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AUTUMN COURSE ON GEOMAGNETISM, THE IONOSPHERE
AND MAGNETOSPHERE

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MATHEMATICS
(continued II)

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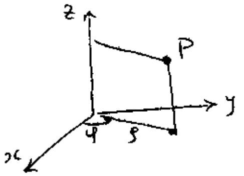
Special cases. The cylindrical polar coordinates:

$$x + iy = \rho e^{i\varphi}$$



Set

$$u + iv = \log(x+iy) = \log\rho + i\varphi$$

Hence

$$u = \log\rho \quad , \quad v = \varphi$$

Usually, the solutions we search for are periodic in φ with period 2π. If we notice, moreover, the identity

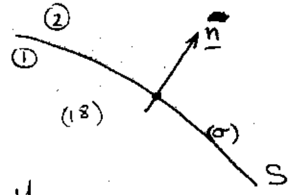
$$e^{n\log\rho} = \rho^n$$, the solution takes the form:

$$V(\rho,\varphi) = A_0 \log\rho + B_0 + \sum_{n=1}^{\infty}(A_n\rho^n + B_n\rho^{-n})(\alpha_n \cos n\varphi + \beta_n \sin n\varphi) \qquad (17)$$

which may also be obtained directly by applying the method of separation of variables to $\nabla^2 V(\rho,\varphi) = 0$.

II-4. Applications. Before dealing with the applications, we briefly mention the conditions that must hold on the surfaces of discontinuity of the electric and magnetic fields(*).

In Dielectrics: At S,



$$V_1 = V_2 \quad , \quad K_1\left(\frac{\partial V}{\partial n}\right)_1 - K_2\left(\frac{\partial V}{\partial n}\right)_2 = 4\pi\sigma \qquad (18)$$

where V denotes the potential, K - the dielectric constant and σ - the density of surface charge on S.

The magnetic ~~Layer~~ Shell (normal)

$$\Omega_2 - \Omega_1 = 4\pi\Psi \quad , \quad \left(\frac{\partial\Omega}{\partial n}\right)_1 = \left(\frac{\partial\Omega}{\partial n}\right)_2 \quad , \qquad (19)$$

where Ω is the magnetic potential, Ψ the intensity of the thin layer, i.e. the moment per unit area directed normal to the ~~surface~~ surface of the layer

The Current sheet.

If Ψ denotes the current function, then the conditions which are satisfied on S are the same as for the normal magnetic layer, provided the currents are steady.

(*) The reader may refer to "Electromagnetic Theory" by V. C. A. Ferraro

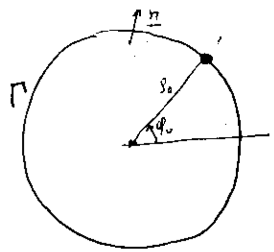
If S is the surface $u_3 = \text{const}$, and $h_3 = 1$ then the current function Ψ is related to the current flowing in S by the formulae

$$i_{u_1} = \frac{1}{h_2}\frac{\partial \Psi}{\partial u_2}, \qquad i_{u_2} = -\frac{1}{h_1}\frac{\partial \Psi}{\partial u_1},$$

where $\underline{i} = (i_{u_1}, i_{u_2})$ is the integrated current density.

Let us now turn to applications.

(i) <u>Electrostatic Potential of a line charge</u> parallel to the z-axis at a point (ρ_0, φ_0).



To solve this problem, consider a surface distribution of charges of density $\sigma(\varphi)$ over a cylindrical surface of radius ρ_0 with axis coinciding with the z-axis. We assume moreover that $\sigma(\varphi)$ is an even function of its argument, i.e. that $\sigma(-\varphi) = \sigma(\varphi)$. Let (V_+), (V_-) be the potential due to such a distribution at points for which $\rho > \rho_0$ $(\rho < \rho_0)$. Clearly,

$$V_+ = A_0 \log(\rho/\rho_0) + \sum_{n=1}^{\infty} A_n (\rho/\rho_0)^{-n} \cos n\varphi \qquad (\rho > \rho_0)$$

$$V_- = B_0 + \sum_{n=1}^{\infty} B_n (\rho/\rho_0)^{n} \cos n\varphi \qquad (\rho < \rho_0)$$

Applying boundary conditions (18), we get:

$$A_n = B_n \quad (n = 1, 2, \dots) \quad \text{and} \quad B_0 = 0.$$

and

$$-\frac{A_0}{\rho_0} + 2\sum_{n=1}^{\infty} n A_n \cos n\varphi = 4\pi\,\sigma(\varphi)$$

This last equation may be regarded as a Fourier expansion of the function $\sigma(\varphi)$, whence the following relations:

$$-\frac{A_0}{\rho_0} = \frac{4\pi}{2\pi}\int_0^{2\pi} \sigma(\varphi)\, d\varphi, \qquad 2nA_n = \frac{4\pi}{\pi}\int_0^{2\pi} \sigma(\varphi)\cos n\varphi\, d\varphi.$$

For the case under consideration (the line charge), it is convenient to transform these last relations as follows:

$$-\frac{A_0}{\rho_0} = \frac{4\pi}{2\pi\rho_0}\int_\Gamma \sigma(\varphi)\, ds, \qquad 2nA_n = \frac{4\pi}{\pi\rho_0}\int_\Gamma \sigma(\varphi)\cos n\varphi\, ds$$

where Γ is the contour of the cylinder and ds – the element of length along Γ.

Let us first suppose that the line charge is located at $(\rho_0, 0)$ (i.e. we set $\varphi_0 = 0$ for the moment). Then $\sigma(\varphi) \equiv 0$ everywhere except at $\varphi = 0$ and, moreover,

$$\lim_{\substack{\Delta s \to 0 \\ \text{at } \varphi = 0}} (\sigma \Delta s) = e \quad \text{– the charge per unit length.}$$

Therefore, the integrations over Γ are in fact confined to a small arc near the point $\varphi=0$. Thus

$$-\frac{A_0}{\rho_0} = \frac{2}{\rho_0}\lim_{\substack{\Delta S\to 0\\ \text{at }\varphi=0}} \sigma\Delta S = \frac{2e}{\rho_0}, \quad 2n A_n = \frac{4}{\rho_0}\lim_{\substack{\Delta S\to 0\\ \text{at }\varphi=0}} (\sigma \cos n\varphi \Delta S) = \frac{4e}{\rho_0}.$$

Hence $\quad A_0 = -2e \quad , \quad A_n = \dfrac{2e}{n} \quad , \quad n=1,2,\dots$

The problem is solved. The case where the line charge is located at (ρ_0, φ_0) may now be deduced from the previous case by ~~making the transformation~~ replacing φ ~~by~~ $\varphi-\varphi_0$

(ii) Magnetic scalar Potential of a line current.

Again, we consider the more general problem of a current sheet in the form of a cylinder of radius ρ_0 and axis coinciding with z-axis. Let $\Psi(\varphi)$ be the current function. we suppose that $\Psi(\varphi)$ is antisymmetric w.r.to φ, i.e. that $\Psi(-\varphi) = -\Psi(\varphi)$ (the reason for that choice is explained later).

As before, the magnetic potential of the sheet is

$$\Omega_+ = A_0 \log(\rho/\rho_0) + \sum_{n=1}^{\infty} A_n (\rho/\rho_0)^{-n} \sin n\varphi \qquad (\rho>\rho_0)$$

$$\Omega_- = \sum_{n=1}^{\infty} B_n (\rho/\rho_0)^n \sin n\varphi$$

The application of the second of conditions (19) gives

$$A_0 = 0 \qquad \text{and} \qquad B_n = -A_n \quad , n=1,2,\dots$$

while the first of conditions (19) reduces to:

$$2\sum_{n=1}^{\infty} A_n \sin n\varphi = 4\pi\,\Psi(\varphi).$$

The current in the sheet is supposed to flow parallel to the z-axis. Let its intensity be denoted by i_z:

$$i_z = \frac{1}{\rho_0}\frac{\partial\Psi}{\partial\varphi} = \frac{1}{2\pi\rho_0}\sum_{n=1}^{\infty} n A_n \cos n\varphi$$

This expression means that i_z is an even function of φ as should be for our purpose. It is now evident that $\Psi(\varphi)$ had to be taken ~~as~~ an odd function in φ.

The last equation may be considered as the Fourier expansion of i_z, whence

$$\frac{1}{2\pi\rho_0} n A_n = \frac{1}{\pi}\int_0^{2\pi} i_z \cos n\varphi \, d\varphi$$

$$= \frac{1}{\pi\rho_0}\int_\Gamma i_z \cos n\varphi \, ds$$

where Γ, as before, is the contour of the cylinder.

Hence

$$A_n = \frac{2}{n}\int_{\Gamma} i_z \, ds$$

To transfer to the case of a current line of intensity I, we suppose that $i_z \equiv 0$ everywhere on the sheet except for a small strip at $\varphi=0$, and that $\lim_{\Delta s\to 0}(i_z \Delta s) = I$.

Thus

$$A_n = \frac{2}{n} \lim_{\substack{\Delta s\to 0 \\ \text{at } \varphi=0}} i_z \Delta s = \frac{2I}{n}.$$

The problem is solved. The case of a current line passing through the point (ρ_0, φ_0) may now be obtained at once by performing the transformation $\varphi \to \varphi - \varphi_0$ in the previous expression for $\Omega_+(\Omega_-)$.

Other applications: As an application to flow of currents around islands and to the effect of solar eclipse on the magnetic field, the reader should try to apply the above mentioned method to find the ~~magnetic~~ electric potential of a current flowing across a uniform plane sheet, apart from a circular disk of different uniform conductivity, when the sheet is subjected to a uniform ~~magnetic~~ electric field.

II-5 Application to Cartesian Coordinates.

The two-dimensional Laplace's equation in cartesian coordinates reads

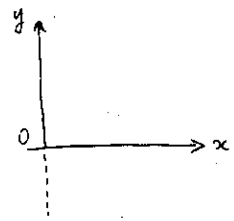
$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0 \qquad (20)$$

If the solution is not periodic in x or y, the solution of (20) may be expressed as

$$V(x,y) = \int_0^\infty \left[A(\lambda)\cosh\lambda x + B(\lambda)\sinh\lambda x\right]\left[\alpha(\lambda)\cos\lambda y + \beta(\lambda)\sin\lambda y\right] d\lambda \qquad (21)$$

(n is no more restricted to integer values). In formula (21), x & y may be interchanged.

As an application, we shall find the magnetic scalar potential due to a line current I parallel to the z-axis and passing through the point (x_0, y_0).



We shall suppose for a moment that $x_0=0$ and $y_0=0$, and introduce a current sheet coinciding with plane $x=0$ and let $\Psi(y)$ be the current function. As for the previous example, we take $\Psi(y)$ an odd function of y.

We may write :

$$\Omega_{\pm}(x,y) = \pm\pi \int_0^\infty f(\lambda)\, e^{\mp\lambda x} \sin\lambda y \, d\lambda \qquad (x \gtrless 0)$$

Hence

$$\Psi(y) = \frac{1}{2\pi}(\Omega_+ - \Omega_-)_{x=0} = \int_0^\infty f(\lambda) \sin\lambda y \, d\lambda$$

The current density is therefore

$$i_z = \frac{\partial\Psi(y)}{\partial y} = \int_0^\infty \lambda f(\lambda) \cos\lambda y \, d\lambda$$

and we see that i_z is symmetrical with respect to the plane $y=0$. The last relation may be viewed as the Fourier integral representation of i_z, whence

$$\lambda f(\lambda) = \frac{2}{\pi} \int_0^\infty i_z(y) \cos\lambda y \, dy$$

For the case of a current line of intensity I, we have : $i_z(y)=0$ everywhere except in a strip around $y=0$ and, moreover,

$$\lim_{\substack{\Delta y \to 0 \\ \text{at } y=0}} (i_z \Delta y) = I$$

Hence

$$\lambda f(\lambda) = \frac{2}{\pi} \lim_{\substack{\Delta y \to 0 \\ \text{at } y=0}} (i_z \cos\lambda y\, \Delta y) = \frac{2I}{\pi}.$$

and therefore

$$f(\lambda) = \frac{2I}{\pi\lambda}.$$

The problem is solved. We may now replace $f(\lambda)$ by its value in the expressions of $\Omega_\pm$ to get :

$$\Omega_\pm(x,y) = \pm 2I \int_0^\infty \frac{1}{\lambda} e^{\mp\lambda x} \sin\lambda y \, dy \qquad (x \gtrless 0)$$

For the current line through point (x_0,y_0) it is now sufficient to make the replacements $x \to x-x_0$, $y \to y-y_0$ and we finally get

$$\Omega_\pm(x,y) = \pm 2I \int_0^\infty \frac{1}{\lambda} e^{\mp\lambda(x-x_0)} \sin\lambda(y-y_0)\, d\lambda .$$

Further applications to two-dimensional solutions

Elliptic cylinders or uniform plane sheet with an elliptic disk of different conductivity.

These cases may be solved by using elliptic coordinates defined by the transformation

$$x + iy = A\cosh(u+iv) ,$$

hence

$$x = A\cosh u \cos v , \qquad y = A \sinh u \sin v .$$