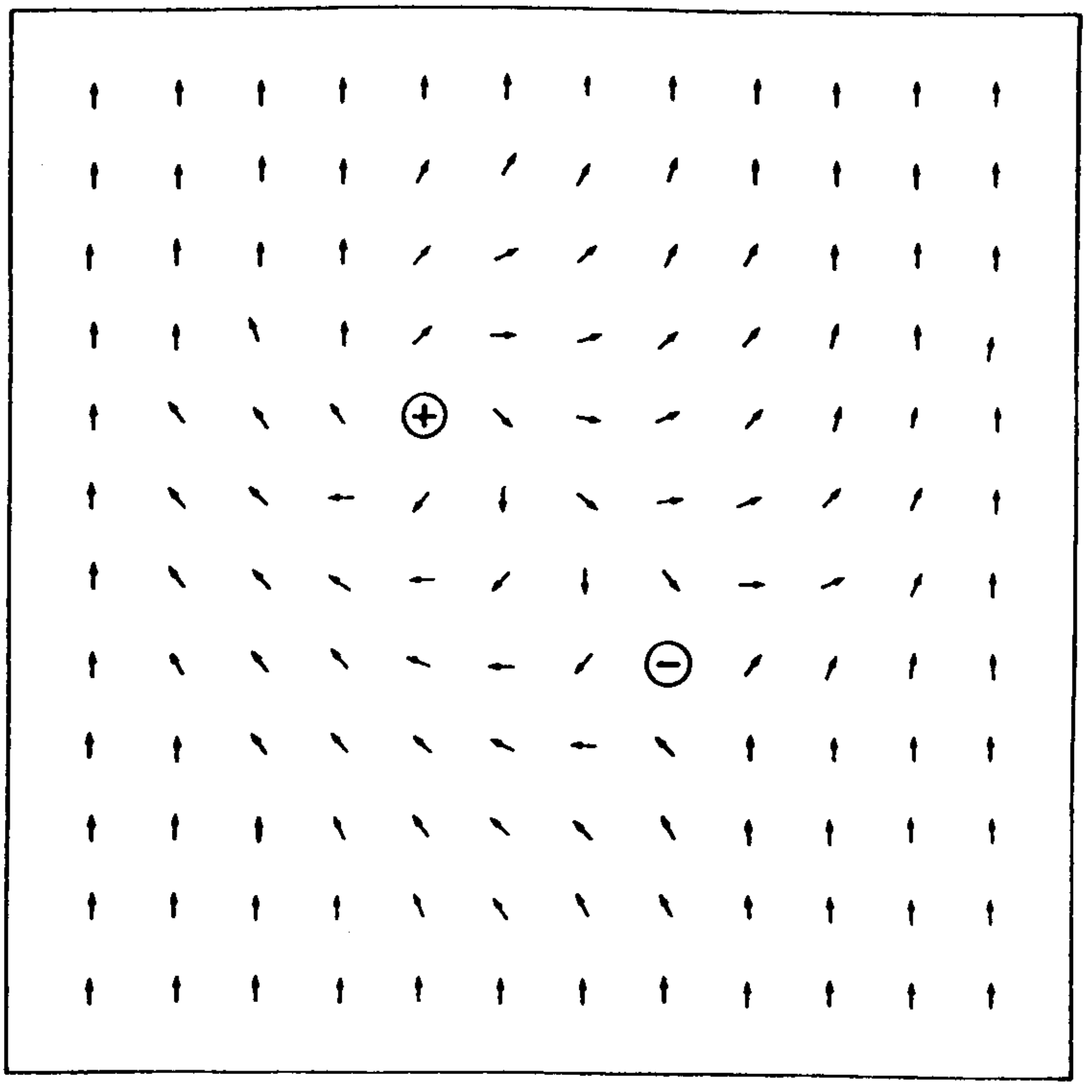
A vortex – antivortex pair can be written as:

$(z = x + iy)$

$$e^{i\theta} = \frac{(z-1)/|z-1|}{(z+1)/|z+1|}$$

at centers +1 (−1) respecti

In general given vortices (or antivortices) of charge $v_j = \pm 1$ located at z_j, the θ configuration is given by
$$\rho e^{i\theta} = \prod_j (z - z_j)^{v_j}$$



Vortex - antivortex pair

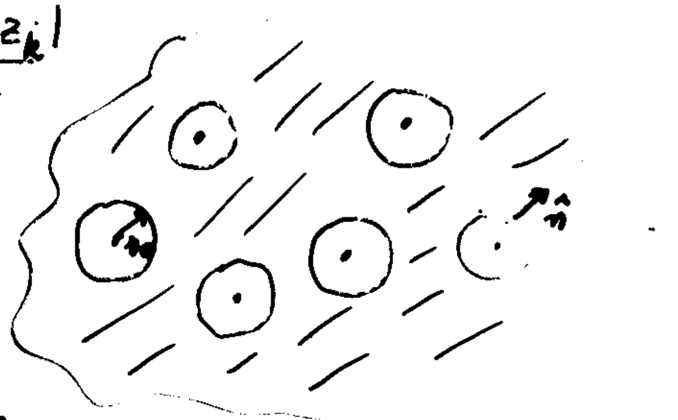
$$\int_{\mathcal{C}_j} \vec{\nabla}\theta \cdot d\vec{\ell} = 2\pi v_j$$

How to rewrite the action in terms of vortices:

$$\rho e^{i\theta} = \prod_{j=1}^{2n} \left(\frac{z-z_j}{L}\right)^{v_j} \qquad v_j = \pm 1$$

vortices centered at z_j

Exclude small circles $\mathcal{C}_j$ of radius r_0 from the domain $\mathcal{D}$



θ and $\ln\rho$ $\left(= \sum_k v_k \ln\left|\frac{z-z_k}{L}\right| \right)$ are harmonic functions in $\mathcal{D} - \bigcup_j \mathcal{C}_j$

Cauchy Riemann conditions give $(\vec\nabla\theta)^2 = (\vec\nabla \ln\rho)^2 = \vec\nabla\cdot(\ln\rho\,\vec\nabla\ln\rho)$

$$S(\{q_j, z_j\}) = -\frac{1}{2}\beta E_J \sum_j \int_{\mathcal{C}_j} r_0\, d\theta_j \ln\rho\; \hat n\cdot\vec\nabla\ln\rho$$

$\vec\nabla \ln|z_i - z_j| \sim 0$ for $i \neq j$. For the gradient only terms with $k=j$ are relevant. Besides

$$\hat n\cdot\vec\nabla\ln\rho\,\Big|_{\mathcal{C}_j} = v_j/r_0$$

Coulomb gas picture :

$$S = -\pi\beta J \sum_{i\neq j} v_i v_j \ln \frac{|z_i - z_j|}{d} + \pi\beta J \left(\sum_i v_i\right)^2 \ln \frac{L}{d}$$

$$+ \mu_0 \sum_i v_i^2$$

Neutral gas : $\sum_i v_i = 0$

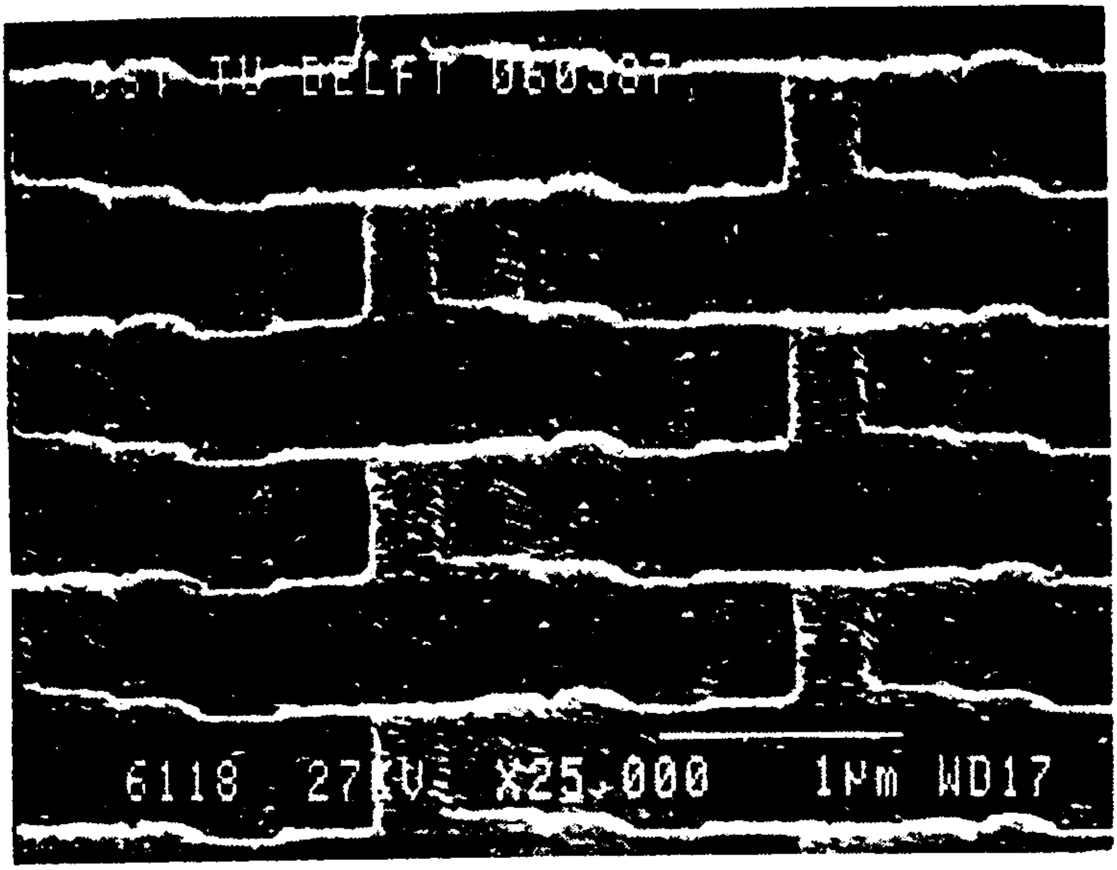
A single vortex can be created in the system when the cost is finite :

$$F_{1\,\text{vortex}} = \pi J \ln \frac{L}{d} - T k_B \ln \left(\frac{L}{d}\right)^2 \rightarrow \text{configurational entropy.}$$

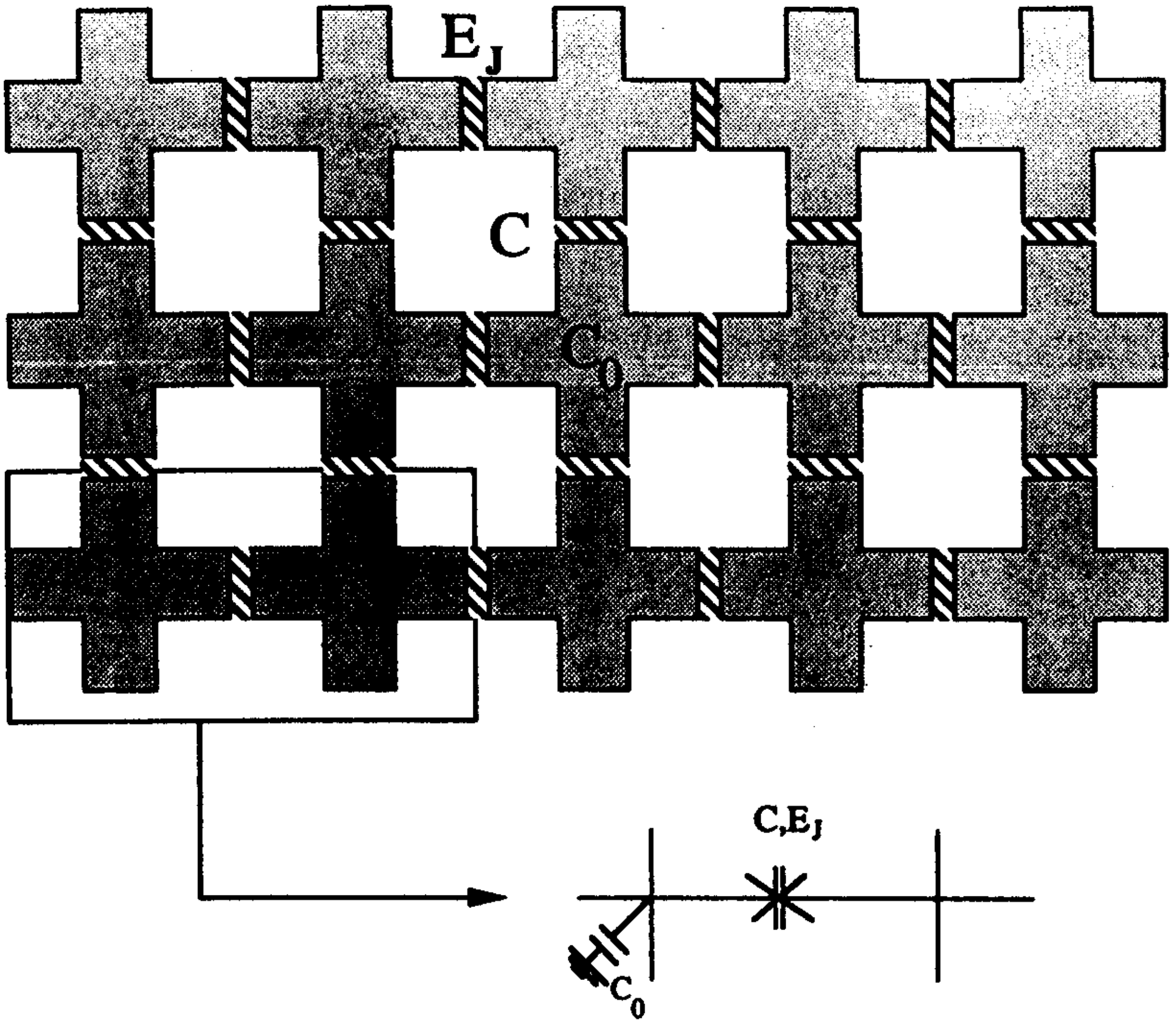
$$= (\pi\beta J - 2)\, k_B T \ln \frac{L}{d}$$

When $L \to \infty$, $F \to c$ for $\beta J < \frac{2}{\pi}$

$$T_{BKT} = \frac{\pi E_J}{2 k_B} \quad : \text{unbinding transition}$$

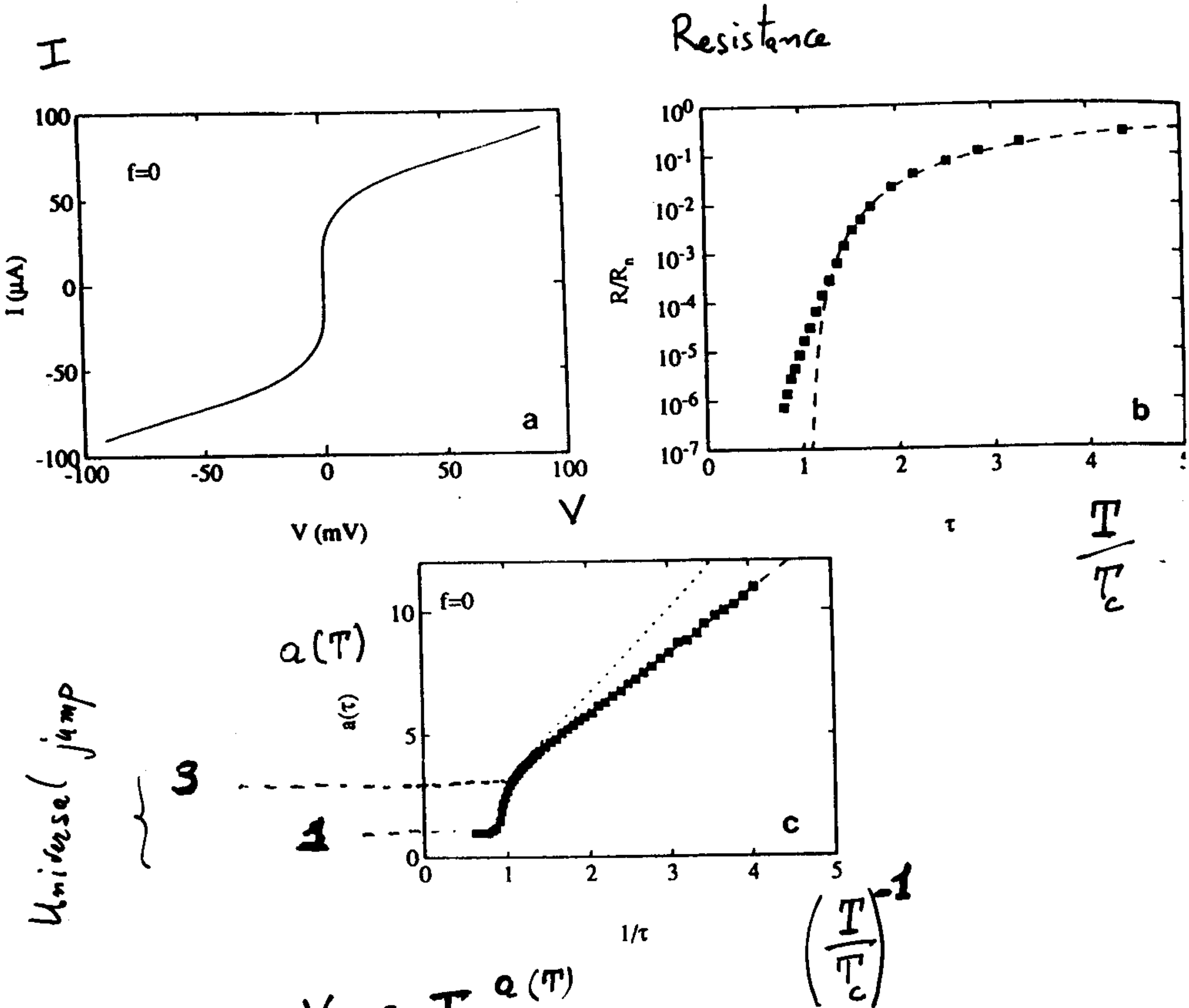


. SEM picture of a fabricated array with junctions with $C \approx 10^{-15} F$.



E_J
C
C_0
C,E_J
C_0

Classical Josephson array



$$V \propto I^{a(T)}$$

non linear resistance

2D Quantum JJA

if junctions are high quality : no shunts

or grains are capacitively coupled : C, C_0

If the charge is not quantized : $\theta_i \in (-\infty, +\infty)$

$$Z = \prod_i \int \mathcal{D}\theta_i(\tau) \; \exp -\frac{1}{\hbar}\int d\tau \left\{ \frac{1}{2}\left(\frac{\hbar}{2e}\right)^2 \sum_{ij} \dot{\theta}_i C_{ij} \dot{\theta}_j + \right.$$

$$\left. - E_J \sum_{(ij)} \cos(\theta_i - \theta_j) \right\}$$

$$C_{ii} = C_0 + 4C$$

$$C_{ij} = -C \quad (i,j \in n.n) \qquad 0 \quad \text{otherwise}$$

$$C(k) = C_0 + C k^2$$

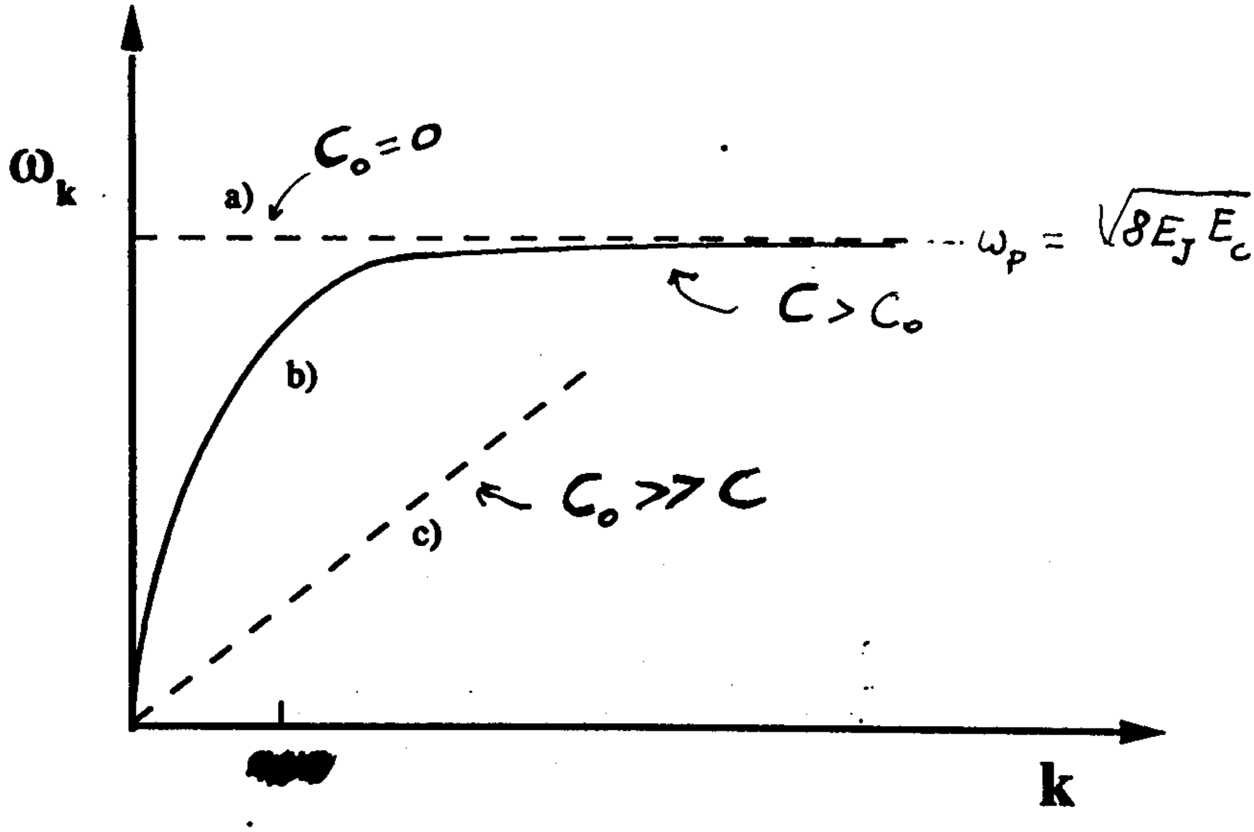
Interaction between charges :

$$C^{-1}(|i-j|) = \frac{1}{2\pi C} K_0(|i-j|/\lambda)$$

$\lambda = \sqrt{C/C_0}$: two possible limits :

short range $\quad C \ll C_0$

long range $\quad C \gg C_0$



Spin waves spectrum

on site inter.: $C >> C_0 \quad \Rightarrow \quad \omega_k = \omega_p$ optical

long range: $C_0 >> C \quad \Rightarrow \quad \omega_k = \bar{\omega}_p \, k$ acustical

$$\bar{\omega}_p = \sqrt{8 E_J E_0}$$

$$E_c = \frac{e^2}{2C} \qquad E_0 = \frac{e^2}{2C_0}$$

[redacted]

Bejond spin waves approximation

Villain approx can be used :

$$e^{E_J \sum_{(ij)} (1-\cos(\theta_i-\theta_j))} \sim$$

$$\sim \sum_{\{v_i\}} e^{-\frac{E_J}{2}\sum_{i,\delta}(\theta_{i+\delta}-\theta_i-2\pi v_i)^2}$$

when θ_i is integrated out $\Longrightarrow$ interaction between vortices mediated by spin waves.

At $T\to 0$ and E_J large vortices are not excited

$\Longrightarrow$ renormalization of E_J

Ex : $C_0 >> C$

$$T\to 0 \quad : \quad \langle e^{i\theta_i(\tau)} e^{-i\theta_j(\tau)} \rangle \sim (|i-j|)^{-1/2\pi\beta E_J^*}$$

$$E_J^* = E_J(1-\beta E_0/24)$$

transition temperature is pushed down to lower values.

(transition is still BKT)

$T=0$: dimensional crossover $\Rightarrow$ now $\langle \cos\theta_i \rangle \neq 0$

$T=0$ $\quad \langle e^{i\theta_i(\tau)} e^{-i\theta_j(\tau)} \rangle \sim \exp - \kappa \sqrt{\frac{8E_0}{E_J}} \frac{1}{|i-j|}$

$|i-j| \to \infty$

Qualitative : mean field approach :

$$H_{MF} = \frac{1}{2} \sum_{ij} Q_i C_{ij}^{-1} Q_j - \frac{z}{2} E_J \langle \cos\theta \rangle \sum_j \cos\theta_j$$

$$\phi \equiv \langle \cos\theta \rangle = \mathrm{Tr} \left\{ \cos\theta_i \, e^{-\beta H_{MF}} \right\}$$

$$\Rightarrow \left(1 - \frac{1}{2} z E_J \int_0^\infty d\tau \langle \cos\theta_i(\tau) \cos\theta_i(0) \rangle_{ch} \right) \phi +$$

$$+ \left(\frac{z E_J}{E_0} \right)^3 ⊗ \phi^3 = 0$$

$$\langle \cdots \rangle_{ch} = \langle \cdots e^{-\frac{1}{2}\beta \sum_{ij} Q_i C_{ij}^{-1} Q_j} \rangle \frac{1}{Z_{ch}}$$

If charging is absent : $\langle \cos\theta_i(\tau) \cos\theta_i(0) \rangle \simeq 1$

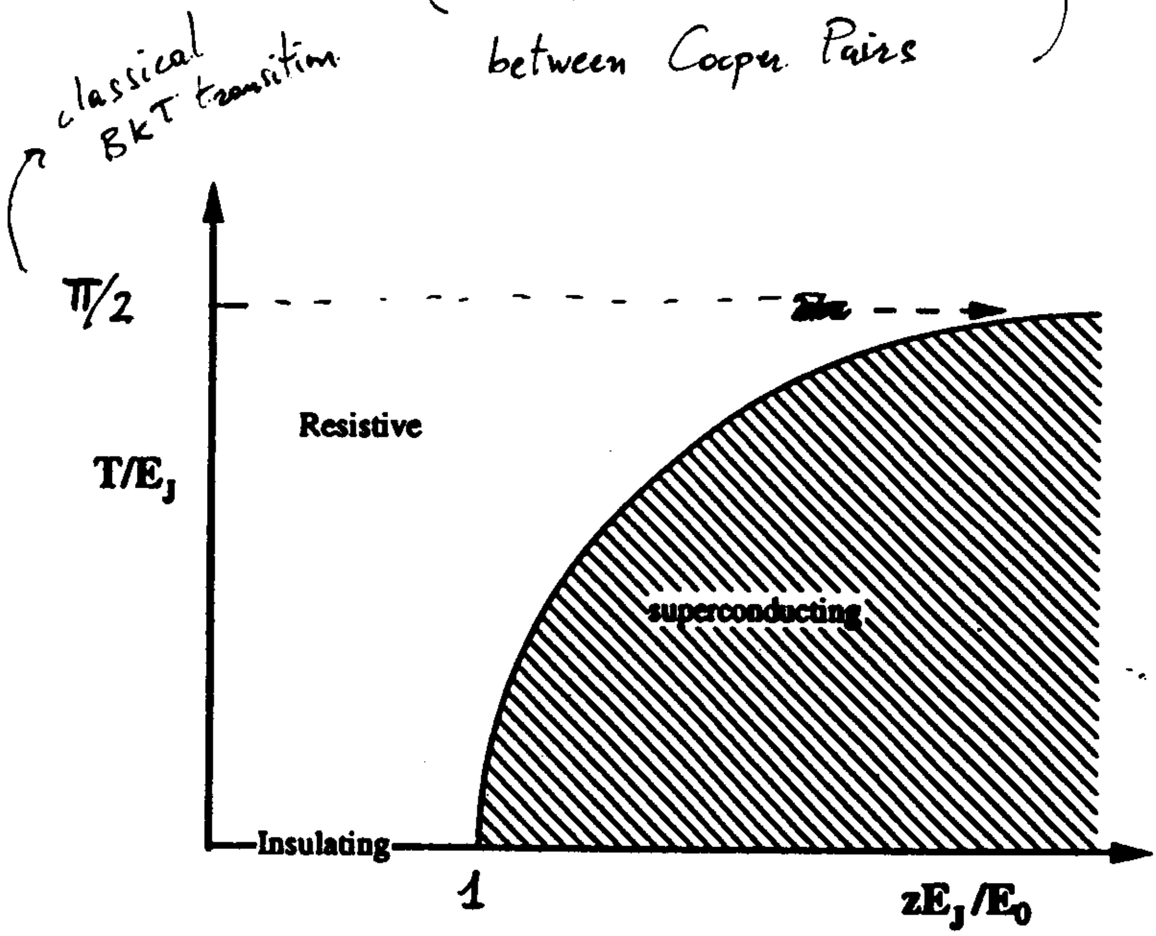
$\Rightarrow \phi$ always non zero

When $\langle \cos\theta_i(\tau) \cos\theta_i(0) \rangle_{ch} = \exp - E_0 \tau$

$\Longrightarrow$ critical point at $z E_J / 2E_0$

Phase diagram for $C_0 >> C$

(onsite Coulomb interaction between Cooper Pairs)



Charge fluctuations also provide a <u>kinetic energy</u> for the vortices :

Think classically : moving vortices generate a voltage pulse in crossing a junction

$V_i = \frac{\hbar}{2e} \dot{\theta}_i$ which, in turn, induces a charge :

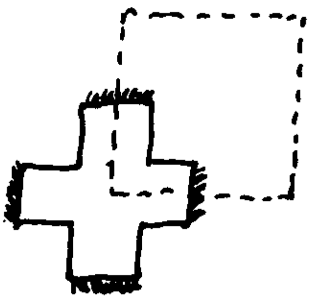
$$\nabla^2 V = - \frac{4\pi}{\epsilon_r} \rho \implies$$

$$\implies d^{-2} \Big[\sum_\delta V_{i+\delta} - 4 V_i \Big] = - \frac{4\pi}{\epsilon_r} \frac{Q_i^v}{\sigma d}$$

(σ → junction area; d → thickness of the insulating layer)

$$\implies Q_i^v = C \Big[\sum_\delta V_{i+\delta} - 4 V_i \Big] \qquad C = \frac{\epsilon_r \sigma}{4\pi d}$$

Note ! Integration over the island area yelds circulation in the dual lattice :



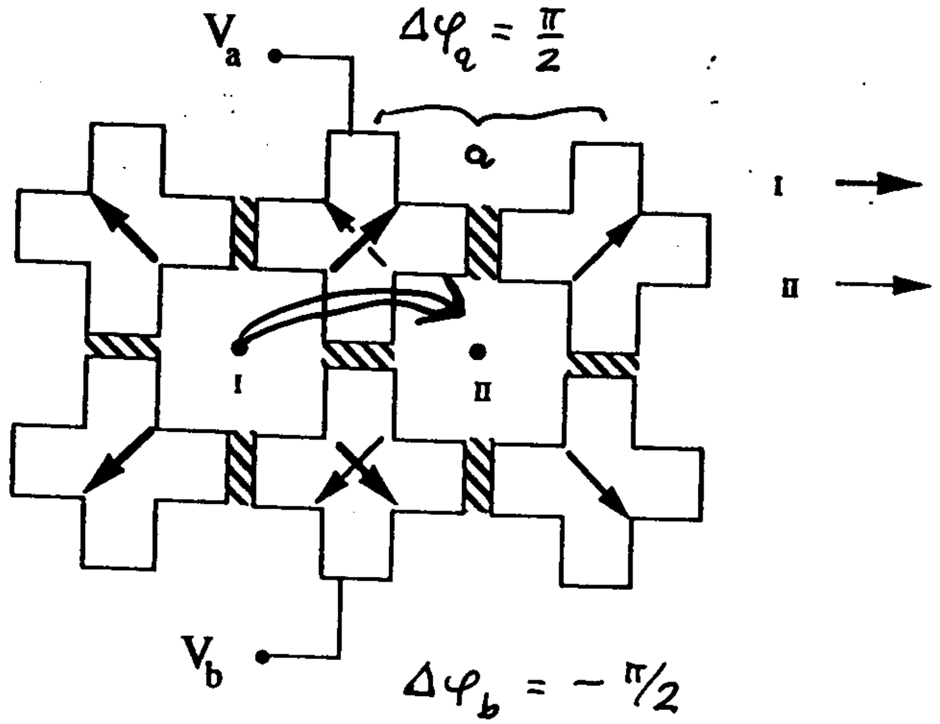
$$\int_{✠} \nabla^2 \dot{\theta} \, d^2 r = \int d^2 r \, \vec{\nabla} \cdot (\vec{\nabla} \dot{\theta}) =$$

$$= \int d\ell \, \hat{n} \cdot \vec{\nabla} \dot{\theta} = \int_{\mathcal{C}_j} d\vec{\ell} \cdot \vec{\nabla} \dot{\theta} = 2\pi \dot{v}_j$$

$$\rightarrow Q_i^v = 2\pi \frac{\hbar}{2e} C \dot{v}_i$$ ($\dot{v}_i$ → moving vortex !)

The charging energy : $\frac{1}{2} \sum_{ij} Q_i^v C_{ij}^{-1} Q_j^v = \frac{1}{2} \left(\frac{\hbar \pi}{e} C \right)^2 \sum_{ij} \dot{v}_i C_{ij}^{-1} \dot{v}_j$

Moving vortex from I to II leaves a phase difference of π behind :



$$V_a - V_b = \frac{\hbar}{2e}\frac{\delta\varphi}{\delta t} = \frac{\hbar}{2e}\pi\,\frac{\dot{x}(t)}{a}$$

$$E_{kin} = \frac{1}{2}\sum_{i\neq j} V_i C_{ij} V_j \sim \frac{\hbar^2}{4e^2} C\left(\frac{\delta\varphi}{\delta t}\right)^2 = \frac{1}{2} M_{ES}\,\dot{x}^2$$

$$M_{ES} = \frac{\hbar^2\pi^2}{4E_c a^2} \sim 10^{-3}\, m_e$$

Full quantum picture:

charge is quantized

Coupled Coulomb gases → vortices, charges

θ_i are defined on a circle: θ_i and $\theta_i + 2\pi m_i$ are equiv

$$Z \propto \sum_{\{m_i\}} \prod_i \int_{\theta_{i0}}^{\theta_{i0}+2\pi m_i} \mathcal{D}\theta_{i\tau} \; e^{-\frac{\epsilon}{\hbar}\sum_\tau \frac{1}{2}\left(\frac{\hbar}{2e}\right)^2 \sum_{ij} \dot\theta_i C_{ij} \dot\theta_j} \cdot e^{-\sum_\tau \epsilon E_J \sum_{(ij)} \cos(\theta_i - \theta_j)}$$

How to deal with the cosine?

In the large E_J limit: $\quad b = \epsilon E_J / \hbar$

$$V(\theta) = b(1-\cos\theta)$$

$$e^{-V(\theta)} = e^{-b} \sum_{J=-\infty}^{+\infty} e^{-iJ\theta - \tilde V(J)} \qquad \text{(J integer)}$$

$$e^{-\tilde V(J)} = \int_0^{2\pi} \frac{d\theta}{2\pi}\, e^{-b}\, e^{iJ\theta + b\cos\theta} = e^{-b} I_J(b)$$

$$\underset{b\to\infty}{\sim} e^{-J^2/2b} \big/ (2\pi b)^{1/2}$$

$$\Rightarrow \exp -\frac{\epsilon E_J}{\hbar} \sum_\tau \sum_{i,\delta} \left(1 - \cos\left(\theta_{i+\delta,\tau} - \theta_{i\tau}\right)\right)$$

$$\sim \sum_{\{J_{i\tau}\}} e^{-\sum_{i\tau}\left[i\, \vec J_{i\tau} \cdot \vec\nabla \theta_{i\tau} - \frac{|J_i|^2 \hbar}{2\epsilon E_J}\right]}$$

Kinetic energy can also be transformed as:

$$e^{-\frac{1}{2}\int d\tau \frac{\hbar}{(2e)^2} \sum_{ij} \dot{\theta}_i C_{ij} \dot{\theta}_j} =$$

$$= \prod_i \int_{n_{i0}}^{n_{i0}} \mathcal{D} n_i(\tau)\, e^{-\int d\tau \frac{2e^2}{\hbar} \sum_{ij} n_i(\tau) (C^{-1})_{ij} n_j(\tau)} \cdot e^{-i \sum_i \int n_i \dot{\theta}_i \, d\tau}$$

Thus, collecting everything:

$$Z = \prod_i \int_{n_{i0}}^{n_{i0}} \mathcal{D} n_i(\tau)\, e^{-\int d\tau \frac{2e^2}{\hbar} \sum_{ij} n_i(\tau) C^{-1}_{ij} n_j(\tau)} \cdot$$

$$\cdot \sum_{\{m_i\}} \prod_i \int_{\theta_{i0}}^{\theta_{i0}+2\pi m_i} \mathcal{D}\theta_{i\tau}\, e^{i \sum_i \int n_i \dot{\theta}_i \, d\tau} \cdot \sum_{\{J_{i\tau}\}} e^{-\sum_{i\tau} \left[i \vec{J}_{i\tau} \cdot \vec{\nabla} \theta_{i\tau} + \frac{|J_{i\tau}|^2 \hbar}{2\epsilon E_J} \right]}$$

But:

$$\prod_i \int_{\theta_{i0}}^{\theta_{i0}+2\pi m_i} \mathcal{D}\theta_{i\tau}\, e^{-i \sum_{i\tau} (\dot{n}_{i\tau} - \vec{\nabla} \cdot \vec{J}_{i\tau}) \theta_{i\tau}} =$$

$$= \prod_i \prod_\tau \delta(\dot{n}_{i\tau} - \vec{\nabla} \cdot \vec{J}_{i\tau})$$

$$\int_0^{2\pi} \frac{d\theta}{2\pi} e^{i(m-n)\theta} = \delta_{m,n}$$

Imposing the constraint on the integers $J_{i\tau}$

$$Z = \prod_i \int_{n_{i0}} \mathcal{D}n_{i\tau} \sum_{\{\vec{J}_{i\tau}\}}{}' e^{-\epsilon \sum_{ij\tau} \frac{2e^2}{\hbar} n_{i\tau} C^{-1}_{ij} n_{j\tau}} \cdot$$

$$\cdot e^{-\sum_{i\tau} \frac{\hbar}{2\epsilon E_J} |J_{i\tau}|^2}$$

Constraint is solved by the condition on each plaquette:

$$J^{(\hat{\delta})}_{i\tau} = \int_i^{i+\hat{\delta}} \dot{n}_{i\tau}\, d\vec{\ell} + \epsilon^{(\delta\nu)} \nabla_\nu A_{i\tau}$$

($A_{i\tau}$ → integers!)

$$|\vec{J}_{i\tau}|^2 = \sum_\delta \left(\int_i^{i+\delta} \dot{n}_{i\tau}\, d\vec{\ell} \right)^2 + 2 \sum_{\delta\nu} \int_i^{i+\delta} \dot{n}_{i\tau}\, d\vec{\ell}\, \epsilon^{(\delta\nu)} \nabla_\nu A_{i\tau}$$

$$+ |\vec{\nabla} A_{i\tau}|^2$$

Using the Poisson resummation formula:

$$\sum_{\{A\}} f(A) = \sum_{\{v_i\}} \int_{-\infty}^{+\infty} dA\, f(A)\, e^{i2\pi v A}$$

and performing the Gaussian integration over A

yelds finally:

$$Z \propto \sum_{\{n_{i\tau}\}} \sum_{\{v_{i\tau}\}} e^{-S_{CCG}[n,v]}$$

Coupled Coulomb gas of charges and vortices.

$$S_{CCG}[n,v] = \int_0^\beta d\tau \sum_{ij} \Big\{ \frac{2e^2}{\hbar} n_{i\tau} C^{-1}_{ij} n_{j\tau} +$$

$$+ \frac{\pi E_J}{\hbar} v_{i\tau} G_{ij} v_{i\tau} + i\, \dot{n}_{i\tau} \Theta_{ij} v_{j\tau} + \frac{\hbar}{4\pi E_J} \dot{n}_{i\tau} G_{ij} \dot{n}_{j\tau} \Big\}$$

$$G(r) = \frac{1}{2\pi} \int d^2q \left(e^{i\vec{q}\cdot\vec{r}} - 1 \right) \frac{1}{q^2} \sim - \ln r$$

interaction between vortices

cutoff in q

$$\Theta_{ij} = \operatorname{arctg} \frac{y_i - y_j}{x_i - x_j}$$

If $C_0 = 0$ also C^{-1}_{ij} is long range and logarithmic.

for $C_0 = 0$ $C^{-1}_{ij} = \frac{1}{2\pi C} G_{ij}$ so that

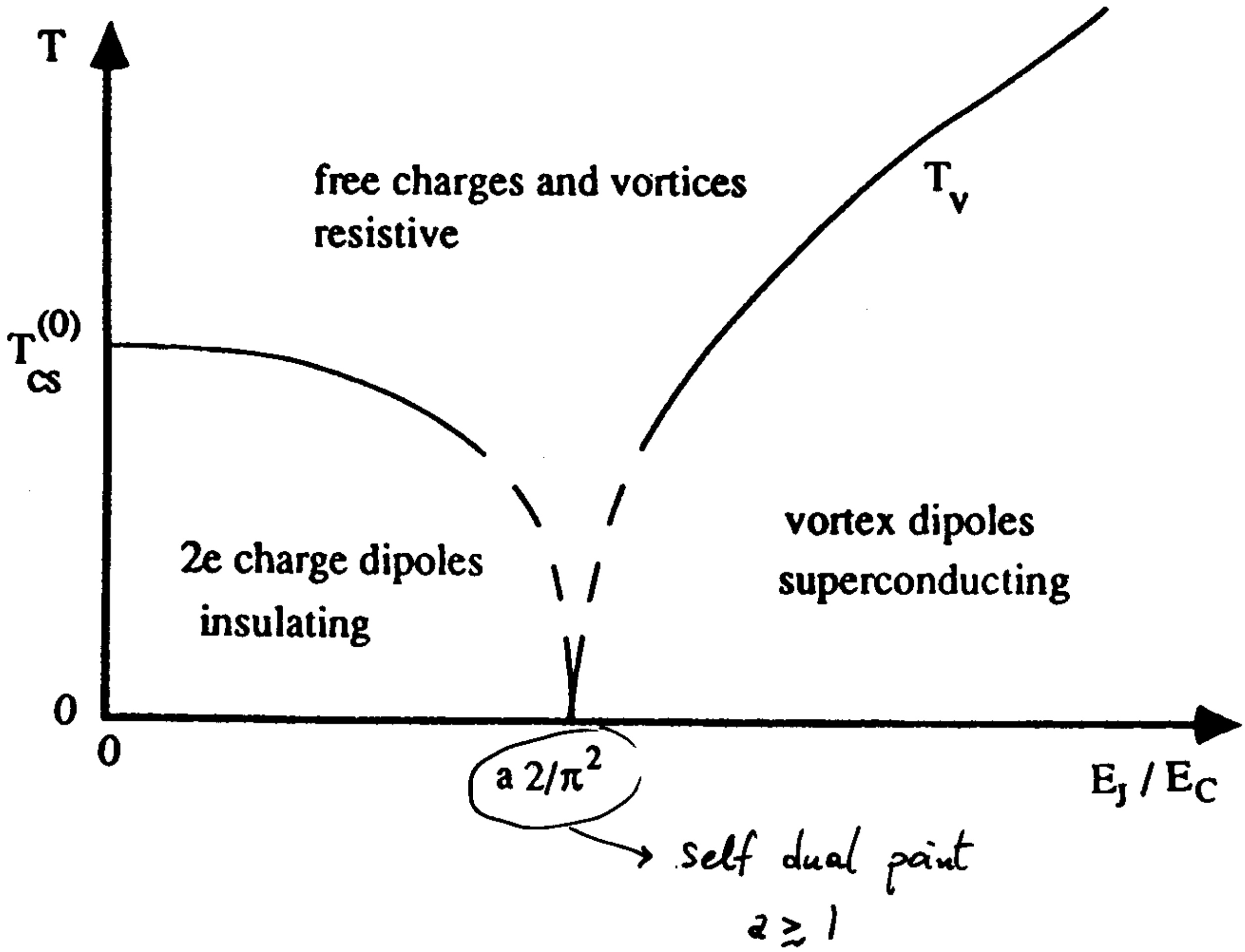
$$\frac{\pi E_J}{\hbar} G_{ij} = 2e^2 C^{-1}_{ij} \cdot \frac{\pi^2 E_J}{2 E_c}$$

If the last term in S can be disregarded, there is

duality vortices $\longleftrightarrow$ charges

Self dual point in the phase diagram is :

$$\frac{\pi^2 E_J}{2 E_c} = 1 \implies \frac{E_J}{E_c} = \frac{2}{\pi^2}$$



MPA Fisher: $V = \frac{h}{2e} \langle \dot{n}_v \rangle$ → # of vortices which cross the sample

$I = 2e \langle \dot{n}_c \rangle$ → # of Cooper pairs

$$R = \frac{V}{I} = \frac{h}{4e^2} \frac{\langle \dot{n}_v \rangle}{\langle \dot{n}_c \rangle}$$

At the selfdual point $\langle \dot{n}_v \rangle = \langle \dot{n}_a \rangle \Rightarrow R = \frac{h}{4e^2}$

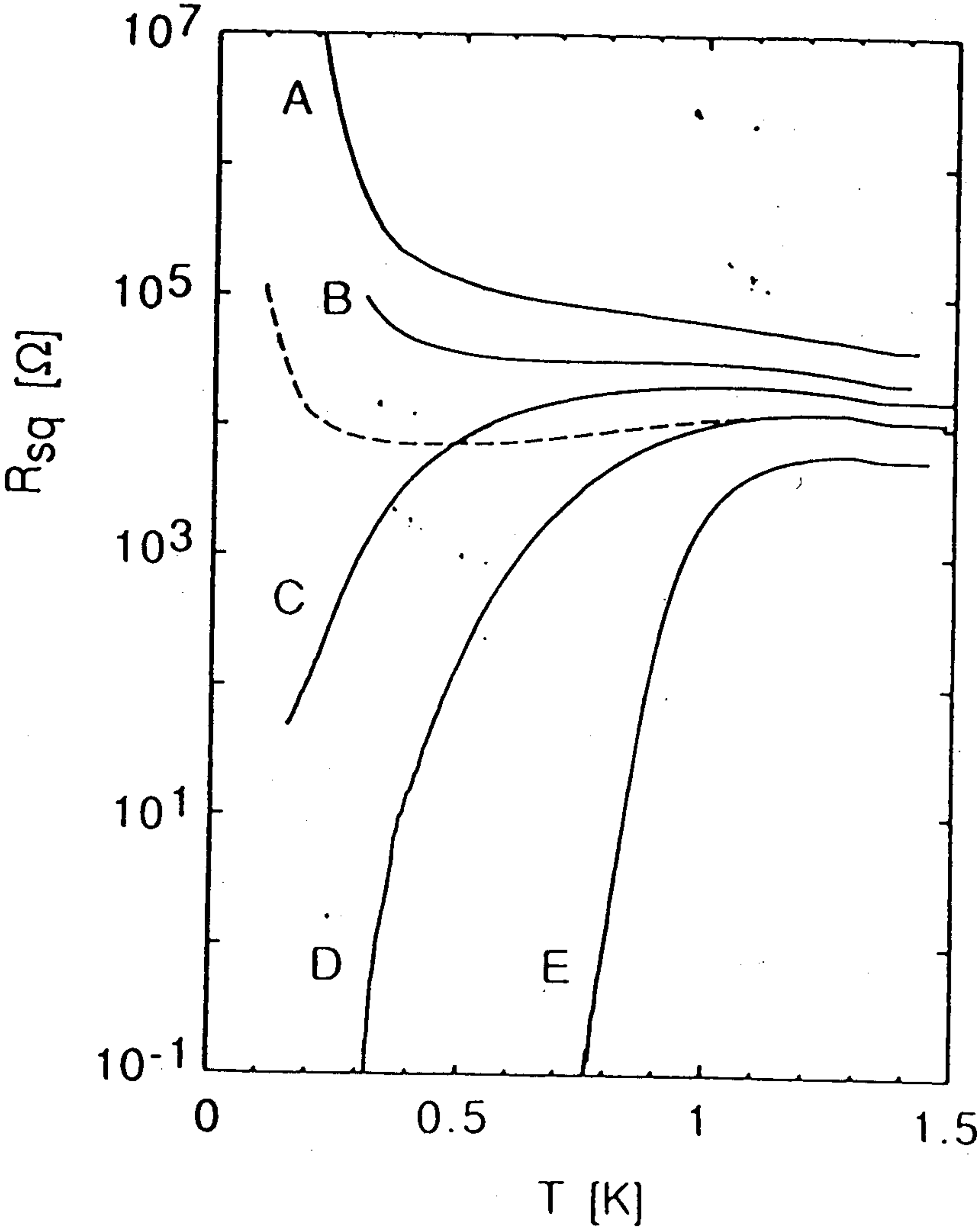


Fig 1.

Follow a different route for the case $C >> C_0$:

Coupled Coulomb gas picture :

$$Z = \sum_{\{n_{i\tau}\}} \sum_{\{v_{i\tau}\}} e^{-S^V - S^C - i \sum_{ij\tau} n_{i\tau} \Theta_{ij} v_{j\tau}}$$

$$S^V = \pi \epsilon \sum_{ij\tau} v_{i\tau} G_{ij} v_{j\tau} + i\epsilon \underbrace{\sum_{ij\tau} \vec{I}_{i\tau} \cdot \vec{\nabla}_i \Theta_{ij} v_{j\tau}}_{\text{external bias term}}$$

background of ext. charges (q_i)

$$S^C = \frac{1}{2} \epsilon^2 \sum_{ij\tau\tau'} (n_{i\tau} - q_i) \boxed{M_{ij,\tau\tau'}} (n_{j\tau'} - q_j)$$

$$M_{ij\tau\tau'} = 4e^2 C_{ij}^{-1} \frac{1}{\epsilon} \delta_{\tau\tau'} - \underbrace{\frac{G_{ij}}{2\pi E_J} \frac{1}{\epsilon^2} \frac{\partial^2}{\partial \tau'^2} \frac{1}{\epsilon} \delta_{\tau,\tau'}}$$

non local contribution to the charge kinetics mediated by spin waves.

$M_{ij\tau\tau'}$ is long range in space : Poisson resummation is useful

Charges disappear in favour of some conjugate integer variables $\phi_{i\tau}$

$$\sum_{\{n_{i\tau}\}} e^{-S^c + i\sum_{ij\tau} n_{i\tau}\theta_{ij}\dot{v}_{j\tau}} =$$

M^{-1} is short range!

$$= \sum_{\{\phi_{i\tau}\}} e^{i\sum_{i\tau} q_i u_{i\tau}} \, e^{-\frac{\epsilon^2}{2}\sum_{ij\tau\tau'} u_{i\tau} M^{-1}_{ij\tau\tau'} u_{j\tau'}}$$

$$u_{i\tau} = 2\pi\phi_{i\tau} + \sum_{e} \theta_{ie}\dot{v}_{e\tau} \equiv \chi_{i\tau}$$

($\phi_{i\tau}$: integers)

The partition function for vortices:

$$Z = Z_0 \sum_{\{v_{i\tau}\}} e^{-S^v + i\sum_{i\tau} q_i \chi_{i\tau} - \frac{\epsilon^2}{2}\sum_{ij\tau\tau'} \chi_{i\tau} M^{-1}_{ij\tau\tau'} \chi_{j\tau'}} \cdot$$

$$\cdot \left< e^{-2\pi\epsilon^2 \sum_{ij\tau\tau'} \phi_{i\tau} M^{-1}_{ij\tau\tau'} \chi_{j\tau'}} \right>_0$$

where

$$<\dots>_0 = \frac{1}{Z_0} \sum_{\{\phi_{i\tau}\}} e^{-S_0[\phi_{i\tau}]} \dots$$

$$S_0[\phi_{i\tau}] = \frac{(2\pi)^2}{2}\epsilon^2 \sum_{ij\tau\tau'} \phi_{i\tau} M^{-1}_{ij\tau\tau'} \phi_{j\tau'} - 2\pi i \sum_{i\tau} q_i \phi_{i\tau}$$

Local approximation to the action S_0 is the SOS model

$$M^{-1}_{ij\tau\tau'} = \frac{1}{16\alpha}(-\nabla^2)\, e^{-\omega_p|\tau-\tau'|}$$

$$\alpha = [E_C/8E_J]^{1/2}$$

$$\sum_{\tau} \sum_{(ij)} (\phi_{i\tau} - \phi_{j\tau})^2$$

Discrete Gaussian

SOS model in the local approximation.

$$S_0[\phi_{i\tau}] = \frac{\pi^2}{8\alpha} \sum_{i\tau} (\nabla \phi_{i\tau})^2 - 2\pi i \sum_{i\tau} q_i \phi_{i\tau}$$

Note : because $\phi_{i\tau}$ are discrete S_0 is invariant when the external charges are increased by multiples of $2e$

infact this is required by physics.

$\langle(\phi_{i\tau} - \phi_{j\tau})^2\rangle \Rightarrow$ average difference in heights between sites

$\Rightarrow \langle \phi_{i\tau} \phi_{j\tau} \rangle$: $\boxed{q_i = 0}$

small α : finite asymptotic value as $|i-j| \to \infty$

($\Rightarrow$ finite interface width)

large α : diverges as $A(\alpha) \ln |i-j|$

$A(\alpha)$ → increasing function of α

$\alpha = \alpha_c$: roughening transition

Corresponds to the superconductor - insulator transition

How is $\langle \phi_{i\tau} \phi_{j\tau} \rangle$ related to the problem?

Evaluate

$$\langle T_\tau \, e^{-2\pi \int_0^\beta d\tau \sum_i \phi_{i\tau} \gamma_{i\tau}} \rangle_0 =$$

$$\gamma_{i\tau} = \int_0^\beta d\tau' \, M^{-1}_{ij\tau\tau'} \, \chi_{j\tau'}$$

$$= \sum_{n=0}^{\infty} (-2\pi)^n \int_0^\beta d\tau_n \cdots \int_0^{\tau_2} d\tau_1 \, \langle \sum_i \phi_{i\tau_n} \gamma_{i\tau_n} \cdots \sum_{i'} \phi_{i'\tau_1} \gamma_{i'\tau_1} \rangle_0$$

because $\langle \int d\tau' \sum_j M^{-1}_{ij\tau\tau'} \phi_{j\tau'} \rangle = 0$, to second order

$$\simeq 1 + (2\pi)^2 \int_0^\beta d\tau_2 \int_0^{\tau_2} d\tau_1 \sum_{ij} \gamma_{i\tau_2} \langle \phi_{i\tau_2} \phi_{j\tau_1} \rangle_0 \, \gamma_{j\tau_1} + \cdots$$

$$\approx \exp (2\pi\epsilon)^2 \sum_{ij\tau\tau'} \gamma_{i\tau} \langle \phi_{i\tau} \phi_{j\tau'} \rangle_0 \, \gamma_{j\tau'}$$

Back to the original notation $Z_c = \sum_{\{n_{i\tau}\}} e^{-S_c[n_{i\tau}]}$

$$Z = Z_c \sum_{\{v_{i\tau}\}} e^{-S^v - S^v_1}$$

$$S^v_1 = \frac{1}{2} \epsilon^2 \sum_{ij\tau} \sum_{k\ell\tau'} Q_{ij\tau\tau'} \, \theta_{ik} \dot{v}_{k\tau} \, \theta_{j\ell} \dot{v}_{\ell\tau'} + $$

$$- i \sum_{ij\tau} \langle n_{i\tau} \rangle \theta_{ij} \dot{v}_{j\tau}$$

$$\langle n_{i\tau} \rangle = q_i + 2\pi i \sum_{j\tau'} M^{-1}_{ij\tau\tau'} \langle \phi_{j\tau'} \rangle$$

$$Q_{ij\tau\tau'} = M^{-1}_{ij\tau\tau'} - (2\pi)^2 \sum_{k\ell,\tau_1\tau_2} M^{-1}_{ik\tau\tau_1} \langle \phi_{k\tau_1} \phi_{\ell\tau_2} \rangle_c M^{-1}_{\ell j\tau_2\tau'}$$

$$Q_{ij\tau\tau'} = \langle (n_{i\tau} - q_i)(n_{j\tau'} - q_j) \rangle_c$$

connected charge-charge correlation function

Calculation of $\langle \phi_{i\tau} \phi_{j\tau'} \rangle_c =$

$$= \langle \phi_{i\tau} \phi_{j\tau'} \rangle - \langle \phi_{i\tau} \rangle \langle \phi_{j\tau'} \rangle$$

$$\alpha = [E_c / 8E_J]^{1/2}$$

Large α:

($q = 0$) $\quad S_0 \propto \frac{1}{\alpha} (\nabla\phi)^2 \Rightarrow \phi$ fluctuate a lot:

Continuum approximation : $\psi = \phi / \sqrt{\alpha}$

$$S_\psi = 2\pi^2 \alpha \sum_{ij\tau\tau'} \psi_{i\tau} M^{-1}_{ij\tau\tau'} \psi_{j\tau'}$$

$$\Rightarrow \langle \phi_{j\tau'} \rangle = 0$$

$$\langle \phi_{i\tau} \phi_{j\tau'} \rangle = \alpha \sum_\psi \psi_{i\tau} \psi_{j\tau'} e^{-S_\psi} \Big/ \sum_\psi e^{-S_\psi}$$

$$= M_{ij\tau\tau'} / (2\pi)^2 \qquad (q \neq 0\ ?)$$

Small α:

lowest excitation is $\phi_{i\tau} = \pm 1$ on site i
$= 0$ on j ($j \neq i$)

degeneracy is N (# of sites)

cost of the unit step $e^{-\pi^2/2\alpha} = s^2$

low α expansion in this parameter s

$\langle \phi_{i\tau} \phi_{j\tau'} \rangle$ is strongly peaked at $i=j$ $\tau=\tau'$, except when q

Charge - Charge correlator $Q_{ij\tau\tau'}$

$\alpha \to 0$ (classical) : only constant $\phi_{i\tau}$ is allowed

$$\langle \phi_{i\tau} \phi_{j\tau'} \rangle_c = 0$$

$$\Longrightarrow \quad Q_{ij\tau\tau'} = M^{-1}_{ij\tau\tau'} = \frac{1}{16\alpha} (-\nabla^2) e^{-\omega_p |\tau-\tau'|}$$

Fourier Transform :

$$M^{-1}_{k\omega} = \frac{\omega_p}{8\alpha} \frac{k^2}{\omega^2 + \omega_p^2}$$

$\frac{\omega_p}{8\alpha} = E_J$

$\alpha \to \infty$ (disordered) : continuum approximation

$$\langle \phi_{i\tau} \phi_{j\tau'} \rangle_c = M_{ij\tau\tau'} / (2\pi)^2$$

$$\Longrightarrow \quad Q_{ij\tau\tau'} \to 0$$

α small : low α expansion

$$\lim_{k \to 0} \left(\mathcal{FT} \{ \langle \phi_{i\tau} \phi_{j\tau'} \rangle_c \} \right) \Big|_{\omega=0} \equiv \xi^2 \frac{2\alpha}{\pi^2 \omega_p} + \mathcal{O}(k^2)$$

ξ^2 → correlation lengt

$$\xi^2 = \frac{\pi^2}{2\alpha} \{ 2 \cos 2\pi q \, s^2 + 16 \cos 4\pi q \, s^3 + \\ + [-20 \cos 4\pi q + 108 \cos 6\pi q + 32 \cos 8\pi q] s^4 \}$$

Critical behavior extracted by Padé - approximant method :

ξ diverges close to $1.2 = \alpha$ $\quad (q=0)$

other results are :

SCHA	$\alpha_c = 1.05$	
duality argument	$\alpha_c = 0.79$	$\alpha = \sqrt{\frac{E_c}{\delta E_J}}$
variational method	$\alpha_c \approx 0.5$	
Monte Carlo	$\alpha_c \sim 0.46$	
exp.	$\alpha_c \sim 0.5$	

Note : SCHA :

$$\sum_{\{\phi_{i\tau}\}} \cdots \simeq \int \prod_{i\tau} d\phi_{i\tau} \cdots e^{K \sum_{i\tau} \cos(2\pi\phi_{i\tau})}$$

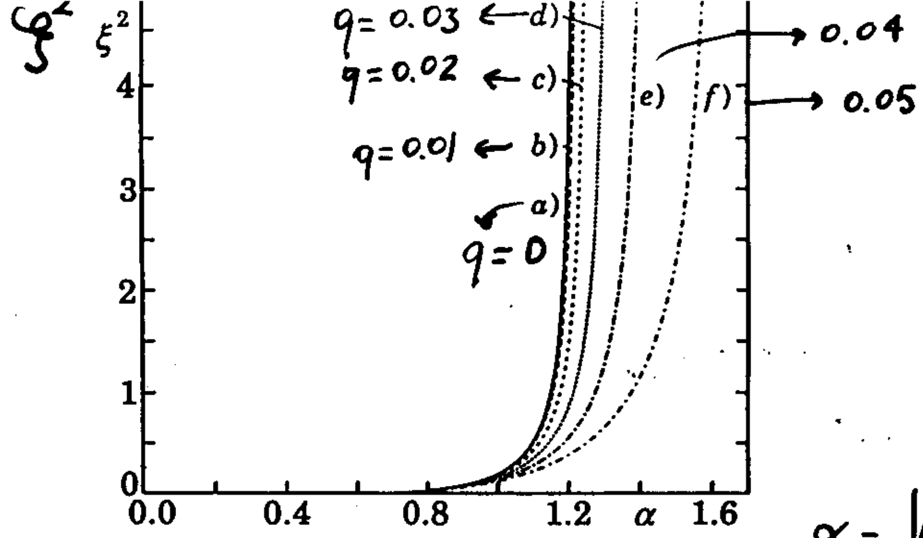
$\{\phi_{i\tau}\}$ → integers; $\phi_{i\tau}$ → continuous

$$\sum_{i\tau}(1-\cos 2\pi\phi_{i\tau}) \simeq + \sum_{ij\tau\tau'} \phi_{i\tau}\, m\, \delta_{ij}\, \delta_{\tau\tau'}\, \phi_{j\tau'}$$

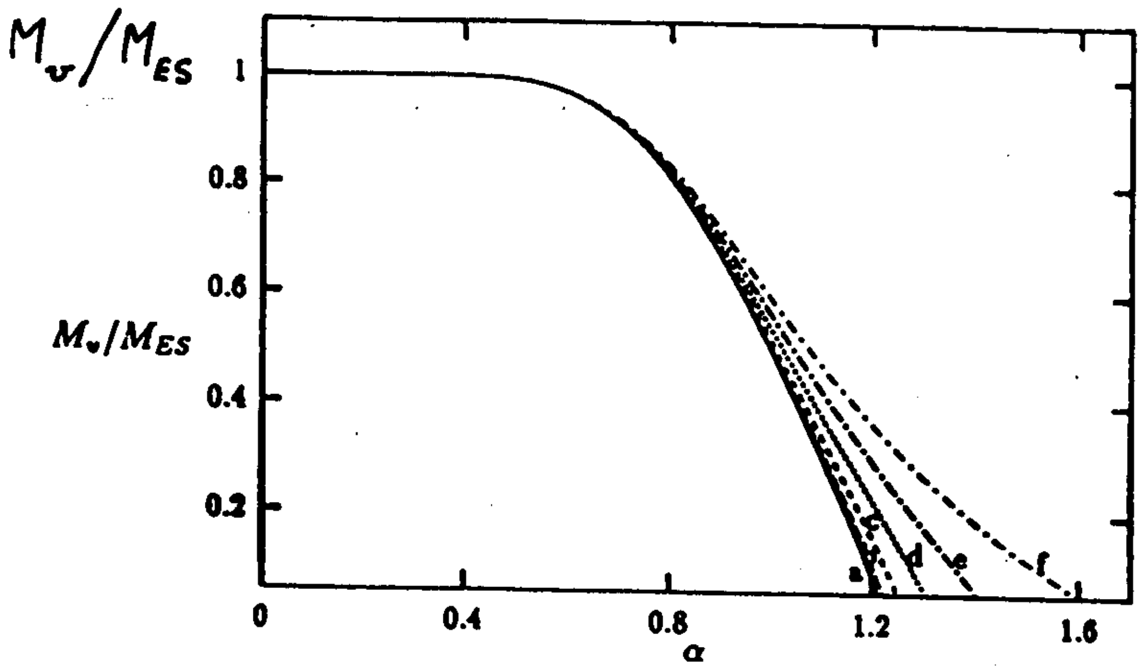
m → to be determined selfconsistently

SCHA: $$\mathcal{F}\mathcal{T}\{\langle\phi_{i\tau}\phi_{j\tau'}\rangle_c\}_{k,\omega} = \frac{2\alpha}{\pi^2\omega_p}\,\frac{\xi^2(\omega^2+\omega_p^2)}{\omega^2+\omega_p^2(1+\xi^2k^2)}$$

$$\xi^2 = 1/8\alpha m$$



$\alpha = \sqrt{\frac{8E_c}{E_J}}$



$\alpha = \sqrt{\frac{8E_c}{E_J}}$

$\alpha \to 0 \qquad M_v \to M_{ES}$

$$M_v = M_{ES} \frac{1}{4\pi\xi^2} \ln\left(1 + 4\pi\xi^2\right)$$

$\alpha \equiv \alpha_c \qquad M_v = 0$

$Q_{k\omega}$: the pole $\pm i\omega = \omega_p$ $(\alpha = 0)$

is shifted at $\pm i\omega = \omega_k$

$$\omega_k^2 = \omega_p^2 (1 + \xi^2 k^2)$$

$$Q_{k\omega} = \frac{\omega_p}{8\alpha} \frac{k^2}{\omega^2 + \omega_k^2} \qquad \text{SCHA}$$

Ansatz : the same occurs in low α expansion

but with the calculated ξ.

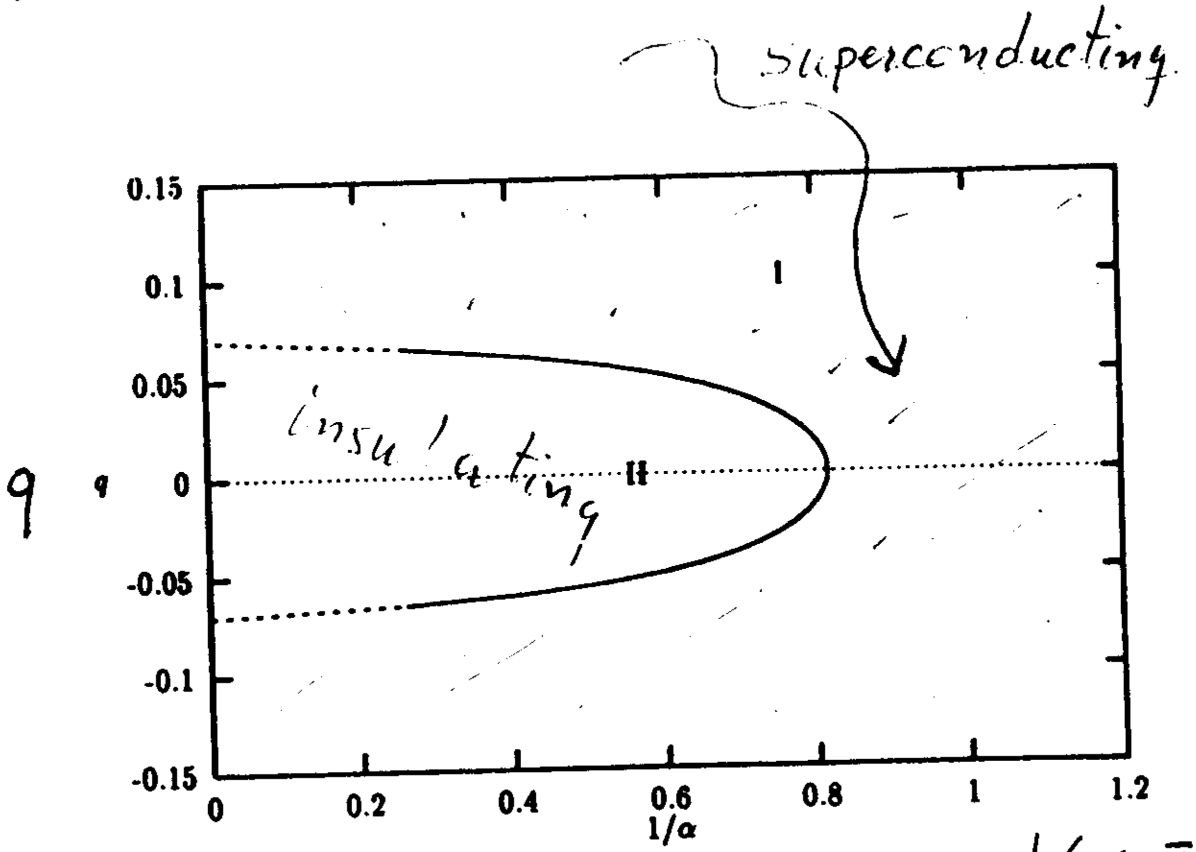
Note $\alpha \to 0 \implies \xi \to 0$: classical limit is recovered

Not so in SCHA.

$\omega_k^2 = \omega_p^2 (1 + \xi^2 k^2)$:

spin wave spectrum is made stiffer by inclusion of charge - charge correlations.

Phase diagram with external charges



$$1/\alpha = \sqrt{\frac{8E_J}{E_c}}$$

To monitor the transition:

follow one single vortex "evaporating"

$$r_{i\tau} = \delta(\vec{r}_i - \vec{r}(\tau))$$

$$Z_{1V} \sim e^{-\frac{1}{2}\epsilon^2 \sum_{\tau\tau'} \dot{\vec{r}}(\tau) \overleftrightarrow{\mathcal{M}}_{\tau\tau'} \dot{\vec{r}}(\tau')} \times$$

$$\times\, e^{i\sum_{i\tau} q_i \dot{\vec{r}}(\tau)\cdot\nabla\theta(\vec{r}(\tau)-\vec{r}_i)} \times$$

$$\times\, e^{-\frac{2\pi}{2e}\epsilon \sum_\tau \vec{I}(\tau)\cdot(\hat{z}\times\dot{\vec{r}}(\tau))}$$

$$\mathcal{M}^{ab}_{\tau\tau'} = \sum_{jk} \nabla^a\theta(\vec{r}_j - \vec{r}(\tau))\, Q_{jk\tau\tau'}\, \nabla^b\theta(\vec{r}_k - \vec{r}(\tau'))$$

$$\equiv \mathcal{M}^{ab}(r(\tau)-r(\tau'),\ \tau-\tau')$$

curvature of the action around the saddle point trajectory

Semiclassically it can be interpreted as the vortex mass:

$$M_v = \epsilon\sum_\tau \mathcal{M}_{xx}(0,\tau) = \frac{1}{2}\int \frac{d^2k}{k^2}\, Q(k,\omega=0)$$

($\dot{\vec{r}}(\tau)$ along x ...)

<u>But dynamical mass requires real time!</u>

2-D Quantum Josephson Junction Array.

$$\mathcal{H} = \frac{1}{2}(2e)^2 \sum_{ij} (n_i - q_i)\, C^{-1}_{ij}\, (n_j - q_j) + \\ - E_J \sum_{(ij)} \cos(\varphi_i - \varphi_j)$$

$$[n_k, \varphi_\ell] = -i\delta_{k\ell}$$

$C_{ii} = C_0 + 4C_1$ (C_0: onsite)

$C_{ij} = -C_1$ (i,j n.n.) (C_1: between islands)

$$C(k) = C_0 + C_1 k^2 \implies C^{-1}(|i-j|) = \frac{1}{2\pi C_1} K_0(|i-j|/\lambda)$$

$$\lambda = \sqrt{C_1/C_0}$$

Take $C_1 \gg C_0 \implies$ long range Coulomb interaction between Cooper pairs

Single vortex dynamics

Propagator for the quantum particle:

$$K(R,0;T) = \int_{R(0)=0}^{R(T)=R} \mathcal{D}R(t)\; e^{\frac{i}{\hbar} S_{eff}[R(t),t]}$$

$S_{eff}[R(t)]$ continued to real times from

$$S^E_{eff}[z(\tau)] = \frac{1}{2}\sum_{\alpha\beta}\int_0^\beta d\tau\, d\tau'\; \dot{z}^\alpha(\tau)\, M_{\alpha\beta}[z(\tau)-z(\tau'),\tau-\tau']\,\dot{z}^\beta(\tau')$$

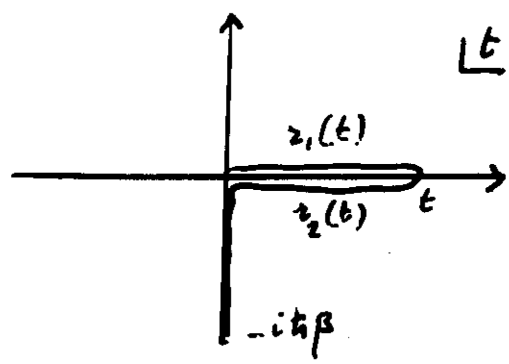
$$+\, i\hbar \sum_i \frac{q_i}{2e}\sum_\alpha \int_0^\beta d\tau\; \nabla_\alpha \theta(z(\tau)-z_i)\,\dot{z}_\alpha(\tau) \;-\; i\int_0^\beta d\tau\, \vec{f}\cdot(\hat{z}\times\vec{z}(\tau))$$

($\vec{f} \to hI/2e$)

$$\theta(i-j) = \mathrm{arctg}\,(y_i-y_j)/(x_i-x_j)$$

$$M_{\alpha\beta}[\vec{z}(\tau)-\vec{z}(\tau');\tau-\tau'] = \sum_{ij}\nabla_\alpha\theta(\vec{z}(\tau)-\vec{z}_i)\,Q_{ij\,\tau-\tau'}\,\nabla_\beta(\vec{z}_j-\vec{z}(\tau'))$$

Use the keldysh contour



Define $\vec{R}(t) = \dfrac{\vec{z}_1(t) + \vec{z}_2(t)}{2}$ $\vec{s}(t) = \vec{z}_1(t) - \vec{z}_2(t)$

$$S^E_{eff}[z(\tau)] \longrightarrow \int_0^t dt' \mathcal{L}[R(t'), s(t'); t']$$

$\delta S = 0 \implies \vec{s}(t) = 0$: keep only this saddle point trajectory

(decoherence effects ignored : $\Delta R \ll \lambda_T \approx \hbar \sqrt{\dfrac{2\beta}{M_{Bs}}}$

For $q = 0$ and $\vec{R}(t) \perp \vec{f}$ ($\vec{R}$ in direction $\vec{x}$, $\vec{f}$ " " $\vec{y}$)

Eq. of motion :

$$f_y(t) = \int d^2k \, \frac{k_y^2}{k^4} \int \frac{d\omega}{2\pi} (-i\omega) \, Q^R_{\vec{k}\omega} \cdot \int_{-\infty}^{+\infty} dt' \, \dot{x}(t') \, e^{i\omega(t'-t) + ik_x(x(t') - x(t))}$$

non linear !

<u>This is non local in time !</u>

retarded

$$Q^R_{\vec{k}\omega} = E_J \frac{k^2}{\omega_k^2 - (\omega + i0^+)^2}$$

$$\omega_k^2 = \omega_p^2 (1 + k^2 \lambda^2)$$

Classical limit : make it local, ignore nonlinearity

$\mathrm{Re}\, Q^R_{k\omega}$ even in ω $\Longrightarrow M\ddot{x}$, $\mathrm{Im}\, Q^R_{k\omega}$ odd in ω $\Longrightarrow \eta \dot{x}$

Classical eq. of motion:

$$M_{dyn}\,\ddot{x}(t) + \eta\,\dot{x}(t) = \frac{h}{2e} I + F_{pinning}$$

$$F_{pinning} \rightarrow \pi U_{barr} \cos(2\pi x(t))$$

$U_{barr} \sim 0.2 E_J$ (Lobb et al. '83)

$$\eta_{BS} = \frac{\phi_0^2}{2\sigma} \frac{1}{R_{subgap}}$$

$$\phi_0 = \frac{hc}{2e}$$

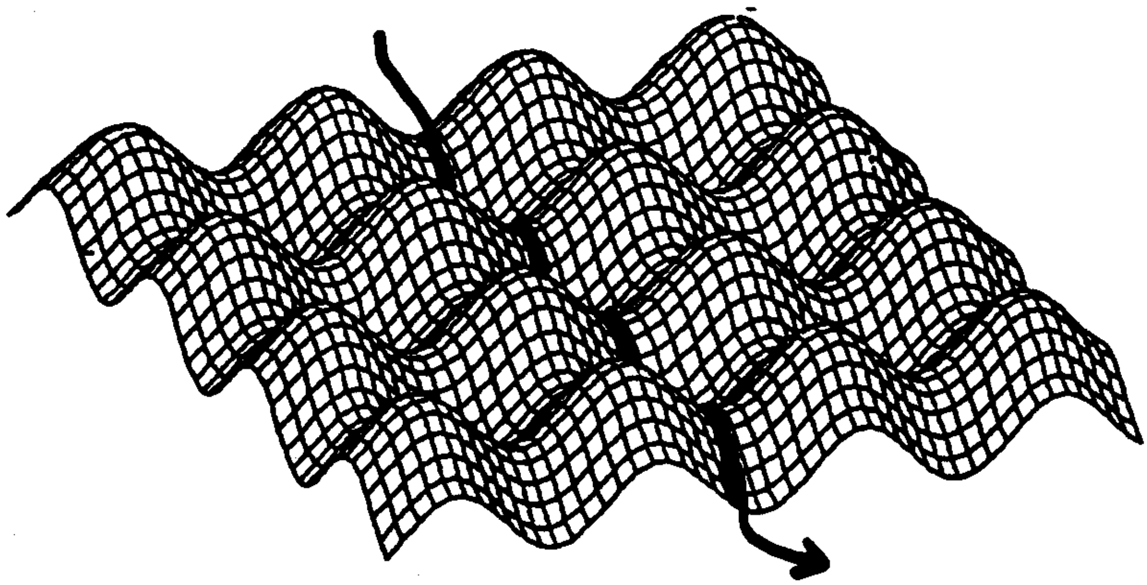


Fig. 1.B.2. A three dimensional plot of the intrinsic junction potential. Minima are found in the center of the cells.

One obtains:

$$\boxed{M_{dyn}\,\ddot{x}(t) + \eta\,\dot{x}(t) = \frac{h}{2e} I} + \pi\, U_{ben} \cos\left(2\pi x(t)\right) +$$

(U_{ben} → $\sim 0.1\, E_J$ (Loll...)

Can motion be ballistic?

$$\dot{x}(t) = v = \text{const}$$

$$I = 0 \qquad (f = 0)$$

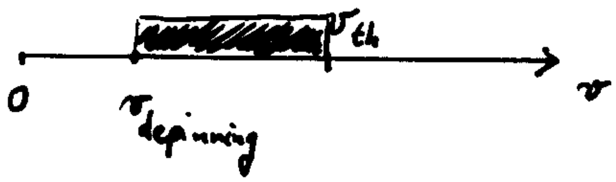
$$f(t) = v \int d^2k \, \frac{k_x^2}{k^4} \int \frac{d\omega}{2\pi} (-i\omega)\, Q^R_{\vec{k},\omega}\, \delta(\omega + k_x v)$$

yes: when $\mathrm{Im}\, Q^R_{k,\omega}$ vanishes.

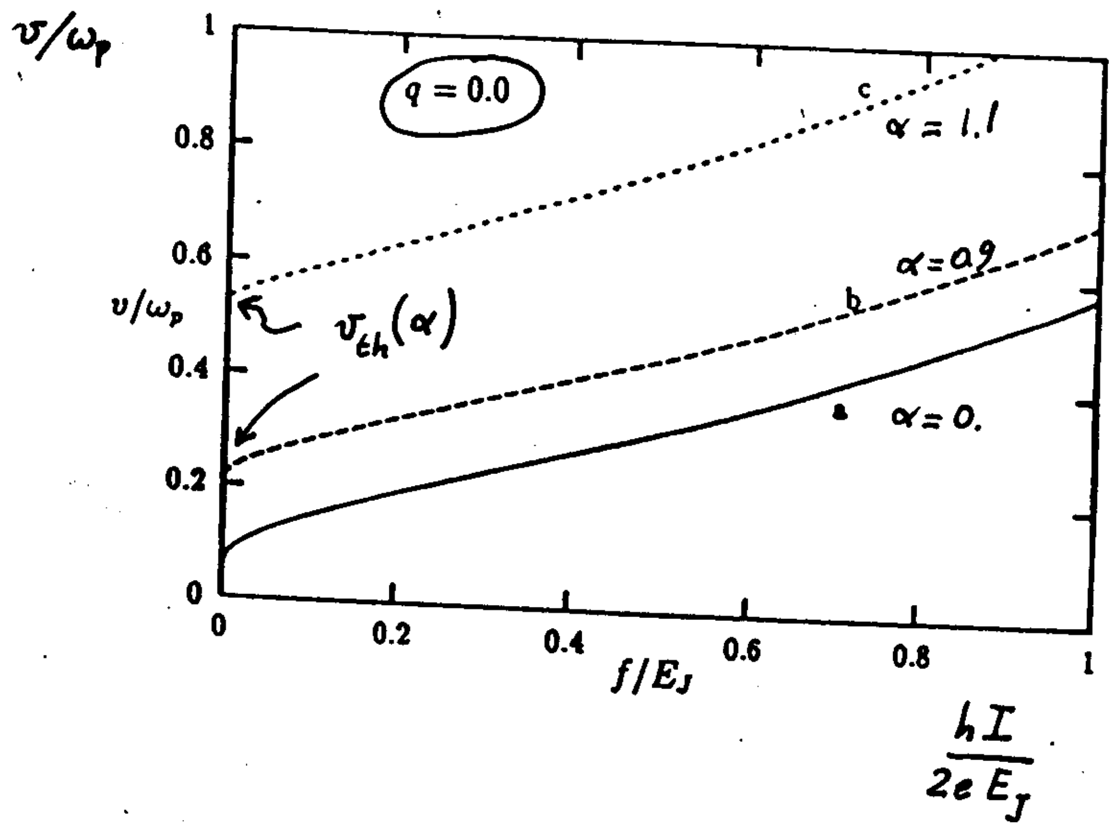
for $v < v_{th} \sim \omega_p \xi$

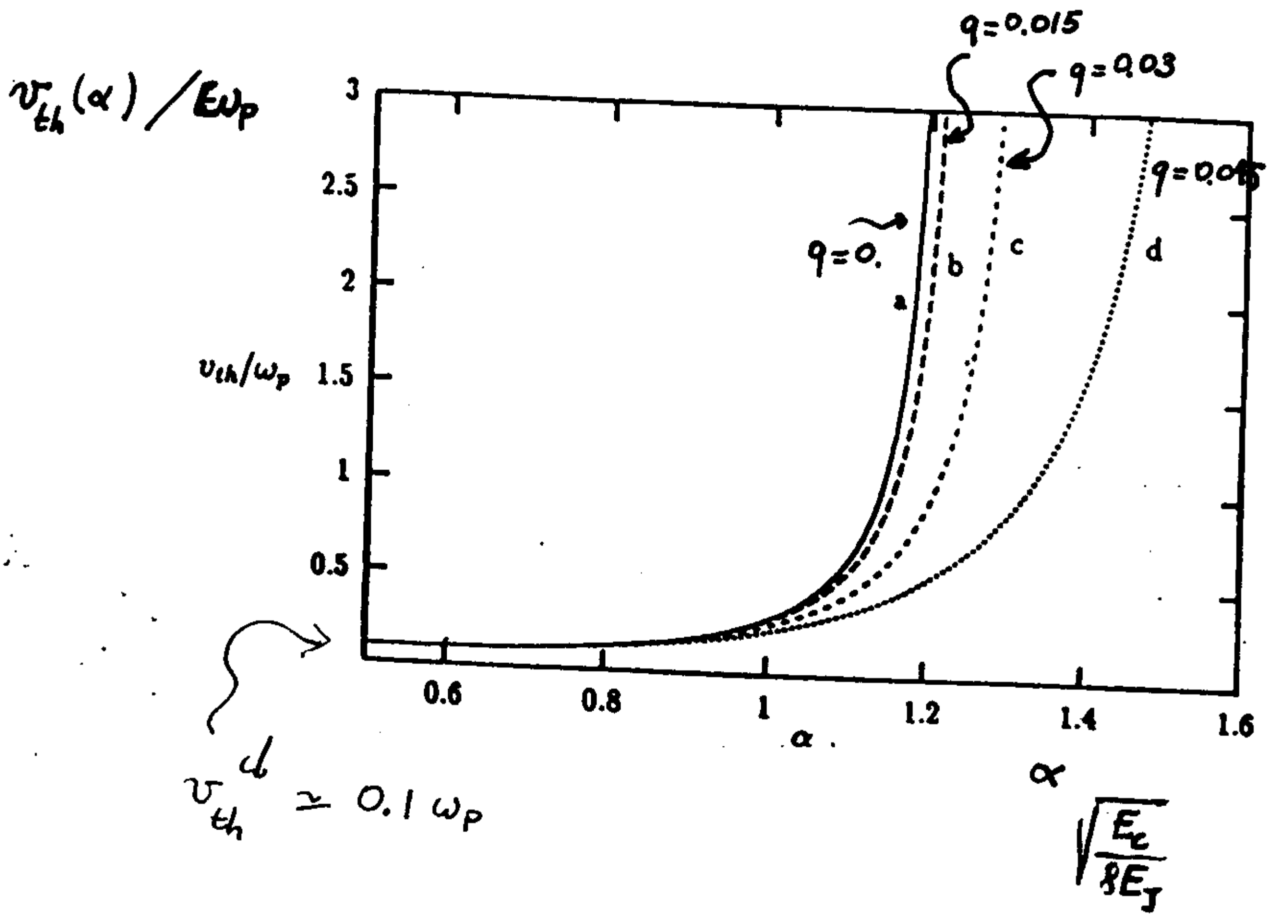
$E_J \gg E_c \implies v_{th} \sim 0.1\, \omega_p \sim$ depinning velocity

Ballistic window



$\alpha = \sqrt{\frac{E_c}{8E_J}}$





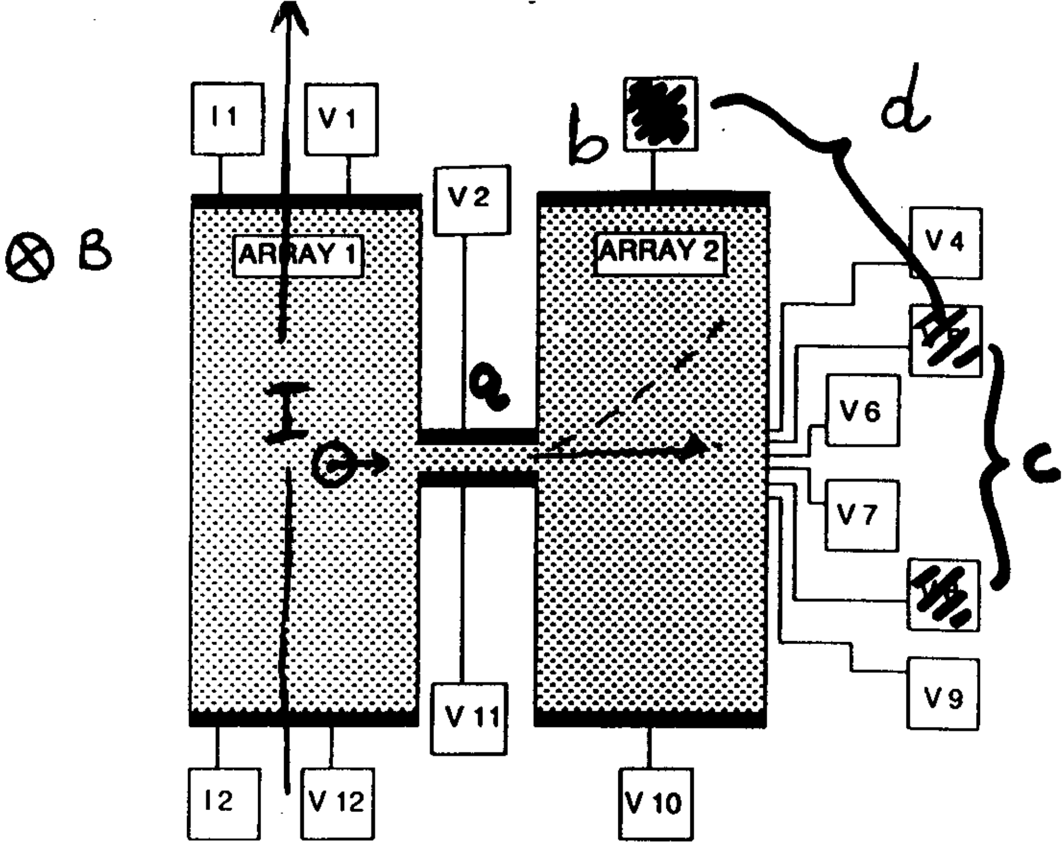


Figure 12. Layout of sample to detect ballistic vortices. A current is imposed in accelerating the vortices. A narrow channel connects to the right, where voltage probes.

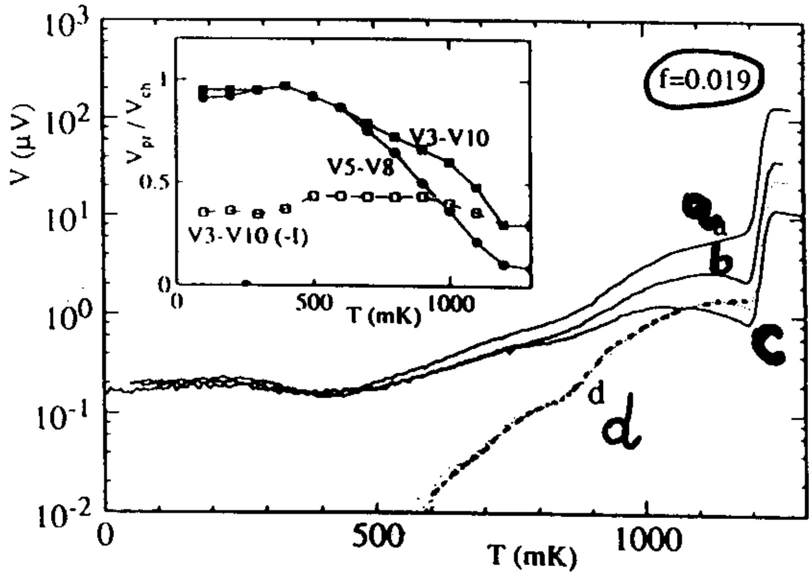


Figure 13. Measurements on the sample of Fig. 12. Plotted are the voltages across of probes as a function of temperature: (a) the voltage across the channel, (b) the volt right array $V3 - V10$, (c) the voltage drop between $V5 - V8$, which counts the vortic propagated ballistically, d) the voltage drop between $V5 - V3$, which is a measure of the have deviated from a ballistic trajectory. The inset shows the ratio of voltages to the ch The open squares give the results after the current in the left array has been reversed.

van der Zant, Fritshy, Mooij Europhys. Lett 19, 541 ('92)
van der Zant, Fritshy, Elion, Geerligs, Mooij PRL 69, 2971 ('92)

The quantum vortex moves in the periodic pinning potential:

Schrödinger eq. gives energy bands

Band mass:

$$\frac{\hbar}{\omega_p a^2} \exp(b)\, \alpha^{-1/2}$$

(b → numerical factor)

$$\alpha = \sqrt{\frac{E_c}{8 E_J}}$$

A larger mass comes from dissipation

Beyond the steady state solution:

linear response to a small perturbation of the constant velocity motion:

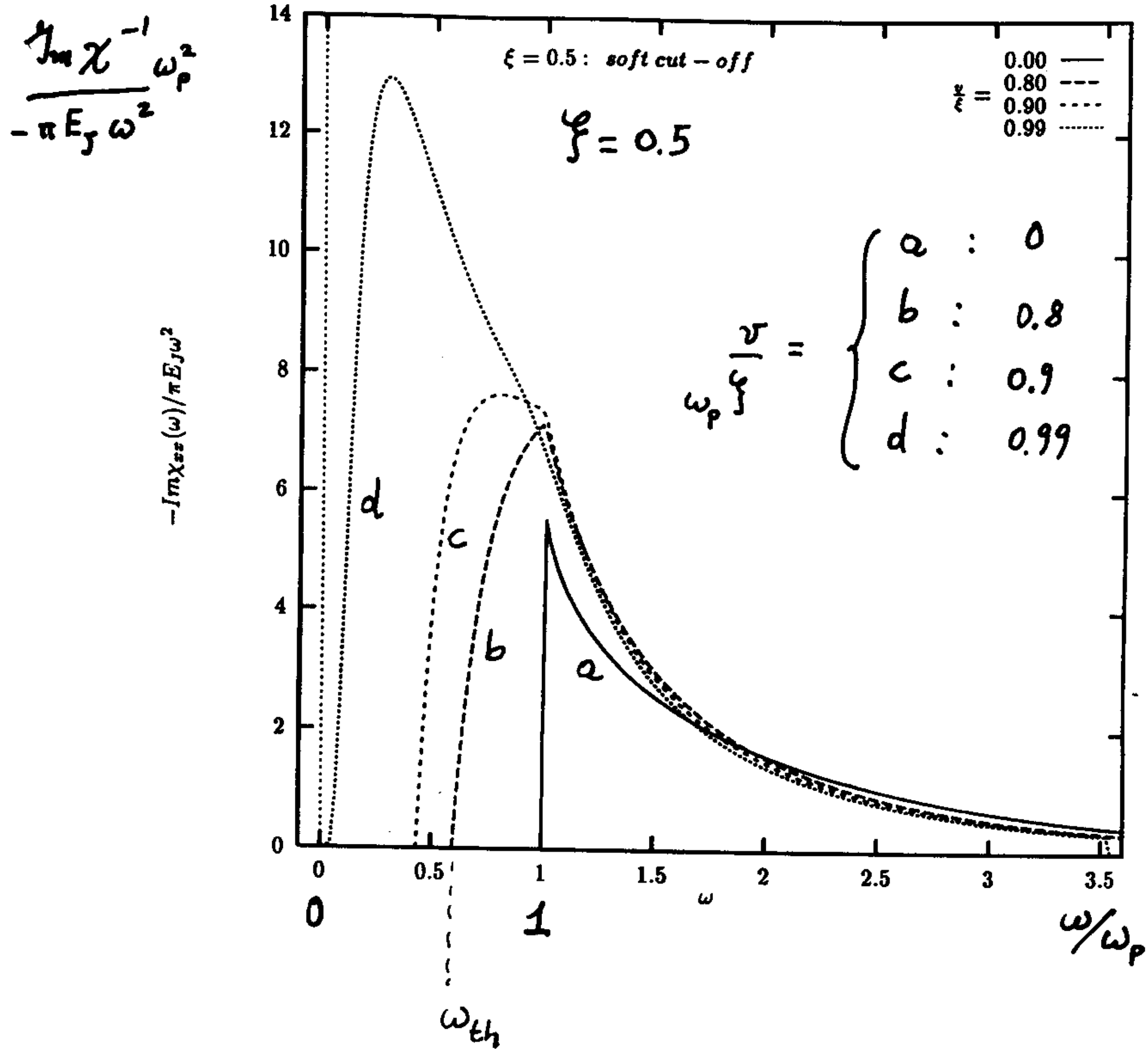
Take $\vec{R}(t) = \hat{x} v t + \delta \vec{R}(t)$ → $(\delta x(t), \delta y(t))$

$$\begin{pmatrix} f_y(\omega) \\ -f_x(\omega) \end{pmatrix} = \begin{pmatrix} \chi_{xx}^{-1}(\omega) & 0 \\ 0 & \chi_{yy}^{-1}(\omega) \end{pmatrix} \begin{pmatrix} \delta x(\omega) \\ \delta y(\omega) \end{pmatrix}$$

$\mathrm{Im}\, \chi_{xx}^{-1}(\omega)$ related to dissipation

$\mathrm{Re}\, \chi_{xx}^{-1}(\omega)$ " " acceleration

$$\frac{\mathrm{Re}\, \chi^{-1}(\omega)}{-\pi E_J} \frac{\omega_p^2}{\omega^2} = \frac{M(\omega)}{M_{ES}} \qquad M_{ES} = \frac{\pi^2 \hbar^2}{4 E_c}$$

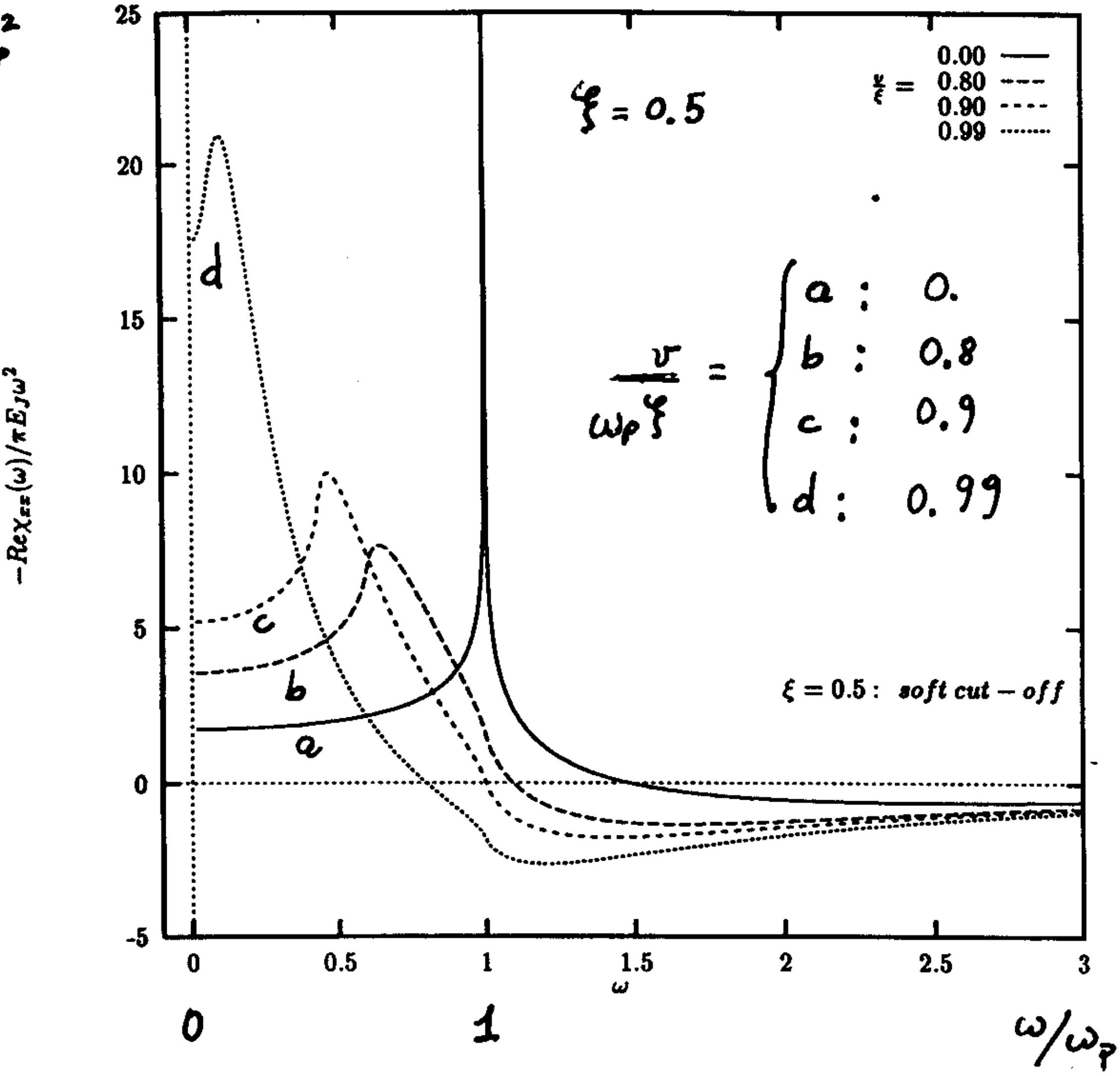


it vanishes for $\omega < \omega_{th}(v)$

at $v = c$ $\omega_{th} = \omega_p$

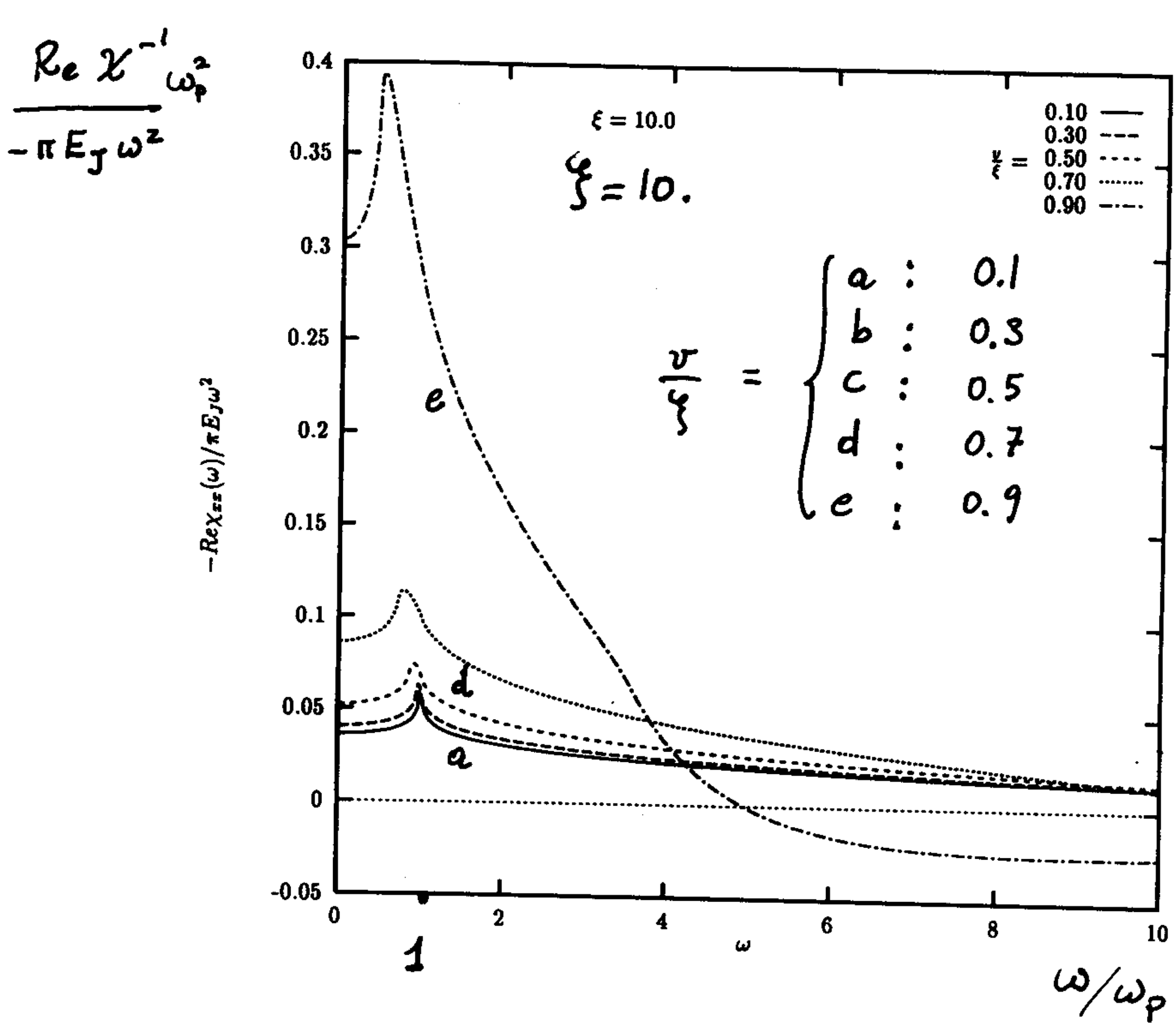
at $v = v_{th}$ $\omega_{th} = c$

$$\frac{\mathrm{Re}\,\chi^{-1}\,\omega_p^2}{-\pi E_J \omega^2}$$

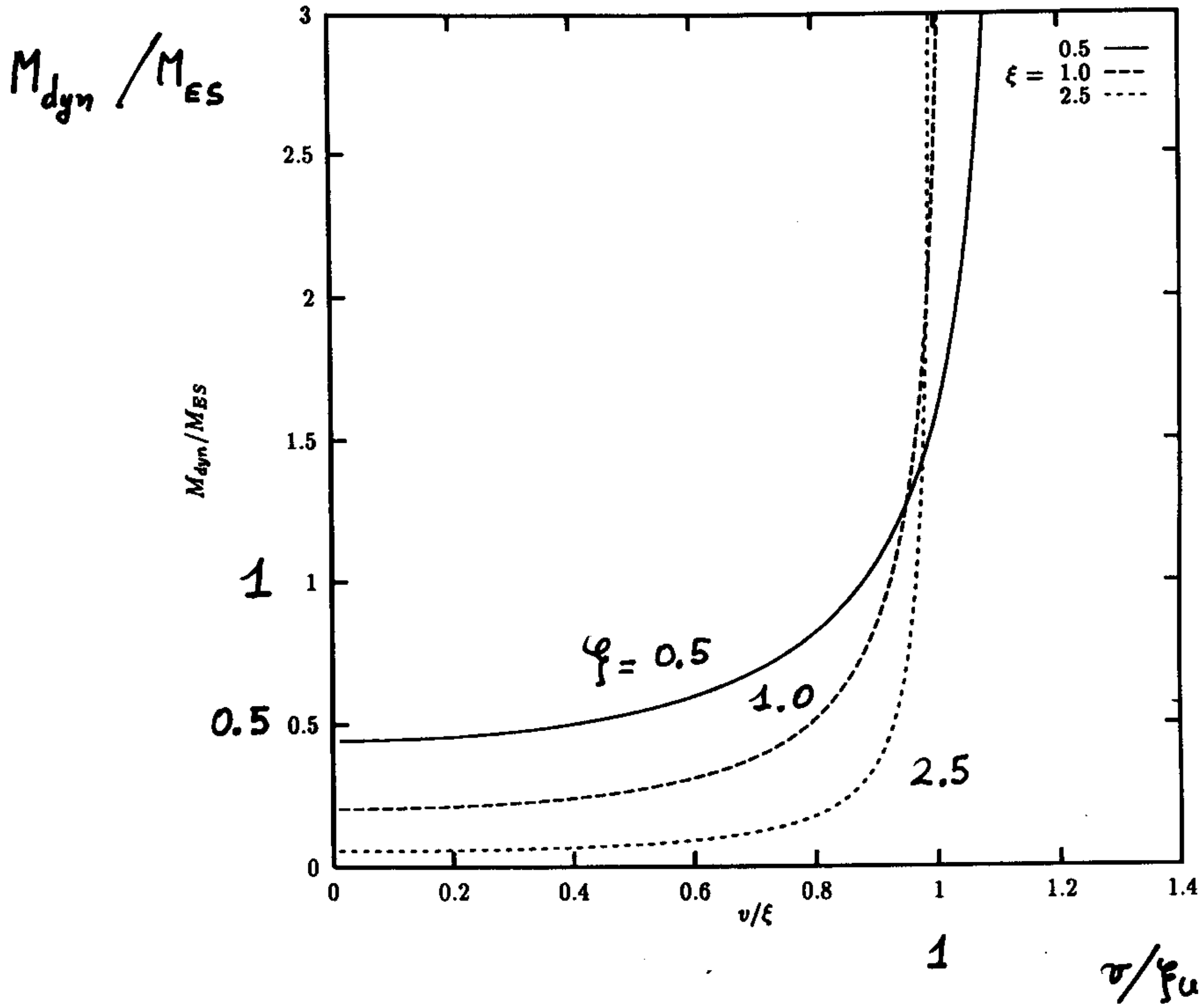


$$\lim_{\omega \to 0} \frac{\mathrm{Re}\,\chi^{-1}(v,\omega)}{-\omega^2} = M_{dyn}(v, \omega = 0)$$

$$= \pi E_J \int dk \, \frac{k\,\omega_k}{\left(\omega_k^2 - (kv)^2\right)^{3/2}}$$



$\frac{Re\,\chi^{-1}\omega_p^2}{-\pi E_J \omega^2}$
$\xi = 10.0$
$\xi = 10.$
0.10
0.30
$\frac{v}{\xi} =$ 0.50
0.70
0.90
$\frac{v}{\xi}$ = a : 0.1, b : 0.3, c : 0.5, d : 0.7, e : 0.9
$-Re\chi_{xx}(\omega)/\pi E_J\omega^2$
0.4
0.35
0.3
0.25
0.2
0.15
0.1
0.05
0
-0.05
0
2
4
6
8
10
ω
1
ω/ω_p
e
d
a



$$M_{dyn}(v, \omega = 0) \sim \frac{M_v}{\sqrt{1 - \frac{v^2}{v_{th}^2}}}$$

1

Only at $v = 0$ is $M_{dyn} = M_v$!

$$M_v = M_{ES} \frac{1}{4\pi\xi^2} \ln(1 + 4\pi\xi^2)$$

The moving vortex loses energy to the junction in its wake and this energy oscillates at ω_p :

Voltage step $V_0 = \frac{\pi \hbar v}{2e}$ in time $\frac{a}{v}$

The response is $V_1(t) = \frac{\pi \hbar \omega_p}{2e} \gamma \cos \omega_p t$

$\Longrightarrow$ vortex oscillates with coordinate

$$x(t) = vt + \gamma \sin \omega_p t$$

Note : $\frac{1}{\pi} \gamma < v$ otherwise $V_0 + V_1(t)$ can be < 0 !

Classically this gives the viscosity :

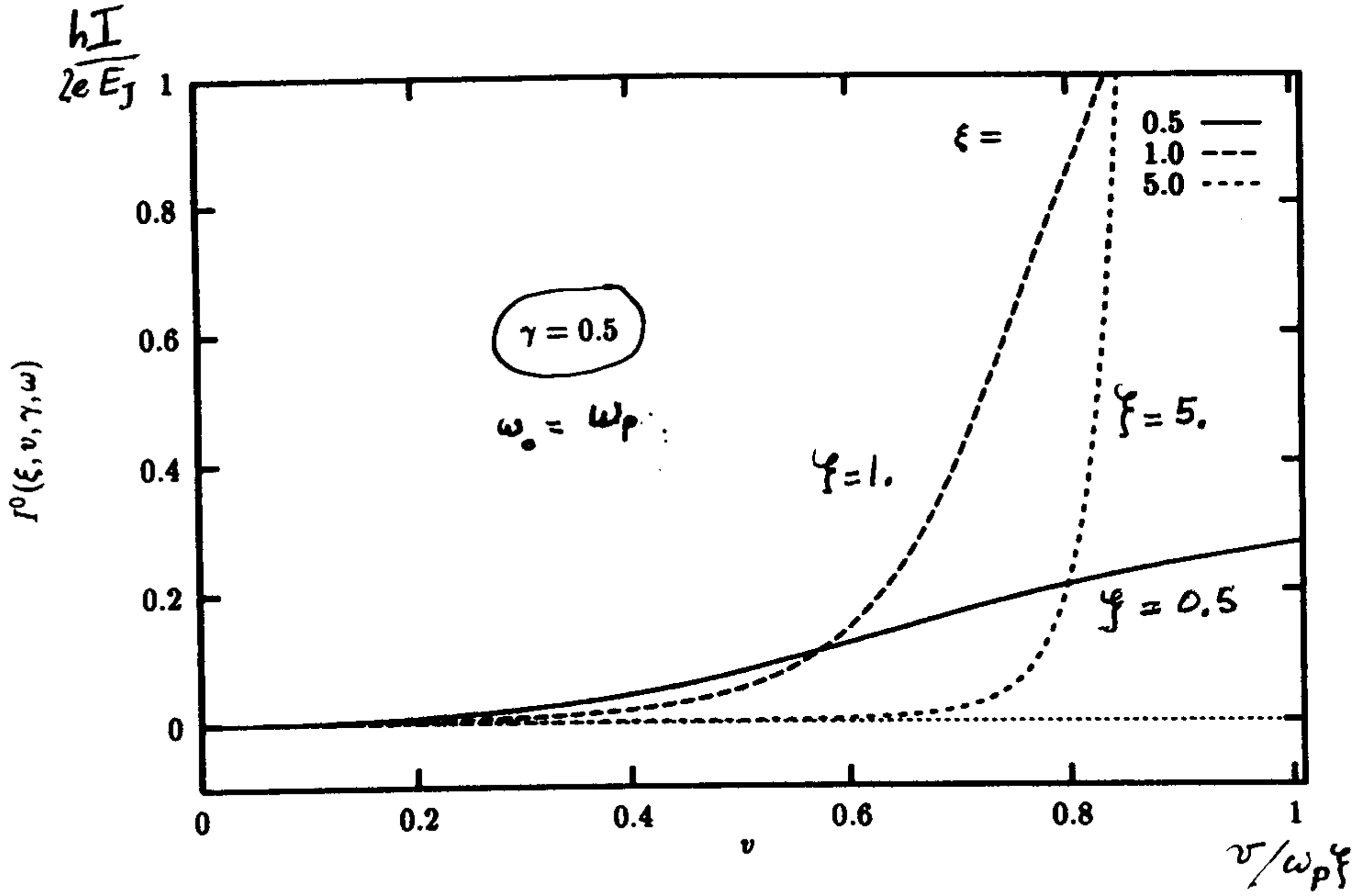
$$\eta_{pl} / \eta_{BS} = \frac{1}{2\pi} \sqrt{\beta_c}$$

$$\beta_c = 2\pi I_c R_N^2 C / \phi_0 \gg 1$$

Put the trajectory in the eq. of motion : $(\omega_0 = \omega_p)$

$$\frac{h}{2e} I(\omega) = 2\pi i \sum_{m\,n} \delta(m\omega_0 - \omega) \int d^2k \, \frac{k_y^2}{k_x k^4} \left[(n+m)\omega_0 - k_x v \right]^2$$

$$\cdot \, i^m J_n(k_x \gamma) \, J_{n+m}(k_x \gamma) \, Q^R_{\vec{k}, ((n+m)\omega_0 - k_x v)}$$



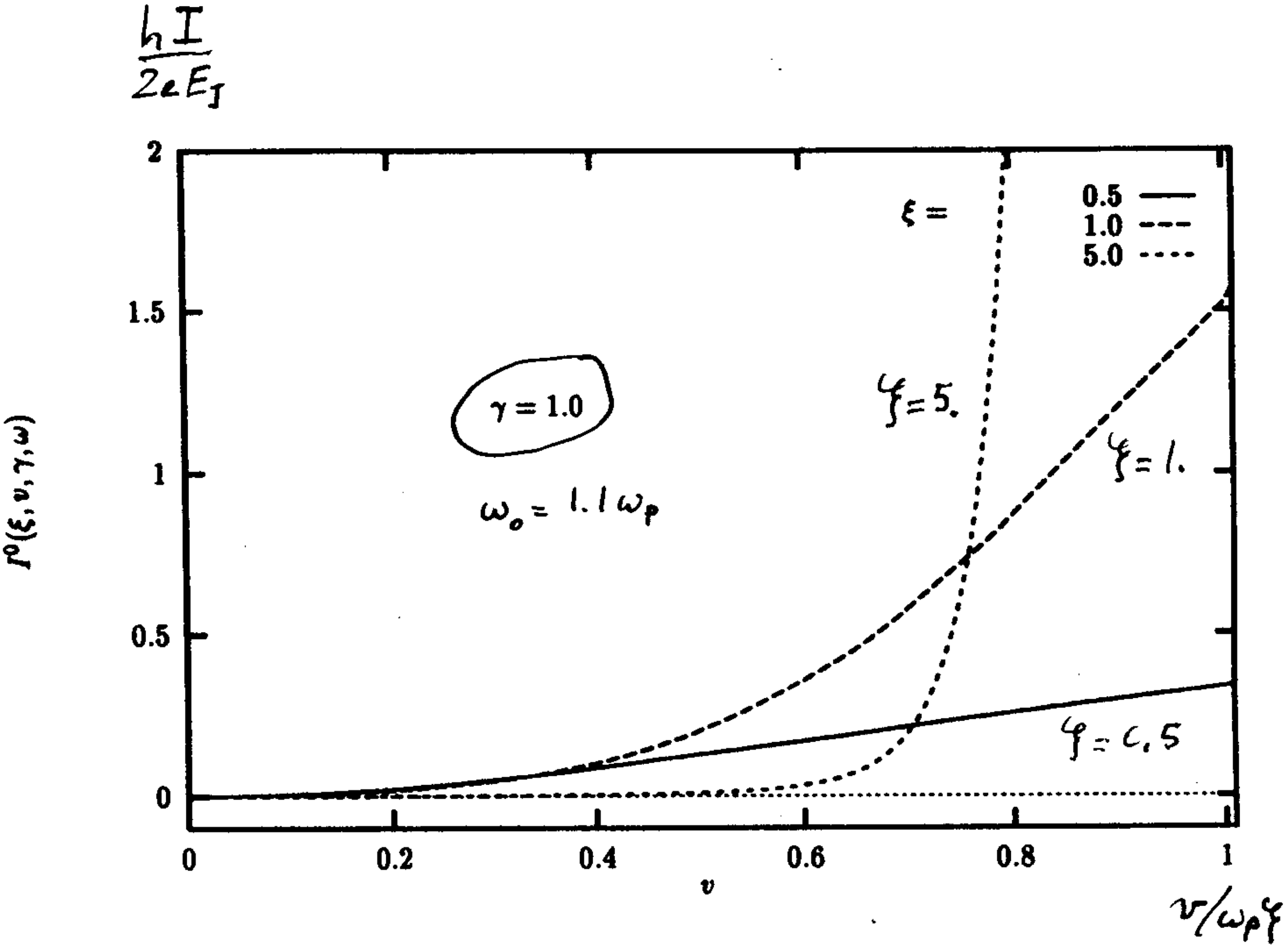
<u>Radiative dissipation of a quantum vortex</u>

$$x(t) = vt + \gamma \sin(\omega_0 t$$

$< \tau_{th}$ $\qquad \omega_p$

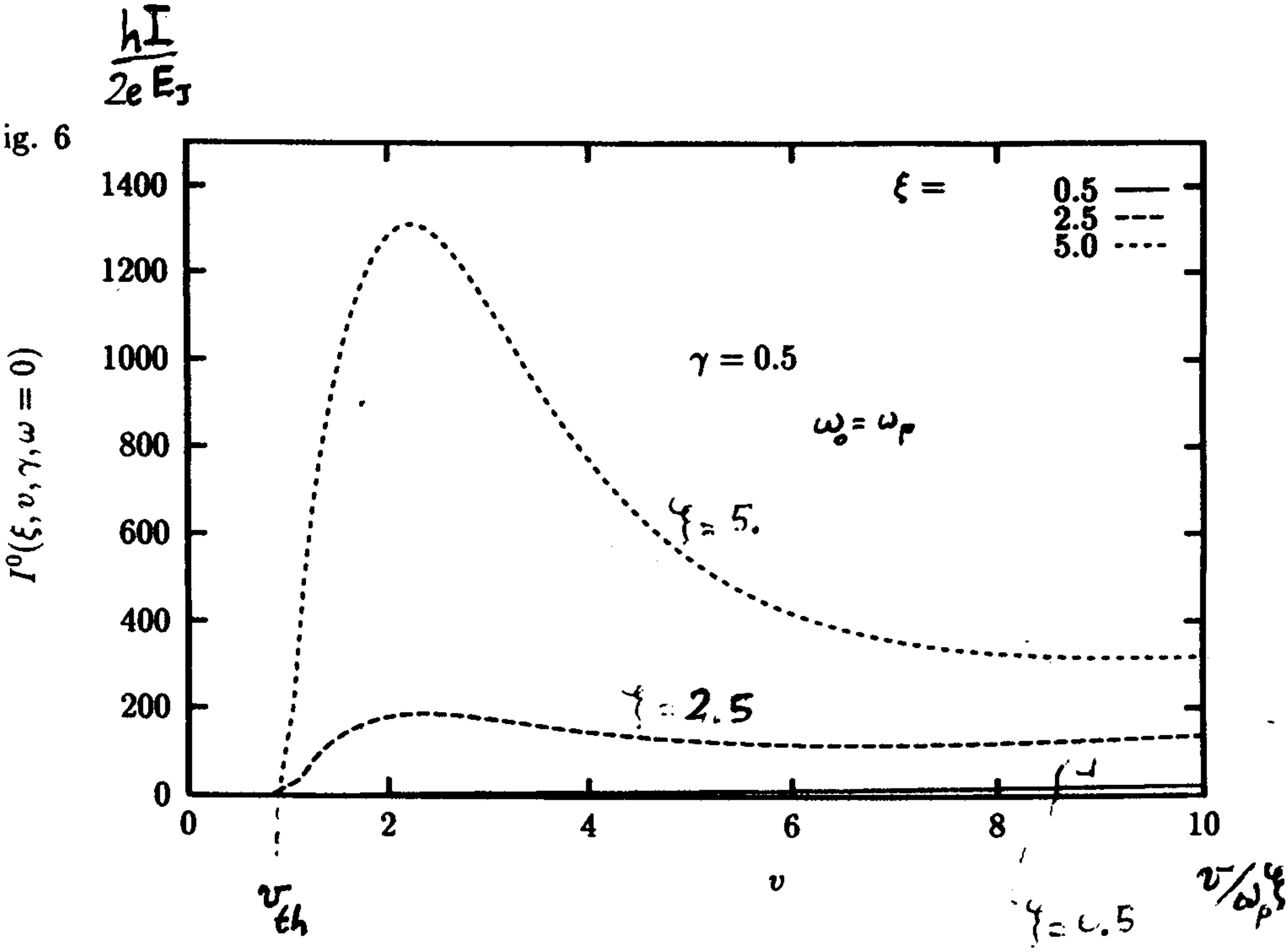
$$V_0 = \frac{\pi \hbar v}{2e} \qquad V_1(t) = \frac{\pi \hbar \omega_0}{2e} \gamma \cos \omega_0 t$$

ballistic motion is fragile !

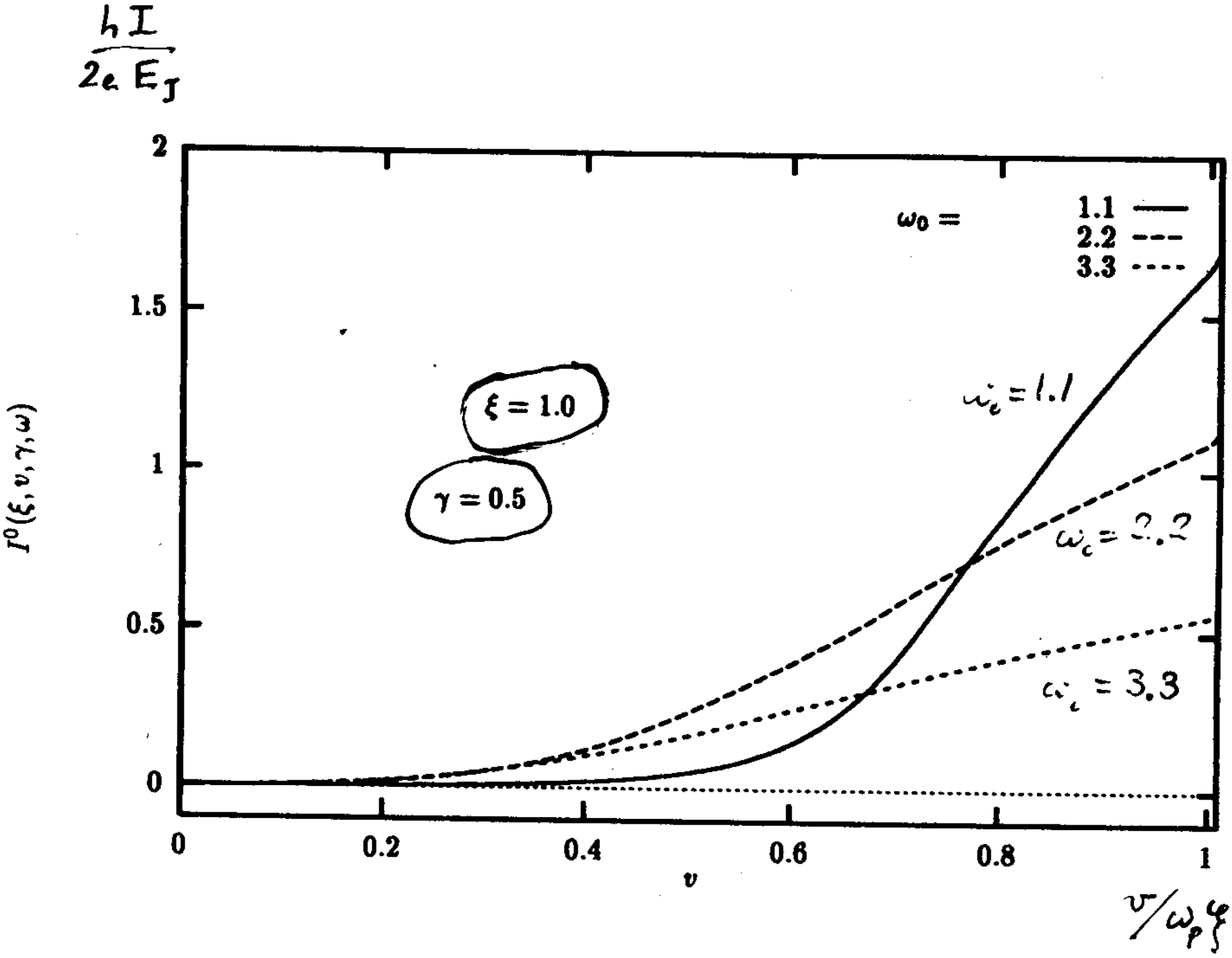


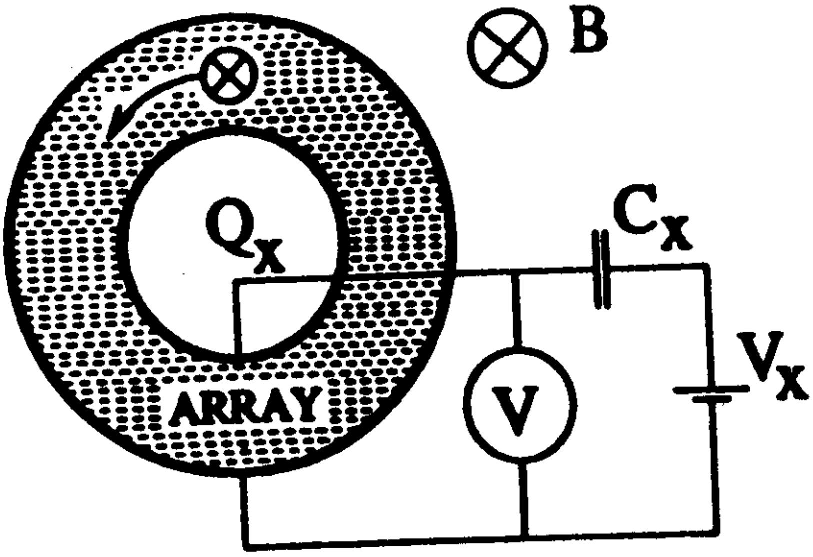
hI / 2eE_J
2
1.5
1
0.5
0
ξ =
0.5
1.0
5.0
γ = 1.0
ω₀ = 1.1 ωp
ξ = 5.
ξ = 1.
ξ = 0.5
I⁰(ξ, v, γ, ω)
0
0.2
0.4
0.6
0.8
1
v
v/ωpξ

ig. 6


hI / 2e E_J
1400
1200
1000
800
600
400
200
0
0
2
4
6
8
10
$I^0(\xi, v, \gamma, \omega = 0)$
$\xi =$
0.5
2.5
5.0
$\gamma = 0.5$
$\omega_0 = \omega_p$
ξ = 5.
ξ = 2.5
ξ = 0.5
v
v_{th}
$v/\omega_p \xi$

$v > v_{th}$


hI / 2e E_J
2
1.5
1
0.5
0
$I^0(\xi, v, \gamma, \omega)$
$\omega_0 =$
1.1
2.2
3.3
$\xi = 1.0$
$\gamma = 0.5$
$\omega_c = 1.1$
$\omega_c = 2.2$
$\omega_c = 3.3$
0
0.2
0.4
0.6
0.8
1
v
v/ω_p ξ



Persistent voltage in ring shaped Josephson arrays (van Wees '90)

Extra term is included:

$$S_{AC} = -i \sum_i \frac{q_i}{2e} \int_0^\beta d\tau \, \dot{\vec{R}}(\tau) \cdot \vec{\nabla} \Theta \left(\vec{r}_i - \vec{R}(\tau) \right)$$

Accumulated phase after 1 loop

$$\chi = \oint_\Gamma d\vec{R} \cdot \int d^2 r \, \frac{\rho(\vec{r})}{2e} \, \vec{A}(\vec{r} - \vec{R}) = 2\pi \frac{Q_x}{2e}$$

$$A(\vec{r}) = \frac{\hat{z} \times \vec{r}}{r^2}$$

Energy levels for the vortex in the ring are

<u>2e-periodic functions of Q_x</u>

The saddle point paths are:

$$\vartheta_n(\tau) = \frac{2\pi n \tau}{\hbar\beta}$$

$\frac{2\pi n}{\hbar\beta} \rightarrow \omega_n$

Dominant contribution:

$$Z = \sum_{n=-\infty}^{+\infty} e^{-S_n + 2\pi i n Q_x/2e}$$

$$\langle V \rangle = \frac{\partial}{\partial Q_x}\left(-\frac{1}{\beta}\ln Z\right)$$

In the classical case $(\alpha = 0)$ $\qquad S_n^{cl} = \frac{1}{2} M_{ES}\, \beta\, \omega_n^2$

Depends on the system size through the ratio $2\pi R/\lambda_T$

$$\lambda_T = \hbar\sqrt{\frac{2\beta}{M_{ES}}} = \frac{a}{\pi}\sqrt{8\beta E_c}$$

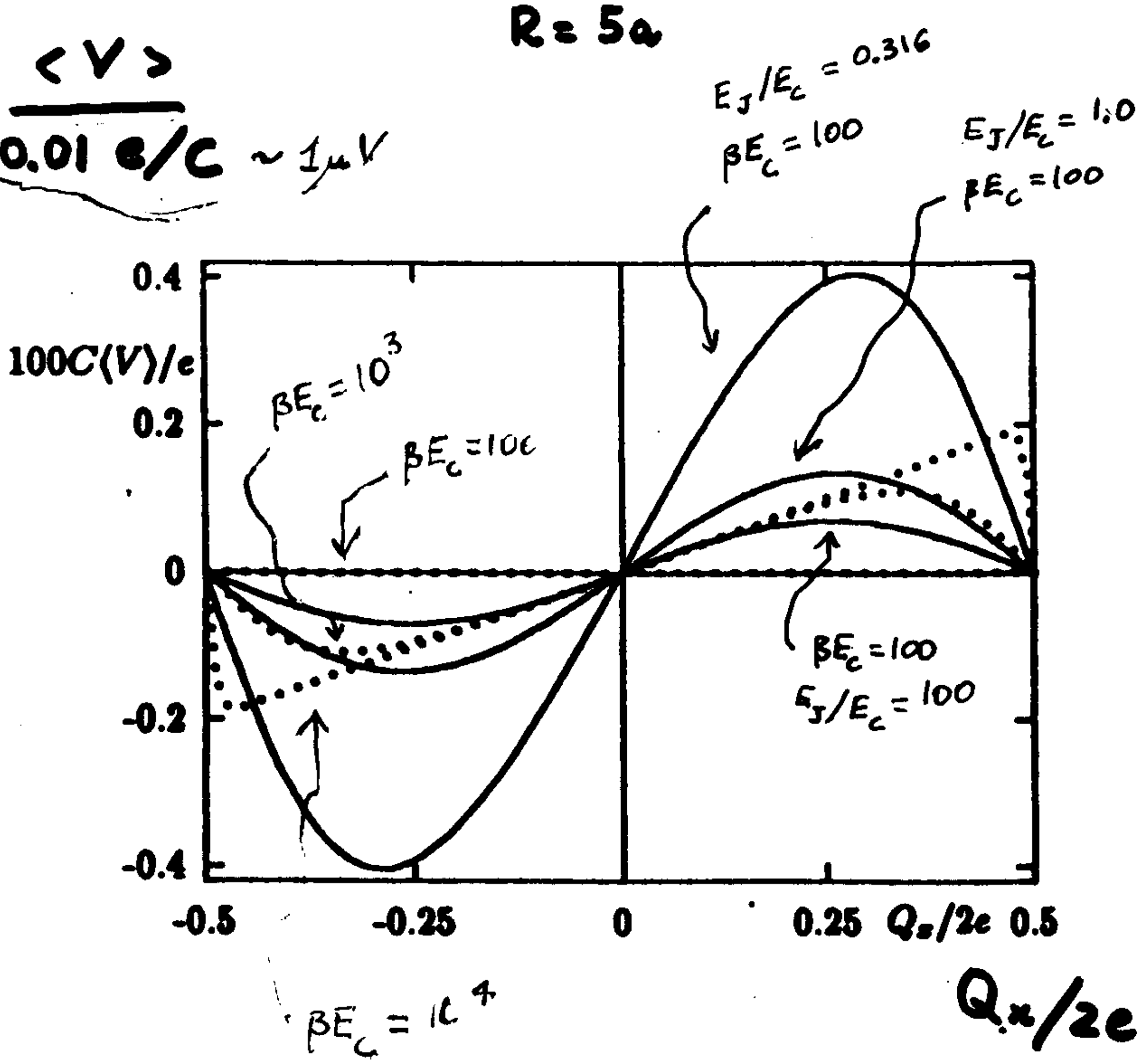
$a \rightarrow$ lattice spacing

$\beta^{-1} = 10\ \text{m}^\circ\text{K} \qquad E_c \sim 1^\circ\text{K} \qquad \lambda_T \sim 10\, a$

too small!

R = 5a

$\frac{\langle V \rangle}{0.01\, e/C} \sim 1\mu V$



S-I transition at $E_J/E_c = 0.12$

For $\alpha \neq 0$ $(\xi \neq 0)$ the saddle point action

$$S_n \simeq \frac{\pi}{4} \frac{\beta E_J \omega_n^2}{\omega_n^2 + \omega_p^2 \xi^2 / R^2} \ln\left[1 + 4\pi\xi^2\left(1 + \frac{\omega_n^2 R^2}{\omega_p^2 \xi^2}\right)\right]$$

Depends on system size through $2\pi R / L_\varphi$

$$\rightarrow L_\varphi = \hbar \beta \omega_p \xi$$

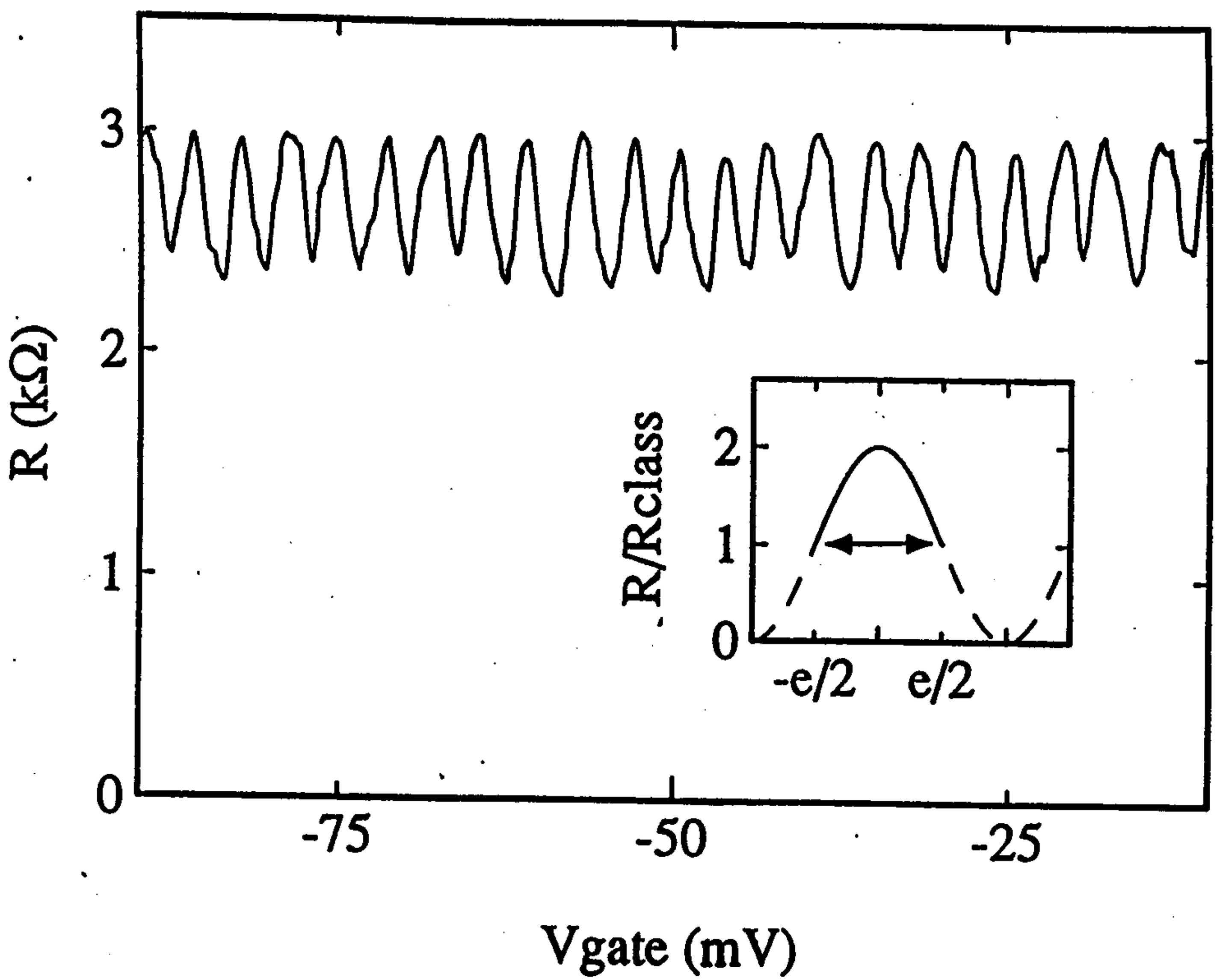
$$\sim 100 \lambda_T \gg R$$

$S_n \rightarrow S_n^d$ for $\xi \ll R$ $\omega_n \ll \omega_p$

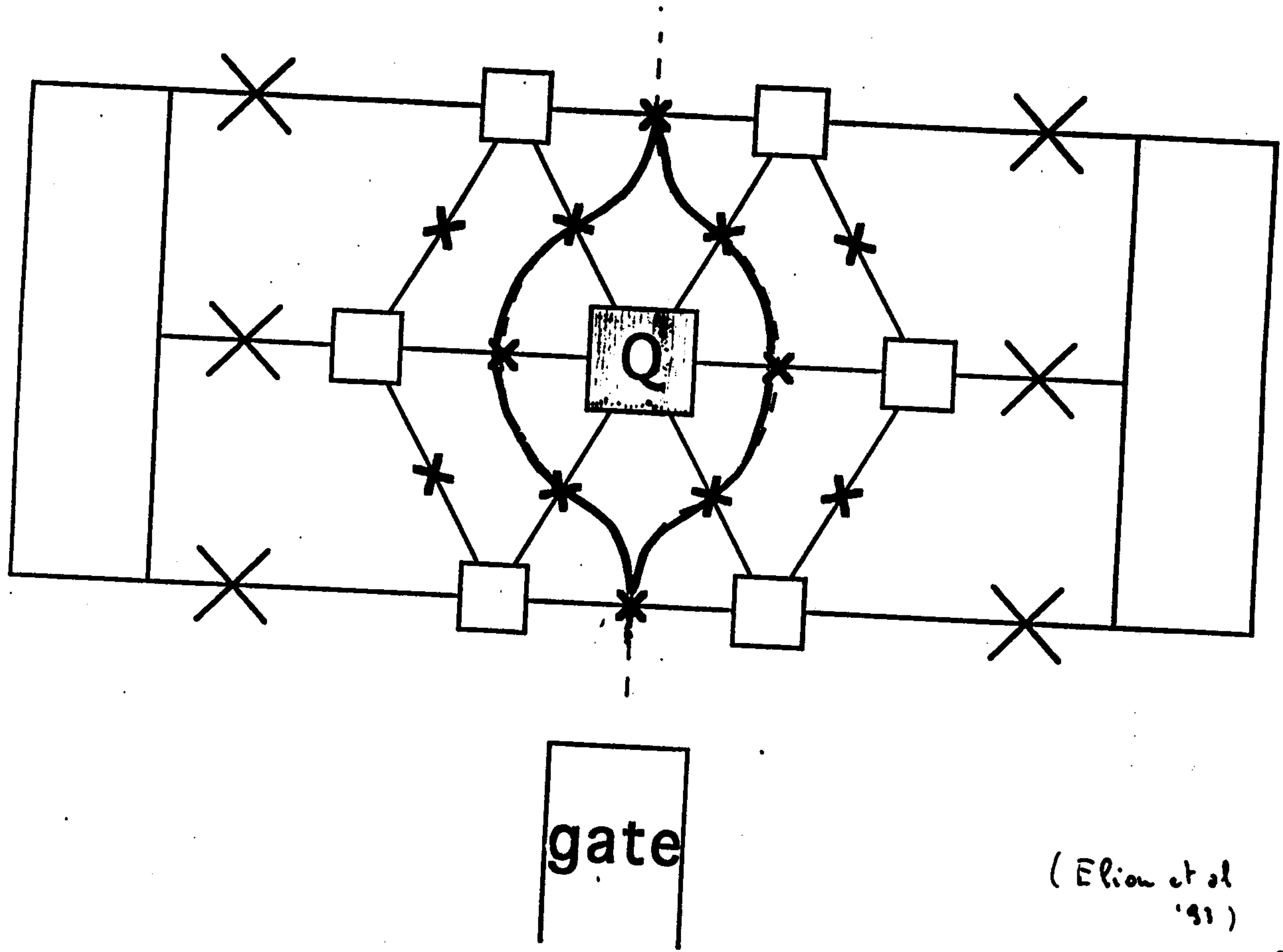
otherwise $S_n \ll S_n^d$

$R \sim 10a$ $E_c \sim E_J \sim 1°K$ $T \sim 10\, m°K$

$$\Rightarrow \bar{V} \sim \mu V$$



(Elion et al '93)



(Elion et al '91)

(64) Fig 1.