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AUTUMN COURSE ON GEOMAGNETISM, THE IONOSPHERE
AND MAGNETOSPHERE

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MATHEMATICS
(Cont. 3)

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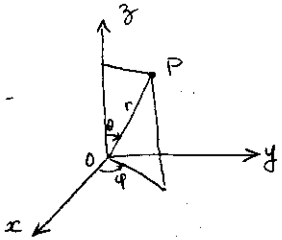
II-6 The Spherical Harmonics

Let us now consider the three-dimensional Laplace's equation in spherical polar coordinates r, θ, φ.



$$\nabla^2 V \equiv \frac{1}{r^2}\left[\frac{\partial}{\partial r}\left(r^2\frac{\partial V}{\partial r}\right) + \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial V}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2 V}{\partial\varphi^2}\right] = 0 \qquad (1)$$

$$0 \le \theta \le \pi \quad , \quad 0 \le \varphi < 2\pi \quad , \quad 0 \le r < +\infty$$

We look for a solution V of the form

$$V(r,\theta,\varphi) = R(r)\, S(\theta,\varphi) \qquad (2)$$

Substituting into (1) we get

$$\frac{1}{R}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) + \frac{1}{S\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial S}{\partial\theta}\right) + \frac{1}{S\sin^2\theta}\frac{\partial^2 S}{\partial\varphi^2} = 0$$

and therefore

$$\frac{1}{R}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) = \text{const} = n(n+1) \text{ , say} \qquad (3)$$

$$\frac{1}{S\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial S}{\partial\theta}\right) + \frac{1}{S\sin^2\theta}\frac{\partial^2 S}{\partial\varphi^2} = -n(n+1) \qquad (4)$$

Equation (3) may be easily integrated for $R(r)$. Its solution is of the form $R_n = A_n r^n + B_n r^{-n-1}$ and hence the general solution of (1) is

$$V = \sum_n (A_n r^n + B_n r^{-n-1}) S_n(\theta,\varphi) \qquad (5)$$

We shall consider first the case where the solution V describes a problem with axial symmetry about the z-axis. This means that V should be independent of φ. In this case, equation (4) reduces to

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{dS}{d\theta}\right) + n(n+1)\, S = 0 \quad , \qquad (6)$$

where now $S = S(\theta)$. Introducing the variable $u = \cos\theta$, equation (6) is transformed into

$$\frac{d}{du}\left[(1-u^2)\frac{dS_n}{du}\right] + n(n+1)\, S_n = 0 \quad , \qquad (7)$$

where we have added the subscript n to function S to emphazise its dependence on n.

The general solution of (1) will then be

$$V = \sum_n (A_n r^n + B_n r^{-n-1}) S_n(\theta)$$

In the following we analyze equation (7) in more detail.

Legendre Polynomials (~~with integral degree~~).

Suppose n is restricted to positive integer values. In this case, it may be shown that equation (7) has a polynomial solution of degree n, called the Legendre Polynomial of degree n, and denoted $P_n(u)$.

$$P_n(u) = \sum_{s=0}^{m} \frac{(-1)^s (2n-2s)!}{2^n (s!)(n-s)!(n-2s)!} u^{n-2s} , \quad (8)$$

where

$$m = \begin{cases} \frac{1}{2}n & \text{if } n \text{ is even.} \\ \frac{1}{2}(n-1) & \text{if } n \text{ is odd.} \end{cases}$$

Legendre Polynomial $P_n(u)$ may be expressed in an equivalent form by Rodrigues's formula:

$$P_n(u) = \frac{1}{2^n n!} \frac{d^n}{du^n} (u^2-1)^n \quad (9).$$

Some properties of $P_n(u)$:

(i) Orthogonality

$$\int_{-1}^{+1} P_n(u) P_m(u)\, du = \begin{cases} 0 & \text{if } n \neq m \\ \frac{2}{2n+1} & \text{if } n = m \end{cases} \quad (10)$$

This property may be used to represent a certain class of functions as ~~power~~ expansions in $P_n(u)$.

Let $f(u)$ be a function which can be represented as

$$f(u) = \sum_{n=0}^{\infty} \alpha_n P_n(u) . \qquad , -1 \le u \le +1$$

Multiplying both terms of this equality by $P_N(u)$ and integrating over the interval $[-1, +1]$ (provided the integration sign may be taken into the summation), we get

$$\int_{-1}^{+1} f(u) P_N(u)\, du = \sum_{n=0}^{\infty} \alpha_n \int_{-1}^{+1} P_n(u) P_N(u)\, du = \frac{2\alpha_N}{2N+1} , \quad (N=0,1,2,\dots)$$

whence

$$\alpha_N = \frac{2N+1}{2} \int_{-1}^{+1} f(u) P_N(u)\, du ,$$

(ii) The first few $P_n(u)$:

$$P_0(u) = 1 , \quad P_1(u) = u , \quad P_2(u) = \frac{1}{2}(3u^2-1)$$

$$P_3(u) = \frac{1}{2}(5u^3-3u) , \quad P_4(u) = \frac{1}{8}(35u^4-30u^2+3)$$

(iii) Numerical values:

$P_n(1) = 1$ for all $n = 0, 1, 2, \dots$

$$P_n(0) = \begin{cases} 0 & \text{if } n \text{ is odd} \\ (-1)^{n/2} \frac{1 \cdot 3 \cdot 5 \cdots (n-1)}{2 \cdot 4 \cdot 6 \cdots (n)} & \text{if } n \text{ is even} \end{cases}$$

$$P_n'(0) = -(n+1)\,P_{n+1}(0)$$

$$P_n'(1) = \tfrac{1}{2}\,n(n+1)$$

(the dash means differentiation w.r.to the argument u).

(iv) Parity:

$$P_n(-u) = (-1)^n P_n(u)$$

(v) Recurrence Relations:

$$(2n+1)u\,P_n(u) = (n+1)P_{n+1}(u) + n\,P_{n-1}(u)$$

$$(1-u^2)\,P_n'(u) = \frac{n(n+1)}{2n+1}\left(P_{n-1} - P_{n+1}\right)$$

$$\int^u P_n(u)\,du = \frac{P_{n+1} - P_{n-1}}{2n+1}$$

Legendre functions $Q_n(u)$. The second solution of Legendre equation (7) is denoted $Q_n(u)$. It is not a polynomial and its expression involves a term $\log\left|\frac{1-u}{1+u}\right|$ and hence $Q_n(u)$ becomes *infinite* for $u=\pm 1$, i.e. when $\theta = 0$ or π (the z-axis).

Solutions having Axial Symmetry.

$$V(r,\theta) = \sum_{n=0}^{\infty}\left(A_n r^n + \frac{B_n}{r^{n+1}}\right)\left(C_n P_n(\cos\theta) + D_n Q_n(\cos\theta)\right) \qquad (11)$$

If the solution is to be finite at at least one point on the axis $\theta=0$ $(u=1)$ or $\theta=\pi$ $(u=-1)$, then $D_n \equiv 0$ for all n.

Hence

$$V(r,\theta) = \sum_{n=0}^{\infty}\left(A_n r^n + \frac{B_n}{r^{n+1}}\right)P_n(\cos\theta) \qquad (12)$$

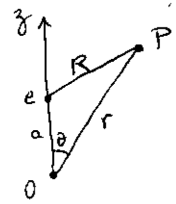
If, further, the function $V(r,0)$ is known and can be expanded as

$$V(r,0) = \sum_{n=0}^{\infty}\left(\alpha_n r^n + \beta_n r^{-n-1}\right) \qquad (13)$$

then comparing (12) and (13) it follows that

$$A_n \equiv \alpha_n \quad \text{and} \quad B_n \equiv \beta_n \quad \text{for all } n.$$

Example: Potential due to a point charge e situated on the z-axis.



Let a point charge e be situated on the z-axis, a distance a away from the center of coordinates O. It is required to find the potential due to this charge at a general point $P \equiv (r,\theta,\varphi)$.

The potential at points on the z-axis is

$$V(r,0)=\begin{cases} e/(z-a) = e/(r-a) & \text{if } r>a \\ e/(a-z) = e/(a-r) & \text{if } r<a \end{cases}$$

or

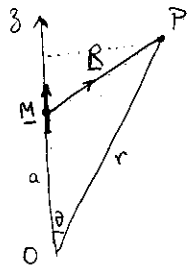
$$V(r,0)=\begin{cases} \frac{e}{r}\sum_{n=0}^{\infty}\left(\frac{a}{r}\right)^n = \frac{e}{a}\sum_{n=0}^{\infty}\left(\frac{a}{r}\right)^{n+1} & \text{if } r>a \\ \frac{e}{a}\sum_{n=0}^{\infty}\left(\frac{r}{a}\right)^n & \text{if } r<a \end{cases}$$

Therefore,

$$V(r,\theta)=\begin{cases} \frac{e}{a}\sum_{n=0}^{\infty}\left(\frac{a}{r}\right)^{n+1}P_n(\cos\theta) & \text{if } r>a \\ \frac{e}{a}\sum_{n=0}^{\infty}\left(\frac{r}{a}\right)^n P_n(\cos\theta) & \text{if } r<a \end{cases} \tag{14}$$

Noting that

$$V(r,\theta)=\frac{e}{R}=\frac{e}{\sqrt{r^2+a^2-2ar\cos\theta}}$$



and comparing this expression with (14), we get an expansion of the inverse distance $\frac{1}{R}$ (setting $e=1$) between two points in terms of the spherical harmonics.

Let us now use these results to deduce the potential due to a ~~magnetic~~ ~~an electric~~ dipole in terms of the spherical harmonics.

Let the ~~electric~~ dipole be placed on the z-axis at a distance "a" from the centre O. In order to ensure axial symmetry, we assume that the moment $\underline{M}$ of the dipole is directed along the positive z-axis. The potential at a point P is

$$\Omega(r,\theta)=\frac{\underline{M}\cdot\underline{R}}{R^3}=M(r\cos\theta-a)(r^2+a^2-2ar\cos\theta)^{-3/2}$$

$$=M\frac{\partial}{\partial a}(r^2+a^2-2ar\cos\theta)^{-1/2}=M\frac{\partial}{\partial a}\left(\frac{1}{R}\right).$$

Hence

$$\Omega(r,\theta)=\begin{cases} M\frac{\partial}{\partial a}\sum_{n=0}^{\infty}\frac{a^n}{r^{n+1}}P_n(\cos\theta) = \frac{M}{a^2}\sum_{n=1}^{\infty}n\left(\frac{a}{r}\right)^{n+1}P_n(\cos\theta) \\ M\frac{\partial}{\partial a}\sum_{n=0}^{\infty}\frac{r^n}{a^{n+1}}P_n(\cos\theta) = -\frac{M}{a^2}\sum_{n=0}^{\infty}(n+1)\left(\frac{r}{a}\right)^n P_n(\cos\theta) \end{cases} \tag{15}$$

<u>General Solution</u>. If there is no axial symmetry, function S defined in (2) will depend on both θ & φ.

We look for solutions $S_n(\theta,\varphi)$ (see eq. 5) in the form

$$S_n(\theta,\varphi)=\mathcal{R}e\sum_{m=0}^{\infty}S_{n,m}(\theta)\,e^{im\varphi}$$

($\mathcal{R}e$ denotes the real part). The equation satisfied by $S_{n,m}(\theta)$ may be easily derived. It reads:

$$\frac{d}{du}\left[(1-u^2)\frac{dS}{du}\right]+\left[n(n+1)-\frac{m^2}{1-u^2}\right]S=0 \qquad (16)$$

where $u \equiv \cos\theta$, and subscripts n, m were dropped.

Equation (16) is called the Associated Legendre equation. It has two independent solutions $P_n^m(u)$ & $Q_n^m(u)$ which are called the associated Legendre functions ~~polynomials~~ of the first and second kinds respectively. The number n is called the degree, while m is called the order of ~~the polynomial~~.

It may be shown that $P_n^m(u)$ & $Q_n^m(u)$ are related to Legendre polynomials $P_n(u)$ & $Q_n(u)$ be the relations

$$P_n^m(u) = (1-u^2)^{m/2}\frac{d^m}{du^m}P_n(u)$$

& $\qquad (17)$

$$Q_n^m(u) = (1-u^2)^{m/2}\frac{d^m}{du^m}Q_n(u)$$

We note that $Q_n^m(\pm 1)$ is still infinite.

Also, $P_n^m(u) \equiv 0$ if $m>n$. Moreover,

$$P_n^0(u) \equiv P_n(u) \qquad (18)$$

Some properties of P_n^m & Q_n^m.

(i) Orthogonality:

$$\int_{-1}^{+1} P_n^m(u)\,P_{n'}^m(u)\,du = \begin{cases} 0 & \text{if } n\neq n' \\ \dfrac{2}{2n+1}\dfrac{(n+m)!}{(n-m)!} & \text{if } n=n' \end{cases}$$

This property may be used to expand functions (belonging to a certain class) in terms of $P_n^m(u)$ for a given integer m:

$$f(u) = \sum_{n=m}^{\infty} A_n P_n^m(u) \quad , -1\le u \le +1 \qquad (19)$$

The coefficients A_n are easily obtained as

$$A_n = \frac{2n+1}{2}\,\frac{(n-m)!}{(n+m)!}\int_{-1}^{+1} f(u)\,P_n^m(u)\,du$$

(ii) Recurrence relations:

$$(2n+1)u\,P_n^m(u) = (n+1-m)P_{n+1}^m(u)+(n+m)P_{n-1}^m(u)$$

$$(2n+1)(1-u^2)\frac{dP_n^m}{du} = -n(n+1-m)P_{n+1}^m(u)+(n+1)(n+m)P_{n-1}^m(u)$$

(iii) Some numerical values:

$$P_n^m(1) = \begin{cases} 0 & \text{if } m\neq 0 \\ 1 & \text{if } m=0 \end{cases}$$

$$P_n^m(0) = \begin{cases} 0 & \text{if } n+m \text{ is odd} \\ \dfrac{1.3.5\cdots(n+m-1)}{2.4.6\cdots(n-m)} & \text{if } n+m \text{ is even.} \end{cases}$$

Separation of internal and external parts of a field, when the potential and normal component of the field are known over a sphere.

Internal & external magnetic distributions.

If an observer is on the Earth's surface (taken to be a sphere of radius a), and if there are magnetic sources inside the Earth, then the potential due to such sources for $r \geq a$ would not include terms involving r^n, since otherwise the potential would be infinite at large distances from the sources. Hence for a field of internal origin the potential is of the form

$$\sum_{n=0}^{\infty} B_n \left(\frac{r}{a}\right)^{-n-1} S_n(\theta,\varphi)$$

Similarly, if the field originates from sources outside the sphere $r=a$, then the potential will not include terms r^{-n-1}, since such terms become infinite as $r \to 0$. Thus the potential of external sources is of the form

$$\sum_{n=0}^{\infty} A_n \left(\frac{r}{a}\right)^{n} S_n(\theta,\varphi)$$

Separation of internal & external parts of a field when potential and normal component of the field are known over a sphere.

The potential of such a field may be written as

$$V(r,\theta,\varphi) = a\sum_{m=0}^{\infty}\sum_{n=m}^{\infty}\left[A_n^m\left(\frac{r}{a}\right)^n + B_n^m\left(\frac{r}{a}\right)^{-n-1}\right]P_n^m(\cos\theta)\,e^{im\varphi} \quad (20)$$

(the real part of V should only be considered).

If V is given for $r=a$, then the sums $A_n^m + B_n^m \equiv C_n^m$ are known.

Let $\frac{A_n^m}{C_n^m} = S_n^m$. Then $\frac{B_n^m}{C_n^m} = 1 - S_n^m$

Therefore,

$$V = a\sum_{m=0}^{\infty}\sum_{n=m}^{\infty} C_n^m\left[S_n^m\left(\frac{r}{a}\right)^n + (1-S_n^m)\left(\frac{r}{a}\right)^{-n-1}\right]P_n^m(\cos\theta)\,e^{im\varphi} \quad (21)$$

Let $\left(\frac{\partial V}{\partial r}\right)_{r=a}$ be given as

$$\left(\frac{\partial V}{\partial r}\right)_{r=a} = \sum_{m=0}^{\infty}\sum_{n=m}^{\infty} D_n^m P_n^m(\cos\theta)\,e^{im\varphi}. \quad (22)$$

The D_n^m are known.

Differentiating (21) w.r.to r and setting $r=a$ and then comparing the result with (22) we get

$$C_m^m\left[n S_n^m - (n+1)(1-S_n^m)\right] = D_n^m$$

This determines S_n^m, whence A_n^m and B_n^m. This determines the internal & external parts.

There are also other methods to do this separation by using integral formulae.

Application to Geomagnetism.

Gilbert (1600) by experiments on spherical pieces of lead-stone, concluded that the Earth's magnetic field has its origin from inside the Earth and is mainly that of a uniformly magnetized sphere.

In the subsequent two centuries, the intensity and magnetic field of the Earth was measured at many points on the Earth. Gauss assumed, what is nearly very correct, that there is no magnetic matter near the ground and no electric currents are flowing from the atmosphere to the ground. Under these conditions, the magnetic field must be derived from a potential satisfying Laplace's equation and representable by a series of spherical harmonics.

We cannot measure the potential directly, but only the field components

$$\underline{H} = -\operatorname{grad} V.$$

Two horizontal components X, Y:

$$X \text{ (north component)} = \frac{1}{r}\frac{\partial V}{\partial \theta} \qquad (\theta\text{ - decreasing})$$

$$Y \text{ (east component)} = -\frac{1}{r\sin\theta}\frac{\partial V}{\partial \varphi} \qquad (\varphi\text{ - increasing})$$

(θ = colatitude, φ = longitude)

and one vertical component Z:

$$Z \text{ (vertically downwards)} = \frac{\partial V}{\partial r} \qquad (r\text{ - decreasing})$$

From observations of $Z(\theta,\varphi)$, we may obtain D_n^m. Also, from observations of X and Y, we can determine $A_n^m + B_n^m = C_n^m$ (for example, from Y by finding $Y\sin\theta$). If X is used, the expansion is more complicated, but still possible.

Gauss has actually pointed out that if X (or Y) is known all over the world, then Y (or X) can be found. If the results of X and Y did not agree, then ~~it~~ this would indicate that the field is not derivable from a potential, i.e the atmosphere is either magnetic or carries electric currents to the Earth. Gauss' results showed that within the accuracy limits which are available, the values of X and Y were consistent. Also, from the Z-data, he aggreed with Gilbert that the cause is internal so that only terms $\left(\frac{a}{r}\right)^{n+1} S_{n,m}(\theta,\varphi)$ occurred. Moreover, the most important term is the term $P_1(\cos\theta)/r^2$. This corresponds to the field of a uniformly magnetized sphere or, what is equivalent at outside points, to that of an infinitesimal magnet at the Earth's center with axis along the axis of harmonics.