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HYDROMAGNETIC WAVES ON A BETA-PLANE: A NUMERICAL STUDY OF THE DISPERSION RELATIONSHIP

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Hydromagnetic Waves on a Beta-plane: A Numerical Study of the Dispersion Relationship

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NOTATION

(Meaning of symbols used unless otherwise stated)

$\mathbf{A}$	Alfvén velocity $(B_x, B_y)/(\mu\rho)^{\frac{1}{2}}$, see equation (1.3)
A	modulus of $\mathbf{A}$
A_1	r.m.s. value of A
a, b, c	coefficients in cubic polynomial, see equation (B4)
$\mathbf{B}$	basic magnetic field vector, see equation (1.3)
B	modulus of $\mathbf{B}$
B_x, B_y	(x,y) components of $\mathbf{B}$
D	discriminant associated with cubic polynomial, see equation (B4)
G	function specifying the latitude dependence of B^2, see equation (A2)
G_1	r.m.s. value of G
k, l	(x, y) components of wavenumber vector $\boldsymbol{\kappa}$, see equation (1.1)
K, L	components of wavenumber vector $\boldsymbol{\kappa}$ parallel and perpendicular to $\mathbf{B}$, see equation (2.4)
k_1, l_1	k_1 is integer characterizing longitudinal dependence of the eigenfunction of a slow mode and $2\pi/l_1$ is angular scale characterizing variation of N with υ, see equation (A3)
N	amplitude of eigenfunction of slow mode, see equation (A1)
Q	$4K(\omega^2 - K^2)[K\omega^{-1}(\omega^2 - K^2) + \cos\theta]/\omega \sin\theta$, see equation (2.8)
r, ϕ	circular polar co-ordinates $(x^2 - y^2)^{\frac{1}{2}}$, $\tan^{-1}(y/x)$, see equation (B1)
R	radius of curvature of thin fluid shell, see equation (1.2)
t	time
$\mathbf{U}$	group velocity $\partial\omega/\partial\boldsymbol{\kappa}$, see equation (3.1)
U_x, U_y	(x, y) components of $\mathbf{U}$, see equations (C1) and (C2)
U_X, U_Y	(X, Y) components of $\mathbf{U}$, see equations (C4) and (C5)
U_r, U_ϕ	(r, ϕ) components of $\mathbf{U}$, see equations (C7) and (C8)
x, y	local eastward and northward Cartesian co-ordinates on the surface of the sphere, see equation (1.1)
X, Y	local co-ordinates parallel and perpendicular to $\mathbf{B}$
β	rate of change with respect to latitude of the radial component of $\boldsymbol{\Omega}$, see equation (1.2)
γ	a hydromagnetic Rossby number, see equation (2.2)
δ	$2\omega + k/\kappa^2$, see equation (C3)
Δ	$2\omega + (K\cos\theta - L\sin\theta)/\kappa^2$, see equation (C6)
δ_1	$2\omega + \cos\phi/\kappa$, see equation (C9)
θ	$\tan^{-1}(B_y/B_x)$, see equation (1.4)
$\boldsymbol{\kappa}$	general two-dimensional vector wavenumber, with components (k, l), (K, L), or (κ, ϕ), see equations (1.1), (2.4) and (B1)
κ	modulus of $\boldsymbol{\kappa}$
μ	magnetic permeability, see equation (1.3)
ρ	density, see equation (1.3)
υ	general latitude, see equation (1.2)
υ_1	latitude at which beta-plane is tangent to the sphere, see equation (1
ϕ	polar angle $\tan^{-1}(y/x)$, see Figure 1
ψ	general longitude, see equation (A1)
ω	general eigenfrequency, see equation (1.1)
ω_m	eigenfrequency of slow mode, see equation (A1)
$\omega_1, \omega_2, \omega_3$	critical frequencies defined by equations (2.10), (2.11)
ω_*	defined by equation (B5)
$\boldsymbol{\Omega}$	basic angular velocity of rotation, see equation (1.2)
Ω	modulus of $\boldsymbol{\Omega}$

Note: Ordinary dimensional quantities measured in SI units are used in equations (1. to (1.10). Quantities used in all other equations are dimensionless, being based $(A/\beta)^{\frac{1}{2}}$ as the unit of time and $(A\beta)^{-\frac{1}{2}}$ as the unit of length unless it is stat otherwise.

Hydromagnetic Waves on a Beta-plane: A Numerical Study of the Dispersion Relationship

by R. Hide, Sc.D., F.R.S. and M. V. Jones, B.Sc.

SUMMARY

The need for a thorough understanding of effects produced by Coriolis forces on hydromagnetic waves in a bounded fluid arises in various theoretical studies of rotating magnetic astronomical bodies (notably the Earth, Jupiter, the Sun and certain stars, including pulsars). Stewartson and Rickard have recently given an exact theoretical analysis of slow waves in a thin rotating spherical shell of an incompressible fluid pervaded by a uniform magnetic field directed (in its undisturbed state) parallel to latitude circles. The present paper shows that an approximate dispersion relationship derived previously by Hide on the basis of a simple physical model agrees satisfactorily with the exact analysis, making it possible to exploit with some confidence the explicit nature of the dispersion relationship and rendering a numerical analysis of the relationship worth while. The numerical analysis is greatly simplified by the circumstance that $\theta = \tan^{-1}(B_y/B_x)$ (where B_y is the northward component of the basic magnetic field and B_x is the eastward component) is the only free parameter when the unit of time is $[R^2\mu\rho/4\Omega^2(B_x^2+B_y^2)]^{\frac{1}{2}}$ and the unit of length is $[R^2(B_x^2+B_y^2)/\mu\rho\Omega^2]^{\frac{1}{4}}$, where μ is the magnetic permeability of the fluid and ρ its density, R is the radius of curvature of the shell and Ω the angular speed of rotation. The complicated properties of a wave of angular frequency ω and wavenumber $\boldsymbol{\kappa} = (k, l)$ depend *inter alia* on whether $\omega < \omega_1$, $\omega_1 < \omega < \omega_2$ or $\omega > \omega_2$ where $\omega_1 = 1{\cdot}6119|\sin\frac{1}{2}\theta|$ and $\omega_2 = 1{\cdot}6119|\cos\frac{1}{2}\theta|$ (in dimensionless units). In the limit when the wavelength is so small that Coriolis forces are negligible (more precisely when a hydromagnetic Rossby number $\gamma = |2(k\cos\theta + l\sin\theta)\kappa^2/k|$ is much greater than unity), $\omega = \pm(k\cos\theta + l\sin\theta)$. This is the dispersion relationship for ordinary non-dispersive Alfvén waves—which are characterized by equipartition between magnetic and kinetic energy—propagating along the magnetic lines of force in opposite directions. In the opposite limit, $\gamma \ll 1$, $\omega = -k/\kappa^2$ or $\omega = \kappa^2(k\cos\theta + l\sin\theta)^2/k$; the former corresponds to a (fast) Rossby–Haurwitz wave, in which hydromagnetic effects are negligible, and the latter to a slow hybrid wave, which is characterized by an exact balance between the torques exerted on individual fluid elements by Coriolis and Lorentz forces. The ratio of magnetic to kinetic energy is γ^2 for the Rossby–Haurwitz wave and γ^{-2} for the hybrid wave.

1. INTRODUCTION

'Hydromagnetics' is the study of the flow of electrically conducting fluids in the presence of magnetic fields ('magnetohydrodynamics', 'M.H.D.' and 'magneto-fluid dynamics' being alternative terms for the subject). Hydromagnetic phenomena are hard to produce with available fluids on the limited scale of the laboratory, but they are common on the enormous scale of cosmical systems and many astrophysical phenomena are utterly incomprehensible except in terms of hydromagnetics. Future progress towards a satisfactory explanation of the Earth's magnetism—which arises in the liquid core of the Earth where the necessary electric currents are produced by fluid motions that are strongly influenced by Coriolis forces due to the Earth's rotation—will be inseparable from developments in the hydromagnetics of rapidly rotating fluids, which is an exciting and comparatively new field.

The problem discussed in the present paper was first studied by Hide,[1]* who propos[ed] a local dispersion relationship for hydromagnetic waves in a rotating spherical hom[o]geneous fluid shell of outer radius R, making use of an approximation which is equival[ent] when the shell is thin to the Rossby–Haurwitz 'beta-plane' used in dynamical meteorolo[gy] and oceanography (Rossby *et alii*[2]). Suppose that the 'beta-plane' is tangent to the sphe[re] at latitude $v = v_1$ and denote by x and y the local eastward and northward Cartesian c[o-]ordinates. When the basic magnetic field $\mathbf{B}$ is independent of x and y and any basic flu[id] flow relative to the rotating frame is slow and uniform, then the dispersion relationsh[ip] for two-dimensional waves propagating in the (x, y) plane relative to that basic flow

$$\omega^2 + \beta k\omega/\kappa^2 - (\mathbf{A} \cdot \boldsymbol{\kappa})^2 = 0. \qquad \ldots (1$$

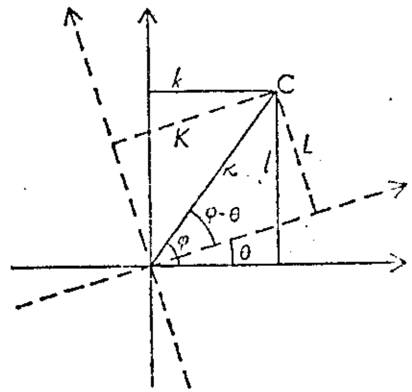


FIGURE 1. Co-ordinates of a point C in wavenumber space

Here ω is the angular frequency of the wave, $\boldsymbol{\kappa} = (k, l)$ is the wavenumber vector (so t[hat] $\kappa = (k^2 + l^2)^{\frac{1}{2}}$ is the total wavenumber, see Figure 1),

$$\beta = 2\Omega\cos v_1/R, \qquad \ldots ($$

the rate of change with respect to latitude of the radial component of the Coriolis pa[ra]meter, $\boldsymbol{\Omega}$ being the basic angular velocity of rotation of the system, and

$$\mathbf{A} \equiv (B_x, B_y)/(\mu\rho)^{\frac{1}{2}}, \qquad \ldots ($$

a two-dimensional vector with magnitude A equal to the Alfvén speed based on $(B_x^2 + B$ inclined at an angle

$$\theta \equiv \tan^{-1}(B_y/B_x) \qquad \ldots ($$

to the x-axis if μ denotes the uniform magnetic permeability of the fluid and ρ its unif[orm] density (all in rationalized SI units).

Equation (1.1) should be valid when the wavelength $2\pi/\kappa$ is much less than the le[ngth] of a great circle, so that

$$\kappa R \gg 1, \qquad \ldots ($$

* Superscript figures refer to the bibliography on p. 13.

and evidence that this is so in the non-hydromagnetic case ($A = 0$) can be found in accurate studies of eigenmodes of a thin spherical shell (e.g. Longuet-Higgins[3]), when equation (1.1) reduces to the well-known Rossby-Haurwitz formula for ordinary planetary waves

$$\omega = -\beta k/\kappa^2 \qquad \ldots (1.6)$$

(Rossby *et alii*[2] and Haurwitz[4]). Comparable studies of the general hydromagnetic case are not available, but the expression

$$\omega = A^2 k\kappa^2/\beta \qquad \ldots (1.7)$$

to which equation (1.4) reduces when $\theta = 0$ and $\omega \ll |\mathbf{A}.\boldsymbol{\kappa}|$, is compatible with accurate analyses by Stewartson[5] and Rickard,[6] as is shown in Appendix A.

Equation (1.1) leads to quite simple expressions for ω in various limiting cases, notably when a 'hydromagnetic Rossby number'

$$\gamma \equiv |2\kappa^2(\mathbf{A}.\boldsymbol{\kappa})/\beta k| \qquad \ldots (1.8)$$

tends to either zero or infinity. In the first such case the two roots are

$$\omega = -\beta k/\kappa^2 \quad \text{and} \quad \omega = (\mathbf{A}.\boldsymbol{\kappa})^2\kappa^2/\beta k, \qquad (\gamma \to 0), \qquad \ldots (1.9\text{a,b})$$

which are the dispersion relationships for ordinary Rossby waves (see equation (1.6)) and hydromagnetic planetary waves (or planetary magnetohydrodynamic waves; see Hide,[1] Malkus,[7] Suffolk and Allan,[8] and Hide and Stewartson[9]) respectively. In the other case the roots are

$$\omega = \mp|\mathbf{A}.\boldsymbol{\kappa}|, \qquad (\gamma \to \infty), \qquad \ldots (1.10)$$

the dispersion relationship for ordinary Alfvén waves (Alfvén[10]). The purpose of the present paper is to examine equation (1.1) in order to find the ranges of validity of equations (1.9) and (1.10) and to clarify the behaviour of the dispersion relationship when γ is neither very large nor very small.

2. ANALYSIS OF DISPERSION RELATIONSHIP

Firstly it can be noted that if $(A/\beta)^{\frac{1}{2}}$ is taken as the unit of length and $(A\beta)^{-\frac{1}{2}}$ as the unit of time then A and β no longer appear explicitly in the dispersion relationship (see equation (1.1)), which reduces to

$$\omega^2 + \omega k/\kappa^2 - (k\cos\theta + l\sin\theta)^2 = 0, \qquad \ldots (2.1)$$

an equation with only one free parameter, θ. In these units, equation (1.8) for γ, the hydromagnetic Rossby number, becomes

$$\gamma = \left|\frac{2(k\cos\theta + l\sin\theta)(k^2+l^2)}{k}\right|. \qquad \ldots (2.2)$$

Because equation (2.1) is unaffected by the transformations

$$(k, l; \theta) \to (k, l; \theta+\pi) \quad \text{and} \quad (k, l; \theta) \to (k, -l; -\theta), \qquad \ldots (2.3)$$

no generality is lost if attention is confined to values of θ in the range from 0 to $\pi/2$.

A useful way of representing the dispersion relationship is by means of sets of 'norm curves', which are contour lines of constant ω in the (k, l) plane, one set for each giv value of θ (see Figures 3–5). Explicit formulae for these contours can be found in pri ciple by solving equation (2.1) for l as a function of k or for k as a function of l, but t equation is quartic in k and l and some simplification of the analysis is required. This c be achieved by rotating the co-ordinate axes in wavenumber space through an angle (see Figure 1), so that $\boldsymbol{\kappa} = (K, L)$ when referred to the new axes, where

$$(K, L) = (k\cos\theta + l\sin\theta, l\cos\theta - k\sin\theta). \qquad \ldots (2$$

Equation (2.1) then becomes

$$\omega^2 + \omega(K\cos\theta + L\sin\theta)/(K^2+L^2) - K^2 = 0 \qquad \ldots (2$$

which leads to the following quadratic in L:

$$L^2 - \frac{\omega L\sin\theta}{\omega^2-K^2} + K\left[K + \frac{\omega\cos\theta}{\omega^2-K^2}\right] = 0. \qquad \ldots ($$

Whence

$$L = \frac{\omega\sin\theta}{2(\omega^2-K^2)}\left[1 \pm (1-Q)^{\frac{1}{2}}\right] \qquad \ldots ($$

where

$$Q \equiv \frac{4K(\omega^2-K^2)}{\omega\sin\theta}\left[\frac{K(\omega^2-K^2)}{\omega} + \cos\theta\right]. \qquad \ldots ($$

By equation (2.7) Q cannot exceed unity, for only real values of L are accepta By considering the variation of Q with K it is readily shown that the general shap the normal curve depends on whether ω lies within certain ranges:

$$\omega \leqslant \omega_1, \qquad \omega_1 < \omega \leqslant \omega_2 \quad \text{or} \quad \omega > \omega_2, \qquad \ldots$$

where

$$\omega_1 = (27/4)^{\frac{1}{2}}|\sin\tfrac{1}{2}\theta|, \qquad \omega_2 = (27/4)^{\frac{1}{2}}|\cos\tfrac{1}{2}\theta|, \qquad \ldots ($$

(see Figure 2 and Appendix B).

The shape of a normal curve $\kappa = \kappa(\phi)$, where ϕ is defined by Figure 1, depends o value of ω (see Figures 2–5). When $\omega > \omega_2$ (see equation (2.10)) there are three roots of κ for all ϕ (see equation (B4)) and the contour may be said to 'surround origin, even though two values of κ tend to $\pm\infty$ as ϕ tends to $\theta - \frac{1}{2}\pi$. In the interme case when $\omega_2 \geqslant \omega > \omega_1$ there is one subrange of ϕ, which includes $\phi = \frac{1}{2}\theta$, in w there are no acceptable roots on the opposite side of the origin, an angular distance π a Finally, when $\omega \leqslant \omega_1$ there are two such subranges, one of which includes $\phi = \frac{1}{2}\theta$ the other of which includes $\phi = \frac{1}{2}(\theta - \pi)$.

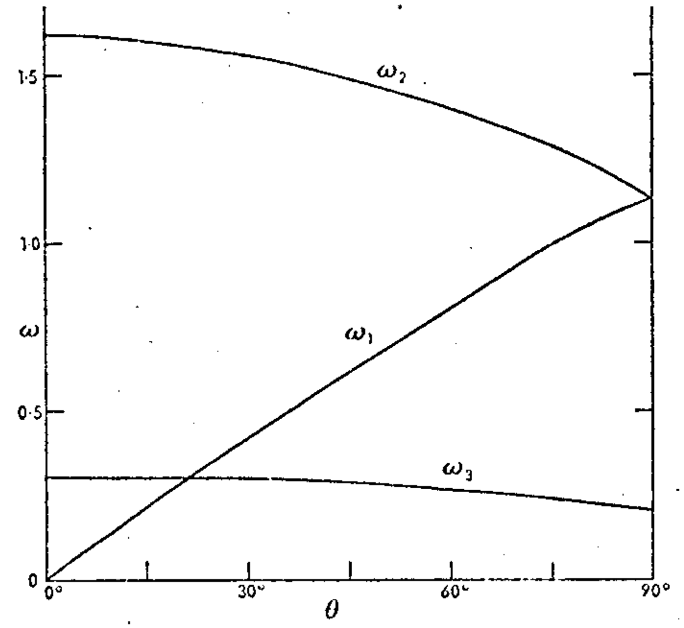


FIGURE 2. The dependence on θ of the critical frequencies ω_1, ω_2 and ω_3

See equations (2.10) and (2.11).

There is a third critical frequency

$$\omega_3 = (1-2^{-\frac{1}{2}})|\cos \tfrac{1}{2}\theta|, \qquad \ldots (2.11)$$

(see Figure 2) such that all curves of constant $\omega \geqslant \omega_3$ have $\gamma > 1$ (see equation (2.2)). Hydromagnetic planetary waves are characterized by $\gamma \leqslant 1$ (and $-\frac{1}{2}\pi < \phi < \frac{1}{2}\pi$), so that such waves with $\omega > \omega_3$ are impossible (see equation (1.9)). The equation of the curve $\gamma = 1$ in a (κ, ϕ) diagram is

$$\kappa^2 = \tfrac{1}{2}|\cos\phi \sec(\phi-\theta)|$$

(see Figure 6), which may be substituted into equation (B2) (see Appendix B) in order to find whether the curve intersects normal curves (of constant ω). Intersections occur when

$$\omega^2 + \omega|2\cos\phi\cos(\phi-\theta)|^{\frac{1}{2}} - \tfrac{1}{2}|\cos\phi\cos(\phi-\theta)| = 0$$

is satisfied. The positive root of this equation is

$$\omega = (1-2^{-\frac{1}{2}})|\cos\phi\cos(\phi-\theta)|^{\frac{1}{2}}$$

giving

$$\phi = \tfrac{1}{2}\{\theta + \cos^{-1}[(2\omega/(\sqrt{2}-1))^2 - \cos\theta]\},$$

which has real solutions when $\omega \leqslant \omega_3$ but not otherwise (see equation (2.11)). It follows, therefore, that those regions in the (k, l) or (κ, ϕ) plane for which $\gamma \leqslant 1$ lie entirely within the normal curve for $\omega = \omega_3$ (see Figures 3–5).

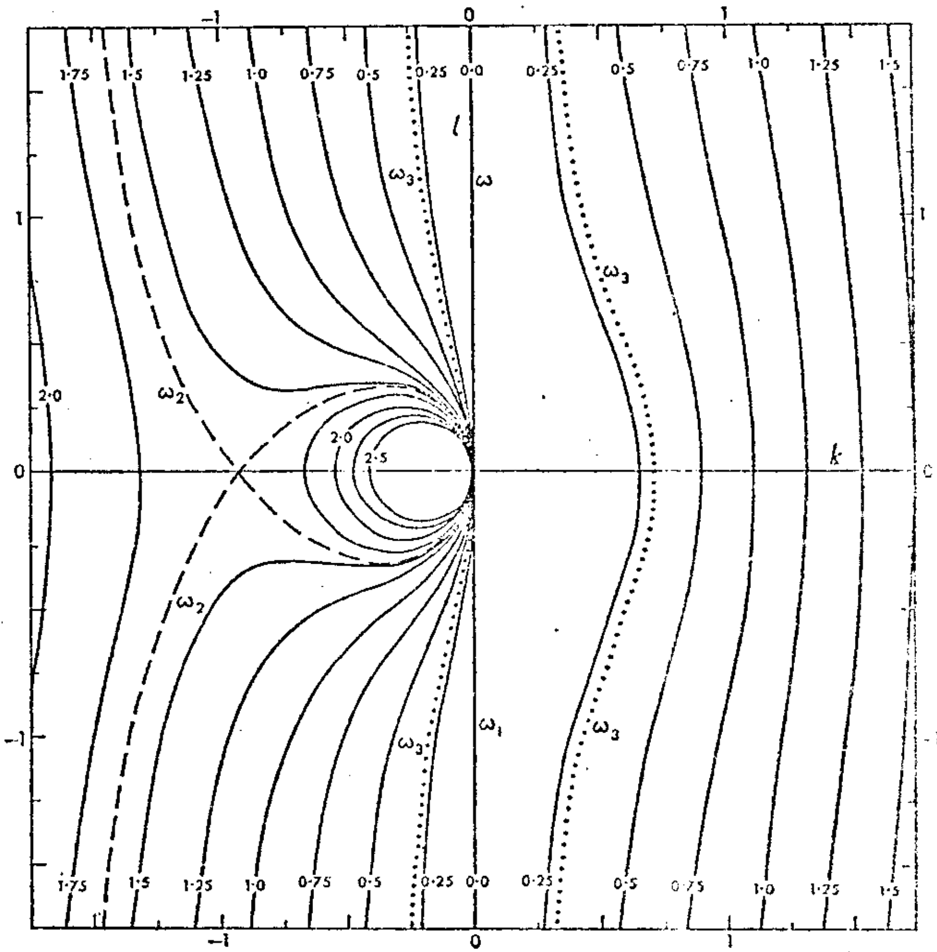


FIGURE 3. Normal curves for the case $\theta = 0°$

See equations (2.5)–(2.8); $\omega_1 = 0$, $\omega_2 = 1{\cdot}6119$, $\omega_3 = 0{\cdot}2929$ (see Figure 2).

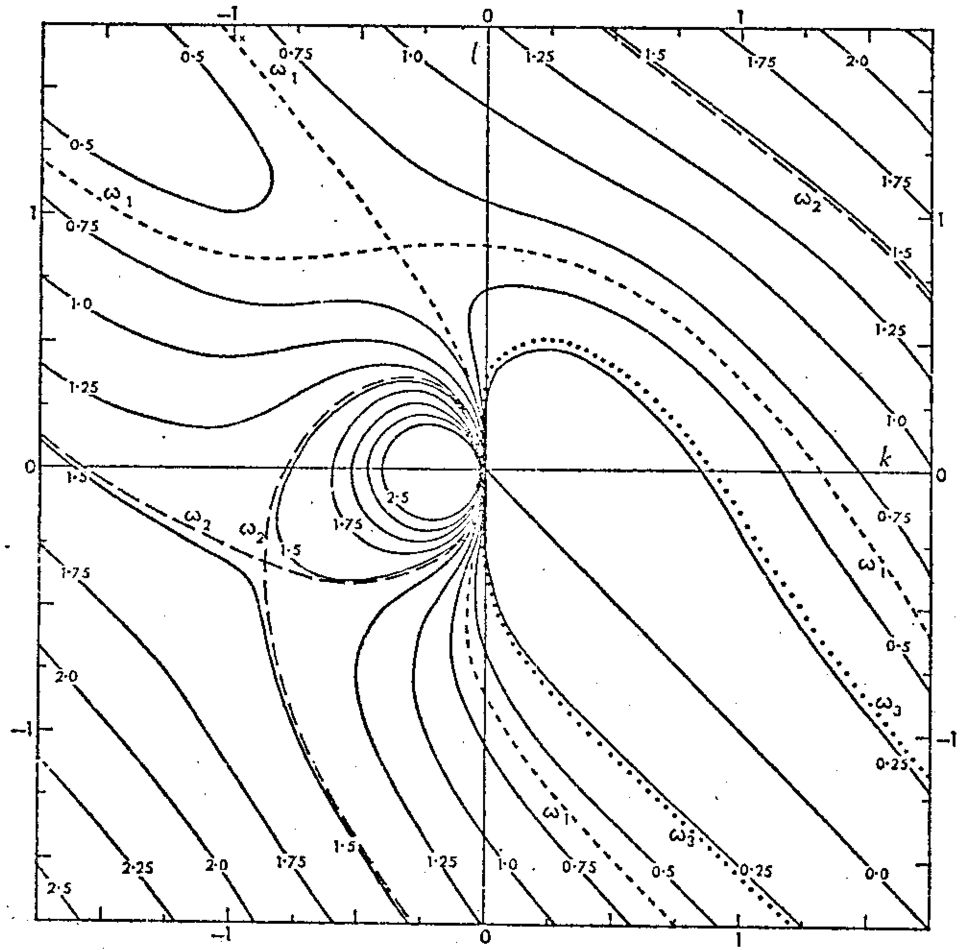


FIGURE 4. Normal curves for the case $\theta = 45°$

See equations (2.5)–(2.8); $\omega_1 = 0{\cdot}6168$, $\omega_2 = 1{\cdot}4881$, $\omega_3 = 0{\cdot}2706$ (see Figure 2).

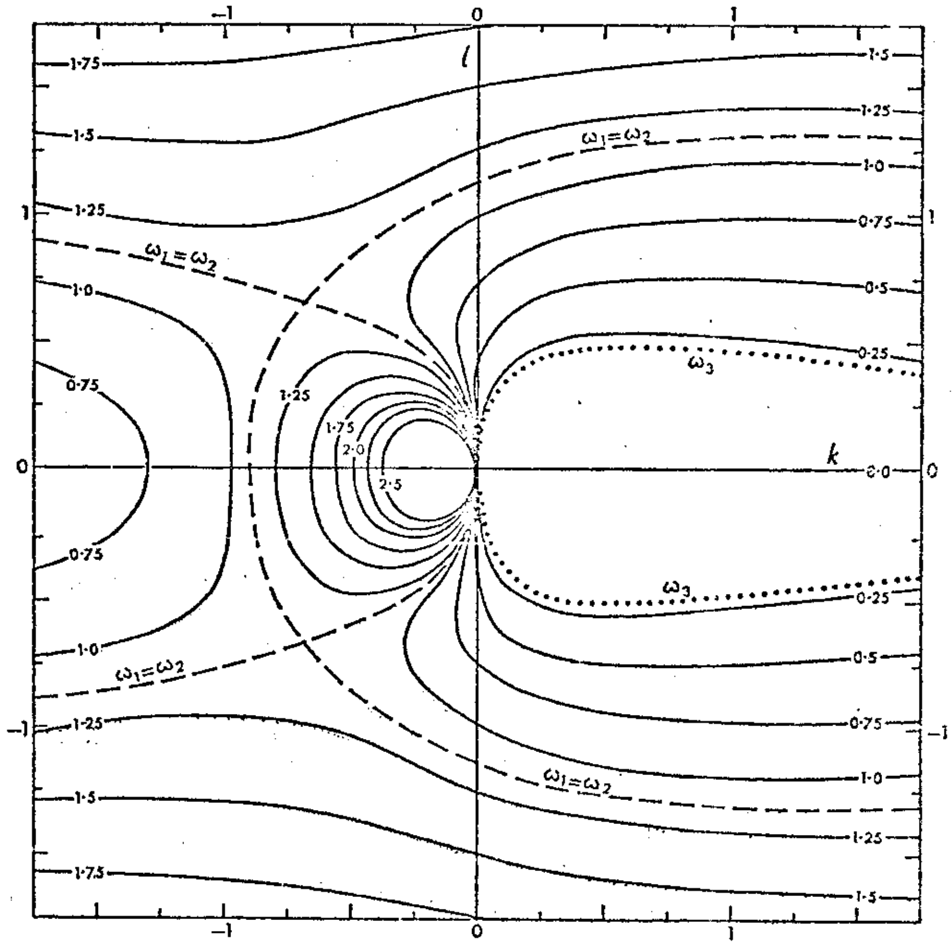


FIGURE 5. Normal curves for the case $\theta = 90°$

See equations (2.5)–(2.8); $\omega_1 = \omega_2 = 1{\cdot}1398$, $\omega_3 = 0{\cdot}2071$ (see Figure 2).

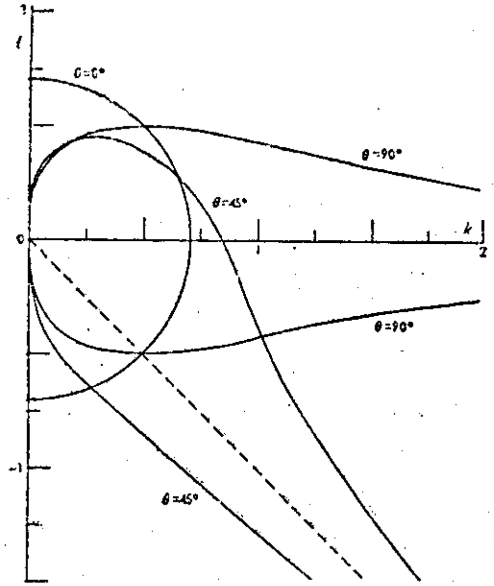


FIGURE 6. The curve in (k, l) plane on which $\gamma = 1$ and within which $\gamma < 1$, for the three cases $\theta = 0°, 45°$ and $90°$

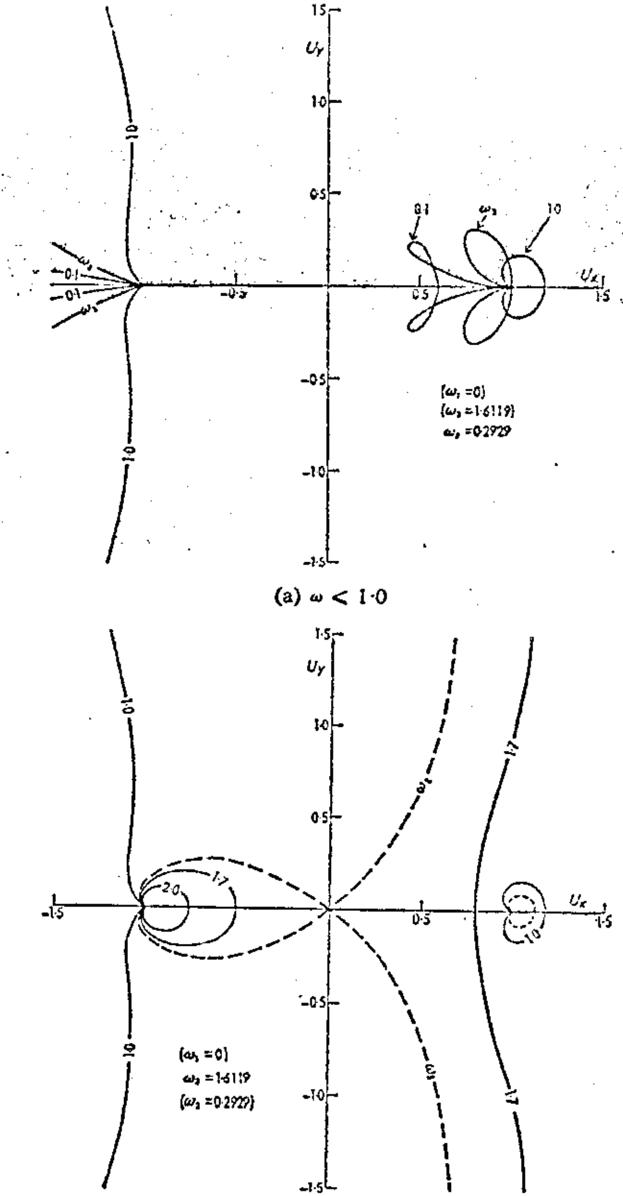
Only that part of the curve for which $k > 0$ is given. All curves γ = constant pass through the origin, have the same general shape, and converge on the line $l = k \tan(\theta - \frac{1}{2}\pi)$ at great distances from the origin (cf. Figures 3, 4 and 5).

Singular points

The dispersion relationship has an apparent singularity at the origin in the (k, l) diagram, where the normal curves crowd together, the corresponding magnitude of the group velocity $\mathbf{U} = \partial\omega/\partial\boldsymbol{\kappa}$ (see equation (3.1)) being infinite, but this has no physical significance because the dispersion relationship is not valid when $\kappa < (A/\beta R^2)^{\frac{1}{2}}$ (see equation (1.5)). The 'saddle points' that occur on the contours $\omega = \omega_1$ and $\omega = \omega_2$ subtend a right angle at the origin and are situated at

$$\left.\begin{aligned} (\kappa, \phi) &= (0{\cdot}9306, \tfrac{1}{2}(\theta+\pi)), \qquad \omega = \omega_1, \\ (\kappa, \phi) &= (0{\cdot}9306, \tfrac{1}{2}(\theta+2\pi)), \qquad \omega = \omega_2; \end{aligned}\right\} \ldots (2.12)$$

$(0{\cdot}9306)^4 \approx 3/4$. The group velocity (see Figures 7–9) vanishes at these saddle points and also along the contour $\omega = 0$, given by the line $\phi = \theta - \frac{1}{2}\pi$ (see Figures 3–5).



(a) $\omega < 1{\cdot}0$

(b) $\omega \geq 1{\cdot}0$

FIGURE 7. Group velocity curves when $\theta = 0°$

Each curve is the locus traced by the tip of the group velocity vector $\mathbf{U} = (U_x, U_y)$ as one proceeds along the corresponding normal curve in Figure 3. When the total wavenumber κ greatly exceeds $[\frac{1}{2}\cos\phi \sec(\phi-\theta)]^{\frac{1}{2}}$ (see Section 2), rotational effects are unimportant and $\mathbf{U}$ is represented by one of the two points $\pm(1, 0)$ in the diagram.

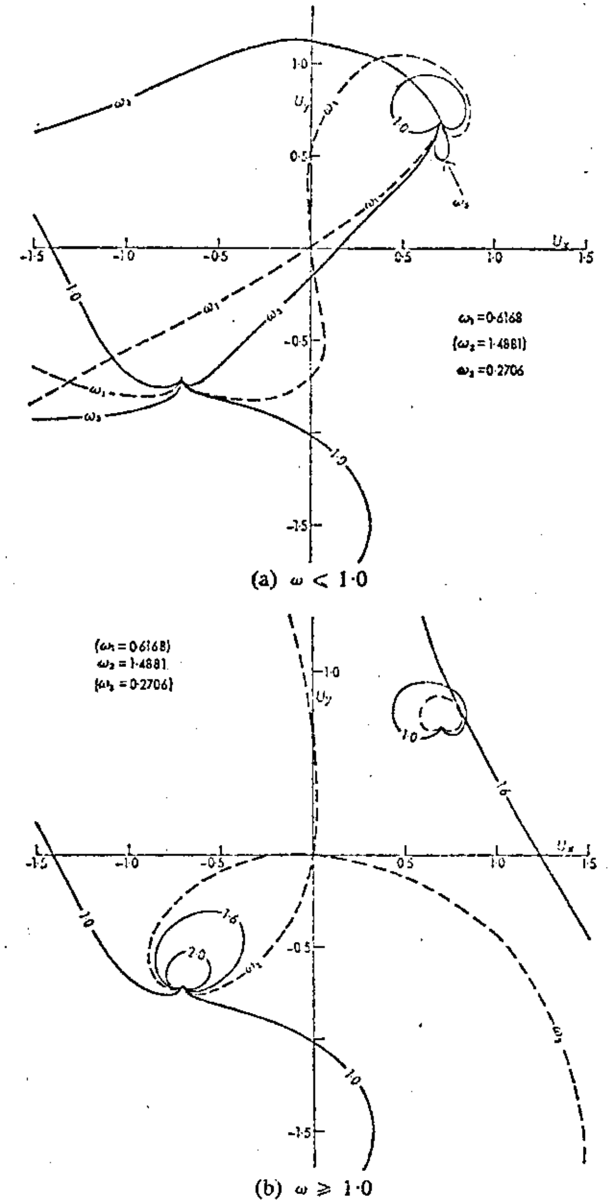


FIGURE 8. Group velocity curves when $\theta = 45°$

Each curve is the locus traced by the tip of the group velocity vector $\mathbf{U} = (U_x, U_y)$ as one proceeds along the corresponding normal curve in Figure 4. When the total wavenumber κ greatly exceeds $[\frac{1}{2} \cos \phi \sec(\phi - \theta)]^{\frac{1}{2}}$ (see Section 2), rotational effects are unimportant and $\mathbf{U}$ is represented by one of the two points $\pm(0{\cdot}7071, 0{\cdot}7071)$ in the diagram.

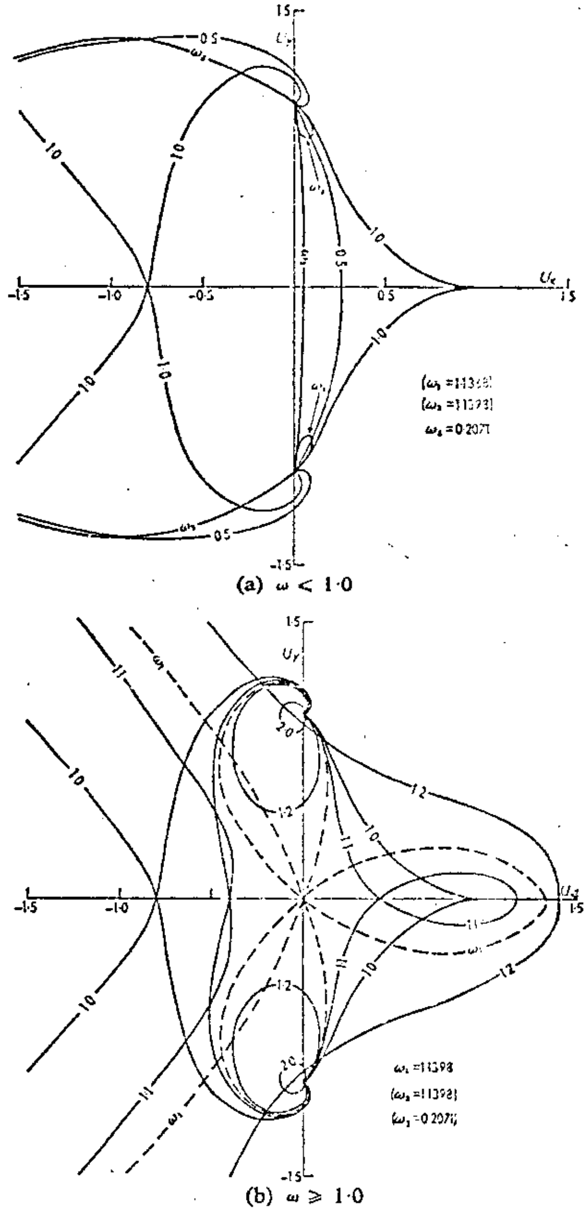


FIGURE 9. Group velocity curves when $\theta = 90°$

Each curve is the locus traced by the tip of the group velocity vector $\mathbf{U} = (U_x, U_y)$ as one proceeds along the corresponding normal curve in Figure 5. When the total wavenumber κ greatly exceeds $[\frac{1}{2} \cos \phi \sec(\phi - \theta)]^{\frac{1}{2}}$ (see Section 2), rotational effects are unimportant and $\mathbf{U}$ is represented by one of the points $\pm(0, 1)$ in the diagram.

3. GROUP VELOCITY

The group velocity

$$\mathbf{U} \equiv \partial\omega/\partial\boldsymbol{\kappa} \qquad \ldots (3.1)$$

has components (U_x, U_y), (U_X, U_Y) or (U_r, U_ϕ) depending on the system of co-ordinates used (see Figure 1 and Appendix C).

When the total wavenumber κ is much greater than $[\frac{1}{2}\cos\phi\sec(\phi-\theta)]^{\frac{1}{2}}$, equation (B2) gives $\omega = \pm\kappa\cos(\phi-\theta)$, the dispersion relationship for pure Alfvén waves (see equation (1.10) and Figure 1). Normal curves are then straight and parallel lines inclined at an angle θ to the line $\phi = \frac{1}{2}\pi$ (the l-axis), which they intersect at distances $\pm\omega\,\mathrm{cosec}\,\theta$ from the origin (see Figures 3–5). The group velocity $\mathbf{U} = \pm(\cos\theta, \sin\theta)$ and can therefore be represented by two single points $(\cos\theta, \sin\theta)$ and $(-\cos\theta, -\sin\theta)$ in a group velocity diagram, in which U_x is the abscissa and U_y the ordinate (see Figures 7–9). When on the other hand the total wavenumber is much less than $[\frac{1}{2}\cos\phi\sec(\phi-\theta)]^{\frac{1}{2}}$, equation (B2) gives $\omega = -\kappa^{-1}\cos\phi$, the dispersion relationship for pure Rossby waves (see equation (1.9a); each normal curve is then a circle of radius $1/2\omega$ centred at $(k, l) = (-1/2\omega, 0)$. Each component of the group velocity varies continuously as one proceeds around any of these circles. The points $(k, l) = (0, 0)$, $(-1/2\omega, 1/2\omega)$, $(-1/\omega, 0)$ and $(-1/2\omega, -1/2\omega)$ are on the circle; at these points the group velocity $\mathbf{U} = (U_x, U_y)$ has the respective values $(-\infty, 0)$, $(0, -2\omega^2)$, $(\omega^2, 0)$ and $(0, 2\omega^2)$. Hence, in contrast to the case of Alfvén waves, whose group velocity diagram degenerates to two points in the (U_x, U_y) plane, the group velocity diagram for hydromagnetic planetary waves consists of a set of curves, one for each value of ω, which are symmetrical about the U_x-axis, intersect the U_x-axis at $-\infty$ and ω^2 and the U_y-axis at $\pm 2\omega^2$.

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APPENDIX A. DISPERSION RELATIONSHIP FOR SLOW MODES WHEN $\theta = 0$

If the eigenfunction of a slow mode of eigenfrequency ω_m (see equation (1.7)) has the form

$$N(\upsilon)\exp i(k_1\psi - \omega_m t)$$

... (A

where ψ denotes longitude, υ latitude, t time and k_1 is a real and positive integer, then the differenti equation for $N(\upsilon)$ given by Rickard[6] can be written as follows:

$$\omega\left(\frac{2\Omega R^2}{A_1^2}\right) = \frac{G}{G_1}\frac{k_1}{\cos^4\upsilon}\left\{k_1^2 - 1 + \cos^2\upsilon\left[\frac{N'' + N'(\tan\upsilon + G'/G)}{N}\right]\right\}.$$

... (A

Here the prime denotes differentiation with respect to υ, the square root of $G = G(\upsilon)$ specifies the latitu dependence of the basic toroidal magnetic field $(B_\lambda, 0)$, G_1 is the r.m.s. average value of G over the interv $-\frac{1}{2}\pi < \upsilon < \frac{1}{2}\pi$ and A_1 is the r.m.s. average value of the Alfvén speed $B_\lambda/(\mu\rho)^{\frac{1}{2}}$ over the same interva The case treated by Stewartson[5] corresponds to $G = G_1 = 1$.

Solutions to the exact equation (A2) are in general very complicated (see Stewartson[5]), and it is n practical to solve the equation for ω and compare the results with values based on the equation (1.7 A much simpler approach is to average the right-hand side of equation (A2) with respect to υ and the compare the form of the resulting equation with the approximate dispersion relationship given by equatic (1.7). The averaging could be carried out formally by integrating equation (A2) with respect to υ, usi suitable weighting functions, but it is more instructive to discuss each term separately.

We shall set G/G_1 equal to unity and $A = A_1$ in equation (A2), which should not lead to serious erro except possibly when G is highly complicated in form. If $2\pi/l_1$ is the angular scale that characteri the variation of N with υ, we can set

$$N''/N \approx -l_1^2;$$

... (A

and if

$$|\tan\upsilon + G'/G| \ll |N''/N| \approx l_1^2$$

... (A

then the term $(\tan\upsilon + G'/G)$ (which vanishes identically when $G = \cos\upsilon$, hardly an extreme case) can ignored. Finally, we replace $\cos\upsilon$ by an average value $\cos\upsilon_1$ (say), relate (k_1, l_1) to (k, l) as follows:

$$k_1 = kR\cos\upsilon_1, \qquad l_1 = lR,$$

... (A

and write

$$\beta = 2\Omega\cos\upsilon_1/R$$

... (A

(cf. equation (1.2)). With these simplifications and the further supposition that $k_1^2 \gg 1$, equivalent equation (1.5), equation (A2) reduces to equation (1.7) exactly, and this is the desired result.

When it cannot be supposed that $k_1^2 \gg 1$, the term κ^2 in equation (1.7) should be replaced $\kappa^2 - (R\cos\upsilon)^{-2}$. This correction—which is too subtle to be represented directly in the physical mo leading to equation (1.1)—would ensure that ω vanishes not only when the lines of magnetic force of basic magnetic field are undisturbed, as in the case of $k_1 = l_1 = 0$ but also when the lines of force merely displaced as a whole without suffering the distortion required to generate magnetic restor forces, as in the cases $k_1^2 + l_1^2 = 1$ and $k_1 = 0$ but $l_1 \neq 0$.

APPENDIX B. THE CRITICAL FREQUENCIES

The critical frequencies ω_1 and ω_2 (see equation (2.10)) enter more naturally if we introduce a p co-ordinate system based on κ, the total wavenumber, and ϕ, the angle of the wavefront normal to x-axis (see Figure 1), so that

$$\left.\begin{aligned} \kappa &= (k^2+l^2)^{\frac{1}{2}} = (K^2+L^2)^{\frac{1}{2}}, \\ \tan\phi &= l/k, \qquad \tan(\phi-\theta) = L/K \end{aligned}\right\}$$

... (

(cf. equations (2.4) and (2.1)). The dispersion relationship (see equation (2.1)) then takes the form:

$$\omega^2 + \omega\cos\phi/\kappa - \kappa^2\cos^2(\phi-\theta) = 0,$$

... (

which may also be written as a cubic equation in κ, namely

$$\kappa^3 \cos^2(\phi-\theta) - \omega^2\kappa - \omega\cos\phi = 0. \quad \ldots \text{(B3)}$$

Because the transformation $(\kappa, \phi) \to (-\kappa, \phi+\pi)$ leaves the equation (B3) unchanged, if for a given value of ϕ there is an acceptable positive root κ, then the corresponding root on the other side of the origin is negative and therefore unacceptable. Conversely, if for a given value of ϕ there is an unacceptable negative root κ, the corresponding root on the other side of the origin is positive and therefore acceptable. Thus, acceptable roots of equation (B3) can be found by first restricting the range of ϕ to two quadrants (and it is convenient to choose $-\frac{1}{2}\pi < \phi < \frac{1}{2}\pi$) and then finding all the real roots for κ, both positive and negative.

Associated with a cubic equation of the form $a\kappa^3+b\kappa+c = 0$ where a, b and c are real, there is a discriminant $D = 4b^3/3a^3+9c^2/a^2$ such that there are

3 distinct roots of κ if $D < 0$,
1 single and 2 coincident real roots of κ if $D = 0$, and
1 real and 2 complex conjugate roots of κ if $D > 0$.

From equation (B3)

$$D = \frac{4\omega^2}{3\cos^6(\phi-\theta)}\left[\frac{27}{4}\cos^2\phi\cos^2(\phi-\theta)-\omega^4\right], \quad \ldots \text{(B4)}$$

so, for example, there are 3 real roots of κ when

$$\omega^4 > 27\cos^2\phi\cos^2(\phi-\theta)/4 = \omega_*^4,$$

where

$$\omega_* = [3\sqrt{3}|\cos\theta+\cos(2\phi-\theta)|/4]^{\frac{1}{2}} \quad \ldots \text{(B5)}$$

(see Figure 10). The critical frequency ω_1 is the maximum of ω_* and the critical frequency ω_2 is the subsidiary maximum (see equations (2.10) and (B5) and Figure 10).

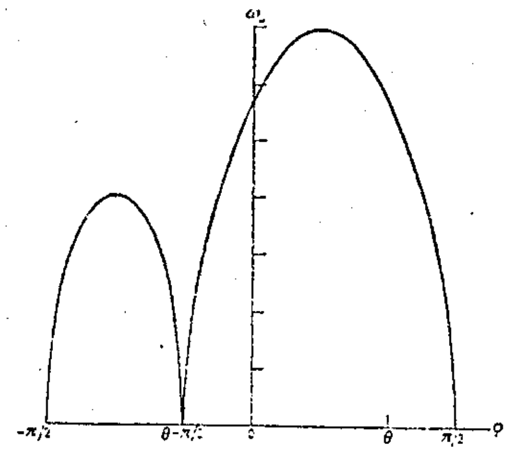

FIGURE 10. The dependence on ϕ of ω_*

$\omega_* = [3\sqrt{3}|\cos\phi\cos(\phi-\theta)|/2]^{\frac{1}{2}}$ ($= [3\sqrt{3}|\cos\theta+\cos(2\phi-\theta)|/4]^{\frac{1}{2}}$,(see equation (B5) and Figure 1). The maxima occur at $(\frac{1}{2}(\theta-\pi),\omega_1)$ and $(\frac{1}{2}\theta, \omega_2)$ (see Figure 2). There are three distinct acceptable roots of κ when $\omega > \omega_*$, one single and two coincident such roots when $\omega = \omega_*$ and one root only when $\omega < \omega_*$ (cf. Figures 3–5).

APPENDIX C. EXPRESSIONS FOR THE GROUP VELOCITY

It may be shown by differentiating equations (2.1), (2.5) and (B2) that

$$U_x = \frac{\partial\omega}{\partial k} = \delta^{-1}\left[2\cos\theta(k\cos\theta+l\sin\theta)-\frac{\omega(l^2-k^2)}{(k^2+l^2)^2}\right], \quad \ldots \text{(C}$$

$$U_y = \frac{\partial\omega}{\partial l} = \delta^{-1}\left[\frac{2\omega kl}{(k^2+l^2)^2}-2\sin\theta(k\cos\theta+l\sin\theta)\right], \quad \ldots \text{(C}$$

where

$$\delta = 2\omega+k/(k^2+l^2); \quad \ldots \text{(C}$$

$$U_X = \frac{\partial\omega}{\partial K} = \Delta^{-1}\left[2K-\frac{\omega}{(K^2+L^2)^2}\left[(L^2-K^2)\cos\theta+2KL\sin\theta\right]\right], \quad \ldots \text{(}$$

$$U_Y = \frac{\partial\omega}{\partial L} = \Delta^{-1}\left[2KL\cos\theta-(L^2-K^2)\sin\theta\right], \quad \ldots \text{(}$$

where

$$\Delta = 2\omega+(K\cos\theta-L\sin\theta)/(K^2+L^2); \quad \ldots \text{(}$$

and

$$U_r = \frac{\partial\omega}{\partial\kappa} = \delta_1^{-1}\left[\frac{\omega\cos\phi}{\kappa^2}+2\kappa\cos^2(\phi-\theta)\right], \quad \ldots \text{(}$$

$$U_\phi = \frac{1}{\kappa}\frac{\partial\omega}{\partial\phi} = \delta_1^{-1}\left[\frac{\omega\sin\phi}{\kappa^2}+2\kappa\cos(\phi-\theta)\sin(\phi-\theta)\right], \quad \ldots \text{(}$$

where

$$\delta_1 = 2\omega+\cos\phi/\kappa; \quad \ldots \text{(}$$

see equation (3.1).

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