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AUTUMN COURSE ON GEOMAGNETISM, THE IONOSPHERE
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Theory of Incoherent Scatter from the Ionosphere.

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Notes for lectures on

Theory of Incoherent Scatter from the Ionosphere

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1. Scattering when electrons are assumed to be independent

(a) Single electron in electric field

$$E = E_o \exp(i\,\omega\, t) \qquad (1)$$

of incident (probing) wave is displaced by

$$x = -Ee/m\omega^2. \qquad (2)$$

It is therefore an oscillating dipole of moment M = ex and re-radiates the scattered power

$$\mu_o M^2\omega^4/12\pi c \;=\; \mu_o e^4E^2/12\pi m^2c. \qquad (3)$$

Energy flux (Poynting) in incident wave is $E_o{}^2/2\mu_o c$ so effective intercepting area of one electron is

$$r_e{}^2/6\pi \qquad (4)$$

where $$r_e \;=\; e^2/\epsilon_o mc^2 \;=\; \mu_o e^2/m \qquad (5)$$

is the "classical electron radius" : about 5 x 10^{-14} m.

(b) Electrons move with thermal velocity

$$v \sim (3KT/m)^{\frac{1}{2}} \qquad (6)$$

If they move independently, the scattered radiation ought to have a spread $\Delta\omega$ in frequency, from the Doppler effect :

$$\Delta\omega/\omega \sim v/c \sim 10^{-3} \quad (T = 1500\ K). \qquad (7)$$

Bowles (1958) found that observed scattered power was about right but Doppler spread was much smaller :

$$\Delta\omega/\omega \sim 10^{-6}.$$

Conclude: Electrons are not independent. Ionospheric plasma behaves more like a continuum. Scattering occurs from fluctuations ΔN_e of electron concentration N_e. These flucutations arise from waves in the plasma, namely (electron) plasma waves and ion acoustic waves.

2. Main properties of electron plasma waves.

Let the electric field in the plasma wave be

$$E_z \;=\; E_1 \exp\{i(\omega t - kz)\}. \qquad (8)$$

The dispersion relation is

$$\omega^2 - k^2\, a_e{}^2 = \omega_{Ne}^2 \qquad (9)$$

where ω_{Ne} is the angular plasma frequency for electrons

$$\omega_{Ne}{}^2 = Ne^2/\epsilon_o m_e \qquad (10)$$

and a_e is the "velocity of sound"

$$a_e^2 = \gamma\, KT_e/m_e. \qquad (11)$$

Here γ is the ratio of specific heats of the electron "gas". Full kinetic treatment shows that γ = 3.

The displacement of an electron in the z direction is

$$\zeta_e \;=\; -\epsilon_o E_z/eN_e. \qquad (12)$$

The flucutation of electron concentration is

$$\Delta N_e = -N_e \partial\zeta_e/\partial z = i\, k\, \varepsilon_o E_z/e. \tag{13}$$

The average energy per unit volume in the wave is

$$\mathcal{E} = \tfrac{1}{2}\, \varepsilon_o\, E_1{}^2. \tag{14}$$

Half this is in the electric field and the other half is the kinetic energy of the moving electrons.

The average value of Δ_{Ne}^2 is

$$|\Delta_{Ne}{}^2| = k^2\, \varepsilon_o{}^2 \langle |E_z|^2 \rangle /e^2 = \tfrac{1}{2} k^2 \varepsilon_o{}^2 E_1{}^2/e^2 = \mathcal{E}\, k^2 \varepsilon_o/e^2. \tag{15}$$

3. Main properties of ion acoustic waves

Use subscripts e for electrons and i for ions. Assume that there is only one kind of positive ion. Then neutrality requires that

$$N_e = N_i = N. \tag{16}$$

Let electric field be (8) as before. Contributions to the electric polarisation P (z component) are

$$P_e = -Ne\zeta_e = Ne^2 E_z/(k^2 \gamma K T_e - \omega^2 m_e) \tag{17}$$

$$P_i = Ne\zeta_i = Ne^2 E_z/(k^2 \gamma K T_i - \omega^2 m_i). \tag{18}$$

Hence

$$-\zeta_i/\zeta_e = (\gamma K T_e - \omega^2 m_e/k^2)/(\gamma K T_i - \omega^2 m_i/k^2). \tag{19}$$

The dispersion relation is

$$\frac{\omega_{Ne}^2}{\omega^2 - k^2 a_e^2} + \frac{\omega_{Ni}^2}{\omega^2 - k^2 a_i^2} = 1. \tag{20}$$

Fig. 1 shows k^2 plotted against ω^2. For ion acoustic waves we are interested in small values of k^2 and ω^2. Multiply (20) by the two denominators and neglect fourth power terms. Then we get for the simplified dispersion relation

$$\frac{\omega^2}{k^2} = \frac{a_i^2 \omega_{Ne}^2 + a_e^2 \omega_{Ni}^2}{\omega_{Ne}^2} \tag{21}$$

where ω_{Ni}^2 has been neglected in the denominator. On using (11), for a_e and the corresponding value of a_i, we get

$$\frac{\omega^2}{k^2} = \frac{\gamma K (T_e + T_i)}{m_i}. \tag{22}$$

We can show, by using (22) in (19) and neglecting small terms, that

$$\zeta_i \approx \zeta_e. \tag{23}$$

The heavy positive ions move and drag the light electrons with them. Since $P_e \approx -P_i$, the space charge is very small and the electric field is very small. Most of the energy in the wave is the kinetic energy of the ions. Hence the average energy per unit volume is

$$\mathcal{E} = \tfrac{1}{2} m_i \langle (\zeta_i \omega)^2 \rangle N = \tfrac{1}{4} N m_i \omega^2 \zeta_o^2 \tag{24}$$

where ζ_o is the amplitude of the oscillations of ζ_i.

The fluctuation ΔN_e of the electron concentration is

$$\Delta N_e = -N\delta\zeta_e/\delta z = i\, k\, N\, \zeta_e. \tag{25}$$

Hence

$$\langle |\Delta N_e|^2 \rangle = k^2 N^2 \langle \zeta_e^2 \rangle = \tfrac{1}{2} k^2 N^2 \zeta_o^2$$

$$= \frac{2N}{m_i} \frac{k^2}{\omega^2} \mathcal{E} = \frac{2N}{\gamma K (T_e + T_i)} \mathcal{E} \tag{26}$$

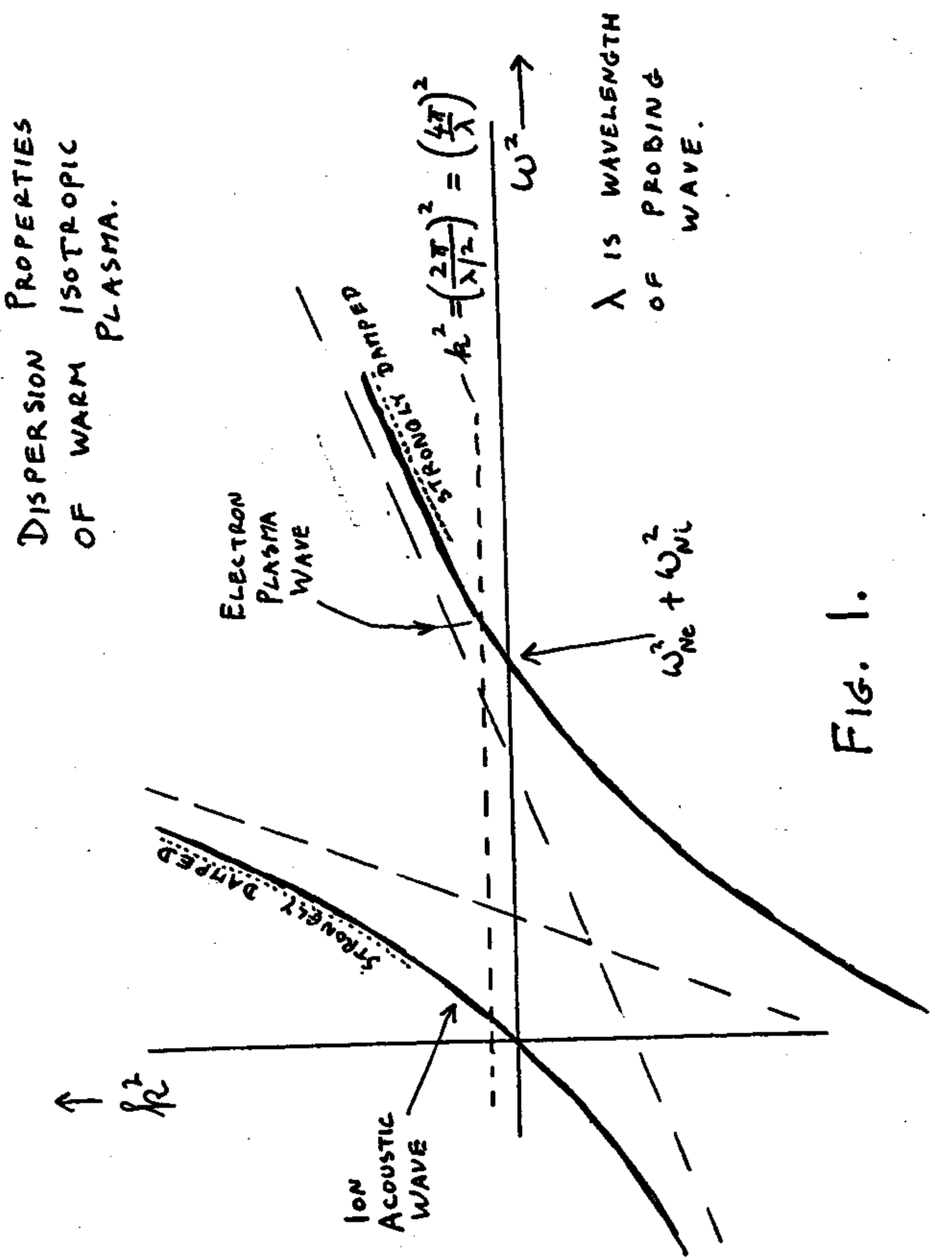

Fig. 1.

where (22) and (24) have been used.

4. The Scattering Theory. I. The normal modes

The theory given here is for the monostatic system as sketched in fig. 2. How much power is scattered back to the receiver from unit volume of the ionosphere in the scattering region, shaded in fig. 2? We shall consider the scattering from a cubical block of plasma of side d. In it there are plasma waves and ion acoustic waves that have a ΔN_e, and this does the scattering. We need to find the amplitudes ΔN_e and the directions of the scattering waves. To do this we use a larger cubical box of side L, with the scattering cube inside it (see fig. 3). We find the normal modes in the L^3 box.

A normal mode is a standing wave in three dimensions in the box. It is composed of 8 obliquely travelling progressive waves which are converted one into another by reflection at the surfaces of the box. A normal mode is a harmonic oscillator of frequency ω (to be found from the dispersion relation). According to the classical principle of <u>equipartition of energy</u>, on the average an oscillator has energy KT. So the average energy in any one component progressive wave is

$$L^3 \mathcal{E} = KT/8 \qquad (27)$$

where $\mathcal{E}$ is the average energy per unit volume.

For a typical progressive scattering wave in the L^3 box

$$\Delta N_e = \phi(\underset{\sim}{k}, t) \cos\{\omega t - \underset{\sim}{k}.\underset{\sim}{r} - \eta(\underset{\sim}{k}, t)\} \qquad (28)$$

where η is a phase angle, $\underset{\sim}{r}$ denotes the coordinates (x, y, z) and

$$\underset{\sim}{k} = (l_x, l_y, l_z)\pi/L. \qquad (29)$$

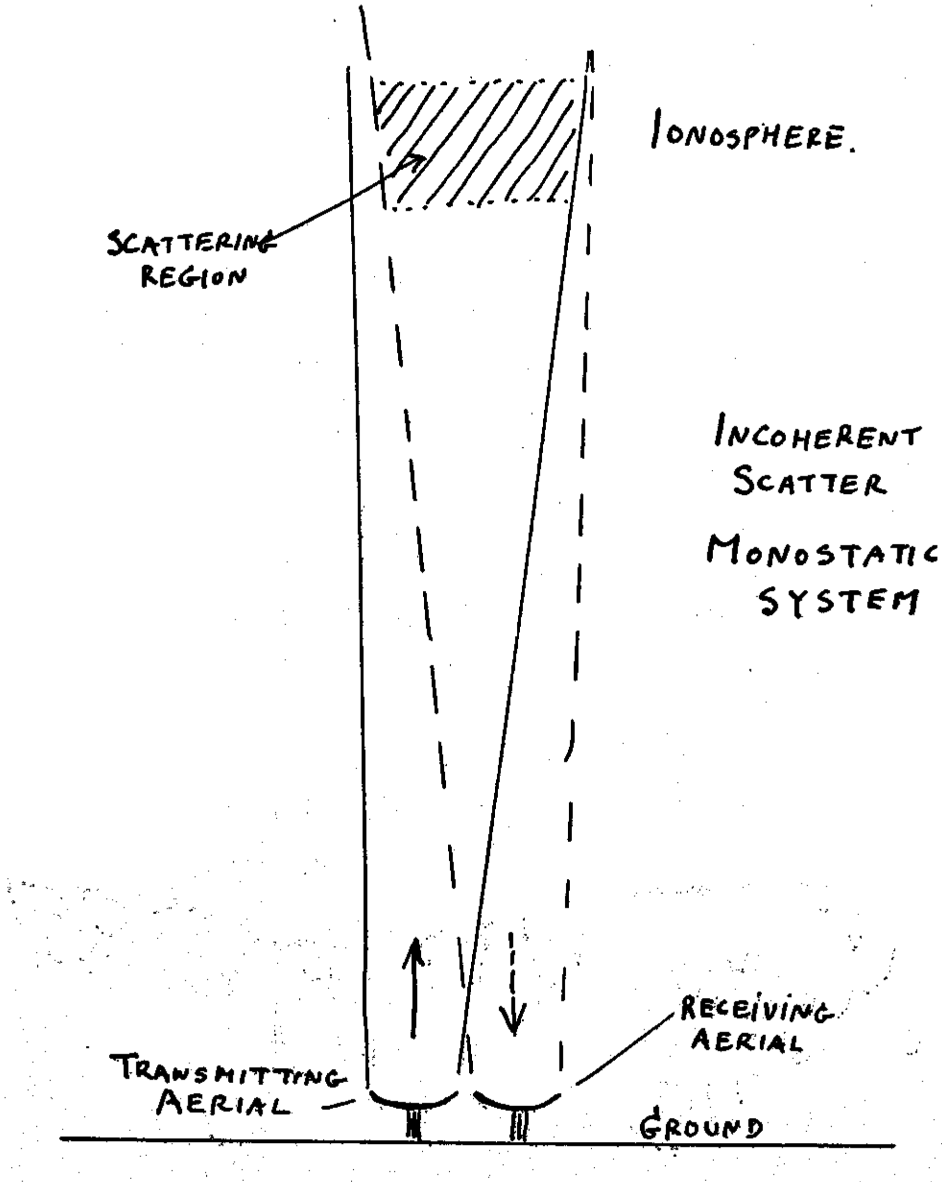

FIG. 2

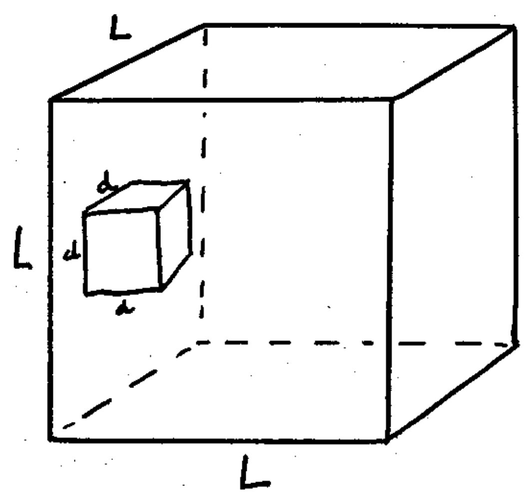

Fig. 3. In the large cube, side L, we find the normal plasma wave modes and the normal ion acoustic wave modes (compare the Debye theory of the specific heats of solids). These waves do the scattering. They are used when finding the scattered power from the smaller cube of side d. The result is independent of both L and d.

Here l_x, l_y, l_z are positive, negative or zero integers.

Note that we use the real (cosine) form for the harmonic oscillation (28). This is because the integrand of (38) below contains a product of two harmonic oscillations, and it would be incorrect to represent them both by exponentials.

(a) Plasma wave

From (15) and (27)

$$\langle |\Delta N_e|^2 \rangle = \langle \phi^2 \rangle = k^2 \varepsilon_0 K T_e / (8 e^2 L^3). \qquad (30)$$

It will be later shown that

$$k \approx \pm 2 k_o \qquad (31)$$

where k_o is $2\pi/\lambda$ for the probing wave. Hence

$$\langle \phi^2 \rangle \approx k_0^2 \varepsilon_0 K T_e / (2 e^2 L^3). \qquad (32)$$

(b) Ion acoustic wave

From (26) and (27)

$$\langle |\Delta N_e|^2 \rangle = \langle \phi^2 \rangle = N T_i / \{ 4 \gamma (T_e + T_i) L^3 \}. \qquad (33)$$

5. The Scattering Theory. II. Scattered power

The incident probing wave has a horizontal electric field

$$E = E_0 \exp\{ i(\omega_0 t - k_0 z) \} \qquad (34)$$

where z is measured vertically upwards from the ground. One electron in this field is an oscillating dipole of moment

$$M = \frac{E_0 e^2}{m \omega^2} \exp\{ i(\omega_0 t - k_0 z) \}. \qquad (35)$$

The re-radiated (scattered) electric field at the receiver from this one electron (see fig. 2) is

$$E_s^{(1)} = [\ddot{M}] / (\varepsilon_0 z c^2). \qquad (36)$$

The retardation [] gives an extra factor $\exp(-i\, k_o\, z)$. So

$$E_s^{(1)} = z^{-1} r_e E_0 \exp\{ i(\omega_0 t - 2 k_0 z) \} \qquad (37)$$

where r_e is given by (5). This is to be multiplied by ΔN_e and integrated over the d^3 box, so

$$E_s = z^{-1} r_e E_0 e^{i\omega_0 t} \iiint \Delta N_e \exp(-2 i k_0 z)\, dx\, dy\, dz. \qquad (38)$$

(Note that z^{-1} is almost constant within the d^3 box and so it can come outside the integral).

The next step applies to either the plasma wave or the ion acoustic wave. From (28) we have for the total ΔN_e

$$\Delta N_e = \sum_{l_x} \sum_{l_y} \sum_{l_z} \phi(\underset{\sim}{k}, t) \cos(\omega t - \underset{\sim}{k} \cdot \underset{\sim}{r} - \eta). \qquad (39)$$

Let $\bar{x}$, $\bar{y}$, $\bar{z}$ be the coordinates of the centre of the d^3 box. The cosine in (39) is the sum of two exponentials and the first of these, when put in (38), gives the contribution

$$E_s = \frac{r_e E_0}{2 \bar{z}} e^{i \omega_0 t} \sum_{l_x l_y l_z} \int_{\bar{z}-d/2}^{\bar{z}+d/2} \int_{\bar{y}-d/2}^{\bar{y}+d/2} \int_{\bar{x}-d/2}^{\bar{x}+d/2} \phi(\underset{\sim}{k}, t) \times$$

$$\times \exp\left[i \left\{ \omega t - \frac{\pi}{L}(x l_x + y l_y + z l_z) - 2 i k_0 z - \eta(\underset{\sim}{k}, t) \right\} \right] dx\, dy\, dz. \qquad (40)$$

This contribution has frequency $\omega_o + \omega$. In the other exponential of (39) the signs of ω, l_x, l_y, l_z are reversed so that the frequency is $\omega_o - \omega$. The integrals can be done and (40) then gives

$$E_s = \frac{r_e E_o}{2\bar{z}} \exp\left[i\{(\omega+\omega_o)t - \psi_1\}\right] d^3 \times$$

$$\times \sum_{l_x l_y l_z} \phi(\underline{k},t) \frac{\sin\alpha}{\alpha} \frac{\sin\beta}{\beta} \frac{\sin\gamma}{\gamma} \tag{41}$$

where ψ_1 is an unimportant phase angle, and

$$\alpha = \tfrac{1}{2}\pi l_x d/L, \quad \beta = \tfrac{1}{2}\pi l_y d/L, \quad \gamma = (k_o + \tfrac{1}{2}\pi l_z/L)d. \tag{42}$$

The three sine terms are largest when

$$l_x = 0, \; l_y = 0, \; k_o + \tfrac{1}{2}\pi l_z/L = 0 \tag{43}$$

that is, from (29), when

$$k_z = -2k_o. \tag{44}$$

These terms pick out the scattering wave that makes the largest contribution. It is the one that is travelling downwards and whose wavelength $2\pi/k$ is half the wavelength of the probing wave. For the other exponential term of (39) it has $k_z = +2\,k_o$ and is travelling upwards. Now the various Fourier components of ΔN_e in (39) are <u>independent</u> oscillators. Their phases are "random". They are <u>incoherent</u>. This means that

$$\left\langle \phi(\underline{k}_1,t)\,\phi(\underline{k}_2,t) \exp\left[i\{\eta(\underline{k}_2,t) - \eta(\underline{k}_1,t)\}\right] \right\rangle$$

$$= 0 \quad \text{if} \quad \underline{k}_1 \neq \underline{k}_2. \tag{45}$$

So from (41) we shall find the average value of $E_S E_S^* = |E_s{}^2|$, proportional to the scattered power. Now (32) and (33) show that the average $\langle\phi^2\rangle$ is independent of $\underline{k}$. The result must be doubled so as to include the contribution from the second exponential of (39). Hence

$$\langle |E_s^2| \rangle = \tfrac{1}{2} r_e^2 E_o^2 \bar{z}^{-2} d^6 \langle\phi^2\rangle \sum_{l_x,l_y,l_z} \frac{\sin^2\alpha}{\alpha^2} \frac{\sin^2\beta}{\beta^2} \frac{\sin^2\gamma}{\gamma^2}. \tag{46}$$

It can be shown that each of the three sums is equal to 2L/d. Hence

$$\langle |E_s^2| \rangle = 4\, r_e^2 E_o^2 \bar{z}^{-2} d^3 L^3 \langle\phi^2\rangle. \tag{47}$$

The average scattered power from unit volume of the scattering cube (divide by d^3 and by $2Z_o$) is

$$P = 2\, r_e^2 E_o^2 \bar{z}^{-2} L^3 \langle\phi^2\rangle / Z_o. \tag{48}$$

6. <u>Results</u>

The result is often quoted in terms of the effective scattering area σ per unit volume of the scattering plasma. So multiply (48) by $\bar{z}^2\cos^2\theta$ and integrate over a sphere. This gives a factor $\frac{4\pi}{3}\bar{z}^2$. Then divide by the average power flux $\frac{1}{2}E_o^2/Z_o$ in the incident probing wave. This gives

$$\sigma = \frac{16\pi}{3} r_e^2 L^3 \langle\phi^2\rangle. \tag{49}$$

For the ion acoustic wave use (33)

$$\sigma = \frac{4\pi}{3} r_e^2 \frac{N T_i}{\gamma(T_e + T_i)}. \tag{50}$$

For the plasma wave use (32)

$$\sigma = \frac{8\pi}{3} r_e^2 k_o^2 \varepsilon_o K T_e / e^2$$

$$= \frac{8\pi}{3} N r_e^2 k_o^2 h^2 \tag{51}$$

where $$h = (\varepsilon_o K T_e / N e^2)^{1/2} \tag{52}$$

is the Debye length.

7. Suggested reading

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