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"ON THE ROLE OF ROTATION IN THE GENERATION OF MAGNETIC FIELDS BY FLUID MOTIONS"

"THE MAGNETIC FLUX LINKAGE OF A RECURRING MEDIUM: A THEOREM AND GEOPHYSICAL APPLICATIONS"

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On the role of rotation in the generation of magnetic fields by fluid motions

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It is generally accepted that the magnetic fields of planets and stars are produced by the self-exciting dynamo process (first proposed by Larmor) and that observed near-alignments of magnetic dipole axes with rotation axes are due to the influence of Coriolis forces on underlying fluid motions. The detailed role of rotation in the generation of cosmical magnetic fields has yet to be elucidated but useful insight can be obtained from general considerations of the governing magnetohydrodynamic equations. A magnetic field $\boldsymbol{B}$ cannot be maintained or amplified by fluid motions against the effects of ohmic decay unless (*a*) the magnetic Reynolds number $R \equiv UL\overline{\mu}\overline{\sigma}$ is sufficiently large, and (*b*) the patterns of $\boldsymbol{B}$ and $\boldsymbol{u}$ are sufficiently complicated (where U is a characteristic flow speed, L a characteristic length and $\overline{\mu}$ and $\overline{\sigma}$ are typical values of the magnetic permeability and electrical conductivity respectively). Axisymmetric magnetic fields will always decay (a result that suggests that palaeomagnetic and archaeomagnetic data might show evidence that departures from axial symmetry in the geomagnetic field are systematically less during the decay phase of a polarity 'reversal' or 'excursion' than during the recovery phase). Dynamo action is stimulated by Coriolis forces, which promote departures from axial symmetry in the pattern of $\boldsymbol{u}$ when $\boldsymbol{B}$ is weak, and is opposed by Lorentz forces, which increase in influence as $\boldsymbol{B}$ grows in strength. If equilibrium is attained when Coriolis and Lorentz forces are roughly equal in magnitude then the system becomes 'magnetostrophic' and the strengths of the internal and external parts of the field, B_i and B_e respectively, satisfy $B_i \lesssim B_s R^{\frac{1}{2}}$ and $B_e \lesssim B_s R^{-\frac{1}{2}}$ if $B_s \equiv (\overline{\rho}(\Omega + UL^{-1})/\overline{\sigma})^{\frac{1}{2}} \approx (\overline{\rho}\Omega/\overline{\sigma})^{\frac{1}{2}}$, ($\overline{\rho}$ being the mean density of the fluid and Ω the angular speed of rotation). The slow and dispersive 'magnetohydrodynamic inertial wave' with a frequency that depends on the square of the Alfvén speed $|\boldsymbol{B}|/(\mu\overline{\rho})^{\frac{1}{2}}$ and *inversely* on Ω exemplifies magnetostrophic flow. Such waves probably occur in the electrically conducting fluid interiors of planets and stars, where they play an important role in the generation of magnetic fields as well as in other processes, such as the topographic coupling between the Earth's liquid core and solid mantle.

1. Introduction

The magnetic fields of the Earth and Sun and of other magnetic planets and stars are thought to be due to electric currents flowing within their interiors. It is now accepted that these currents are largely maintained against the effects of ohmic dissipation by electromotive forces due to motional induction, as Larmor first pointed out in his pioneering paper on self-exciting fluid dynamos. The fluid motions involved are produced in most (if not all) cases by the action of gravity on density inhomogeneities.

Theoretical studies of the flow of electrically conducting fluids – 'magnetohydrodynamics' or 'hydromagnetics' – are based on the highly nonlinear equations of hydrodynamics, thermodynamics and electrodynamics (see § 3). Dynamo models treated on the basis of all these equations are often referred to us 'magnetohydrodynamic dynamos'. But most progress to date has been made with the study of 'kinematic dynamos' for which the field of fluid flow is postulated *a priori* and non-decaying solutions sought of the electrodynamic equations alone (see § 2).

The mathematical analysis of dynamo models is complicated by the finding that suitable departures from axial symmetry are required for dynamo action to occur (see (2.10)). This follows from existence theorems in kinematic dynamo theory (for reference see, for example, Moffatt 1978; Parker 1979) and the recent extension of Cowling's theorem (Hide & Palmer 1982) showing quite generally:

Fluid motions cannot prevent the ohmic decay of a magnetic field that retains an axis of symmetry. (1.1)

This result (which suggests, incidentally, that palaeomagnetic and archaeomagnetic data might show evidence that departures from axial symmetry are systematically less during the decay phase of a geomagnetic polarity 'reversal' or 'excursion' than during the growth or recovery phase (see Hide 1981*a*)) provides a useful starting point for discussing how rotation affects the generation of magnetic fields by the dynamo process.

Order of magnitude estimates of the various terms in the dynamical equations (see § 3 below) show that flows associated with typical natural dynamos are strongly influenced by Coriolis forces due to general rotation. It is through the analysis of this influence of Coriolis forces that the explanation of the near-alignment of the rotation and magnetic dipole axes of the Earth, Jupiter, Saturn, etc., and other properties of the magnetic fields must be sought. But further work on the magnetohydrodynamics of rapidly rotating fluids will be needed before the role of rotation in the production of cosmical magnetic fields can be fully elucidated. Details of investigations in this important but highly mathematical branch of geophysical fluid dynamics together with references to early work can be found in several recent publications (see, for example, Moffatt 1978; Roberts & Soward 1978; Busse 1978, 1979; Parker 1979; Stevenson 1979; Braginskiy 1980; Krause & Rädler 1980; Soward 1982). The purpose of the present paper is to outline certain general properties of flows that are strongly influenced by Coriolis forces due to general rotation and Lorentz forces due to the presence of electric currents within the fluid. These properties follow more or less directly from the governing equations (see §§ 3 and 4) and they can serve among other things as a guide to the more technical literature.

Possibly the most significant of these properties for dynamo studies is the result (see (3.10) below):

Rapid rotation promotes departures from axial symmetry in the pattern of fluid motions when the magnetic field is weak. (1.2)

Coriolis forces can thus stimulate the amplification of a weak magnetic field by producing departures from axial symmetry in the pattern of fluid motions. As the magnetic field increases in strength so does the Lorentz force (see equation (3.4)), and it is possible that the amplification of the magnetic field cannot continue beyond the point at which the Lorentz force is typically comparable in magnitude with the Coriolis force. This hypothesis provides a basis for estimating the ultimate strength obtained by the magnetic field and leads to predictions that are concordant with observations (Hide 1974; see also § 4).

2. Basic equations: electrodynamics

Consider a connected body of electrically conducting fluid V_0 bounded by a surface S_0 with surface element $\mathrm{d}\boldsymbol{S}$. The linkage with S_0 of a magnetic field $\boldsymbol{B}$ that pervades the conducting fluid and the surrounding space is defined as the essentially non-negative quantity

$$N(S_0; t) \equiv \iint_{S_0} |\boldsymbol{B}\cdot \mathrm{d}\boldsymbol{S}|. \tag{2.1}$$

In the absence of permanent magnets, $\boldsymbol{B}$ is due entirely to electric currents of density $\boldsymbol{j}$ and in the self-exciting dynamo the electromotive forces that produce these electric currents are provided by motional induction, involving fluid motions within V_0 with Eulerian velocity $\boldsymbol{u}$. By this means some weak adventitious seed field can be amplified and maintained against the effects of ohmic decay. If the fluid motions were suddenly to cease, $N(S_0; t)$ would decay on a timescale $O(\tau_d)$ where τ_d is the ohmic decay time based on a characteristic length L of the order of the dimensions of V_0 (see equation (2.11) below). For the Earth's liquid electrically conducting core τ_d lies somewhere between 10^4 and 10^5 years; for the cores of Jupiter and Saturn τ_d could be somewhat longer, possibly 10^6 or 10^7 years, but still short compared with the presumed ages of the magnetic fields of these planets, in excess of 10^9 years.

Dynamo action can be said to occur in a theoretical model when the magnitude and configuration of $\boldsymbol{u}$ and $\boldsymbol{B}$ are such that over the long but otherwise arbitrary interval $t = t_1$ to $t = t_2$ (where $t_2 - t_1$ greatly exceeds the ohmic decay time τ_d (see equation (2.11)),

$$(t_2 - t_1)^{-1} \int_{t_1}^{t_2} |\mathrm{d}N(S_0; t)/\mathrm{d}t|\, \mathrm{d}t \ll 0. \tag{2.2}$$

This criterion has advantages over proposals based on total magnetic energy or equivalent magnetic moment, which can be ambiguous when $\boldsymbol{B}$ has toroidal as well as poloidal components or when the conducting fluid is not incompressible (see Hide 1981*b*; Hide & Palmer 1982). We consider first the electrodynamic equations required in the formulation of theoretical models. These are Gauss's law

$$\nabla\cdot\boldsymbol{B} = 0 \tag{2.3}$$

(which implies, of course, that

$$\iint_S \boldsymbol{B}\cdot\mathrm{d}\boldsymbol{S} = 0$$

taken over any closed surface S; cf. equation (2.1)), Faraday's law

$$\partial\boldsymbol{B}/\partial t + \nabla\times\boldsymbol{E} = 0, \tag{2.4}$$

and Ampère's law

$$\nabla\times(\mu^{-1}\boldsymbol{B}) = \boldsymbol{j}, \tag{2.5}$$

together with Ohm's law

$$\boldsymbol{j} = \sigma(\boldsymbol{E} + \boldsymbol{u}\times\boldsymbol{B}), \tag{2.6}$$

where $\boldsymbol{E}$ is the electric field in the basic frame of reference and $\boldsymbol{E}+\boldsymbol{u}\times\boldsymbol{B}$ is the electric field experienced by a fluid element moving with velocity $\boldsymbol{u}$ relative to that frame. The magnetic permeability μ and electrical conductivity σ are scalars but they may be functions of position and time.

When $\boldsymbol{E}$ and $\boldsymbol{j}$ are eliminated from these equations it is found that

$$\partial\boldsymbol{B}/\partial t = -\nabla\times(\sigma^{-1}\nabla\times(\mu^{-1}\boldsymbol{B})) + \nabla\times(\boldsymbol{u}\times\boldsymbol{B}). \tag{2.7}$$

By multiplying this equation scalarly by $\mathrm{d}\boldsymbol{S}$ integrating the resulting expression for $\partial|\boldsymbol{B}\cdot\mathrm{d}\boldsymbol{S}|/\partial t$ over the general closed surface S it may be shown that

$$\mathrm{d}N(S; t)/\mathrm{d}t = -2\sum\oint_C [\sigma^{-1}\nabla\times(\mu^{-1}\boldsymbol{B}) + (\boldsymbol{v}-\boldsymbol{u})\times\boldsymbol{B}]\cdot\mathrm{d}\boldsymbol{C} \tag{2.8}$$

if $\boldsymbol{v}$ denotes the motion of a general point on S (see Hide 1981*b*). The line integrals are taken over all the one or more closed curves C, vector element of length $\mathrm{d}\boldsymbol{C}$, on S where $\boldsymbol{B}\cdot\mathrm{d}\boldsymbol{S} = 0$, in the sense that keeps the neighbouring region where $\boldsymbol{B}\cdot\mathrm{d}\boldsymbol{S}$ is positive (negative) on the left (right) when moving in the direction of $\mathrm{d}\boldsymbol{C}$. When S is any material surface S' we have $(\boldsymbol{v}-\boldsymbol{u})\cdot\mathrm{d}\boldsymbol{S} = 0$ everywhere on S'; whence $(\boldsymbol{v}-\boldsymbol{u})\times\boldsymbol{B}\cdot\mathrm{d}\boldsymbol{C} = 0$ and

$$\mathrm{d}N(S'; t)/\mathrm{d}t = -2\sum\oint_C \sigma^{-1}\nabla\times(\mu^{-1}\boldsymbol{B})\cdot\mathrm{d}\boldsymbol{C} = -2\sum\oint_C \sigma^{-1}\boldsymbol{j}\cdot\mathrm{d}\boldsymbol{C}. \tag{2.9}$$

Dynamo action requires high electrical conductivity (see equation (2.13) below), but equation (2.9) shows that it cannot occur in a perfect conductor, since $\mathrm{d}N(S'; t)/\mathrm{d}t = 0$ when $\sigma^{-1} = 0$.

As we have already noted, the mathematical difficulties presented by the full magnetohydrodynamic dynamo problem are so severe that most studies to date have been concerned with kinematic dynamo problems of obtaining non-decaying solutions of equation (2.7) when $\boldsymbol{u}$ is specified *a priori*. Such solutions can be found, but only when (*a*)

$\boldsymbol{u}$ and $\boldsymbol{B}$ are sufficiently complicated in form (there being only decaying solutions when the configurations of $\boldsymbol{u}$ and $\boldsymbol{B}$ possess a common axis of symmetry) (2.10)

(cf. (1.1)), and (*b*) the ohmic decay time τ_d is so long in comparison with the advective time scale τ_a, where

$$\tau_d \equiv L^2\bar{\mu}\bar{\sigma}; \quad \tau_a \equiv L/U, \tag{2.11}$$

that the so-called 'magnetic Reynolds number'

$$R \equiv UL\bar{\mu}\bar{\sigma} = \tau_d/\tau_a \tag{2.12}$$

satisfies

$$R > R_*, \tag{2.13}$$

where R_* is typically between 10 and 10^2. (Here U, L, $\bar{\mu}$ and $\bar{\sigma}$ are typical value of the flow speed, length scale, magnetic permeability and electrical conductivity, respectively.)

Kinematic dynamo studies have called attention to the role of the helicity of the motion

$$H \equiv \boldsymbol{u}\cdot\nabla\times\boldsymbol{u} \tag{2.14}$$

in the amplification process. This pseudo-scalar quantity is easy to visualize when it is expressed as the sum of three contributions, each proportional to the rate of change with respect to one of the Cartesian coordinates x_i ($i = 1, 2, 3$) of the direction made by the projection of $\boldsymbol{u}$ on the local (x_j, x_k) ($j = 2, 3, 1$; $k = 3, 2, 1$) plane perpendicular to the x_i axis. Thus

$$H = \sum_{i=1}^{3} H_i \quad \text{where} \quad H_i = -(u_j^2 + u_k^2)\frac{\partial}{\partial x_i}\tan^{-1}\left(\frac{u_k}{u_j}\right). \tag{2.15}$$

This form of H also has certain physical advantages when dealing with large-scale fluid motions that depart but little from rigid body rotation relative to an inertial frame having angular velocity $\boldsymbol{\Omega}$ about one of the coordinate axes, say x_3, and are therefore dominated by Coriolis forces (see § 3). A major contribution to H is then provided by H_3, which satisfies the following equation (Hide 1976):

$$H_3 \approx \frac{1}{2\Omega^2}\{(\boldsymbol{u}\cdot\boldsymbol{g})(\boldsymbol{\Omega}\cdot\nabla\theta) - (\boldsymbol{u}\cdot\nabla\theta)(\boldsymbol{\Omega}\cdot\boldsymbol{g})\} + \frac{\boldsymbol{u}\cdot(\boldsymbol{\Omega}\times \mathrm{d}\boldsymbol{\Omega}/\mathrm{d}t)}{\Omega^2} + \frac{(\boldsymbol{\Omega}\times\boldsymbol{u})}{\bar{\rho}}\cdot(\nabla\times[\nabla\times(\mu^{-1}\boldsymbol{B})\times\boldsymbol{B}]) \tag{2.16}$$

(where $\boldsymbol{g}$ denotes the acceleration due to gravity and centripetal effects, $\boldsymbol{u}$ the fluid motion relative to the rotating frame and $\bar{\rho}\theta$ the departure of the nearly uniform density from its mean value $\bar{\rho}$). The first term on the right-hand side of equation (2.16) is zero in the absence of buoyancy forces associated with density variations; the second term is zero when the precessional term vanishes (i.e. when $\mathrm{d}\boldsymbol{\Omega}/\mathrm{d}t$ is either zero or parallel to $\boldsymbol{\Omega}$); and the third term is zero when there are no Lorentz forces.

3. Basic equations: magnetohydrodynamics

The full dynamo problem requires the simultaneous solutions of the equations of electrodynamics, thermodynamics and hydrodynamics. The equations of electrodynamics give

$$\nabla\cdot\boldsymbol{B} = 0 \tag{3.1}$$

and

$$\partial\boldsymbol{B}/\partial t + (\boldsymbol{u}\cdot\nabla)\boldsymbol{B} - (\boldsymbol{B}\cdot\nabla)\boldsymbol{u} = (\mu\sigma)^{-1}\nabla^2\boldsymbol{B} \tag{3.2}$$

when σ and μ are constant (see equations (2.3) and (2.7)). The equations of thermodynamics (see, for example, Gubbins & Masters 1979) comprise an equation of state relating the density $\rho = \bar{\rho}(1+\theta)$ to the pressure p, temperature and chemical composition, together with equations governing the advection and diffusion of heat and variations in chemical composition. The equations of hydrodynamics express continuity of matter and momentum balance of individual fluid elements. The first of these, $\mathrm{D}\rho/\mathrm{D}t + \rho\nabla\cdot\boldsymbol{u} = 0$ (where $\mathrm{D}/\mathrm{D}t \equiv \partial/\partial t + \boldsymbol{u}\cdot\nabla$), reduces to

$$\nabla\cdot\boldsymbol{u} = 0, \tag{3.3}$$

when dynamical effects of fluid compressibility are negligible. The momentum equation

$$\rho\left(\frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t} + 2\boldsymbol{\Omega}\times\boldsymbol{u} - \boldsymbol{r}\times\frac{\mathrm{d}\boldsymbol{\Omega}}{\mathrm{d}t}\right) = -\nabla p + \boldsymbol{g}\rho - \nabla\times(\nu\rho\nabla\times\boldsymbol{u}) + \nabla\times(\mu^{-1}\boldsymbol{B})\times\boldsymbol{B}$$

reduces to its Boussinesq version

$$2\boldsymbol{\Omega}\times\boldsymbol{u} + \nabla P = \boldsymbol{A}, \tag{3.4a}$$

where

$$\nabla P \equiv \nabla(P/\bar{\rho}) - \boldsymbol{g}$$

and

$$\boldsymbol{A} \equiv -\frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t} + \boldsymbol{r}\times\frac{\mathrm{d}\boldsymbol{\Omega}}{\mathrm{d}t} + \boldsymbol{g}\theta + \nu\nabla^2\boldsymbol{u} + \frac{\{(\boldsymbol{B}\cdot\nabla)\boldsymbol{B} - \nabla(B^2/2)\}}{\mu\bar{\rho}} \tag{3.4b}$$

when the kinematic viscosity ν is constant, $\boldsymbol{g}$ greatly exceeds the other acceleration terms and fractional density variations θ are very much less than unity. Taking the curl of equation (3.4a) gives the vorticity equation expressing the local balance of angular momentum of an individual fluid element; thus

$$(2\boldsymbol{\Omega}\cdot\nabla)\boldsymbol{u} = -\nabla\times\boldsymbol{A}, \tag{3.5a}$$

where

$$\nabla\times\boldsymbol{A} = -\mathrm{D}\boldsymbol{\xi}/\mathrm{D}t + (\boldsymbol{\xi}\cdot\nabla)\boldsymbol{u} - 2\,\mathrm{d}\boldsymbol{\Omega}/\mathrm{d}t - \boldsymbol{g}\times\nabla\theta + \nu\nabla^2\boldsymbol{\xi} + (\mu\bar{\rho})^{-1}\nabla\times(\boldsymbol{B}\cdot\nabla)\boldsymbol{B} \tag{3.5b}$$

if $\boldsymbol{\xi} \equiv \nabla\times\boldsymbol{u}$.

Order of magnitude estimates of the various terms in equations (3.4) and (3.5) applied to motions in the Earth's liquid core indicate that the relative acceleration term $\mathrm{D}\boldsymbol{u}/\mathrm{D}t \equiv \partial\boldsymbol{u}/\partial t + (\boldsymbol{u}\nabla)\boldsymbol{u}$ and the viscous term $\nu\nabla^2\boldsymbol{u}$ are many orders of magnitude less than the Coriolis term $2\boldsymbol{\Omega}\times\boldsymbol{u}$. When the precessional term is also negligible we have the case of *magnetostrophic flow*, with

$$\boldsymbol{A} = \boldsymbol{A}_{\mathrm{m}} \equiv \boldsymbol{g}\theta + (\mu\bar{\rho})^{-1}\{(\boldsymbol{B}\cdot\nabla)\boldsymbol{B} - \nabla(B^2/2)\} \tag{3.6a}$$

and

$$\nabla\times\boldsymbol{A} = \nabla\times\boldsymbol{A}_{\mathrm{m}} = -\boldsymbol{g}\times\nabla\theta + (\mu\bar{\rho})^{-1}\nabla\times(\boldsymbol{B}\cdot\nabla)\boldsymbol{B}. \tag{3.6b}$$

When, in addition, the Lorentz term is negligible in comparison with the Coriolis term we have the case of *geostrophic flow*, with

$$\boldsymbol{A} = \boldsymbol{A}_{\mathrm{g}} \equiv \boldsymbol{g}\theta \tag{3.7a}$$

and

$$\nabla\times\boldsymbol{A} = \nabla\times\boldsymbol{A}_{\mathrm{g}} = -\boldsymbol{g}\times\nabla\theta. \tag{3.7b}$$

The vorticity equation (3.5) then yields the 'thermal wind equation'

$$(2\boldsymbol{\Omega}\cdot\nabla)\boldsymbol{u} = \boldsymbol{g}\times\nabla\theta \tag{3.8}$$

in the 'baroclinic' case ($\boldsymbol{g}\times\nabla\theta \neq 0$), which reduces to the Proudman–Taylor theorem

$$(2\boldsymbol{\Omega}\cdot\nabla)\boldsymbol{u} = 0 \tag{3.9}$$

in the 'barotropic' case ($\boldsymbol{g}\times\nabla\theta = 0$), when surfaces of equal density coincide with geopotential surfaces.

Equations (3.8) and (3.9) are succinct expressions of the powerful gyroscopic constraints on the motion of a fluid of low viscosity that departs but little from solid body rotation with steady angular velocity $\boldsymbol{\Omega}$ when Lorentz forces are negligibly small. Studies of such flows are important in dynamo theory because they provide insight into the initial stages of the amplification process, when Lorentz forces would indeed be small. Equation (3.4) with $\boldsymbol{A} = \boldsymbol{A}_{\mathrm{g}}$ (see equation (3.7)) leads to the important result (cf. (1.2)):

> *The hydrodynamical motion of a fluid of low viscosity that departs only slightly from steady rapid rigid-body rotation will not in general be symmetric about the rotation axis, even when the boundary conditions are axisymmetric.* (3.10)

The validity of this result (which provides the most direct explanation of the occurrence of large-scale non-axisymmetric disturbances in the Earth's atmosphere and other natural systems) is readily verified by laboratory experiments. The result can be deduced as follows. In cylindrical co-ordinates (r, ϕ, z) where $\boldsymbol{\Omega} = (0, 0, \Omega)$ the second component of equation (3.4) is:

$$u_r = (2\Omega)^{-1}\{-r^{-1}\partial P/\partial\phi + (\boldsymbol{A})_\phi\} \tag{3.11}$$

(since $(\boldsymbol{g})_\phi = 0$ by the assumption of axial symmetry in the boundary conditions), where $(\boldsymbol{A})_\phi$ denotes the ϕ component of $\boldsymbol{A}$. Now, over any cylindrical surface of radius r the rate of advective transport $M(r, t; Q)$ of any quantity Q (per unit volume), such as heat, angular momentum, etc., is given by

$$M(r, t; Q) = \int_{z_1}^{z_2}\int_0^{2\pi} u_r Q r\,\mathrm{d}\phi\,\mathrm{d}z = \frac{1}{2\Omega}\int_{z_1}^{z_2}\int_0^{2\pi}\left\{-\frac{\partial P}{\partial\phi} + r(\boldsymbol{A})_\phi\right\} Q\,\mathrm{d}\phi\,\mathrm{d}z. \tag{3.12}$$

Since the contribution $(\boldsymbol{A})_\phi$ to equation (3.11) decreases rapidly with increasing Ω, advective transport perpendicular to the axis of rotation, as measured by $M(r, t; Q)$, will be negligible unless the flow pattern departs significantly from axial symmetry. In the axisymmetric case we have $\partial P/\partial\phi = 0$ and $M(r, t; Q)$ of the order of the small ageostrophic contribution.

This argument is the basis of (3.10). There may be singular cases when the flow remains axisymmetric and in consequence advective transfer perpendicular to the rotation axis is negligible. Indeed, such cases can be realized in the laboratory by taking certain special precautions, but the general conclusion from laboratory experiments is that (3.10) is a correct inference from the geostrophic equation.

There is a further property of equation (3.4) with $A = A_g$ that leads to a useful general prediction. The equation is mathematically degenerate; being lower in order than the full equation to which it is a leading approximation when Ω is large, it cannot be solved under all the necessary boundary conditions. For this to be possible every term in A must be included in the analysis, which implies:

> *Regions of highly ageostrophic flow occurring not only on the boundaries of the system but also in localized regions (detached shear layers, jet streams, etc.) of the main body of the fluid are necessary concomitants of geostrophic motion.* (3.13)

Within these highly ageostrophic regions, $\rho\, \mathrm{D}\boldsymbol{u}/\mathrm{D}t + \nabla \times (\rho\nu\nabla \times \boldsymbol{u})$ is comparable in magnitude with $2\rho\boldsymbol{\Omega} \times \boldsymbol{u}$; the corresponding relative vorticity $\boldsymbol{\zeta} \equiv \nabla \times \boldsymbol{u}$ can be comparable with or even exceed $2\boldsymbol{\Omega}$ in magnitude. Many examples of such vorticity concentrations are found in the laboratory and in Nature.

We have seen that slow relative hydrodynamical flow in a rotating fluid of low viscosity will in general be non-axisymmetric (see (3.10)). Laboratory studies show that there are two non-axisymmetric régimes of thermal convection in a rotating fluid annulus subject to differential heating in the horizontal, one highly regular (i.e. spatially and temporally periodic) and the other, which is reminiscent of large-scale flow in the Earth's atmosphere, irregular. Thus when the basic rotation rate Ω of the fluid annulus exceeds a certain value Ω_R, Coriolis forces inhibit axisymmetric overturning motions in meridian planes and promote a completely different kind of motion, which has been termed 'sloping convection'. The motion is then non-axisymmetric and largely confined to jet-streams, with typical trajectories of individual fluid elements inclined at small but essentially non-zero angles to the horizontal. The kinetic energy of the non-axisymmetric flow derives from the interaction of slight upward and downward motions in these sloping trajectories with the potential energy field produced by the action of gravity on the density variations produced by the applied differential heating. The kinetic energy of the motion is dissipated by friction arising in boundary layers on the walls of the container and in the main body of the fluid. The critical value Ω_R of the rotation speed is of course dependent on many parameters, including the acceleration of gravity, the shape and dimensions of the apparatus, the coefficients of thermal expansion, thermal conductivity and viscosity of the fluid and its mean density, and the distribution and intensity of applied differential heating. This dependence has been determined by extensive laboratory studies and interpreted on the basis of stability theory.

Provided that Ω, though greater than Ω_R, does not exceed a second critical value Ω_I, the main features of the non-axisymmetric motion are characterized by great regularity and the heat flow is virtually independent of Ω and some 20% less than when $\Omega = 0$. This regular flow is either steady (apart from a slow steady drift of the horizontal flow pattern relative to the walls of the container) or it exhibits periodic 'vacillation' in amplitude, shape and other characteristics. The number of 'waves' m around the annulus is not uniquely determined by the impressed

conditions; the flow is found to be 'intransitive' owing to the occurrence of what are now called 'multiple equilibrium states'. But the most likely value of m tends to increase with increasing Ω, and when $\Omega = \Omega_I$, m has that value for which the azimuthal scale of the horizontal flow pattern is about 1.5 times the radial scale and the flow undergoes a transition to irregular flow or 'geostrophic turbulence'. When $\Omega > \Omega_I$ we have the irregular flow régime, for which heat flow decreases with increasing Ω.

The behaviour just described is now known to be typical of a wide variety of dynamical systems, where large-scale motions can be highly regular under some impressed conditions and highly irregular under other conditions (see, for example, Haken 1981; Hide 1981*c*, 1982). Both types of large-scale flow can occur in natural systems and are therefore of interest in the theory of magnetic field generation by dynamo action. Underlying mechanisms are not yet fully understood but it is likely that one important role of Coriolis forces is to render the flow highly anisotropic. Energy transfer between different scales of motion within such flows contrasts sharply with that which occurs in isotropic flows, where nonlinear interactions can produce a 'cascade' of energy towards the smallest scales of motion and render the system highly chaotic (i.e. turbulent). Such cascades cannot occur in typical anisotropic flows unless they are accompanied by a simultaneous energy transfer to the largest scales available.

4. Magnetostrophic flows

Setting $Q = 1$ in equation (3.12) leads to a useful general result which, when applied to the magnetostrophic case, reduces to the expression for the constraint on $\boldsymbol{B}$ first noted by Taylor (1963), namely that over any cylindrical surface coaxial with the rotation axis, $\boldsymbol{B}$ must be such that

$$\int_{z_1}^{z_2}\int_0^{2\pi} \{(\nabla \times \boldsymbol{B}) \times \boldsymbol{B}\}_\phi\, r\, \mathrm{d}\phi\, \mathrm{d}z = 0. \tag{4.1}$$

The term in equation (3.12) involving P vanishes when $Q = 1$ because P is single valued. $M(r, t; Q)$ is negligibly small when $Q = 1$ because, by considerations of continuity, $M(r, t; 1)$ is equal to the volume flux across the surfaces $z = z_1(r)$ and $z = z_2(r)$; this depends on boundary layer suction, which vanishes in the limit of zero viscosity. It follows that

$$\int_{z_1}^{z_2}\int_0^{2\pi} (2\Omega)^{-1}(A)_\phi\, r\, \mathrm{d}\phi\, \mathrm{d}z = 0 \tag{4.2}$$

and this reduces to Taylor's result (see equation (3.1)) in the magnetostrophic case when $A = A_m$ (see equation (3.6*a*)), since $\boldsymbol{g}$ has no ϕ component.

Let us now consider the problem of deducing from first principles the strength of the magnetic field produced by dynamo action, denoting by B_e the average field strength just outside the dynamo region and by B_i the average strength of the field within the dynamo region. This difficult problem has not yet been solved but it has been discussed by several investigators (for references see Jacobs 1975; Parker 1979). In an attempt to set useful limits on B_e and B_i Hide (1974) has argued on the basis of general considerations of equations (3.2) and (3.4) that the magnetic field is unlikely to build up beyond that value for which the Lorentz torque $\bar{\rho}\nabla \times (\nabla \times (\mu^{-1}\boldsymbol{B}) \times \boldsymbol{B})$ acting on an individual fluid element is unlikely to exceed the acceleration term $\nabla \times (\mathrm{D}\boldsymbol{u}/\mathrm{D}t + 2\boldsymbol{\Omega} \times \boldsymbol{u})$, which reduces to $\nabla \times (2\boldsymbol{\Omega} \times \boldsymbol{u})$ in the magnetostrophic limit. From this, B_i satisfies

$$B_i \lesssim B_s R^{\frac{1}{2}}, \tag{4.3}$$

where R is the magnetic Reynolds number (see equation (2.12)) and B_s is the 'scale magnetic field strength',

$$B_s \equiv \{\bar{\rho}(\Omega + UL^{-1})/\bar{\sigma}\}^{\frac{1}{2}}, \tag{4.4}$$

which reduces to $(\bar{\rho}\Omega/\bar{\sigma})^{\frac{1}{2}}$ in the magnetostrophic limit when $U/L\Omega \ll 1$. The ratio of the magnetic to kinetic energy implied by equation (4.3) is given by

$$B_i^2/\mu U^2 = \Omega L/U + 1. \tag{4.5}$$

This is very much greater than unity when $U/L\Omega \ll 1$, as in the case of a typical planetary dynamo.

Now, although the rate of generation of total magnetic energy by the dynamo mechanism, and hence B_i, is expected to *increase* with increasing electrical conductivity σ, the strength B_e of the *external* magnetic field produced by the dynamo (and this is the only part of the field we are able to observe) should *decrease* with increasing σ when σ is large, with B_e vanishing altogether when σ is infinite, for it is impossible to change the magnetic flux linkage of a perfect conductor (see equation (2.9)). It can therefore be supposed that

$$B_e/B_i \approx R^{-q}, \tag{4.6}$$

where the index q is essentially positive and possibly close to unity. Hence

$$B_e \lesssim B_s R^{(1-2q)/2} \equiv \hat{B}_e. \tag{4.7}$$

$\hat{B}_e$ is thus an upper limit to the strength of the magnetic field just above the fluid region where dynamo action is taking place. If $q \approx 1$, then $\hat{B}_e$ satisfies

$$B_s R^{-\frac{1}{2}} \approx \hat{B}_e \ll B_s R^{\frac{1}{2}}. \tag{4.8}$$

Corresponding expressions for the equivalent magnetic moment can be obtained from equations (4.7) and (4.8) by multiplying by the cube of an appropriate length. Taking as typical values for the core of the Earth

$\Omega \approx 10^{-4}\,\mathrm{rad\,s^{-1}}$, $\bar{\rho} \approx 10^4\,\mathrm{kg\,m^{-3}}$, $\bar{\sigma} \approx 3 \times 10^5\,\mathrm{S\,m^{-1}}$, $U \approx 10^{-4}\,\mathrm{m\,s^{-1}}$, and $L \approx 10^6\,\mathrm{m}$,

we find that $U/L\Omega \approx 10^{-6}$ and $B_s \approx (\bar{\rho}\Omega/\bar{\sigma})^{\frac{1}{2}} = 2 \times 10^{-3}\,\mathrm{T}$ (20 G). Accordingly, by equations (4.3) and (4.8) we have

$$B_i \lesssim 10^{-2}\,\mathrm{T}\,(100\,\mathrm{G}); \quad B_e \lesssim 4 \times 10^{-4}\,\mathrm{T}\,(4\,\mathrm{G})$$

if we assume that, in accordance with kinematic dynamo studies, $R \approx 25$. Now, the mainly dipolar field of 5×10^{-5} (0.5 G) T at the surface of the Earth implies that the average field strength just outside the core, $\hat{B}_e$, is less than about $10^{-3}\,\mathrm{T}$ (10 G). This falls within the preferred range of B_e implied by the above discussion. So far as the value of B_i for the Earth's core is concerned, this cannot be inferred directly from geomagnetic observations. But various lines of evidence indicate that the magnetic field throughout the main body of the core might be largely toroidal in configuration, with the lines of force lying approximately on horizontal surface, and *ca.* $10^{-2}\,\mathrm{T}$ (100 G) in strength (see Hide & Roberts 1979), which is also concordant with the above calculation.

By Faraday's law the magnetic flux linkage of a perfect conductor cannot change so that effects due to ohmic dissipation are central to dynamo theories of the generation of the external magnetic field (see equation (2.9) above). On the other hand, such effects may not be of primary importance when dealing with some aspects of the motions themselves and their neglect leads to a considerable simplification of equation (2.7) (see also equation (3.2)), which then reduces to

$$\partial \boldsymbol{B}/\partial t + (\boldsymbol{u}\cdot\nabla)\boldsymbol{B} - (\boldsymbol{B}\cdot\nabla)\boldsymbol{u} = 0. \tag{4.9}$$

Alfvén's celebrated 'frozen field' theorem. The mathematical difficulties involved are still severe, especially when realistic boundary conditions are taken into account, and their discussion lies beyond the scope of this article (for references see Soward 1982). Fortunately, some of the main dynamical processes are exemplified by the properties of small-amplitude plane waves with angular frequency ω and vector wavenumber $\boldsymbol{k}$ propagating relative to a fluid that rotates uniformly with steady angular velocity $\boldsymbol{\Omega}$, is pervaded by a uniform magnetic field $\boldsymbol{B}_0$, and within which there is a uniform vertical density gradient $\bar{\rho}\,\mathrm{d}\theta_0/\mathrm{d}z$, where z is the downward vertical coordinate. The dispersion relationship for these waves is

$$\omega^2 = \omega_m^2 + \tfrac{1}{2}[\omega_s^2 + \omega_r^2 \pm \{(\omega_s^2 + \omega_r^2)^2 + 4\omega_m^2\omega_r^2\}^{\frac{1}{2}}] \tag{4.10}$$

where $$\omega_m^2 \equiv (\boldsymbol{B}_0\cdot\boldsymbol{k})^2/\mu\bar{\rho}; \quad \omega_s^2 \equiv -(\boldsymbol{g}\times\boldsymbol{k})^2(\mathrm{d}\theta_0/\mathrm{d}z)/k^2; \quad \omega_r^2 \equiv (2\boldsymbol{\Omega}\cdot\boldsymbol{k})^2/k^2. \tag{4.11}$$

In the three cases when all but one of the quantities ω_m^2, ω_s^2 and ω_r^2 is equal to zero, we have

$$\omega^2 = \omega_m^2, \quad \omega^2 = \omega_r^2 \quad \text{or} \quad \omega^2 = \omega_s^2. \tag{4.12}$$

The first of these expressions is the dispersion relation for ordinary Alfvén waves, where the restoring force is provided entirely by the magnetic field and there is on average equipartition between kinetic and magnetic energy. These waves are non-dispersive and linearly polarized. The second expression is the dispersion relation for inertial waves, where the restoring force is provided by Coriolis effects. These waves are highly dispersive, circularly or elliptically polarized and less than or equal to 2Ω in frequency. The third expression in the dispersion relation for internal gravity waves, where buoyancy forces provide the restoring force when $\mathrm{d}\theta_0/\mathrm{d}z$ is positive (and promote convective overturning when $\mathrm{d}\theta_0/\mathrm{d}z$ is negative). These waves are highly dispersive and linearly polarized and less than or equal to $(-g\,\mathrm{d}\theta_0/\mathrm{d}z)^{\frac{1}{2}}$ in frequency.

In general there are two modes according to whether the upper or lower sign is taken in equation (4.10); we designate these as the 'fast' and 'slow' modes and their frequencies by ω_+ and ω_- respectively, which satisfy

$$\omega_+^2\omega_-^2 = \omega_m^2(\omega_m^2 + \omega_s^2) \tag{4.13}$$

for all values of ω_r^2. When $\omega_s^2 = 0$ we have

$$\omega^2 = \omega_m^2 + \tfrac{1}{2}[\omega_r^2 \pm \{\omega_r^4 + 4\omega_m^2\omega_r^2\}^{\frac{1}{2}}] \tag{4.14}$$

and

$$\omega_+^2\omega_-^2 = \omega_m^4. \tag{4.15}$$

In the case when rotational effects are weak, notably when $\omega_r^2 \ll 2\omega_m^2$, equation (4.14) gives

$$\omega_+^2 = \omega_m^2(1 + |\omega_r/\omega_m|); \quad \omega_-^2 = \omega_m^2(1 - |\omega_r/\omega_m|), \tag{4.16}$$

which correspond to ordinary Alfvén waves, very slightly modified by Coriolis forces.

At the other extreme, when $\omega_r^2 \gg 2\omega_m^2$, and this is the case of most interest when dealing with waves in the core of the Earth on scales of more than a few hundred kilometres, Coriolis forces are so strong that the two roots of equation (4.14) can have quite different values:

$$\omega_+^2 = \omega_r^2; \quad \omega_-^2 = \omega_m^4/\omega_r^2. \tag{4.17}$$

This extreme 'frequency splitting' due to rotation is accompanied by other effects, notably wave dispersion, circular or elliptical polarization of the trajectories of individual fluid elements, and imbalance of kinetic energy (the whole of which is now associated with the fast inertial wave) and magnetic energy (now entirely in the slow 'magnetohydrodynamic inertial' wave).

When equations (4.17) are satisfied, the period of the inertial mode $2\pi/\omega_-$ is then typically more than about π/Ω (i.e. a few days), whereas that of the magnetic mode $2\pi/\omega_-$ is *ca.* $2\pi\Omega L^2\mu\bar{\rho}/B_0^2$, which for the Earth's core when $L \approx 10^6$ m is 10^{10} s (300 years) and therefore comparable with the timescale of the geomagnetic secular variation. This is the quantitative basis of the theory of the geomagnetic secular variation that interprets its general timescale and westward drift in terms of magnetohydrodynamic oscillations of the liquid core. (The electrical conductivity of the overlying 'solid' mantle, though weak, would be sufficient to prevent magnetic changes in the core on the shortest timescale of the inertial modes from penetrating to the Earth's surface.) These oscillations should play an important role in the dynamo process and also in the electromagnetic and topographic coupling between the liquid core and overlying solid mantle that has been invoked to account for the so-called decade variations in the length of the day (for references see Braginskiy 1980; Hassan & Eltayeb 1982).

The root corresponding to the magnetohydrodynamic inertial wave could have been obtained directly by using the magnetostrophic version of equations (3.4), a procedure that eliminates solutions corresponding to inertial waves from the system of equations *ab initio* (just as the use of equation (3.3) acts as a filter for sound waves in the analysis). When $\boldsymbol{g} \times \nabla\theta = 0$, there is a balance of Coriolis and Lorentz couples acting on individual fluid elements (see equations (3.5) and (3.6)), so that the ratio of amplitudes of the velocity and magnetic fields associated with the wave is $B_0/\Omega L\mu\bar{\rho}$. By equation (4.9) this ratio is also equal to $LB_0\tau_-$, where $\tau_- = 2\pi/\omega_-$, the period of the wave; whence

$$\tau_- \approx \Omega L^2\mu\bar{\rho}/B_0^2, \tag{4.18}$$

which is *ca.* $\{\Omega L/B_0(\mu\bar{\rho})^{-\frac{1}{2}}\}^2$ (about 10^5 for the core of the Earth) rotation periods ('days') of the system and *ca.* $\Omega L/B_0(\mu\bar{\rho})^{-\frac{1}{2}}$ multiplied by the time taken for an ordinary Alfvén wave to traverse a distance equal to the characteristic length scale L.

5. Concluding remarks

I have outlined certain general properties of flows that are strongly influenced by Coriolis forces and Lorentz forces which can be deduced in a fairly straightforward way from the basic equations of magnetohydrodynamics. In preparing this brief survey, no attempt has been made to do full justice to the extensive important work that has been done on the magnetohydrodynamics of rotating fluids and dynamo theory, but references to articles describing original work and the development of ideas can be found in the reviews cited in the reference list. My remarks here are addressed primarily to those participants in this Discussion Meeting who are concerned with the structure and evolution of the Earth's core and more practical aspects of the study of geomagnetism, in the hope that the remarks can serve as a guide to the more technical and often highly mathematical literature describing the theory of the generation of cosmical magnetic fields by the self-exciting dynamo process.

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THE MAGNETIC FLUX LINKAGE OF A MOVING MEDIUM: A THEOREM AND GEOPHYSICAL APPLICATIONS

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Abstract. The time of change $\dot{N}(S;t)$ of the 'number of intersections' $N(S;t)$ of the lines of force of a magnetic field B with a closed surface S satisfies a certain theorem. This leads directly to a necessary and sufficient condition for self-exciting hydromagnetic dynamo action (involving inductive interactions between B and fluid motions) in the electrically conducting fluid 'core' of a 'planet,' in terms of the structure of B at the core surface S_0. The condition shows the crucial importance of representing that structure with great accuracy in numerical or analytical models of dynamos, and it suggests that dynamo action might be associated with certain topological properties of the pattern of the normal component of B on S_0. The condition also indicates that axisymmetric (but not necessarily steady) magnetic fields cannot be maintained by fluid motions; such fields must ultimately decay away, even in a compressible fluid (contrary to a recent claim in the literature to the contrary). The condition has to be modified when thermoelectric emf's are present, but when these emf's are proportional to B, they cannot prevent the decay of axisymmetric magnetic fields. This result refutes a recent claim that such fields can be maintained by Nernst-Ettinghausen effects. The theorem for $N(S;t)$ also shows that while fluctuations in B can take place on time scales down to that (τ_a) associated with core motions (decades to centuries in the case of the earth), the quantity $N(S_0;t)$ cannot change on time scales shorter than the Ohmic decay time τ_d ($> 10^4$ years). This result provides the basis of a method for solving the inverse problem of determining the radius of the core of the planet from observations of secular changes in B near the surface of the planet. The method has been applied to the earth and Jupiter with encouraging results.

Introduction

This survey outlines recent theoretical work on the electrodynamics of a moving medium and its application to various geophysical problems. Its basis is a theorem [Hide, 1979a] governing the time rate of change $\dot{N}(S;t)$ of the essentially nonnegative quantity

$$N(S;t) \equiv \iint_S |B \cdot dS| \equiv \iint_S q B \cdot dS \qquad (1)$$

which measures the number of intersections of the lines of force of a magnetic field B with a general closed surface S, area element dS, enclosing a volume K. Thus

$$\dot{N}(S;t) = -2\Sigma \oint_C [\sigma^{-1} j + (v - u) \times B] \cdot dC \qquad (2)$$

Here t denotes time and $q \equiv \mathrm{sgn}(B \cdot dS)$ is equal to $+1$ in regions where lines of force emerge from K and to -1 in regions where lines of force enter K. By the laws of Ampère and Ohm, respectively, the electric current density j, satisfies

$$\nabla \times (\mu^{-1} B) = j \qquad (3)$$

and

$$j = \sigma [E + u \times B] \qquad (4)$$

where μ is the magnetic permeability and σ the electrical conductivity. E is the electric field in the frame of reference with respect to which the medium moves with Eulerian velocity u and a general point on the surface S moves with velocity v. The line integrals are taken around all the one or more closed curves C, vector element of length dC, on S where B is either zero or tangential to S, in the sense that keeps the neighboring region where $B \cdot dS$ is positive (negative) on the left (right) when moving in the direction of dC. Singularities associated with the discontinuous change in q on crossing a C-line do not arise because S is defined in such a way as to exclude the C lines from the area of surface integration. When S is a material surface S_m (say), i.e., one on which $(v-u) \cdot dS = 0$, the term $(v-u) \times B \cdot dC = 0$ in equation (2), which then reduces to

$$\dot{N}(S_m;t) = -2\Sigma \oint_C \sigma^{-1} j \cdot dC \qquad (5)$$

Equation (2) follows directly from the laws of Gauss and Faraday, respectively,

$$\nabla \cdot B = 0 \qquad (6)$$

(which implies that

$$\iint_S B \cdot dS = 0$$

cf. equation (1)) and

$$\nabla \times E = -\partial B/\partial t \qquad (7)$$

and Ohm's law applied to a moving medium (equation (4)). Relativistic effects and departures from Ohm's law due to Hall, thermoelectric, and other effects (see below) give rise to additional terms in equations (2), (3), and (4) [see Hide, 1979b; Palmer, 1979], but the nonrelativistic case when Ohm's law holds has a wide range of application in earth and planetary sciences and astrophysics. Equation (2) leads to some novel general results concerning the self-exciting dynamo mechanism invoked in the explanation of the magnetic fields of the earth and other cosmical bodies, such as the planets Jupiter and Saturn. Equation (5) provides the basis of a new method [Hide, 1978] for determining from observations of the magnetic field in the accessible region at or near the surface of the planet $r = r_s$ the radius r_c of the electrically conducting fluid core of a planet in which the magnetic Reynolds number is so high that self-exciting dynamo action can occur. Indeed, it was the need to place this method on a firm theoretical footing that led to the development of the theorem and its application to the dynamo problem.

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Definition of Self-Exciting Dynamo Action in a Connected Body of Fluid

Theoretical investigations of self-exciting dynamos (for references see Moffatt [1978] and Parker [1979]) are concerned with the mathematical problem of finding, on the basis of the laws of electrodynamics and hydrodynamics, within a connected body K_0 of electrically conducting fluid, velocity fields $\underset{\sim}{u}$ that through inductive interactions with the magnetic field $\underset{\sim}{B}$ pervading the fluid and extending into the surrounding medium are capable of maintaining or amplifying the field outside K_0 against the action of dissipative effects. When S_0, the outer surface of K_0, is a material surface, we have, by equation (5) [see Hide, 1979b],

$$\dot{N}(S_0;t) = -2\Sigma \oint_C \sigma^{-1} \underset{\sim}{j}\cdot d\underset{\sim}{C} \quad (8)$$

By virtue of equations (3) and (6), Stokes's theorem and a well-known vector identity, this equation may also be written as follows:

$$\frac{d}{dt}\iint_{S_0} |\underset{\sim}{B}\cdot d\underset{\sim}{S}| = \iint_{S_0} \sigma^{-1}\,[\nabla^2(\mu^{-1} q \underset{\sim}{B}) - \nabla(q\underset{\sim}{B}\cdot\nabla\mu^{-1})]\cdot d\underset{\sim}{S} \quad (9)$$

where $q \equiv \operatorname{sgn} \underset{\sim}{B}\cdot d\underset{\sim}{S}$, see equation (1).

In the absence of fluid motions and external sources the magnetic field $\underset{\sim}{B}$ would decay with Ohmic decay time $\tau_d = L^2\mu\sigma$ where L is a characteristic length; $\tau_d = R\tau_a$ if R is the 'magnetic Reynolds number' $UL\mu\sigma$ and $\tau_a = L/U$ is an advective time scale, U being a typical relative flow speed. We are concerned here with a magnetic field $\underset{\sim}{B}$ that emanates from within a conducting body of fluid K_0 and which endures for a length of time $t = t'$ to $t = t''$ (say) which greatly exceeds $\max(\tau_d, \tau_e)$, where τ_e is the length of any interval during which are present permanent magnets or currents generated by electromotive forces due to noninductive processes.

There is some slight arbitrariness in the literature concerning the definition of dynamo action. Some authors require that magnetic energy, and others that the effective magnetic dipole moment of the system, shall not decay [see Moffatt, 1978]. There are advantages in defining self-exciting dynamo action in a fluid bounded by S_0 over the time interval $t = t'$ to $t = t''$ as requiring that the average value of $N(S_0;t)$ shall be nonnegative (see Figure 1), i.e.,

$$(t''-t')^{-1}\int_{t'}^{t''} \dot{N}(S_0;t)\,dt \geq 0;\quad t''-t' \gg \max(\tau_d, \tau_e) \quad (10)$$

(According to this definition introduced by Hide [1979b], the process of magnetic field amplification directly associated with the gravitational collapse of a conducting fluid does not constitute dynamo action. In the limit of perfect conductivity, while $N(S_0;t)$ remains constant during the collapse, the magnetic energy increases, but the magnetic moment decreases, the former being inversely proportional and the latter directly proportion to the size of the system.) Thus, by equations (10), (8) and (9) a necessary and sufficient condition for dynamo action is that on S_0 we have

$$(t''-t')^{-1}\int_{t'}^{t''} (\Sigma \oint_C \sigma^{-1} \underset{\sim}{j}\cdot d\underset{\sim}{C})\,dt \leq 0;\quad t''-t' \gg \max(\tau_d, \tau_e) \quad (11)$$

or, equivalently, since

$$\nabla\times\nabla\times(\mu^{-1}\underset{\sim}{B}) = \nabla(\nabla\cdot(\mu^{-1}\underset{\sim}{B})) - \nabla^2(\mu^{-1}\underset{\sim}{B})$$

and $\nabla\cdot\underset{\sim}{B} = 0$,

$$(t''-t')^{-1}\int_{t'}^{t''} \{\iint_{S_0} \sigma^{-1}[\nabla^2(\mu^{-1} q\underset{\sim}{B}) - \nabla(q\underset{\sim}{B}\cdot\nabla\mu^{-1})]\,d\underset{\sim}{S}\}\,dt \geq 0; \quad (12)$$

$$t''-t' \gg \max(\tau_d, \tau_e).$$

We can show that equation (12) has direct implications for the structure of $\underset{\sim}{B}$ near S_0 when self-exciting dynamo action occurs [see Hide, 1979b], but a full investigation of these implications lies beyond the scope of the present paper. One simple but illuminating result follows directly by applying equation (9) to a fluid bounded by a spherical surface S_0 and introducing spherical polar coordinates (r, θ, ϕ) with $\underset{\sim}{B} = \underset{\sim}{B}(r,\theta,\phi,t) = (B_r, B_\theta, B_\phi)$. In the case when $\nabla\mu = 0$ we have

$$\frac{d}{dt}\iint_{S_0} q B_r\,dS = \mu^{-1}\iint_{S_0} \frac{\sigma^{-1}}{r^2\sin\theta}\left\{\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial B_r}{\partial\theta}\right) + \frac{1}{\sin\theta}\frac{\partial^2 B_r}{\partial\phi^2}\right\} q\,dS + \mu^{-1}\iint_{S_0} \frac{\sigma^{-1}}{r^2}\frac{\partial^2}{\partial r^2}(r^2 B_r)\,q\,dS \quad (13)$$

where $dS = r^2\sin\theta\,d\theta\,d\phi$. The first term on the right-hand side is readily shown to be negative (since $\mu\sigma > 0$). Indeed, the individual contribution from each of the (two or more) separate regions into which S_0 is divided by the (one or more) C lines is negative. This follows because qB_r is positive everywhere within the region and vanishes only on the bounding C line. The surface integral over each separate region is equal to the area of the region times the average value of the 'horizontal' divergence of the 'horizontal' gradient of qB_r over the region, and this is equal to the length of the bounding C line multiplied by minus value of the essentially positive 'horizontal' gradient of qB_r averaged aound the C line. Any amplification of the external field must therefore be associated entirely with the second term, which measures the net outward diffusion of B_r across the surface S_0. This term involves radial derivatives of B_r and shows the importance in the investigation of specific dynamo models by analytical or numerical methods of ensuring that careful attention is given to the radial structure of $\underset{\sim}{B}$ near the outer boundary of the system. If we suppose that the left-hand side of equation (13) varies exponentially with t, as $\exp(-t/\tau)$, and associate (complex) length scales k_θ^{-1} and k_ϕ^{-1} with the first integral on the right-hand side of equation (13) and k_r^{-1} with the second integral, then

$$\tau^{-1} = (-\mu\sigma)^{-1}(k_\theta^2 + k_\phi^2 + k_r^2) \quad (14)$$

Dynamo action (corresponding to $\mathrm{Re}(\tau^{-1}) \geq 0$) cannot occur unless $\mathrm{Re}(k_r^2)$ is sufficiently negative, less than the essentially negative quantity $-\mathrm{Re}(k_\theta^2 + k_\phi^2)$.

2

Can Dynamo Action Occur in a Perfectly Conducting Fluid?

The basic amplification process in dynamo action is, of course, due to the electromotive forces produced by motional induction, involving the $\underset{\sim}{u} \times \underset{\sim}{B}$ term on the right-hand side of equation (4) and requiring that the magnetic Reynolds number of the system $R \equiv U L \mu \sigma$ should be large enough for motional induction to enhance the magnetic energy in the system at a rate sufficient to compensate for degradation due to Ohmic decay. However, whether dynamo action might be possible in the limit when $\sigma \to \infty$ is a matter of interest (for references see Moffatt [1978]), since the magnetic flux linkage of a perfect conductor ($\sigma = \infty$) cannot change. Equation (13), in which the velocity $\underset{\sim}{u}$ does not appear explicitly, shows that outward radial diffusion across S_0 is the sole process by which $N(S_0;t)$ can increase. By that equation (or equation (8)), dynamo action in the limiting case when $\sigma \to \infty$ implies the existence of current sheets where $|\underset{\sim}{j}| \to \infty$. The magnitude of the concomitant mechanical stress $\underset{\sim}{j} \times \underset{\sim}{B}$ within such a sheet would also in general tend to infinity, but (as in the case of vortex sheets studied in the theory of the dynamics of inviscid fluids) this need not necessarily be associated with an infinitely rapid acceleration if the stress can be supported by a finite pressure drop across the sheet [Hide, 1979b]. But even if dynamo action is possible in the limit when $\sigma \to \infty$, it is unlikely to be particularly efficient.

Fig. 1. Illustrating the criterion for self-exciting dynamo action expressed by equation (10), where the time interval $t''-t'$ greatly exceeds $\max(\tau_d, \tau_e)$, τ_d being the Ohmic decay time of the system and τ_e the length of any interval of time within $t = t'$ and $t = t''$ during which are present permanent magnets or currents generated by electromotive forces due to noninductive processes. $N(S_0;t)$ is the total number of intersections of lines force of the magnetic field B with the outer surface S_0 of the volume of fluid K_0 within which, by equation (10), dynamo action is occurring in cases (b) and (c), but not in case (a).

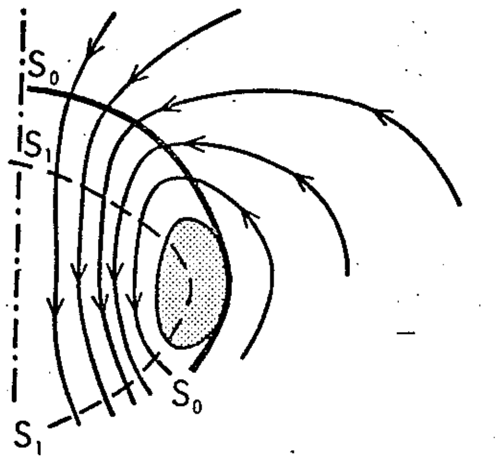

Fig. 2. Illustrating part of the meridional cross section of the closed axisymmetric surfaces S_0 and S_1 when there is only one C line on S_0, at the 'equator.' The stippled area is bounded by the field line that is tangential to S_0. Within the stippled area there exists in general a set of s O-type neutral points and p X-type neutral points (see Figure 3). In the case illustrated here and in Figure 4a (cf. Figure 4b), s = 1 and p = 0.

Decay of Axisymmetric Magnetic Fields

Equation (2) leads directly to a novel proof and extension to the nonsteady case of the well-known 'antidynamo' theorem of Cowling and others that no magnetic field B that remains symmetrical about an axis can be maintained or amplified in the region external to the conducting fluid by motions within the fluid (for references see Moffatt [1978]; Parker [1979] and James et al. [1980]). This can be demonstrated as follows, using slightly different and somewhat fuller arguments than those given by Hide [1979b].

When $\underset{\sim}{B} = (B_r, B_\theta, B_\phi)$ has an axis of symmetry through the center of K_0 (where ϕ is now the azimuthal direction referred to the axis of symmetry and θ is the 'co-latitude'), $\underset{\sim}{B} = \underset{\sim}{B}(r, \theta, t)$ and the family of C lines on S_0 is a set of one or more coaxial 'latitude' circles. By assumption B is due entirely to electric currents, any effects due to permanent magnetism being considered negligible. Associated with each of these C lines on S_0 there exists in any meridional (r, θ) cross section through the axis of symmetry (see Figures 2, 3, and 4) a closed meridional field line which is tangential to S_0 on the C line (see, e.g., Figure 2). By equation (3) (Ampere's law), the line integral of the tangential component of $\mu^{-1}\underset{\sim}{B}$ around this limiting field line is equal to the total azimuthal current crossing the area enclosed by the field line. The distribution of this azimuthal current over the area will not in general be uniform, and there may be regions where the azimuthal current density $j_\phi(r, \theta, t)$ vanishes or even has the opposite sign to that of the total current. The corresponding configuration of the meridional field lines within the area enclosed by the limiting closed field line will include a set of neutral points at which the meridional components (B_r, B_θ) of B vanish (although the azimuthal component B_ϕ will in

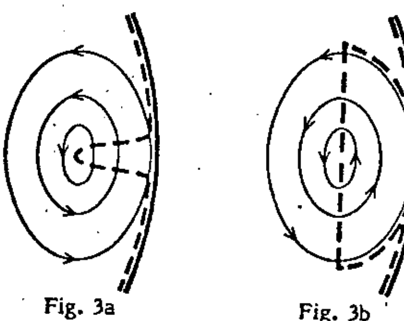

Fig. 3a Fig. 3b

Fig. 3. Illustrating a meridional cross section of other possible closed axisymmetric control surfaces S_1 (dashed lines) in the vicinity of a neutral point. These surfaces S_1 are entirely equivalent to each other as well as to that shown in Figure 2. The solid line corresponds to the surface S_0.

general be nonzero). Both types of neutral point will be present in general, namely those of the O–type, where the neighboring field lines defined by (B_r, B_θ) in the (r, θ) plane are ellipitical in shape, and those of the X-type, where the neighboring field lines are hyperbolic in shape. Denote by s the total number of O–type neutral points and by p the total number of X–type neutral points in the set. In the simplest case of all we have a single O-type neutral point ($s = 1$) and none of the X–type ($p = 0$), as in Figure 2 and 4a. Slightly more complicated would be a set consisting of the two O-type neutral points (i.e., $s = 2$) and an associated X-type neutral point ($p = 1$), and so on (cf. Figure 4b).

The meridional field lines can be regarded as height contours on a map of an island whose coast line corresponds to the limiting field line that touches S_0 (see Figure 2). Peaks of hills or 'summits' correspond to O-type neutral points where j_ϕ has the same sign as that of the total azimuthal current through the area enclosed by the limiting field line, the bottoms of hollows or 'immits' correspond to O-type neutral points where j_ϕ has the opposite sign to that of the total azimuthal current, and cols or 'saddle points' correspond to X-type neutral points where j_ϕ can be zero or have either sign. According to a theorem due to Cayley and Maxwell [see, e.g., Agoston, 1976], the number of summits plus the number of immits minus the number of saddle points is always equal to unity, and since there must be at least one summit, the numbers of O- and X-type neutral points, s and p, respectively, must satisfy $s \geq 1$ and $p = s-1 \geq 0$.

For each C line on S_0 there is set of circular neutral lines (corresponding to the neutral points in the meridional cross section), coaxial with the axis of symmetry. We can define an axisymmetric surface, S_1 (where $r = r(\theta, t)$) within the volume K_0 enclosed by S_0 such that the C lines on S_1 (see equation (2)) are the sets of neutral circles defined by the C lines on S_0 (see Figures 2 and 3). Now $\underset{\sim}{B} \times d\underset{\sim}{C} = 0$ on a neutral circle, so that $(\underset{\sim}{v}-\underset{\sim}{u}) \times \underset{\sim}{B} \cdot d\underset{\sim}{C} = 0$ and, in consequence, by equation (2),

$$\dot{N}(S_1;t) = -2 \sum_{O} \oint_{C} \sigma^{-1} \underset{\sim}{j} \cdot d\underset{\sim}{C} - 2 \sum_{X} \oint_{C} \sigma^{-1} \underset{\sim}{j} \cdot d\underset{\sim}{C}) \qquad (15)$$

(Note that S_1 is not in general a material surface, cf. equation (8).) The first term on the right-hand side represents contributions from those C lines on S_1 that coincide with O-type neutral circles and the second term represents contributions from C lines that coincide with X-type neutral circles. It is readily shown by Ampere's law (equation (3)) that $\underset{\sim}{j} \cdot d\underset{\sim}{C} > 0$ on an O-type neutral cricle, so that the first term on the right-hand side of equation (15) is essentially negative. It is less straightforward to evaluate the second term (except in the case $s = 1$, see Figure 2, when it vanishes identically), but by a judicious choice of an open control surface related to S_1 (R. Hide and T. N. Palmer, Generalization of Cowling's Theorem, Geophys. Astrophys. Fluid Dyn. (in press), 1981) its contribution can be rendered unimportant. Thus, the right-hand side of equation (15) is essentially negative.

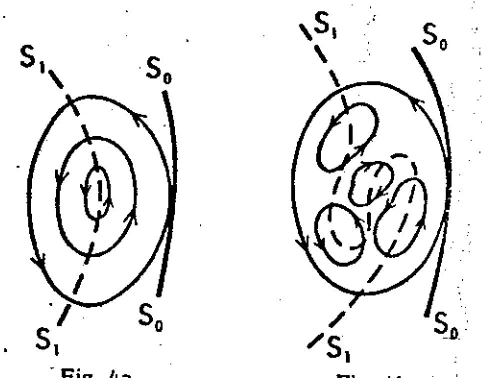

Fig. 4a Fig. 4b

Fig. 4. Illustrating meridional cross sections of S_1 surfaces (cf. Figure 3) in the vicinity of neutral points in two cases; namely, $s = 1$ and $p = 0$, (b) $s = 4$ and $p = 3$, where s and p ($= s-1$) are the numbers of O- and X-type neutral points, respectively.

Now, every line of magnetic force that intersects S_0 also intersects S_1 (see Figures 2 and 3), so that $N(S_1;t) \geq N(S_0;t)$. When the configuration of $\underset{\sim}{B}$ is time-independent and the surface S_1 consequently remains fixed in shape and position (i.e., $r = r(\theta)$ on S_1 so that $\underset{\sim}{v} = 0$), it follows immediately that $\dot{N}(S_0;t) < 0$ when $\dot{N}(S_1;t) < 0$. But in the more general case when changes in the configuration of S_1 have to be taken into account there are concomitant changes in the shape and position of S_1 (i.e., $r = r(\theta, t)$ on S_1 so that $v \neq 0$). These temporal changes in S_1 can, over short intervals of time, produce temporary increases in $N(S_0;t)$ even though $\dot{N}(S_1;t) < 0$. But over long periods of time the inexorable decay of $N(S_1;t)$, which tends to zero as $t \to \infty$, will be accompanied by the decay of $N(S_0;t)$ (see Figure 5). Thus, the left-hand side of equation (10) is negative when $\underset{\sim}{B}$ has an axis of symmetry.

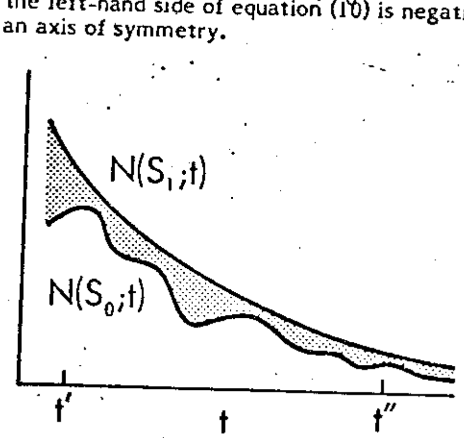

Fig. 5. Schematic diagram illustrating the decay of axisymmetric magnetic fields. $N(S_0;t)$ is the flux linkage of the outer surface S_0 of the 'core' K_0, and $N(S_1;t)$ is the flux linkage of the control surface S_1 upon which the main neutral circles of the magnetic field $\underset{\sim}{B}$ are situated (see Figures 2, 3, and 4).

4

In some treatments of the behavior of axisymmetric systems, mathematical simplifications are introduced by confining attention to incompressible fluids, for which $\nabla \cdot u = 0$ [see, e.g., Moffatt, 1978]. This had led to a recent speculation in the literature that axisymmetric magnetic fields in a compressible fluid might be amplified by fluid motions [Todoeschuck and Rochester, 1980]. The speculation is at variance with the conclusions of the discussion presented here, which are independent of the compressibility of the fluid. (The axisymmetric or radially symmetric gravitational collapse of a (necessarily) compressible fluid can amplify the magnetic energy - but not the magnetic moment, see above - associated with an axisymmetric magnetic field, but this does not constitute dynamo action as defined by equation (10).)

Topology of C Lines on S_0 When Dynamo Action Occurs: A Conjecture

It will be of interest to investigate on the basis of equations (8) and (9) whether or not a useful criterion for dynamo action can be formulated in terms of the topological properties of the family of C lines on the boundary S_0 between K_0 and the surrounding medium, and I hope that some suitably qualified mathematician will consider this problem. We have seen that $N(S_0;t) < 0$ when B is axisymmetric (and fixed in configuration), implying that dynamo action as defined by equation (10) cannot then occur. The C lines on S_0 in this case are 'latitude' circles, all of which surround a single axis. If the configuration of B is such that its departure from axial symmetry corresponds to a slight distortion of the C lines on S_0 then all the C lines still surround a single axis, closed and nearly circular neutral lines would still exist in K_0, and by the arguments used above in the strictly axisymmetrical case, B would ultimately decay away to zero. It is conjectured here (see Figure 6) that $N(S_0;t)$ cannnot be positive unless the pattern formed by the family of C lines on S_0 is such that is it is impossible to find an axis (defined by any pair of points on S_0) surrounded by all the C lines. It is always possible to find such an axis when the number of C lines is 1 or 2, so that should conjecture prove to be correct, $N(S_0;t)$ cannot be positive unless the number of C lines is 3 or more.

Thermoelectric Effects

The dynamo mechanism has received a large amount of attention from theoretical geophysicists and astrophysicists because (as Larmor was the first to recognize) electromotive forces due to motional induction are typically of the right order of magnitude to account for cosmical magnetic fields. But temperature gradients are present in most natural systems, and these will give rise to thermoelectric emf's of various kinds. Such emf's are typically very feeble, but Hibberd [1979], reviving an earlier quantitatively doubtful proposal, has recently suggested that planetary magnetic fields might be due to the Nernst-Ettinghausen (N-E) effect, whereby the flow of heat down a temperature gradient ∇T across a magnetic field gives rise to an electromotive force $-\alpha \nabla T \times B$, where α is the N-E coefficient.

Departures from Ohm's law can be represented by replacing $u \times B$ by $u \times B - Q$ in equations (4) and (2) see Hide, 1979b, where $Q = -G \times B$ (with $G = -\alpha \nabla T$) when these departures are due to the N-E effect. Since $((v-u) \times B + Q) \cdot dC$ vanishes on the neutral circles when Q is proportional to B, which includes the N-E case, it

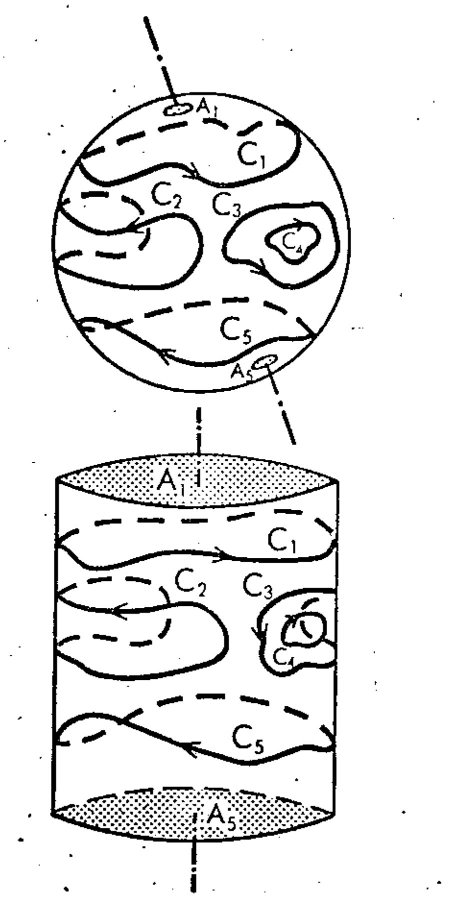

Fig. 6. Illustrating the conjecture that $N(S_0;t)$ cannot be positive unless the pattern formed by the family of C lines on S_0 is such that it is impossible to find an axis (formed by joining any two points on S_0) surrounded by all the C lines. The cylindrical surface of Figure 6b is topologically equivalent to the spherical surface minus the small areas surrounding A_1 and A_5 of Figure 6a. C_1 and C_5 surround the axis A_1-A_5 but C_2, C_3 and C_4 do not.

follows from the above discussion of axisymmetric nonsteady magnetic fields that, contrary to Hibberd's claim, such fields cannot be maintained by the N-E effect. This result effectively extends to the nonsteady case the demonstration by Braginsky [1964] (which P. H. Roberts has recently brought to my attention), using a different method, that N-E effects cannot maintain steady axisymmetric fields.

A Method for Finding the Size of the Core of a Planet

No one has yet suggested a satisfactory scaling law for planetary dynamos, let alone one that would lead to an estimate of the mean radius r_c of the conducting core of the planet from, say, the strength of the dipole component of B. However, a method for finding r_c from determination of secular changes in B in the accessible region near the surface of the planet, mean radius r_s, follows from equation (8) [see Hide, 1978]. This method (see Figure 7) exploits the fact that when B is produced by self-exciting dynamo action, the magnetic Reynolds number $R = U L \mu \sigma = \tau_d / \tau_a$ (where τ_d is the Ohmic decay time $L^2 \mu\sigma$ and τ_a the advective time scale, see

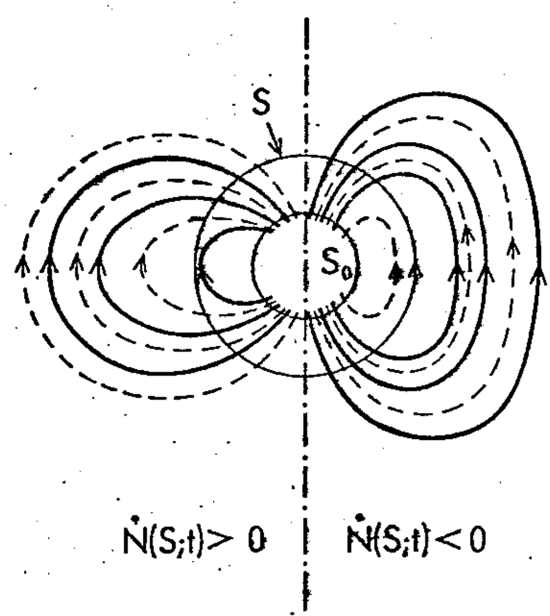

Fig. 7. Illustrating the principle of the method introduced by Hide [1978] for finding the radius of the electrically conducting fluid core of a planet (outer surface S_0) from observations of secular changes in the magnetic field B in the accessible region at or above the surface of the planet. As S approaches S_0, $\dot{N}$ (S;t) becomes very small, $O(\tau_d^{-1} N)$ (see equation (16)), where τ_d is the Ohmic decay time of the core.

the discussion following equation (9) above) is much greater than unity, but the method is independent of the details of the dynamo mechanism.

Denote by F (B) any function of B, such as its magnitude at a point, |B|, the amplitude or phase of any of its individual centered multipole components as determined by the traditional spherical harmonic representation of B in $r > r_c$, or the quantity N (S;t) (see equation (1)). In general, the most rapid fluctuations in F will be associated with the advective time scale τ_a when $\tau_a << \tau_d$, that is to say

$$\dot{F} = O(\tau_a^{-1} F)$$

For example, $\dot{N}(S;t) = O(\tau_a^{-1} N(S;t))$ in general (as is evident from equation (2)), but in the special case when S is a material surface, such as the outer surface S_0 of the core, we have, by equation (8),

$$\dot{N}(S_0;t) = O(\tau_d^{-1} N(S_0;t)) \qquad (16)$$

N (S_0;t) changes on the long time scale τ_d (rather than τ_a) and remains constant (i.e., $\dot{N} = O$), when $\tau_d \to \infty$. This is the case of perfect conductivity when, as several writers have pointed out [see, e.g., Backus, 1968; Bondi and Gold, 1950; Roberts and Scott, 1965], lines of magnetic force can be redistributed by fluid motions but they cannot be created or destroyed. When dealing with changes in B on time scales very much less than τ_d we can treat N (S_0) as an invariant to leading order in the small quantity $\tau_a/\tau_d = R^{-1}$.

Thus, the method [Hide, 1978] for finding r_c involves the downward extrapolation of B and its secular variation $\dot{B}$ measured in the accessible region $r = r_s$ until that level is reached at which $\dot{N}$ and all higher time derivatives effectively vanish. In the case when $\sigma = 0$ and $\nabla\mu = 0$ in $r > r_c$ the downward extrapolation is straightforward, for B in $r > r_c$ can then be expressed as the gradient of a scalar potential satisfying Laplace's equation (see equations (3), (4) and (6)). Otherwise r_c has to be defined as that value of r at which the magnetic lines of force are effectively tied to the material on time scales very much shorter than the Ohmic decay time associated with the material below $r = r_c$. Though soundly based physically and capable of formulation in clear mathematical terms, the method requires for reliable practical applications quite detailed observations of $\dot{B}$ at different epochs, and such observations are not yet available. When applied to the earth using the best available models for the main geomagnetic field and secular variation [Hide, 1978; Hide and Malin, 1981], the method gives for r_c/r_s a value which agrees within the errors involved, around 10 to 15%, with the much more accurate 'seismological' value of about 0.55. It was expected that when applied to the planet Jupiter using even the best magnetic field determinations then available, namely, those made during the 'Pioneer 10' and 'Pioneer 11' space probe encounters with the planet in December 1973 and December 1974, respectively, the method would give highly implausible or even physically impossible values of r_c/r_s. Remarkably and possibly fortuitously, this turned out not to be the case, for reasons which remain to be elucidated [Hide and Malin, 1979]. (It might be thought that the method can be tested using observations of the solar magnetic field, but contributions from the term (v-u) x B in equation (2) complicate, and also add interest to, the interpretation of fluctuations in N evaluated at the edge of the photosphere, a study of which will be reported in due course.)

Acknowledgments. Sir Edward Bullard's death on April 3, 1980, robbed geophysics of a versatile and outstanding practitioner and colorful personality, who made important contributions to the application of electrodynamics to problems of the earth's interior and geomagnetism. I am grateful to the organizers of the symposium in his honor in January 1980 for their invitation to attend and to several of the participants, including Sir Edward Bullard himself, for useful comments on those parts of this survey that were presented at the meeting. I also wish to acknowledge helpful discussions with participants in the International Workshop on 'Dynamo theory and the generation of the earth's magnetic field' (organized by the Geophysical Institute of the Czechosolovak Academy of Sciences) at Alsovice, Czechosolovakia, in 1979, where most of this survey was first presented, particularly P. H. Roberts, and also T. N. Palmer, J. W. Dungey and J. Simkin.

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