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AUTUMN COURSE ON GEOMAGNETISM, THE IONOSPHERE
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RADIO PROPAGATION THEORY

The influence of boundary structures on v.l.f. and U.L.F. waves

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1. Alfvén waves in the magnetosphere

1.1 Introduction

Alfvén waves occur for frequencies well below the ion gyrofrequency. They occur in the magnetosphere in many forms. Generally they have long periods ($\sim 10^{-1} - 10^{3}$ s) and the wavelength may often be comparable to the size of the magnetosphere, so that in studying them we may deal with resonant cavities rather than propagating waves. The geometry of the medium is thus important in discussing them. We begin by discussing the propagation of such waves in uniform media, then consider oscillations in more realistic magnetic field geometries.

1.2 Constitutive relations and Maxwell's equations

Hydromagnetic waves are often thought of as somehow more like hydrodynamic than electromagnetic waves. They are, however, perfectly good electromagnetic waves as we shall emphasize by deriving their dispersion relation in the same way as was done by Budden for high frequencies. We shall see that in the low frequency limit we obtain hydromagnetic waves.

Consider a uniform single species cold plasma with uniform magnetic field $\vec{B}$. Define

$$\vec{\Omega}_i = \vec{B}_0 |e| / m_i$$
$$\vec{\Omega}_e = \vec{B}_0 |e| / m_e$$
$$\omega_{pi}^2 = Ne^2/\varepsilon_0 m_i$$
$$\omega_{pe}^2 = Ne^2/\varepsilon_0 m_e$$

We shall be considering the conditions where

$$\omega \ll \omega_{pe}, \omega_{pi}, \Omega_{pe}, \Omega_{pi} \;;\quad \Omega_e \gg \Omega_i \;;\quad \omega_{pe} \gg \omega_{pi}$$

Assume $\partial/\partial t = i\omega$. Then the linearized equations of motion of the ions and electrons may be written

$$i\omega\vec{v}_e = -|e|\vec{E}/m_e - \vec{v}_e \times \vec{\Omega}_e \qquad (1)$$
$$i\omega\vec{v}_i = |e|\vec{E}/m_i + \vec{v}_i \times \vec{\Omega}_i$$

Take the scalar and vector products of $\vec{\Omega}_e$ with (1) getting

$$i\omega\vec{\Omega}_e.\vec{v}_e = -|e|(\vec{\Omega}_e.\vec{E})/m_e \qquad (3)$$
$$i\omega\Omega_e \times \vec{v}_e = -|e|(\vec{\Omega}_e \times \vec{E})/m_e - \Omega_e^2\vec{v}_e + \vec{\Omega}_e(\vec{\Omega}_e.\vec{v}_e)$$
$$= -|e|(\vec{\Omega}_e \times \vec{E})/m_e - \Omega_e^2\vec{v}_e + (i\Omega_e/\omega)|e|(\Omega_e.\vec{E})/m_e \qquad (4)$$

(Equation (3) has been used in deriving (4)). Equation (4) can be used in (1) to get

$$\vec{v}_e = \frac{|e|}{m_e(\omega^2-\Omega_e^2)}\left\{i\omega\vec{E} - \frac{i\vec{\Omega}_e}{\omega}(\vec{\Omega}_e.\vec{E}) + (\vec{\Omega}_e \times \vec{E})\right\} \qquad (5)$$

Similarly

$$\vec{v}_i = \frac{|e|}{m_i(\omega^2-\Omega_i^2)}\left\{i\omega\vec{E} - \frac{i\Omega_i}{\omega}(\vec{\Omega}_i.\vec{E}) - (\vec{\Omega}_i \times \vec{E})\right\} \qquad (6)$$

The ion and electron currents are $\vec{j}_i = N|e|\vec{v}_i$, $\vec{j}_e = -N|e|\vec{v}_e$

Thus

$$\vec{j}_e = -\frac{\varepsilon_0\omega_{pe}^2}{(\omega^2-\Omega_e^2)}\left\{i\omega\vec{E} - \frac{i\vec{\Omega}_e}{\omega}\vec{\Omega}_e.\vec{E} + \vec{\Omega}_e \times \vec{E}\right\} \qquad (7)$$
$$\vec{j}_i = -\frac{\varepsilon_0\omega_{pi}^2}{(\omega^2-\Omega_i^2)}\left\{-i\omega\vec{E} + \frac{i\vec{\Omega}_i}{\omega}\vec{\Omega}_i.\vec{E} + \vec{\Omega}_i \times \vec{E}\right\}$$

So far no approximations have been made. Note that $\omega^2 \ll \omega_e^2, \omega_i^2$, $\varepsilon_0\omega_{pe}^2/\Omega_e^2 = (N/B_0^2)m_e \ll (N/B_0^2)m_i = \varepsilon_0\omega_{pi}^2/\Omega_i^2$.

Also note that

$$\varepsilon_0\omega_{pe}^2/\Omega_e = Ne/B_0 = \varepsilon_0\omega_{pi}^2/\Omega_i .$$

Then, using these results

$$\vec{j} = \vec{j}_e + \vec{j}_i$$
$$= \frac{\varepsilon_0\omega\omega_{pi}^2}{\Omega_i^2}\vec{E} - i\varepsilon_0\frac{\omega_{pe}^2}{\omega}\frac{\vec{\Omega}_e}{\Omega_e}\frac{\vec{\Omega}_e.\vec{E}}{\Omega_e}$$
$$= \frac{i\omega Nm_i}{B_0^2}\vec{E} - i\frac{Ne^2}{m_e\omega}\hat{n}(\hat{n}.\vec{E})$$

where $\hat{n}$ is the unit vector parallel to $\vec{B}$. The parallel component of $\vec{j}$ is then

$$j_\parallel = \hat{n}.\vec{j} = \left\{\frac{i\omega Nm_i}{B_0^2} - \frac{iNe^2}{m_e\omega}\right\}E_\parallel$$
$$\simeq -i(Ne^2/m_e\omega)\vec{E} \qquad (8)$$

since

$$\omega^2/\Omega_i^2 \ll 1, \quad \omega_{pi}^2 \ll \omega_{pe}^2 .$$

The perpendicular component is

$$\vec{j}_\perp = \frac{i\omega}{\mu_0 V_A^2}\vec{E}_\perp \qquad (9)$$

where

$$V_A = B/\sqrt{\mu_0 Nm}$$

and is called the Alfvén speed.

Consider Maxwell's equations:

$$\nabla\times\vec{E} = -\partial\vec{b}/\partial t = -i\omega\vec{b} \qquad (10)$$
$$\nabla\times\vec{b} = \mu_0\vec{j} + \mu_0\varepsilon_0\,\partial\vec{E}/\partial t$$
$$= \mu_0(\vec{j} + i\omega\varepsilon_0\vec{E}) \qquad (11)$$

Here $\vec{b}$ is the perturbation magnetic field. From (8) and (9)

$$E_\parallel / E_\perp \sim (\omega^2 / \Omega_e \Omega_i)\, j_\parallel / j_\perp \ll 1$$

unless $j_\parallel \gg j_\perp$. The ratio of the two terms on the right of (11) is $c^2/V_A^2 \gg 1$ for the pependicular case, and $\omega_{pe}^2/\omega^2 \gg 1$ for the parallel case. Thus we can write (11) as

$$\nabla \times \vec{b} = \mu_0 \vec{j} \qquad (12)$$

We then combine (8) (10) and (12) to give

$$\nabla \times \vec{E}_\perp = -i\omega \vec{b} \qquad (13)$$

$$(\nabla \times \vec{b})_\perp = (i\,\omega / V_A^2)\, \vec{E}_\perp \qquad (14)$$

$$(\nabla \times \vec{b})_\parallel = \mu_0 j_\parallel \qquad (15)$$

where (13) and (14) are sufficient to determine the fields and (15) fixes $j_\parallel$.

It should also be noted that, if we consider the mean velocity of the plasma,

$$\vec{v} = (m_e \vec{v}_e + m_i \vec{v}_i) / (m_e + m_i) \qquad (16)$$

then, if

$$\omega \ll \Omega_e, \Omega_i$$

equations (1) and (2) give, on multiplication by m_e^2, m_i^2 respectively

$$(m_e + m_i)\, \vec{E} + (m_e \vec{v}_e + m_i \vec{v}_i) \times \vec{B} = 0$$

or

$$\vec{E} + \vec{v} \times \vec{B} = 0 \qquad (17)$$

The implication of this is that the perturbation velocity of the plasma is

$$\vec{v} = \vec{E} \times \vec{B} / B^2$$

as can be seen by taking the cross product of (17) with $\vec{B}$.

1.3 Uniform medium Dispersion relation

In a uniform medium we assume spatial variation of the form $\exp\{-i\, k.\vec{r}\}$. Then the ∇ operator may be replaced by $-ik$ and equations (1. 2.13), 1.2.14) become

$$\vec{k} \times \vec{E}_\perp = \omega \vec{b} \qquad (1)$$

$$(\vec{k} \times \vec{b})_\perp = -(\omega / V_A^2)\, \vec{E}_\perp \qquad (2)$$

Let $\vec{k}$ be in the x-z plane with $\hat{z}$ in the $\vec{B}$-direction. Then the components split into two sets.

$$\left.\begin{array}{l|l} -k_z E_y = \omega b_x & k_z E_x = \omega b_y \\ k_z b_x - k_x b_z = -(\omega/V_A^2)\, E_y & k_z b_y = -(\omega/V_A^2)\, E_x \\ k_x E_y = \omega b_z & \end{array}\right\} \qquad (3)$$

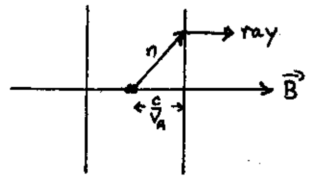
The first set has a dispersion relation

$$k_x^2 + k_z^2 \equiv k^2 = \omega^2 / V_A^2$$

It is an isotropic wave travelling with speed V_A, and is generally known as the *fast m.h.d. wave*. The other wave has the dispersion relation

$$k_z^2 = \omega^2 / V_A^2$$

This is remarkable because it does not involve k_x. That is the wave can have any transverse behaviour whatsoever. The magnetic and electric perturbations are at right angles to B and for this reason it is known as the *transverse* Alfvén wave. This means that the Poynting vector $\vec{E} \times \vec{B} / \mu_0$ is always exactly along $\vec{B}$ no matter what the direction of $\vec{k}$. From another point of view the refractive index surface is as shown, so that the ray direction is parallel to B. The result of this is that magneto-hydrodynamic waves with very long wavelengths can be very closely confined to a particular magnetic field line, on a length scale very much shorter than the wavelength measured along the field. To see this it is useful to discuss the propagation of such waves in non-uniform media.



1.4 Magnetic coordinates

In studying Alfvén waves in the magnetosphere we note that there are substantial advantages in taking the magnetic field direction as one of the coordinate directions. In the special case of a current-free region, for which there are advantages in choosing these coordinates in a special way. In such a region we can express $\vec{B}$ as the gradient of a scalar magnetic potential Φ_m

$$\vec{B} = -\nabla \Phi_m \qquad (1)$$

Then choose the w-coordinate of a curvilinear system as

$$w = A^{-1} \Phi_m \qquad (2)$$

where A^{-1} is constant. Since $\vec{B}$ is normal to equipotentials, u and v axes will lie in equipotential planes. Now

$$\vec{B} = -h_w^{-1}\, \partial \Phi_m / \partial w = A / h_w$$

where h_w is the scale factor, i.e.

$$h_w = A / B \qquad (3)$$

Now consider a flux tube of cross section du dv. The flux is conserved and thus

$$B\, h_u h_v\, du\, dv = \text{constant} \times du\, dv$$

The constant may be taken as A so that in such a system

$$h_u h_v = h_w$$

An example of such a system is the set of magnetic dipole coordinates. In a magnetic dipole field

$$\Phi_m \propto \sin\theta / r^2$$

where r, θ, φ are spherical polar coordinates, with θ the latitude *not* the polar angle.

The equation of a field line can be shown to be

$$r = aL\sin^2\theta \tag{5}$$

where L defines the radius at which the field line cuts the equatorial plane. Then take ν in the meridian measured outwards and μ in the direction of $\vec{B}$, with

$$\nu = -\cos^2\theta/r = (aL)^{-1} \tag{6}$$

$$\varphi = \varphi \tag{7}$$

$$\mu = \sin\theta/r^2 = \sin\theta/a^2L^2\cos^4\theta \tag{8}$$

The scale factors can be shown to be

$$h_\nu = \{r^2/\cos\theta\}\{4-3\cos^2\theta\}^{-1/2} = a^2L^2\cos^3\theta\{4-3\cos^2\theta\}^{-1/2} \tag{9}$$

$$h_\varphi = r\cos\theta = aL\cos^3\theta \tag{10}$$

$$h_\mu = r^3\{4-3\cos^2\theta\}^{-1/2} = a^3L^3\cos^6\theta\{4-3\cos^2\theta\}^{-1/2} \tag{11}$$

1.5 Alfvén waves confined to a field line

Fejer [J. Geophys. Res. 86, 5614, 1981] has considered this problem from a ray tracing point of view. We take a different approach using the field equations. We shall consider a current-free medium which is non-uniform. Choose magnetic coordinates u,v,w. Then the field equations (1.3.13), (1.3.14) are

$$\frac{1}{h_v h_w}\left\{\frac{\partial}{\partial v}(h_w b_w) - \frac{\partial}{\partial w}(h_v b_v)\right\} = i\omega E_u/V_A^2 \tag{1}$$

$$\frac{1}{h_w h_u}\left\{\frac{\partial}{\partial w}(h_u b_u) - \frac{\partial}{\partial u}(h_w b_w)\right\} = i\omega E_v/V_A^2 \tag{2}$$

$$\frac{1}{h_v h_w}\frac{\partial}{\partial w}(h_v E_v) = i\omega b_u \tag{3}$$

$$-\frac{1}{h_w h_u}\frac{\partial}{\partial w}(h_u E_u) = i\omega b_v \tag{4}$$

$$-\frac{1}{h_u h_v}\left\{\frac{\partial}{\partial u}(h_v E_v) - \frac{\partial}{\partial v}(h_u E_u)\right\} = i\omega b_w \tag{5}$$

Assume that the phase variation perpendicular to the field is very rapid compared to the variation of the density, and magnetic field. In this case we may write $h_u^{-1}\partial/\partial u \equiv -ik_u$, $h_v^{-1}\partial/\partial v \equiv -ik_v$ where $k_u, k_v \gg \ell^{-1}$. ℓ is the length scale on which B and ρ vary. Using these results and eliminating b_u b_v from (1) - (5) we get

$$\frac{1}{h_v h_w}\frac{\partial}{\partial w}\left\{\frac{h_v}{h_u h_w}\frac{\partial}{\partial w}(h_u E_u)\right\} + \omega^2 E_u/V_A^2 = \omega k_v b_w \tag{6}$$

$$\frac{1}{h_w h_u}\frac{\partial}{\partial w}\left\{\frac{h_u}{h_v h_w}\frac{\partial}{\partial w}(h_v E_v)\right\} + \omega^2 E_v/V_A^2 = -\omega k_u b_w \tag{7}$$

$$k_u E_v - k_v E_u = \omega b_w \tag{8}$$

In these equations it is to be expected that k_u, k_v are slowly varying functions of u,v,w and that the variation of field components with u and v are of phase integral type:

$$\exp\left[-i\left\{\int^u k_u h_u\,du + \int^v k_v h_v\,dv\right\}\right] \tag{9}$$

If this is the case then, locally, u and v may be chosen so that u is in the direction in which the phase varies most rapidly perpendicular to the magnetic field. We can then make $k_u \gg k_v$. Assume that the scale length for variation along the magnetic field, V_A/ω, is comparable with ℓ i.e. $k_u \gg \omega/V_A$. Then (7) implies

$$E_v/b_w \sim (k_u/\omega)V_A^2 = (k_u V_A/\omega)V_A \gg V_A$$

and (8) implies

$$E_v/b_w \sim \omega/k_u \ll V_A$$

unless

$$E_v \simeq (k_v/k_u)E_u \ll E_u$$

Thus the only way that the equations can be consistent is for E_u to be small and b_v to be small. To the order of approximation we are using equation (6) then becomes

$$\frac{1}{h_v h_w}\frac{\partial}{\partial w}\left\{\frac{h_v}{h_u h_w}\frac{\partial}{\partial w}(h_u E_u)\right\} + \omega^2 E_u/V_A^2 = 0 \tag{10}$$

which is decoupled from E_v. The resulting oscillation has rapid phase variation in the u direction; the magnetic perturbation is transverse to this,

$$b_v = (i/\omega h_w h_u)\frac{\partial}{\partial w}(h_u E_u) \tag{11}$$

and the variation along the field line is given by (10). To this order of approximation the other field components are zero. Comparison with (1.4.3) shows that, in a uniform medium, the wave is identical with a transverse Alfvén wave. The plasma velocity is then $\vec{v} = \vec{E}\times\vec{B}/B^2$

i.e.

$$v_v = -E_u/B \tag{12}$$

Because of its confinement to a field line such a mode is often called a "guided" mode.

1.6 Dipolar coordinates

Two special cases are of importance when we use magnetic dipolar coordinates (eqs. (1.4.6) - (1.4.11)).

(i) "Guided" toroidal mode

If the phase variation is in the ν direction equation (1.5.10) becomes

$$\frac{h_\nu}{h_\varphi h_\mu}\left\{\frac{\partial}{\partial\mu}\left(\frac{h_\varphi}{h_\mu h_\nu}\frac{\partial \mathcal{E}_\nu}{\partial\mu}\right)\right\} + \omega^2\mathcal{E}_\nu/V_A^2 = 0 \qquad (1)$$

where we have written

$$\mathcal{E}_\nu = h_\nu E_\nu \qquad (2)$$

Assume that

$$B^2 = (a^6 L^6/r^6)\, B_{eq}^2 (4-3\cos^2\theta) \qquad (3)$$

and

$$\rho = \rho_{eq}(aL/r)^p \qquad (4)$$

Then

$$V_A^2 = (B_{eq}^2/\mu_0\rho_{eq})\cos^{2p-12}\theta\,(4-3\cos^2\theta) \qquad (5)$$

where (1.4.5) has been used. Now make the substitution $z=\cos\theta$ and use (1.4.9) - (1.4.11) for the scale factors. Equation (1) becomes

$$\frac{d^2\mathcal{E}_\nu}{dz^2} + q^2(1-z^2)^{6-p}\mathcal{E}_\nu = 0 \qquad (6)$$

where

$$q^2 = \mu_0\omega^2 a^2 L^8 \rho_{eq}/B_{eq}^2 \qquad (7)$$

The only requirement for this equation to hold is that the phase variation in the ν direction is on a length scale small compared with all other length scales. It is interesting that this equation is often derived on another assumption. Because of the cylindrical symmetry of the dipole field a cylindrically symmetric solution with $\partial/\partial\varphi = 0$ is possible. Equation (1.5.6) then has $k_\nu \simeq k_\varphi = 0$ and is decoupled to give an identical equation for the toroidal mode. In this solution the oscillation is seen to be that of a complete shell. This is unnecessarily restrictive; it is only necessary to have the phase variation in the meridian much larger than that in azimuth.

(ii) "Guided" poloidal mode

If we carry out exactly the same procedure, but choose the φ direction as that for which phase varies most rapidly, we get for $\mathcal{E}_\varphi$

$$\frac{d^2\mathcal{E}_\varphi}{dz^2} - \frac{6z}{1+3z^2}\frac{d\mathcal{E}_\varphi}{dz} + k^2(1-z^2)^{6-p}\mathcal{E}_\varphi = 0 \qquad (8)$$

where

$$\mathcal{E}_\varphi = h_\varphi E_\varphi \qquad (9)$$

which is the equation for the guided poloidal mode. This equation is important in discussing waves propagating in the azimuthal direction.

1.7 The ionosphere as a boundary

In the previous sections we have shown that long-period hydromagnetic waves are guided accurately by a magnetic field line. These field lines are bounded at each end by the ionosphere. Frequently the wavelength is comparable with the length of the field line, and the possibility of standing waves (like those on a guitar string) arises.

Above about 120 km the ionospheric collision frequency is negligible at the frequencies of interest. In the E-region between 100 - 120 km the ion collision frequency is comparable with, or much greater than, the ion gyrofrequency, while electron collisions can still be neglected. The net effect is that the ionosphere has an anisotropic conductivity, which is a function of height. We shall not derive expressions for this conductivity here, but it can be done by including a collision term in the basic equations of motion (1.2.1) and (1.2.2) and making appropriate approximations. This results in a tensor conductivity so that

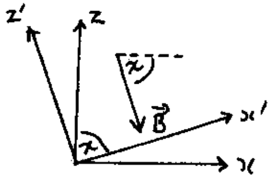
$$\vec{j} = \underline{\underline{\sigma}}\cdot\vec{E} \qquad (1)$$

where

$$\underline{\underline{\sigma}} = \begin{pmatrix} \sigma_P & -\sigma_H & 0 \\ \sigma_H & \sigma_P & 0 \\ 0 & 0 & \sigma_\parallel \end{pmatrix} \qquad (2)$$

Here σ_P and σ_H are the Pederson and Hall conductivities.

Now let $\vec{B}$ be in the x-z plane and have dip angle χ as shown. Then equation (1) may be written



$$\begin{pmatrix} j_x' \\ j_y' \\ j_z' \end{pmatrix} = \begin{pmatrix} \sigma_P & -\sigma_H & 0 \\ \sigma_H & \sigma_P & 0 \\ 0 & 0 & \sigma_\parallel \end{pmatrix}\begin{pmatrix} E_x' \\ E_y' \\ E_z' \end{pmatrix}$$

Transform to the (x,y,z) system getting

$$\begin{pmatrix} j_x \\ j_y \\ j_z \end{pmatrix} = \begin{pmatrix} \sigma_P\sin^2\chi + \sigma_\parallel\cos^2\chi & -\sigma_H\sin\chi & (\sigma_P-\sigma_\parallel)\sin\chi\cos\chi \\ \sigma_H\sin\chi & \sigma_P & \sigma_H\cos\chi \\ (\sigma_P-\sigma_\parallel)\sin\chi\cos\chi & -\sigma_H\cos\chi & \sigma_P\cos^2\chi + \sigma_\parallel\sin^2\chi \end{pmatrix}\begin{pmatrix} E_x \\ E_y \\ E_z \end{pmatrix}$$

If we now regard the ionosphere as a region in which no vertical current can flow then $j_z = 0$ and

$$E_z = \{-(\sigma_P - \sigma_{\parallel}) \sin\chi \cos\chi \, E_x + \sigma_H \cos\chi \, E_y\}\{\sigma_P \cos^2\chi + \sigma_{\parallel} \sin^2\chi\}^{-1}$$

Using this expression for E_z we can express j_x, j_y in terms of E_x, E_y:

$$\begin{pmatrix} j_x \\ j_y \end{pmatrix} = \begin{pmatrix} \sigma_{xx} & \sigma_{xy} \\ \sigma_{yx} & \sigma_{yy} \end{pmatrix} \begin{pmatrix} E_x \\ E_y \end{pmatrix}$$

where

$$\sigma_{xx} = \sigma_P \sigma_{\parallel} / \{\sigma_P \cos^2\chi + \sigma_{\parallel} \sin^2\chi\} \simeq \sigma_P / \sin^2\chi$$

$$\sigma_{xy} = -\sigma_{yx} = -\sigma_H \sigma_{\parallel} \sin\chi / \{\sigma_P \cos^2\chi + \sigma_{\parallel} \sin^2\chi\} \simeq -\sigma_H / \sin\chi$$

$$\sigma_{yy} = \sigma_P + \sigma_H \cos^2\chi / \{\sigma_P \cos^2\chi + \sigma_{\parallel} \sin^2\chi\} \simeq \sigma_P$$

The approximations hold if $\sigma_{\parallel} \gg \sigma_P, \sigma_H$ unless $\tan\chi \ll \sigma_P / \sigma_H$
They are then valid except very near the equator.

Thus

$$\begin{pmatrix} j_x \\ j_y \end{pmatrix} \simeq \begin{pmatrix} \sigma_P / \sin^2\chi & -\sigma_H / \sin\chi \\ \sigma_H / \sin\chi & \sigma_P \end{pmatrix} \begin{pmatrix} E_x \\ E_y \end{pmatrix} \qquad (3)$$

Now often the thickness of the ionosphere is small compared with the wavelength along a field line and it is convenient to work in terms of a thin sheet current. We assume that over the thickness of the ionosphere $\vec{B}$ is constant and thus so is $\vec{E}$ (why?). Then

$$\begin{pmatrix} I_x \\ I_y \end{pmatrix} = \int_{z_1}^{z_2} \begin{pmatrix} j_x \\ j_y \end{pmatrix} dz = \left(\int \begin{Bmatrix} \sigma_P / \sin^2\chi & -\sigma_H / \sin\chi \\ \sigma_H / \sin\chi & \sigma_P \end{Bmatrix} dz \right) \cdot \begin{pmatrix} E_x \\ E_y \end{pmatrix} \qquad (4)$$

or

$$\begin{pmatrix} I_x \\ I_y \end{pmatrix} = \begin{pmatrix} \Sigma_P / \sin^2\chi & -\Sigma_H / \sin\chi \\ \Sigma_H / \sin\chi & \Sigma_P \end{pmatrix} \qquad (5)$$

where Σ_P, Σ_H are height-integral Pedersen and Hall conductivities.

In order to discuss the effect of the ionosphere on a signal which originates above it, we follow closely Walker *et al* [J.G.R. 84, 3373-1979]

It was first recognised by Dungey [1963] that for signals varying in the horizontal direction, on terrestrial length scales, currents flowing in the ionosphere would effectively screen that part of the magnetic field that has a vertical component of its curl from the ground. This can be most easily seen by considering Maxwell's equations in the free space region below the ionosphere. We now adopt a new rectangular coordinate system (x', y', z'). This is rotated about the y axis of the (x,y,z) system, so that in the northern hemisphere the z' axis is vertically downward, and the x' and y' axes point magnetic north and east, respectively. The ionosphere fills the region $z' < 0$, and the region $z' > 0$ is free space. For the time being, we ignore the ground plane. We assume that there is a hydromagnetic wave incident from the ionosphere on the $z' = 0$ plane. It is partially reflected and partially transmitted there. This wave can be Fourier transformed spatially. The resulting Fourier components are plane waves with wave normals in vertical planes making various angles with the magnetic meridian (the x'-z' plane). Consider one of these Fourier components. We adopt a local coordinate system (ξ, η, ζ), chosen so that the ζ axis coincides with the z' axis and is vertically downward and so that the wave normal lies in the $(\xi - \zeta)$ plane. The assumed time variation is $\exp(i\omega t)$, and the dependence on ξ is $\exp(-i k_\xi \xi)$. Below the ionosphere, where $\zeta > 0$ the dependence on ζ of the transmitted wave will be $\exp(-i k_\zeta \zeta)$, where

$$k_\zeta^2 + k_\xi^2 = (\omega/c)^2 \qquad (6)$$

Now, in Maxwell's equations, $\partial/\partial t = i\omega$, $\partial/\partial\xi = -ik_\xi$, $\partial/\partial\zeta = -ik_\zeta$, $\partial/\partial\eta = 0$
The equations, curl $E = -\partial \vec{b}/\partial t$ and curl $\vec{b} = c^{-2} \partial\vec{E}/\partial t$ may be written

$$\begin{aligned} ik_\zeta E_\eta &= -i\omega b_\xi & ik_\zeta b_\eta &= i(\omega/c^2) E_\xi \\ -ik_\zeta E_\xi + ik_\xi E_\zeta &= -i\omega b_\eta & -ik_\zeta b_\xi + ik_\xi b_\zeta &= i(\omega/c^2) E_\eta \\ -ik_\xi E_\eta &= -i\omega b_\zeta & -ik_\xi b_\eta &= i(\omega/c^2) E_\zeta \end{aligned} \qquad (7)$$

These separate into two sets, one involving b_ξ, b_ζ, E_η and the other E_ξ, E_ζ, b_η:

$$\left.\begin{aligned} k_\zeta E_\eta = -\omega b_\xi \qquad & k_\xi E_\eta = \omega b_\zeta \\ -k_\zeta b_\xi + k_\xi b_\zeta &= (\omega/c^2) E_\eta \end{aligned}\right\} \qquad (8)$$

$$\left.\begin{aligned} k_\zeta b_\eta = (\omega/c^2) E_\xi \qquad & k_\xi b_\eta = -(\omega/c^2) E_\zeta \\ k_\zeta E_\xi - k_\xi E_\zeta &= \omega b_\eta \end{aligned}\right\} \qquad (9)$$

The first of these sets (the E_η polarization) has $\vec{E}$ and curl $\vec{b}$ perpendicular to the plane of incidence; the second (the b_η polarization) has $\vec{b}$ and curl $\vec{E}$ perpendicular to the plane of incidence.

Now, for periods of the order of hundreds of seconds the quantity $(\omega/c)^2$ on the right-hand side of (6) is about 10^{-20} m^{-2} (corresponding to free space wavelengths of about 10^5 km). The quantity k_ξ, however, depends on the horizontal length scale imposed by the boundary conditions at $\zeta = 0$. This is typically of the order of hundreds or thousands of kilometers. For any length scale less than, say, 10 000 km, $k_\xi^2 < 4 \times 10^{-13}$ m^{-2}
Thus for all terrestrial length scales the right-hand side of (6) is negligible and

$$k_\zeta \simeq i k_\xi$$

with $|k_\zeta|$ and $|k_\xi|$ both very much greater than ω/c.

Now consider (9). The value of b_η is of the order of $\omega E_\xi/c^2 k_\xi$ which for length scales less than 10 000 km is less than 4×10^{-13} E. E_ξ is the horizontal component of the field in the plane of incidence below the ionosphere, and boundary conditions at the ionosphere require that E_ξ be continuous across the boundary. In the ionosphere its value is at most about 10^{-1} V m^{-1}, and thus below the ionosphere, $b_\eta \sim 4 \times 10^{-5}$ nT It is reasonable to assume that b_η within the ionosphere is typically 10 - 100 nT, so it is clear that this field must be reduced essentially to zero at the boundary. The implication is that ionospheric currents flow to screen b_η from the ground. We note also that (8) implies that $|E_\eta| \sim |\omega b_\xi / k_\xi|$ Even with b_ξ as large as 100 nT, E_η does not exceed 4 mV^{-1} for length scales of the order of 100 km. Thus both E_η and b_η are negligible below the ionosphere. The signals below the ionosphere are dominated by components of the electric and magnetic field in the plane of incidence, and these have horizontal curl only. The electric field below the ionosphere is a quasi- electrostatic fringing field matched to the electric field E_ξ in the ionosphere.

If the ionospheric conductivity were isotropic, this would be the end of the story so far as magnetometer observations of the previously described hydromagnetic resonances were concerned. Since $b_\xi \ll b_\eta$ and is screened by from the ground, the signals observed on the ground would be negligible. However, several authors have noted that the effect of the anisotropic ionospheric conductivity is to rotate the horizontal component of the magnetic field through 90°, thus supplying a b_ξ which can be observed on the ground. We present here a simplified picture of the theory, whichis adequate for our purposes and show the essential features of the problem.

Consider that the E region of the ionosphere may be regarded as a thin layer in which sheet currents may flow. It is characterized by a height-integrated conductivity tensor of the form (5). The closure current which shields b_η from the ground is the Pedersen current. It is continuous with the field-aligned currents that must flow because curl $\vec{b}$ has a component along the magnetic field. Associated with it is a Hall current flowing at right angles to the electric field. It is this divergence-free Hall current that is the source of the b_ξ seen below the ionosphere. Thus $\vec{b}$ changes from being essentially perpendicular to the plance of incidence above the E-region to being essentially within the plane of incidence below it. This is the essence of the explanation of the rotation of the polarization of $\vec{b}$. The implication is that if one measures the Hall current in the ionosphere, one should be able to deduce the magnetic field on the ground without considering the Pedersen current, or any field-aligned current.

To sum up, the horizontal component of the dominant magnetic field is rotated through 90° with respect to that above the E region, while the horizontal component of the dominant electric field is not so rotated. Additional electric and magnetic fields are present, perpendicular to the plane of incidence, but these are negligibly small.

1.8 Standing waves

As an example of a standing wave we consider a pure toroidal oscillation with $p = 3$. As a start we consider the problem with an infinitely conducting ionosphere. Then $\mathcal{E}_\nu = 0$ at the boundary. We will solve this by a standard perturbation technique (e.g. Morse and Feshbach, p.1002). With a slight change of notation equation (1.6.6) becomes

$$\frac{d^2 \mathcal{E}}{dz^2} + q^2 \left\{ 1 - (z - \tfrac{1}{2} z_0)^2 \right\}^3 \mathcal{E} = 0 \qquad (1)$$

Here the origin of z is at the foot of the field line in the southern hemisphere rather than at the equator. $\mathcal{E}$ represents the scaled toroidal electric field and $z = \sin\theta$. The equation may be written

$$\frac{d^2 \mathcal{E}_n}{dz^2} + \left[q^2 - q^2 \eta(z) \right] \mathcal{E}_n = 0 \qquad (2)$$

where

$$\eta(z) = 3(z - \tfrac{1}{2} z_0)^2 - 3(z - \tfrac{1}{2} z_0)^4 + (z - \tfrac{1}{2} z_0)^6 \qquad (3)$$

The exactly soluble problem is

$$\frac{d^2 \Phi_n}{dz^2} + q_n^2 \Phi_n = 0 \qquad (4)$$

with the same boundary conditions, having solutions

$$\Phi_n(z) = \sqrt{2/z_0} \sin(n\pi z / z_0), \quad n = 1, 2, 3 \cdots \qquad (5)$$

with the eigenvalues q_n being given by

$$q_n^2 = (n\pi / z_0)^2 \qquad (6)$$

and normalization s.t.

$$\int_0^{z_0} \Phi_n \Phi_m \, dz = \delta_{nm}$$

The first order perturbation solution is

$$q^2 \simeq q_n^2 + \eta_{nn} q^2 \quad \text{or} \quad q^2 = q_n^2 / (1 - \eta_{nn}) \qquad (7)$$

and

$$\mathcal{E}_n = \Phi_n + \sum_{p \neq n} \frac{q^2 \eta_{pn} \Phi_p}{q_n^2 - q_p^2} \simeq \Phi_n + \sum_{p \neq n} \frac{\eta_{pn}}{1 - (p/n)^2} \Phi_p \qquad (8)$$

where

$$\eta_{pn} = \int_0^{z_0} \Phi_p(z) \, \eta(z) \, \Phi_n(z) \, dz \qquad (9)$$

It is easy to see that this integral is bounded. Let A be the upper bound of η_{pn} Then for sufficiently large p the terms in the series in (8) are less than A/p^2 and the series converges.

Now consider the n = 1 mode:

$$\eta_{11} = \frac{2}{z_0} \int_0^{z_0} \sin^2\left(\frac{\pi z}{z_0}\right) \left\{ 3(z - \tfrac{1}{2} z_0)^2 - 3(z - \tfrac{1}{2} z_0)^4 + (z - \tfrac{1}{2} z_0)^6 \right\} dz \qquad (10)$$

Let $\qquad s = z - \tfrac{1}{2}z_0 = z - s_0$

$$\therefore ds = dz$$

$$\therefore \eta_{11} = \frac{1}{s_0}\int_{-s_0}^{s_0} \sin^2\left(\frac{\pi s}{2s_0} + \frac{\pi}{2}\right)\left\{3s^2 - 3s^4 + s^6\right\} ds$$

$$= \frac{1}{s_0}\int_{-s_0}^{s_0} \cos^2\left(\frac{\pi s}{2s_0}\right)\left\{3s^2 - 3s^4 + s^6\right\} ds \qquad (11)$$

Let $\quad \theta = \pi s/2s_0$

$$\therefore ds = (2s_0/\pi)\, d\theta$$

$$\eta_{11} = \frac{2}{\pi}\int_{-\pi/2}^{\pi/2} \cos^2\theta \left\{3\left(\frac{2s_0}{\pi}\right)^2\theta^2 - 3\left(\frac{2s_0}{\pi}\right)^4\theta^4 + \left(\frac{2s_0}{\pi}\right)^6\theta^6\right\} d\theta \qquad (12)$$

Now

$$\int_{-\pi/2}^{\pi/2} \theta^2\cos^2\theta\, d\theta = \left\{\frac{1}{6}\theta^3 + \frac{1}{4}\theta\cos 2\theta + \text{terms in } \sin 2\theta\right\}\Big|_{-\pi/2}^{\pi/2} = \frac{\pi^3}{24} - \frac{\pi}{4} \qquad (13)$$

$$\int_{-\pi/2}^{\pi/2} \theta^4\cos^2\theta\, d\theta = \left\{\frac{1}{10}\theta^5 + \left(\frac{1}{2}\theta^3 - \frac{3}{4}\theta\right)\cos 2\theta + \text{terms in } \sin 2\theta\right\}\Big|_{-\pi/2}^{\pi/2} = \frac{\pi^5}{160} - \frac{\pi^3}{8} + \frac{3\pi}{4} \qquad (14)$$

$$\int_{-\pi/2}^{\pi/2} \theta^6\cos^2\theta\, d\theta = \left\{\frac{1}{14}\theta^7 + \left[\frac{3}{4}\theta^5 - \frac{15}{4}\theta^3 + \frac{45}{8}\theta\right]\cos 2\theta + \text{terms in } \sin 2\theta\right\}\Big|_{-\pi/2}^{\pi/2}$$

$$= \frac{1}{896}\pi^7 - \frac{3}{64}\pi^5 + \frac{15}{16}\pi^3 - \frac{45}{8}\pi \qquad (15)$$

Thus, from (12)

$$\eta_{11} = \left(1 - \frac{6}{\pi^2}\right)s_0^2 - \left(\frac{3}{4} - \frac{12}{\pi^2} + \frac{72}{\pi^4}\right)s_0^4 + \left(\frac{1}{7} - \frac{6}{\pi^2} + \frac{120}{\pi^4} - \frac{720}{\pi^6}\right)s_0^6 \qquad (15a)$$

Thus the first eigenvalue is given, from (6) and (7) as

$$q^2 = (\pi/2s_0)^2\left\{1 - \left(1 - \frac{6}{\pi^2}\right)s_0^2 + \left(\frac{3}{5} - \frac{12}{\pi^2} + \frac{72}{\pi^4}\right)s_0^4 - \left(\frac{1}{7} - \frac{6}{\pi^2} + \frac{120}{\pi^4} - \frac{720}{\pi^6}\right)s_0^6\right\}^{-1} \qquad (16)$$

$$= \frac{2.467}{s_0^2\left\{1 - 0.3921 s_0^2 + 0.1233 s_0^4 - 0.0180 s_0^6\right\}} \qquad (17)$$

The first order perturbation for the n = 1 eigenfunction is, from (8) and (5)

$$\mathcal{E}_1(z) = \sqrt{\frac{2}{z_0}}\left\{\sin\frac{\pi z}{z_0} - \frac{\eta_{13}}{8}\sin\frac{3\pi z}{z_0} - \frac{\eta_{15}}{24}\sin\frac{5\pi z}{z_0} + \cdots\right\} \qquad (18)$$

(The even order times in the series are zero from the symmetry of the problem). The series converges rapidly, and a high degree of accuracy is obtained by taking only the first term. In order to assess this accuracy we shall evaluate η_{13}

$$\eta_{13} = \frac{2}{z_0}\int_0^{z_0} \sin\frac{\pi z}{z_0}\sin\frac{3\pi z}{z_0}\,\eta(z)\, dz$$

$$= \frac{2}{z_0}\int_0^{z_0} \eta(z)\sin\frac{\pi z}{z_0}\left[\sin\frac{\pi z}{z_0}\cos\frac{2\pi z}{z_0} + \cos\frac{\pi z}{z_0}\sin\frac{2\pi z}{z_0}\right] dz$$

$$= \frac{2}{z_0}\int_0^{z} \eta(z)\sin\frac{\pi z}{z_0}\left[\sin\frac{\pi z}{z_0}\cos^2\frac{\pi z}{z_0} - \sin^3\frac{\pi z}{z_0} + 2\sin\frac{\pi z}{z_0}\cos^2\frac{\pi z}{z_0}\right] dz$$

$$= \frac{2}{z_0}\int_0^{z_0} \eta(z)\sin^2\frac{\pi z}{z_0}\left[3 - 4\sin^2\frac{\pi z}{z_0}\right] dz$$

$$= 3\eta_{11} - \frac{8}{z_0}\int_0^{z_0}\left(3\left(z - \tfrac{1}{2}z_0\right)^2 - 3\left(z - \tfrac{1}{2}z_0\right)^4 + \left(z - \tfrac{1}{2}z_0\right)^6\right)\sin^4\frac{\pi z}{z_0}\, dz$$

$$= 3\eta_{11} - \frac{8}{\pi}\int_{-\pi/2}^{\pi/2}\left[3\left(\frac{2s_0}{\pi}\right)^2\theta^2 - 3\left(\frac{2s_0}{\pi}\right)^4\theta^4 + \left(\frac{2s_0}{\pi}\right)^6\theta^6\right]\cos^4\theta\, d\theta \qquad (19)$$

Now

$$\int_{-\pi/2}^{\pi/2} \theta^2\cos^4\theta\, d\theta = \left[\frac{\theta\cos^3\theta}{16}\left(2\cos\theta + 4\theta\sin\theta\right)\right]_{-\pi/2}^{\pi/2} + \frac{3}{4}\int_{-\pi/2}^{\pi/2}\theta^2\cos^2\theta\, d\theta - \frac{1}{8}\int_{-\pi/2}^{\pi/2}\cos^4\theta\, d\theta$$

$$= \frac{3}{4}\left(\frac{\pi^3}{24} - \frac{\pi}{4}\right) - \frac{1}{8}\left[\frac{3}{8}\theta + \frac{3}{8}\sin\theta\cos\theta + \frac{1}{4}\sin\theta\cos^3\theta\right]_{-\pi/2}^{\pi/2}$$

$$= \frac{\pi^3}{32} - \frac{3\pi}{16} - \frac{3\pi}{64}$$

$$= \frac{\pi^3}{32} - \frac{15\pi}{64} \qquad (20)$$

$$\int_{-\pi/2}^{\pi/2} \theta^4\cos^4\theta\, d\theta = \left[\frac{\theta^3\cos^3\theta}{16}\left(4\cos\theta + 4\theta\sin\theta\right)\right]_{-\pi/2}^{\pi/2} + \frac{3}{4}\int_{-\pi/2}^{\pi/2}\theta^4\cos^2\theta\, d\theta - \frac{3}{4}\int_{-\pi/2}^{\pi/2}\theta^2\cos^2\theta\, d\theta$$

$$= \frac{3}{4}\left[\frac{\pi^5}{160} - \frac{\pi^3}{8} + \frac{3\pi}{4}\right] - \frac{3}{4}\left[\frac{\pi^3}{32} - \frac{15\pi}{64}\right]$$

$$= \frac{3}{4}\left[\frac{\pi^5}{160} - \frac{5\pi^3}{32} + \frac{63\pi}{64}\right] \qquad (21)$$

$$\int_{-\pi/2}^{\pi/2}\theta^6\cos^4\theta\,d\theta = \left[\frac{\theta^5\cos^3\theta}{16}\left(6\cos\theta+4\theta\sin\theta\right)\right]_{-\pi/2}^{\pi/2} + \frac{3}{4}\int_{-\pi/2}^{\pi/2}\theta^6\cos^2\theta\,d\theta - \frac{15}{8}\int_{-\pi/2}^{\pi/2}\theta^4\cos^4\theta\,d\theta$$

$$= \frac{3}{4}\left[\frac{\pi^7}{896} - \frac{3\pi^5}{64} + \frac{15\pi^3}{16} - \frac{45\pi}{8}\right] - \frac{3}{4}\times\frac{15}{8}\left[\frac{\pi^5}{160} - \frac{5\pi^3}{32} + \frac{63\pi}{64}\right]$$

$$= \frac{3}{4}\left[\frac{\pi^7}{896} - \frac{15\pi^5}{256} + \frac{315\pi^3}{256} - \frac{3825\pi}{512}\right] \qquad (22)$$

Combining (19), (15a), (20), (21), (22) we get

$$\eta_{13} = \frac{9}{2\pi^2}s_0^2 - \left(\frac{9}{\pi^2} - \frac{135}{2\pi^4}\right)s_0^4 + \left(\frac{9}{\pi^2} - \frac{225}{2\pi^4} + \frac{2835}{4\pi^6}\right)s_0^6 \qquad (23)$$

$$= \underset{0.456}{\cancel{2.736}}\,s_0^2 - 0.219\,s_0^4 + 0.494\,s_0^6 \qquad (23a)$$

Thus

$$\mathcal{E}_1(z) = \sqrt{\frac{2}{z_0}}\left\{\sin\frac{\pi z}{z_0} - \frac{1}{8}\left(0.456 - 0.219\,s_0^2 + 0.494\,s_0^4\right)s_0^2\sin\frac{3\pi z}{z_0} - \dots\right\}$$

$$= \sqrt{\frac{2}{z_0}}\left\{\sin\frac{\pi z}{z_0} - \left(0.0570 - 0.0274\,s_0^2 + 0.0618\,s_0^4\right)s_0^2\sin\frac{3\pi z}{z_0} - \dots\right\} \qquad (24)$$

The value of the coefficient of $\sin(3\pi z/z_0)$ is always less than 0.0570 - 0.0274 + 0.0618 = 0.0914. Thus even the unperturbed eigenfunction is accurate to 10%.

Now suppose the boundary has a finite Pedersen conductivity σ. Ignore the coupling into the other mode due to Hall conductivity.

At the boundary

$$b_\varphi = \mu_0\Sigma E_\nu\sin\alpha \qquad (25)$$

where $\sin\alpha$ is the dip angle, Σ the height integrated Pedersen conductivity and α the dip angle. Thus

$$b_\varphi - (h_\varphi/h_\nu)\,\mu_0\Sigma\sin\alpha\;\mathcal{E}_\nu = 0$$

or, since $\quad b_\varphi = -(i/\omega)\,\nu^2\,\partial\mathcal{E}_\nu/\partial z \qquad (26)$

$$\mathcal{E}_\nu + \frac{i}{\omega}\frac{\nu^2 h_\nu}{h_\varphi}\frac{1}{\mu_0\Sigma\sin\alpha}\frac{\partial\mathcal{E}_\nu}{\partial z} = 0 \qquad (27)$$

Now

$$\frac{h_\nu}{h_\varphi} = -\frac{1}{\nu}\frac{1}{\sqrt{1+3\sin^2\theta}} = -\frac{1}{\nu}\frac{1}{\sqrt{1+3(z-\frac{1}{2}z_0)^2}}$$

$$\tan\alpha = 2\tan\theta$$

$$\therefore\quad \sin\alpha = \frac{2\sin\theta}{\sqrt{1+3\sin^2\theta}}$$

$$\therefore\quad \frac{h_\nu}{h_\varphi}\frac{1}{\sin\alpha} = -\frac{1}{2\nu\sin\theta}$$

Thus

$$\mathcal{E}_\nu - \frac{i\nu}{2\omega\mu_0\Sigma\sin\theta}\frac{\partial\mathcal{E}_\nu}{\partial z} = 0$$

or

$$\mathcal{E}_\nu - \frac{i\nu}{2\omega\mu_0\Sigma(z-\frac{1}{2}z_0)}\frac{\partial\mathcal{E}_\nu}{\partial z} = 0$$

At z = 0

$$\mathcal{E}_\nu(0) + \frac{i\nu}{\omega\mu_0\Sigma z_0}\left(\frac{\partial\mathcal{E}_\nu}{\partial z}\right)_0 = 0$$

At $z = z_0$

$$\mathcal{E}_\nu(z_0) - \frac{i\nu}{\omega\mu_0\Sigma z_0}\left(\frac{\partial\mathcal{E}_\nu}{\partial z}\right)_{z_0} = 0$$

Let

$$\xi = \frac{\nu}{\omega\mu_0\Sigma z_0}$$

Then

$$\mathcal{E}_\nu(0) + i\xi\left(\frac{\partial\mathcal{E}_\nu}{\partial z}\right)_0 = 0$$

$$\mathcal{E}_\nu(z_0) - i\xi\left(\frac{\partial\mathcal{E}_\nu}{\partial z}\right)_{z_0} = 0$$

are the boundary conditions, and ξ is small if Σ is sufficiently large.

Suppose now that the first mode solution for $\xi = 0$ is

$$\mathcal{E}_\nu^{(0)} = \sin k^{(0)}z$$

This satisfies diff eqn with appropriate q

$$q = \pi/z_0$$

Look for a solution with perturbed q, $\mathcal{E}_\nu$

$$\mathcal{E}_\nu = \sin[q^{(0)}(1+\zeta)z] + \eta\cos q^{(0)}z$$

where ζ and η are small and terms of second order are neglected. Thus

$$\mathcal{E}_\nu = \sin q^{0}(z) + (\eta + q^{0}\zeta z)\cos q^{(0)}(z)$$

$$\frac{\partial\mathcal{E}_\nu}{\partial z} = q^{(0)}\cos q^{(0)}z + O(\eta,\zeta)$$

Substituting in the boundary conditions

$$z=0: \quad \eta + i\xi q^{(0)} = 0$$

$$\text{i.e.} \quad \eta = -iq^{(0)}\xi$$

$$z=z_0: \quad -(\eta + q^{(0)}\zeta z_0) + i\xi q^{(0)} = 0$$

$$q^{(0)}\zeta z_0 = 2iq^{(0)}\xi$$

$$\therefore \quad \zeta = 2i\xi/z_0$$

Thus

$$q = q^{(0)}(1+\zeta) = \frac{\pi}{z_0}\left(1 + \frac{2i\xi}{z_0}\right)$$

or

$$q = (\pi/z_0)\left\{1 + \frac{2i}{z_0}\frac{\nu}{\omega\mu_0\Sigma z_0}\right\}$$

$$\therefore \quad q_1 = \frac{2i\pi\nu}{\omega\mu_0\Sigma z_0^3}$$

$$\mathcal{E}_\nu = \sin\frac{\pi z}{z_0} + \left(-\frac{i\pi\xi}{z_0} + \frac{\pi z}{z_0}\frac{2i\xi}{z_0}\right)\cos\frac{\pi z}{z_0}$$

$$= \sin\frac{\pi z}{z_0} + \frac{i\pi\xi}{z_0}\left(\frac{2z}{z_0} - 1\right)\cos\frac{\pi z}{z_0}$$

$$= \sin\frac{\pi z}{z_0} + \frac{i\pi\nu}{\omega\mu_0\Sigma z_0^2}\left(\frac{2z}{z_0} - 1\right)\cos\frac{\pi z}{z_0}$$

Now

$$q^2 = \omega^2\frac{aL\mu_0\rho_{eq}}{B_0^2} = \frac{\omega^2\mu_0\rho_{eq}}{\nu^2 B_0^2}$$

Thus

$$\frac{\pi\nu}{\omega\mu_0\Sigma} = \frac{\pi\nu}{\mu_0\Sigma q B_0}\sqrt{\frac{\mu_0\rho_{eq}}{\nu^2}}$$

$$= \frac{\rho_{eq}^{1/2} z_0}{\mu_0^{1/2}\Sigma B_0}$$

$$\therefore \quad q_1 = \frac{2i\rho_{eq}^{1/2}}{\mu_0^{1/2}\Sigma B_0 z_0^2} \qquad (28)$$

$$\mathcal{E}_\nu = \sin\frac{\pi z}{z_0} + \frac{i\rho_{eq}^{1/2}}{\mu_0^{1/2}\Sigma B_0 z_0}\left(\frac{2z}{z_0} - 1\right)\cos\frac{\pi z}{z_0} \qquad (29)$$

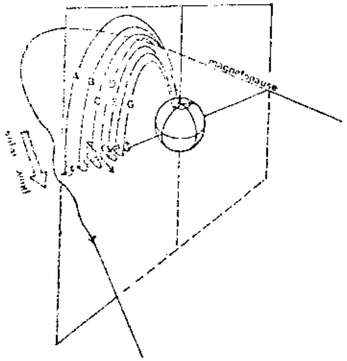
These can be combined with the expressions for infinite conductivity.

1.9 Resonance theory

So far we have assumed that, if $\vec{k}_\perp$ is small enough, we can completely decouple a field line oscillation. We describe now a problem in which there is coupling between adjacent field lines.

Work done using magnetometer chains has shown that the polarization of a Pc 5 geomagnetic pulsation is a function of local time and of station latitude. This led Southwood (1974) and Chen and Hasegawa (1974) independently to propose a theory of hydromagnetic resonances to explain the magnetometer data. This theory (hereafter SCH theory after the initials of its originators) was successful in explaining the magnetometer results and made further predictions which could not be verified by magnetometers because of an inherent lack of spatial resolution due to the effect of the earth-ionosphere cavity on the signal.

The basis of SCH theory is as follows: It can easily be shown that, in a cylindrically symmetrical geometry, magnetic shells can oscillate toroidally with cylindrical symmetry independently of adjacent shells. The plasma motion is thus everywhere in a direction perpendicular to the magnetic meridian plane. The frequency of this toroidal oscillation is determined by the plasma density and by the length of a field line. In an ideal case the ionosphere is a node of the electric field but if the ionosphere has a finite conductivity the oscillation is damped. The natural frequency of oscillation is a monotonic decreasing function of L except in regions where the plasma density changes rapidly, such as the plasmapause. It is assumed, in SCH theory, that, as illustrated in the figure, the solar wind sets up monochromatic waves on the magnetopause through the Kelvin-Helmholtz instability. These waves drive an oscillatory motion in the magnetic meridian, decaying with distance from the magnetopause, but penetrating deep into the magnetosphere. Because of the dipole geometry this motion is coupled to the toroidal motion. If the natural frequency of toroidal oscillation matches that of the Kelvin-Helmholtz wave, resonance takes place. The resulting motion of the plasma in the equatorial plane is shown in the figure and maps down to the ionosphere.



Schematic diagram of SCH mechanism. The solar wind causes a surface wave on the magnetopause. The field lines A, B, C, D, E, F, G move as shown. The toroidal frequency of D matches the wave frequency leading to a large toroidal component. The polarization changes across the resonance.

Let us return to equations (1.5.1) - (1.5.5). Assume a dipolar geometry with phase varying with φ as exp $(-im\varphi)$. Use the substitutions (1.6.2), and (1.6.9) and eliminate b_ν, b_μ, b_φ getting

$$\frac{h_\nu}{h_\varphi h_\mu}\frac{\partial}{\partial\mu}\left(\frac{h_\varphi}{h_\mu h_\nu}\frac{\partial\mathcal{E}_\nu}{\partial\mu}\right) + (\omega^2/V_A^2)\,\mathcal{E}_\nu = -\frac{imh_\nu}{h_\varphi h_\mu}\left\{im\,\mathcal{E}_\nu + \frac{\partial\mathcal{E}_\varphi}{\partial\nu}\right\} \qquad (1)$$

$$\frac{h_\varphi}{h_\mu h_\nu}\frac{\partial}{\partial\mu}\left(\frac{h_\nu}{h_\mu h_\varphi}\frac{\partial\mathcal{E}_\varphi}{\partial\mu}\right) + (\omega^2/V_A^2)\,\mathcal{E}_\varphi = -\frac{h_\varphi}{h_\mu h_\nu}\frac{\partial}{\partial\nu}\left\{im\,\mathcal{E}_\nu + \frac{\partial\mathcal{E}_\varphi}{\partial\nu}\right\} \qquad (2)$$

We assume that we are near a resonance in the toroidal mode, occurring over a narrow latitude range and as a first approximation we assume that the equation (1) decouples. The left-hand side is then identical with the guided toroidal equation (1.6.1), and has an eigenvalue given by (1.8.16) and (1.8.28), and eigenfunction given by (1.8.29). To the next order of approximation we assume that the right-hand side of (1) is not exactly zero, but small. The actual eigenvalue will then differ from that obtained by a small amount so that equation (1) becomes

$$(k^2 - k_0^2)\,\mathcal{E}_\nu = -\frac{imh_\nu}{h_\varphi h_\mu}\left\{im\,\mathcal{E}_\nu + \frac{\partial\mathcal{E}_\varphi}{\partial\nu}\right\} \qquad (3)$$

where k^2 is related to q² and depends on ν. Equation (2) has the terms on the left-hand side negligible compared with the right-hand side so that

$$\frac{\partial}{\partial\nu}\left\{im\,\mathcal{E}_\nu + \frac{\partial\mathcal{E}_\varphi}{\partial\nu}\right\} = 0 \qquad (4)$$

Now if h_ν, h_φ, h_μ are all slowly varying functions of ν compared with $\mathcal{E}_\nu, \mathcal{E}_\varphi, k^2$ we substitute from (3) into (4) to get

$$\frac{\partial}{\partial\nu}\left\{(k^2 - k_0^2)\,\mathcal{E}_\nu\right\} = 0 \qquad (5)$$

Thus

$$\mathcal{E}_\nu = A(z)/K^2(\nu) \qquad (6)$$

where $K^2 = k^2 - k_0^2$ and A(z) is the eigenfunction (1.8.29). There is a small error due to the neglect of the dependence of the scale factors on ν. Now, to this order of approximation

$$\frac{\partial\mathcal{E}_\varphi}{\partial\nu} \simeq -im\,\mathcal{E}_\nu$$

$$= -im\,A(z)/K$$

$$\therefore\quad \mathcal{E}_\varphi = -im\,A(z)\left\{\int^\nu \frac{d\nu}{K^2} + B(z)\right\}$$

If the equations are linearized so that

$$K^2 \simeq a(\nu - \nu_0)$$

where ν_0 is complex because of losses in the ionosphere,

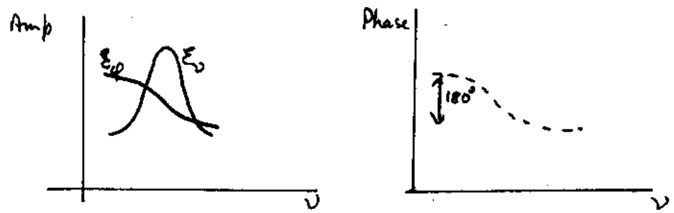
$$\mathcal{E}_\varphi \propto \ln(\nu - \nu_0)$$

and

$$\mathcal{E}_\nu \propto \frac{1}{\nu - \nu_0}$$

if B is zero.

As ν varies through $\mathcal{R}(\nu_0)$ this leads to a resonance behaviour in $\mathcal{E}_\nu$ as illustrated where suitable assumptions have been made about A and B.



2. The earth-ionosphere waveguide

2.1 Natural waveguides

Waveguide mode theory can be applied to radio wave propagation along elongated structures whose transverse dimensions are comparable with the wavelength and whose properties do not change substantially in the direction of the elongation for many wavelengths. Familiar examples in the laboratory are:

(i) Microwave waveguides.
(ii) Coaxial cables.
(iii) Transmission lines.
(iv) Optical fibres*.

Examples of natural waveguides are:

(i) The earth ionosphere cavity. (Wait,1976 , Budden,1961b)
Between the conducting ground and the ionosphere v.l.f. ($f = 2 - 100$ kHz; $\lambda = 150 - 3$ km) radio waves can be trapped and propagated to large distances.
(ii) Tropospheric ducts
Very small vertical gradients of density in the troposphere can provide refractive index gradients which guide v.h.f. waves ($\sim$100 MHz) over the horizon.
(iii) Whistler ducts
Field aligned ducts of ionization in the plasmasphere.
(iv) Guiding of waves by electron density gradients in the ionosphere or at the plasmapause.

In this section we shall consider the theory of guiding of radio waves in such natural waveguides under idealized conditions. Before embarking on the study of guiding we summarize some properties of waves propagated in a uniform medium.

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* The transverse dimensions of an optical fibre are large compared with a wavelength but they are included here as an example of a dielectric guide, which has analogies in geophysics.

2.2 Maxwell's equations and constitutive relations

In all that follows we assumetime variation $e^{i\omega t}$ so that $\frac{\partial}{\partial t} \equiv i\,\omega$.

We consider a non-magnetic dielectric medium with dielectric tensor $\underline{K}$. Maxwell's equations, in the absence of current and charge densities may be written

$$\nabla \times \vec{E} = -\,i\,k\,\mathcal{B} \tag{1}$$

$$\nabla \times \vec{\mathcal{B}} = i\,k\,\underline{K}\,E \tag{2}$$

S.I. units are used but, following Budden (1968), we make use of a magnetic field variable

$$\vec{\mathcal{B}} = c\,\vec{B} \tag{3}$$

which has the same physical dimensions as $\vec{E}$, thus retaining some advantages of Gaussian units. We have defined

$$k = \omega/c. \tag{4}$$

The dielectric tensor depends on the medium. In free space it is simply

$$\underline{K} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{5}$$

so that equation (1.2.2) becomes

$$\nabla \times \vec{\mathcal{B}} = ik\,\vec{E} \tag{6}$$

Similarly in an isotropic dielectric, characterized by dielectric constant K

$$\nabla \times \vec{\mathcal{B}} = i\,k\,K\,\vec{E} \tag{7}$$

For a magneto-ionic medium

$$\underline{K} = \underline{I} + \underline{M} \tag{8}$$

where $\underline{I}$ is the unit tensor and $\underline{M}$ the susceptibility tensor given by Budden (1961a, eq. 3.24).

2.3 Plane waves in a uniform isotropic dielectric

We assume a harmonic wave propagating in a direction defined by a refractive index vector $\vec{\mu}$, so that spatial variation is of the form

$$\exp\{-ik\,\vec{\mu}.\vec{r}\}. \tag{1}$$

Where it is necessary to consider a particular coordinate system we shall denote the components of $\vec{\mu}$ by

$$\vec{\mu} = \hat{x}\,S_1 + \hat{y}\,S_2 + \hat{z}q. \tag{2}$$

From (1.3.1) the operator ∇ may be written

$$\nabla \equiv -\,i\,k\,\vec{\mu} \tag{3}$$

Thus equations (2.2.1) and (2.2.6) become

$$\vec{\mu} \times \vec{E} = \vec{\mathcal{B}} \tag{4}$$

$$\vec{\mu} \times \vec{\mathcal{B}} = -\,K\,\vec{E} \tag{5}$$

We can find the value of μ in terms of K by first taking the scalar product of $\vec{\mu}$ with (1.3.5) getting

$$\vec{\mu}.\vec{E} = 0 \tag{6}$$

This implies that $\vec{E}$ is perpendicular to $\vec{\mu}$ while (4) shows that $\vec{\mathcal{B}}$ is perpendicular to $\vec{\mu}$ and $\vec{E}$ and $\vec{E}, \vec{\mathcal{B}}, \vec{\mu}$ form a right handed triad. Now take the vector product of $\vec{\mu}$ and (1.3.4) getting

$$\vec{\mu} \times \vec{\mathcal{B}} = -\,\mu^2\,\vec{E}\,. \tag{7}$$

where we have used the vector identity $\vec{\mu} \times (\vec{\mu} \times \vec{E}) = (\vec{\mu}.\vec{E})\,\vec{\mu} - \mu^2\,\vec{E}$ and equation (6). For consistency of (5) and (7)

$$\mu^2 = \sqrt{\varepsilon}\,. \tag{8}$$

In an isotropic plasma $\varepsilon = 1 - X$ (Budden, 1962) where

$$X = \omega_p^{\,2}/\omega^2. \tag{9}$$

Very often the boundary conditions determine the spatial variation on a plane. In such cases S_1 and S_2 can be regarded as given. Equations (2) and (8) require

$$q^2 + S_1^{\,2} + S_2^{\,2} = \varepsilon = 1 - X$$

or

$$q^2 = 1 - (S_1^{\,2} + S_2^{\,2}) - X \tag{10}$$

Thus if the x and y spatial variation is given the z variation is determined.

2.4 Plane waves in a uniform anisotropic dielectric

When K is a tensor we can carry out an analogous procedure. The details will not be given but can be found in Budden's (1961a) book. For a magneto-ionic medium we find that μ depends on X and Y where

$$Y = \omega_H/\omega\,. \tag{1}$$

When S_1 and S_2 are given we can find q from the *Booker quartic equation* (Budden, 1962, eq.13.13).

2.5 Angular spectrum of plane waves

Consider a field component represented on the plane $z = 0$ by a suitable function f(x,y). Then f(x,y) can be Fourier analysed in space so that

$$f(x,y) = \frac{k^2}{4\pi^2} \int\!\!\int_{-\infty}^{\infty} F(S_1,S_2)\; e^{-ik(S_1x + S_2y)}\, dS_1\, dS_2 \tag{1}$$

where F is the double Fourier transform of f. Equation (1) represents a superposition of quantities which vary harmonically with x and y on the $z = 0$ plane. We have seen that the z variation of each of these quantities is determined through (2.3.10) or its equivalent in an anisotropic medium. Thus the field component varies throughout all space as

$$f(x,y,z) = \frac{k^2}{4\pi^2} \int\!\!\int_{-\infty}^{\infty} F(S_1,S_2)\, e^{-ik\{S_1x + S_2y + q(S_1,S_2)z\}}\, dS_1\, dS_2 \qquad (2)$$

This is a superposition of plane waves propagating in all the possible directions defined by the values of S_1 and S_2. It is called an angular spectrum of plane waves (Clemmow, 1966). Because q is not single valued and there are branch points on the S_1,S_2 axes the contour of integration must be appropriately chosen to satisfy causality and the boundary conditions at infinity.

2.6 Inhomogeneous and evanescent waves

In equation (2.5.2) S_1 and S_2 take all values between $-\infty$ and ∞. Thus it is quite possible that q^2, given by (2.3.10) could be negative if S_1^2 or S_2^2 is sufficiently large. Thus some of the plane waves in the spectrum (1.5.2) may have imaginary values of q. This will happen unless F is zero in the range where this occurs. Let us take, as an example, a wave in free space for which $S_1 = S > 1$, $S_2 = 0$. Then

$$q = \pm\, i(S^2 - 1)^{\frac{1}{2}}$$
$$= \pm\, ip \ ,$$

say, where p is real. The wave then behaves in space as

$$\exp\{-ik\, S_x \mp k\, p\, z\}, \qquad (1)$$

Because of the sign ambiguity this wave cannot exist through all space as the field components will become infinite as $z \to \pm\infty$. Suppose it is defined in the half space $z > 0$. Then, to satisfy boundary conditions as $z \to \infty$, we must choose the upper sign. The resulting wave has phase depending on x and amplitude depending on z; planes of constant amplitude are parallel to the x-y plane and planes of constant phase are parallel to the y-z plane. Such a wave is called an *inhomogeneous* wave. Sometimes the medium is such that q can be imaginary when S is zero. Such a wave varies in amplitude in the z direction but has no phase variation. It is called an evanescent wave.

2.7 Reflection and Transmission coeffcients for sharp boundaries in and isotropic medium

In considering reflection and transmission at a plane boundary we can, without loss of generality choose the orientation of the boundary so that the normal to the boundary is parallel to $\hat{z}$, and the direction of the wavenormal to be in the x-z plane (the *plane of incidence*). Then equations (2.3.4) and (2.3.5), with $K = \mu^2$, may be written

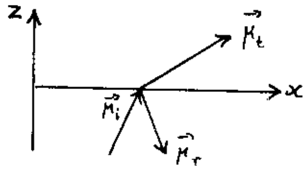
$$\begin{array}{r|r} \mp q\, E_y = \mathcal{B}_x & \mp q\, \mathcal{B}_y = -\mu^2\, E_x \\ S\, E_y = \mathcal{B}_z & S\, \mathcal{B}_y = -\mu^2\, E_z \\ \pm q\, \mathcal{B}_x - S\, \mathcal{B}_z = \mu^2\, E_y & \pm q\, E_x - S\, E_z = \mathcal{B}_y \end{array} \qquad (1)$$

i.e. there are two independent polarizations, one involving $\mathcal{B}_x$, $\mathcal{B}_z$, E_y and the other involving E_x, E_z, $\mathcal{B}_y$. The first is called *transverse magnetic* or TM polarization because $\vec{\mathcal{B}}$ is perpendicular to the plane of incidence and the second is called *transverse electric* or TE polarization. If equation (2.3.10) holds, the equations are self-consistent, and the last equation of each set is redundant.

Let us now consider reflection from a sharp boundary located at $z = 0$ for various media. In each case the upper sign of q corresponds to a wave propagating upwards and the lower sign to a wave propagating downwards.

In considering reflection and transmission at a sharp boundary we note that, in order to match the boundary conditions across the boundary, the incident, reflected, and transmitted waves must each behave in the same way with respect to variation with x and y at the boundary. This implies that S_1 and S_2 (or S in the case of $S_2 = 0$) must be the same for each. This is an expression of *Snell's Law*.

(i) Perfect conductor



The boundary condition is $E_x, E_y = 0$

TE polarization

$$E_y^i + E_y^r = 0$$

Define

$$R_{TE} = E_y^r / E_y^i$$
$$T_{TE} = E_y^t / E_y^i$$

Here

$$R_{TE} = -1$$
$$T_{TE} = 0 \qquad (2)$$

TM polarization

$$E_x^i + E_x^r = 0$$

i.e. $\quad - q \mathcal{B}_y^i + q \mathcal{B}_y^r = 0$

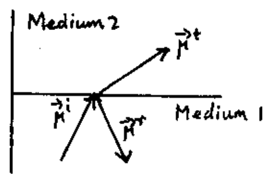
Define

$$R_{TM} = \mathcal{B}_y^r / \mathcal{B}_y^i$$
$$T_{TM} = \mathcal{B}_y^t / \mathcal{B}_y^i$$

Then

$$R_{TM} = 1$$
$$T_{TM} = 0 \qquad (3)$$

(ii) Two dielectrics



The boundary condition is that the transverse components of $\vec{E}$ and $\vec{\mathcal{B}}$ are continuous.

/..........

TE polarization

$$E_y^i + E_y^r = E_y^t$$
$$- q_1 E_t^i + q_1 E_y^r = - q_2 E_y^t$$

i.e. $\quad 1 + R = T$

$$- q_1 (1 - R) = - q_2 T$$

so that

$$R_{TE} = \frac{q_1 - q_2}{q_1 + q_2}$$
$$T_{TE} = \frac{2q_1}{q_1 + q_2} \qquad (4)$$

These are the *Fresnel formulae* for TE polarization. The are usually written in terms of μ and θ, the angle of incidence (see Budden 1961a, eq. 8.23, 8.24).

TM polarization

It can easily be shown that

$$R_{TM} = \frac{(q_1/\mu_1^2) - q_2/\mu_2^2)}{(q_1/\mu_1^2 + (q_2/\mu_2^2)}$$
$$T_{TM} = \frac{2q_1/\mu_1^2}{(q_1/\mu_1^2) + (q_2/\mu_2^2)}$$

2.8 Reflection coefficients as a function of z

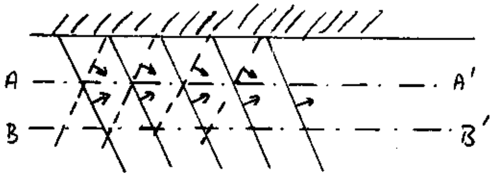
In the previous section we have defined R at the boundary $z = 0$. We could equally well have defined the ratio of two field quantities for an upgoing and downcoming wave at any other level. Consider the example the TE wave at some value. If the incident wave is E_y^i at $z = 0$ then at $z = -h$ it is $E_y^i \exp(- i k q z)$. Similarly the reflected wave is $E_y^r \exp(+ i k q z)$. Thus at z

$$R(z) = E_y^r e^{ikqz} / E^i e^{-ikqz}$$
$$= R(0) e^{2ikqz} \qquad (1)$$

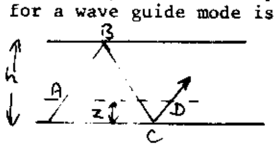
This is the rule for finding field ratios at various levels z.

2.9 Waveguide modes - Heuristic ideas

As an example of how waveguide modes arise consider the case of a perfectly conducting sheet. A wave is reflected from it. The diagram illustrates the wavefronts of incident and reflected waves. The phase difference between illustrated wavefronts is 2π.



Now along the lines AA', BB', the resultant field components have the same relationship as at the conducting surface. A sheet of conductor could be inserted to consider with AA', BB' or other similar surfaces without changing the field components between the plates. The boundary conditions at AA' or BB' are automatically satisfied. The resultant disturbance can be regarded as propagating between the plates. Clearly the separation of the plates depends on the angle of incidence of the plane wave on the boundaries. Conversely, if we have plates separated by a distance h, the condition for a wave guide mode is that as the wave is reflected backwards and forwards across the guide, as illustrated by the ray ABCD, the phase change in moving from A to D must be an integral multiple of 2π. This will in general be achieved for a discrete set of wave-normal angles, corresponding to the waveguide modes.



Another way of finding the waveguide modes is to note that a field component in the wave has undergone reflection at B and C. If R is the reflection coefficient at B and $\tilde{R}$ the reflection coefficient at C, then the field component at D is given by

$$\begin{aligned} \mathcal{F}_D &= \mathcal{F}_D^r e^{-ikqz} \\ &= \tilde{R}\, \mathcal{F}_D^i e^{-ikqz} \\ &= \tilde{R}\, \mathcal{F}_B^r e^{-ikqh} e^{-ikqz} \\ &= R\tilde{R}\, \mathcal{F}_B^i e^{-ikqh} e^{-ikqz} \\ &= R\tilde{R}\, \mathcal{F}_A e^{-ikq(h-z)} e^{-ikqh} e^{-ikqz} \end{aligned}$$

$$\therefore \quad \mathcal{F}_D = R\tilde{R} e^{-2ikqh} \mathcal{F}_A$$

Since we require $\mathcal{F}_D = \mathcal{F}_A$ this implies

$$R\tilde{R} e^{-2ikqh} = 1$$

e.g. Two perfectly conducting plates. TE polarization:

$$R = -1 = \tilde{R}$$

$$\therefore \quad e^{-2ikqh} = 1$$

$$\therefore \quad -2kqh = 2n\pi$$

$$q = 1 - S^2 = n\pi/kh = n\lambda_0/2h$$

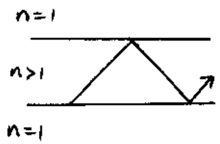
The x-dependence of the wave is then of the form

$$\exp\left\{-ik\sqrt{1 - n\pi/kh}\; x\right\}$$

Exercise Find the values of S corresponding to modes for a TM mode between conducting plates and for TE modes in a dielectric slab.

2.10 Locked and leaky modes

Consider a slab of dielectric with $n > 1$. Such a medium can sustain guided modes. If we imagine crossing plane waves with angle of incidence greater than the critical angle, it is possible to have a trapped mode.



Let $\quad C^2 = 1 - S^2 \qquad (1)$

$\qquad q^2 = n^2 - S^2 \qquad (2)$

Then the reflection coefficients for TE propagation (for example) are

$$R = (q - C)/(q + C)$$

$$T = 2q/(q + C)$$

The critical angle occurs for $S = 1$ and total reflection for $S > 1$. The mode condition can again be written

$$R\tilde{R} e^{-2ikqh} = 1$$

i.e.

$$\frac{(q-C)^2}{(q+C)^2} e^{-2ikqh} = 1 \qquad (3)$$

This is an equation determining S. There is no straightforward solution for this equation. It is best solved numerically. In general there will be a number of roots. We shall not do a detailed computation but make some general points.

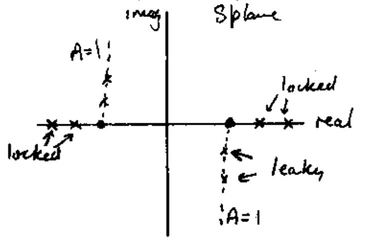
Suppose S is real and greater than 1. Then C is pure imaginary and q is real. In this case, when we can write

$$\left\{\frac{q-C}{q+C}\right\}^2 = A e^{i\varphi} \qquad (4)$$

A is equal to unity. Thus along the real S axis, where $|S| > 1$, the mode condition is

$$\varphi - 2kqh = 2n\pi \qquad (5)$$

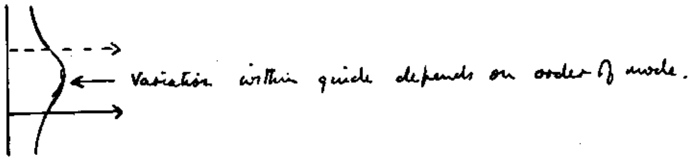
We thus find a number of modes in the positions shown. These are locked modes, analogous to those for conducting plates.



When S is less than unity and real then A < 1 and we cannot find a solution for (3). There are, however, solutions which have S complex. These will lie on the contour A = 1 in the S-plane at values of the phase given by (5). They correspond to leaky modes and we briefly indicate the physical characteristics of locked and leaky modes.

Locked modes:

If we examine the field components they all vary as exp (-ikSx) where S is real. In the medium above the slab, they vary as exp $\{-k|(1-S^2)^{\frac{1}{2}}|z\}$ where the -sign is chosen to satisfy boundary conditions at Z = +∞. Similarly, below the slab they vary as $\exp\{+k|(1-S^2)^{1/2}|z\}$. A sketch of E is shown.



Leaky modes:

If S is complex we can write $S = S_r - iS_i$

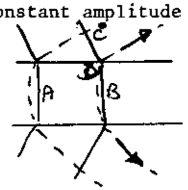
$$C = \pm\{1-(S_r^2-S_i^2) - 2i\,S_i S_r\}^{1/2}$$

$$= \pm\{C_r + i\,C_i\}, \text{ say}.$$

Variation in the x direction has behaviour

$$\exp(-kS_i x)\,\exp(-ikS_r x)$$

Thus the amplitude decays with increasing x. From boundary conditions at Z = +∞ we might expect that above the slab it is necessary to choose the negative sign for C so that the wave decays exponentially with distance from the boundary. If this is done, consideration of the energy flux shows that energy propagates from infinity towards the guide which is unreasonable. The opposite choice satisfies energy flux requirement. We sketch planes of constant amplitude (dotted) and phase (full lines). In the guide the amplitude at A is larger than that at B. It is possible to show that the Poynting vector is aligned with the planes of constant amplitude. The amplitude at C is then determined by the amplitude at A and that at D by the amplitude at B. This explains why the amplitude must grow with distance from the guide. Of course at great distances from the guide interference with other

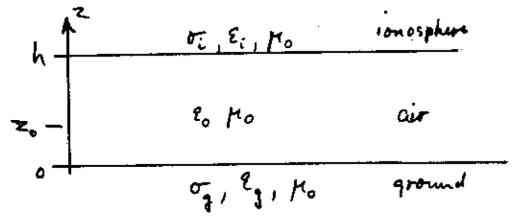


modes must ultimately reduce the signal to zero at z = ∞

3. Excitation of waveguide modes

3.1 Introduction

This section gives an example of the techniques used to study the excitation of waveguide modes. We shall consider the particular examples illustrated below and outline the treatment necessary. It is beyond the scope of these lectures to treat numerous realistic cases. Our treatment follows that of Wait (1970).



The case to be treated is a simple model of the earth-ionosphere cavity as illustrated.

3.2 Use of the Hertz vector

We shall work in terms of the Hertz vector $\vec{\Pi}$. (Stratton 1941 §1.11). In free space this satisfies the wave equation

$$\nabla^2\vec{\Pi} + (\omega^2/c^2)\,\vec{\Pi} = -\vec{p}/\varepsilon_0 \qquad (1)$$

where $\vec{p}$ is a dipole moment density and the fields are derived from $\vec{\Pi}$ through

$$\vec{E} = \mu_0\omega^2\vec{\Pi} + \varepsilon_0^{-1}\,\nabla(\nabla.\vec{\Pi}) \qquad (2)$$

$$\vec{B} = i\omega\,(\mu_0/\varepsilon_0)^{1/2}\,\nabla\times\vec{\Pi} \qquad (3)$$

Suppose we have a vertical point dipole source. We can write

$$\vec{p} = \hat{z}\,M\,\delta^3(\vec{r}) \qquad (4)$$

The z component of equation (1) is

$$\nabla^2\Pi_z - (i\omega/c^2)\,\Pi_z = -M\,\delta^3(r)/\varepsilon_0 \qquad (5)$$

and the horizontal component of $\vec{\Pi}$ is zero. In free space the solution of this is well known to be

$$\Pi_z = \frac{M}{4\pi r}\exp(-ikr) \qquad (6)$$

Thus if the dipole is located at z = z_o, a general solution of (1) will in cylindrical coordinates have the form

$$\Pi_z = \frac{M}{4\pi r}\exp(-ikr) + f(r, z, \rho) \qquad (7)$$

$$r^2 = \rho^2 + (z-z_0)^2$$

where f is a solution of the honogeneous equation - equation (5) with the r.h.s.

set equal to zero – chosen to make Π_z fit the boundary conditions.

3.3 Angular spectrum representation

Consider a region of free space.

In the y-z plane the dipole moment per unit area is

$$P(y,z) = M\,\delta(y)\,\delta(z-z_0)$$

If this is Fourier transformed we get

$$P(y,z) = \frac{k^2M}{4\pi^2}\iint_{-\infty}^{\infty} \exp\{-ik\,S_2y + C(z-z_0)\}\,dS_2\,dC \qquad (1)$$

The Hertz vector in the space x > 0 is

$$\Pi_z = \iint_{-\infty}^{\infty} A\exp\{-ik\,S_1x + S_2y + C(z-z_0)\}\,dS_2\,dC \qquad (2)$$

where

$$S_1^2 = 1 - S_2^2 - C^2$$

and S_1 is either positive or negative imaginary. Now the surface current density in x = 0 is $i\omega P$ and the y-component of B is $\tfrac12\mu_0 I$, i.e., near x = 0,

$$\bar{\mathcal{B}}_y = c\mathcal{B}_y = \tfrac12\sqrt{\mu_0/\epsilon_0}\,(i\omega k^2/4\pi^2)\iint_{-\infty}^{\infty}\exp\{-ik[S_2y + C(z-z_0)]\}\,dS_2\,dC \qquad (3)$$

But from 3.2.3

$$\mathcal{B}_y(x=0) = -i\omega\sqrt{\mu_0/\epsilon_0}\;\partial\Pi_z/\partial x$$

$$= -\omega k\,S_1\sqrt{\mu_0/\epsilon_0}\;A\iint_{-\infty}^{\infty}\exp\{-ik[S_2y+C(z-z_0)]\}\,dS_2\,dC \qquad (4)$$

Comparing (3) and (4)

$$A = -\tfrac12 ik/4\pi^2S_1 \qquad (5)$$

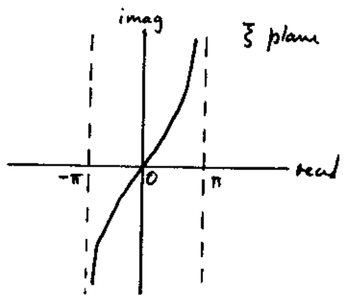
Thus

$$\frac{4\pi}{M}\Pi_z = \frac{e^{-ikr}}{r} = -\frac{ik}{2\pi}\iint_{-\infty}^{\infty} \frac{1}{S_1}\exp\{-ik[S_1x+S_2y+C(z-z_0)]\}\,dS_2\,dC \qquad (6)$$

This can also be written in terms of the zero order Hankel function $H_0^{(2)}$ defined by

$$H_0^{(2)}(\zeta) = \frac{1}{\pi}\int_c \exp(-i\zeta\cos\xi)\,d\xi$$

where the contour is as shown, i.e. it runs from $-\pi-i\infty$ to $\pi+i\infty$.



Consider $\quad H_0^{(2)}(kS\rho) = \frac{1}{\pi}\int_c \exp(-ikS\rho\cos\xi)\,d\xi$

Let $\quad x=\rho\cos\theta,\ y=\rho\sin\theta,\ \xi = u-\theta \quad$ Then

$$H_0^{(2)}(kS\rho) = \frac{1}{\pi}\int_{c'}\exp\{-ikS(x\cos u + y\sin u)\}\,du$$

Let $\quad S\cos u = S_1,\ S\sin u = S_2 \quad$ i.e. $\tan u = S_2/\sqrt{1-C^2-S_2^2}$

$$du = dS_2/S_1$$

$$\therefore H_0^{(2)}(kS\rho) = \frac{1}{\pi}\int_{c''}\frac{1}{S_1}\exp[-ik(S_1x+S_2y)]\,dS_2 \qquad (7)$$

Consideration of the mapping of the contour shows it lies along the real S_2 axis in the negative direction. Comparing this with (6) we see

$$\Pi_z = \frac{ikM}{8\pi}\int_\Gamma H_0^{(2)}(kS\rho)\exp[-ikC|z-z_0|]\,dC \qquad (8)$$

where, in the S plate, Γ extends along the real axis from $-\infty$ to ∞.

3.4 Solution for plane guide

Now we write the solution in the guide in the form (3.2.7) and that in the ground and ionosphere in similar form but without the source term.

$0 \le z \le h$: $$\Pi_z = \Pi_z^{(p)} + \int_\Gamma [A(C)e^{-ikCz} + B(C)e^{ikCz}]\,H_0^2(kS\rho)\,dC \qquad (1)$$

$z < 0$: $$\Pi_z^{(g)} = \int_\Gamma G(C)\,e^{ik_gC_gz}\,H_c(kS\rho)\,dC \qquad (2)$$

$z > 0$: $$\Pi_z^{(i)} = \int_\Gamma I(C)\,e^{-ik_iC_iz}\,H_0^{(2)}(kS\rho)\,dC \qquad (3)$$

The generalization of Snell's law to a medium with conductivity requires that

$$n_g(1-C_g^2)^{1/2} = (1-C^2)^{1/2} = n_i(1-C_i^2)^{1/2} \qquad (4)$$

with

$$n_g^2 = (\sigma_g + i\epsilon_g\omega)/i\epsilon_g\omega \qquad (5)$$

$$n_i^2 = (\sigma_i + i\epsilon_i\omega)/i\epsilon_i\omega \qquad (6)$$

We can now apply boundary conditions at the ground and ionosphere to relate A,B,G,I through reflection coefficients. The result in the free space region is

$$\Pi_z = \frac{ikM}{8\pi}\int_\Gamma F(C)\,H_0^{(2)}(k\rho S)\,dC$$

where

$$F(C) = \frac{e^{-ikCh}\left(e^{ikCz}+R_g e^{-ikCz}\right)\left(e^{ikC(h-z_0)}+R_i e^{-ikC(h-z_0)}\right)}{\left(1-R_gR_i e^{-2ikCh}\right)} \qquad (7)$$

$$R_g = (n_gC - C_g)/(n_gC+C_g) \qquad (8)$$

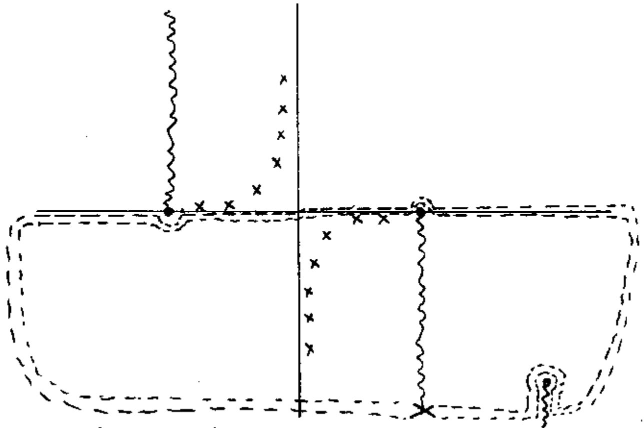
$$R_i = (n_iC - C_i)/(n_iC + C_i) \qquad (9)$$

Equation (9) is just the mode condition and the integrand has poles at each of the modes. The problem of obtaining the fields is solved if we can evaluate this integral. We shall show how this is done in principle.

First transform the integral to the S-plane.

$$\Pi_z = -\frac{ikM}{8\pi}\int_\Gamma F(C)\, H_0^{(2)}(kS\rho)\,(S/C)\, dS \qquad (10)$$

There is a branch point at $S = \pm 1$ and we make branch cuts as shown. There may be other branch points. One is shown



The contour must be closed in the negative half plane. It has to run across two Riemann sheets as shown. This means that the integral can be reduced to a portion round the branch cuts and a residue series - the sum of the modes.

If the branch points other than $S = 1$ are well below the real axis the branch cut integral is small.

The purpose of this section has been to show how, in a typical problem, we are lead to a solution in the form a series which represents a summation over modes with different amplitudes and phases. We shall not continue further with the analysis. Further details are given by Wait (1970, p.139).

4. Excitation of earth-ionosphere waveguide by an external source.

4.1 Introduction

We consider here a simple model which relates to coupling between downcoming whistlers and the earth-ionosphere wave guide. This follows the paper by Walker (1974).

A full treatment of the problem is algebraically complicated and the physical principles tend to be buried in mathematical and numerical detail. Here an analogous but simpler problem is discussed. The object is to treat a case which is analytically tractable, but not particularly realistic, in order to emphasize the physical interpretation of the mathematical results.

The problem discussed is this:

A spatually confined electromagnetic wave is propagated downwards in a semi-infinite isotropic dielectric representing the ionosphere. It is incident on the plane boundary of a region of free space. Below the free space region is a plane perfect conductor representing the ground, a distance h below the boundary. It is assumed that h is comparable with or not much greater than a wavelength. The refractive index of the dielectric is assumed to be large

The incident wave is represented by a Fourier synthesis of plane waves with different wave normal directions. Transverse electric (TE) polarization (electric field perpendicular to the plane of incidence) is chosen as this allows a simpler analysis.

The initial stages of the treatment are similar to a straight-forward treatment of interference of plane waves in a thin film. The relatively unfamiliar features arise from the spatial confinement of the incident signal.

4.2 Plane wave case

Suppose that an incident plane wave has its normal in the x-z plane. For the TE case $E_x = E_0 = 0$ and $\mathcal{B}_y = 0$. In the dielectric Maxwell's curl equations with $\partial/\partial t = i\omega$, $\partial/\partial x = -ikS$, and $\partial/\partial y = 0$ yield,

$$\frac{1}{ik}\frac{d}{dz}\begin{pmatrix}E_y\\ \mathcal{B}_x\end{pmatrix} = \begin{pmatrix}0 & 1\\ \mu^2 - S^2 & 0\end{pmatrix}\begin{pmatrix}E_y\\ \mathcal{B}_x\end{pmatrix} \qquad (1)$$

Here E_z and $\mathcal{B}_z$ have been eliminated for convenience because the field components E_y and $\mathcal{B}_x$ are continuous across any boundary in the x-y plane whereas E_z and $\mathcal{B}_z$ are not. The quantity S defines the angle of incidence θ_i such that

$$S = \mu \sin\theta_i \qquad (2)$$

Clearly S is the sine of the angle made by the wave normal with the z axis when the wave is refracted into the free space region.

Two independent solutions of equation (1) are

$$\begin{pmatrix}E_y\\ \mathcal{B}_x\end{pmatrix} = \begin{pmatrix}1\\ q\end{pmatrix} e^{ikqz} \quad \text{and} \quad \begin{pmatrix}1\\ -q\end{pmatrix} e^{-ikqz} \qquad (3)$$

where

$$q = (\mu^2 - S^2)^{1/2} \qquad (4)$$

and it is assumed that the boundary of the dielectric is at $z = 0$.

The first of the solutions (3) represents a wave propagated towards the boundary, and the second a wave reflected upwards. The solution in the dielectric is then

$$\begin{pmatrix}E_y\\ \mathcal{B}_x\end{pmatrix} = \begin{pmatrix}1\\ q\end{pmatrix} e^{ikqz} + R\begin{pmatrix}1\\ -q\end{pmatrix} e^{-ikqz}$$

where R is a reflexion coefficient to be found from the boundary conditions, and the incident wave has unit amplitude.

Inthe free space region equation (1) is obeyed with $\mu = 1$. The only solution which satisfies the boundary condition $E_y = 0$ at $z = -h$ (the position of the ground plane) is

$$\begin{pmatrix}E_y\\ \mathcal{B}_x\end{pmatrix} = 2T\begin{pmatrix}i \sin kC(z+h)\\ C \cos kC(z+h)\end{pmatrix}$$

where
$$C = (1 - S^2)^{1/2}$$

and T is a coupling coefficient to be determined from the boundary conditions. The factor 2 is introduced for later convenience; it arises because the sinusoidal signal is a superposition of two crossing plane waves each of amplitude T.

The boundary condition that E_y and $\mathcal{B}_x$ are continuous requires that the right hand side of equations (5) and (6) are equal when z = 0. The resulting expressions for R and T are

$$R = \frac{iq \sin kCh - C \cos kCh}{iq \sin kCh + C \cos kCh} \qquad (8)$$

$$T = \frac{q}{iq \sin kCh + C \cos kCh} \qquad (9)$$

These may be compared with the Fresnel reflexion and transmission coefficients which arise when the ground plane is absent:

$$R_F = \frac{q - C}{q + C} \qquad (10)$$

$$T_F = \frac{2q}{q + C} \qquad (11)$$

An obvious difference in the interpretation of T is that $-R_F + T_F = 1$ but $-R + T \neq 1$ Another difference is that, for $|S| < 1$, R_F and T_F are both real so that their phase is constant. This is not so for R and T. The dependence of R and T on S is of great importance.

The behaviour of R

Clearly
$$|R| = 1 \qquad (12)$$

This is not surprising since the final reflexion takes place at a perfect conductor and there are no losses. If we assume that $\mu >> 1, |S|$ then $q >> C$ so that, except near

$$kCh = n\pi \qquad (13)$$
$$R \simeq 1 \qquad (14)$$

and
$$pha(R) \simeq 2n\pi. \qquad (15)$$

Thus R and R_F approach the same behaviour as q becomes large compared with C unless (13) is true; the wave does not 'see' the conductor but behaves almost as if it were incident on the boundary between two semi-infinite media. When condition (13) holds, however,

$$R = -1 \qquad (16)$$

and
$$pha(R) = (2n+1)\pi \qquad (17)$$

In the neighbourhood of kCH = nπ the phase of R changes rapidly. The greater the difference between q and C the more rapid is this phase change. This behaviour is illustrated in Figure 1a, and is important in the discussion which follows.

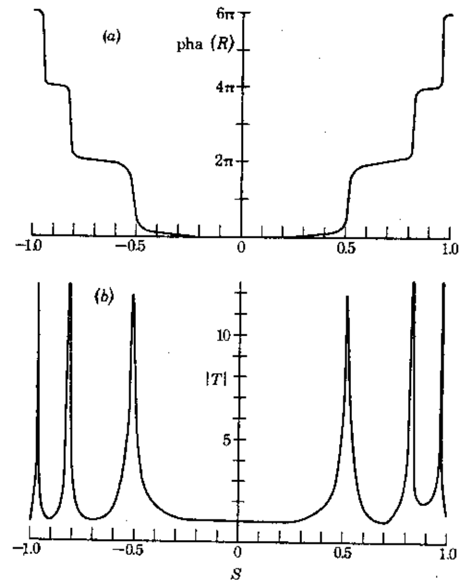


Figure 1(a) Dependence on the phase of the reflexion coefficient on S
(b) Dependence of the modulus of the coupling coefficient on S
$(\mu = 10;\ h = 11\lambda/2\pi)$

The behaviour of T

If kCH is not near $n\pi$ and q >> then from (9)

$$T \simeq 1/(i \sin kCh) \qquad (18)$$

When kCH= $n\pi$

$$|T| = q/C \qquad (19)$$

Again the phase varies rapidly near kCH. The behaviour of the amplitude of T is shown in Figure 1b. There are sharp peaks in the values of $|T|$ at $kCh = n\pi$ which become sharper if q/C is increased. In particular it should be noted that $|T|$ can be arbitrarily large if q is sufficiently large. This is somewhat surprising in view of the fact that the incident wave is of unit amplitude. The interpretation becomes clear when the incident signal is spatially confined. This is discussed in §§4 and 5.

4.3 The reflected signal

Suppose that the incident signal can be represented by a Fourier synthesis of plane waves propagating at different angles to the z axis. We may write for the field components

$$\begin{pmatrix} E_y(x,z) \\ B_x(x,z) \end{pmatrix} = \int_{-\infty}^{\infty} A(S) \begin{pmatrix} 1 \\ q \end{pmatrix} \exp\{-ik[S(x-x_0) - q(z-z_0)]\}\, dS \qquad (20)$$

where we have assumed that the signal originates from some point (x_0, y_0) in the dielectric. A(S) dS is the amplitude of the plane wave in the angular spectrum with direction defined by S. When the wave is sufficiently far from the source Rayleigh's method of stationary phase (see for example Jeffreys and Jeffreys, 1956) may be used to evaluate the integral. This assumes that for a particular value of S(S_0, say) the component plane waves in the angular spectrum interfere destructively for most x and z. Interference is constructive, however, at those values of x and z where the rate of change of the phase of the waves with respect to the direction of the wavenormal (i.e. with respect to S) is zero. This occurs when

$$\frac{\partial}{\partial S}\left[\mathrm{pha}(A) - S(x-x_0) + q(z-z_0) \right] = 0 \qquad (21)$$

i.e. where

$$x - x_0 = \left[\frac{\partial q}{\partial S}(z-z_0) - \frac{d}{dS}(\mathrm{pha}\ A) \right]_{S=S_0} \qquad (22)$$

Equation (22) defines the ray path of the incident signal corresponding to $S = S_0$.

Suppose now that, for $S = S_0$, the ray intersects the boundary at (0,0). Then

$$x_0 - \left(\frac{\partial q}{\partial S}\right)_0 z_0 = \left[\frac{d}{dS}\,(\mathrm{pha}(A))\right]_0 \qquad (23)$$

so that we may write

$$\begin{pmatrix} E_y(x,z) \\ B_x(x,z) \end{pmatrix} = \int_{-\infty}^{\infty} \mathcal{Y}(S) \begin{pmatrix} 1 \\ q \end{pmatrix} \exp\{-ik(Sx - qz)\}\, dS \qquad (24)$$

where

$$\mathcal{Y}(S) = A(S)\exp[ik(Sx_0 - qz_0)] \qquad (25)$$

and the phase of $\mathcal{Y}$ is independent of S. By choosing the time origin suitably $\mathcal{Y}$ may be made purely real.

The reflected field components are then found by using the formula (8) for the reflexion coefficient and changing the sign of q in the exponential:

$$\begin{pmatrix} E_y(x,z) \\ B_x(x,z) \end{pmatrix} = \int R(S)\, \mathcal{Y}(S) \begin{pmatrix} 1 \\ -q \end{pmatrix} \exp[-ik(Sx + qz)]\, dS \qquad (26)$$

A crucial point in the argument follows. We apply the method of stationary phase to the reflected signal in order to determine its ray path. If the phase of R is $\psi(S)$ we obtain for the ray path

$$\frac{d\psi}{dS} = k\left(x + \frac{\partial q}{\partial S} z\right)$$

$$\text{or} \quad x = -\frac{\partial q}{\partial S} z + \frac{1}{k}\frac{d\psi}{dS} \qquad (27)$$

We recall that the incident ray intersects the boundary at x = 0. The reflected ray intersects it where

$$x = \frac{1}{k}\frac{d\psi}{dS}$$

As shown in Figure 1a, $d\psi/dS$ is small for most values of S. Thus incident and reflected rays intersect the boundary at the same point. This is in agreement with the behaviour of the infinite plane wave discussed in 3. The behaviour is as if the conductor were absent. However, for those values of S defined by condition (13), $d\psi/dS$ is large, and, if $\mu \gg 1$, $d\psi/dS$ is very large. The reflected ray intersects the boundary at a great distance from the point at which the incident signal arrives. This is interpreted as meaning that the signal is trapped in the waveguide formed between the dielectric and the conductor and travels a great distance before emerging from it. The interpretation is strengthened by considering the condition for a wave to be trapped in a waveguide mode between two conductors. If the signal is treated as two crossing plane waves then the total phase change as the wave is reflected back and forth over one complete traverse of the guide, including phase changes of on each reflexion, is an integral multiple of 2π, i.e.

$$2kCh + \pi + \pi = 2m\pi$$

or

$$kCh = n\pi \qquad (29)$$

which is the same as condition (13).

4.4 The signal in the guide

A field component within the waveguide can be represented by

$$\begin{pmatrix} E_y(x,z) \\ B_x(x,z) \end{pmatrix} = \int_{-\infty}^{\infty} \mathcal{Y}(S)\, T(S)\, e^{-ikSx} \begin{pmatrix} i \sin kC(z+h) \\ C \cos kC(z+h) \end{pmatrix} dS \qquad (30)$$

where the contour of integration is the real axis (distorted slightly if necessary to avoid any singularities). Here the method of stationary phase is less convenient. In order to evaluate the integral it is more convenient to use techniques standard in waveguide theory. The behaviour of the integrand in the region of interest requires discussion.

Assume that $\mathcal{Y}(S)$ which is specified by the source, is an analytic function of S in the region of interest. The function T(S) has branch points at $S = \pm 1$ and $S = \pm\mu$. Assume that at large values of S the amplitude of $\mathcal{Y}$ becomes very small so that where $S = \pm\mu$ it is essentially zero and there is no contribution to the integral for values of S greater than this. At $S = \pm 1$ the functions T(S) sin kC(z+h) and $CT(S)\cos kC(z+h)$ are analytic even though this is not true for T(S). The contour of integration of the integral (30) can then be taken to coincide with the real axis.

Figure 1b shows the behaviour of T(S) on the real axis for a typical case. It is because of the sharp peaks that the method of stationary phase is not convenient.

It can be shown that these peaks are associated with poles of the function T which lie just below the real axis for S   0. The positions of these poles are given by the zeros of the denominator of expression (9).

$$\tan kCh = iC/q$$

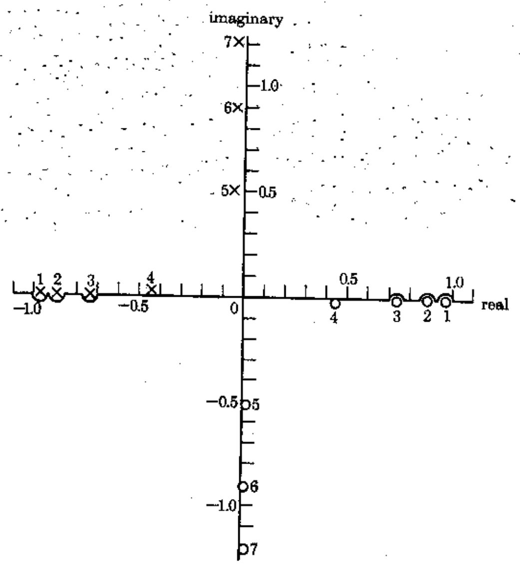


Figure 2 The poles of T(S) in the complex S-plane (= 10; H =
the contour is closed in the lower half plane; poles   contribute to the integral. x   0 : the contour is closed in the upper half plane; the poles (x) contribute to the integral.

For $|C/q| \ll 1$ we note that kCH must lie near $n\pi$ i.e. S must lie near $(1-n^2\pi^2/k^2h^2)^{1/2}$
To first order in S/μ condition (31) yields for the value of S corresponding to the nth pole.

$$S_n = \pm \left[(1 - n^2\pi^2/k^2h^2)^{1/2} - \frac{i\, n^2\pi^2}{\mu k^3h^3 (1 - n^2\pi^2/k^2h^2)^{1/2}} \right]. \qquad (32)$$

When $n^2\pi^2/k^2h^2$ this condition is better written

$$S_n = \pm \left[\frac{n^2\pi^2}{\mu k^3h^3 (n^2\pi^2/k^2h^2 - 1)^{1/2}} - i\,(n^2\pi^2/k^2h^2 - 1)^{1/2} \right] \qquad (33)$$

The expression breaks down when $|S_n|$ is large, i.e. when $k^2h^2 \simeq n^2\pi^2$.
The poles are shown for reasonable parameters in figure 2. The integral (30) can now be evaluated by closing the contour in the lower half plane for $x > 0$ and in the upper half plane for $x < 0$. The value of the integral becomes a residue series

$$\begin{pmatrix} E_y(x,z) \\ B_x(x,z) \end{pmatrix} = -2\pi i \sum_{m=1}^{\infty} \mathcal{G}(S_m)\, \mathrm{res}(T(S_m))\, e^{-ikS_m x} \begin{pmatrix} i \sin kC_m(z+h) \\ C \cos kC_m(z+h) \end{pmatrix} \qquad (34)$$

for $x > 0$ or a similar series summed from $-\infty$ to -1 for $x < 0$. Each term corresponds to a waveguide mode varying in the x direction as $\exp(-ikS_m x)$ and if $\mathcal{G}(S)$ is known, the amplitudes of each mode can be computed.

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