INTERNATIONAL ATOMIC ENERGY AGENCY
UNITED NATIONS EDUCATIONAL, SCIENTIFIC AND CULTURAL ORGANIZATION

INTERNATIONAL CENTRE FOR THEORETICAL PHYSICS
34100 TRIESTE (ITALY) - P.O.B. 586 - MIRAMARE - STRADA COSTIERA 11 - TELEPHONES: 224281/2/3/4/5/6
CABLE: CENTRATOM - TELEX 460392-I

SMR/98 - 21

AUTUMN COURSE ON GEOMAGNETISM, THE IONOSPHERE
AND MAGNETOSPHERE

(21 September - 12 November 1982)

BASIC EQUATIONS AND THEOREMS OF M.H.D. OF ROTATING FLUIDS

R. HIDE
Geophysical Fluid Dynamics Laboratory
Meteorological Office
Bracknell
Berks RG12 2SZ
U.K.

2. Basic equations

2.1. *Introduction*

The basic equations of hydromagnetics can be found in several textbooks and reviews (eg Alfvén and Fälthammar 1963, Cowling 1957, 1962, Roberts 1967, Shercliff 1965); here we follow the treatment of Hide and Roberts (1962). Non-relativistic equations suffice for most problems and as the relativistic equations are quite complicated their full presentation here would serve no useful purpose (but see § 2.4 and Hide and Roberts 1962).

2.2. *Hydrodynamic and thermodynamic equations*

The continuity and momentum equations governing the flow of a Newtonian fluid of density ρ, coefficient of shear viscosity $\rho\nu$, and coefficient of bulk viscosity $\rho\nu'$ relate the values of pressure (p), fluid velocity ($\boldsymbol{u}$) and body force ($\boldsymbol{F}$) at a general point in space with position vector $\boldsymbol{r}$ at time t. They are, when $\boldsymbol{r}$ is measured relative to an inertial frame, the following:

$$\frac{\mathrm{D}\rho}{\mathrm{D}t} \equiv \frac{\partial\rho}{\partial t} + \boldsymbol{u}.\nabla\rho = -\rho\nabla.\boldsymbol{u} \tag{2.1}$$

and

$$\rho\frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t} \equiv \rho\left(\frac{\partial\boldsymbol{u}}{\partial t} + (\boldsymbol{u}.\nabla)\boldsymbol{u}\right)$$
$$= -\nabla\{p + \rho(\nu' - \tfrac{2}{3}\nu)\operatorname{div}\boldsymbol{u}\} - \nabla\times(\rho\nu\nabla\times\boldsymbol{u}) + \boldsymbol{F}. \tag{2.2}$$

We shall be largely concerned with conditions when the speed of fluid flow is much less than that of sound in the medium and when accelerations are slow compared with those associated with sound waves. The continuity equation (2.1) then reduces to that appropriate to the case of an 'incompressible' fluid, namely

$$\nabla.\boldsymbol{u} = 0 \tag{2.3}$$

and the first term on the right-hand side of equation (2.2) reduces to $-\nabla p$.

In the case of a conducting fluid carrying an electric current, density $\boldsymbol{j}$, in the presence of a magnetic field $\boldsymbol{B}$, to the usual body force we must add the Lorentz force $\boldsymbol{j}\times\boldsymbol{B}$. Thus

$$\boldsymbol{F} = -\rho\nabla\Phi + \boldsymbol{j}\times\boldsymbol{B} \tag{2.4}$$

where Φ is the gravitational potential (cf equation (2.19)).

The seven scalar equations to which (2.1), (2.2) and (2.4) are equivalent contain fourteen unknowns, and must therefore be supplemented with further mathematical relations. These relations stem from thermodynamic and electrodynamic considerations. The thermodynamic relations comprise an equation of state together with statements concerning irreversible transport processes, such as diffusion, thermal conduction and radiation. The electrodynamic relations are Maxwell's equations (or, more precisely, the 'pre-Maxwell' equations, see equations (2.10)–(2.13)) together with a statement concerning the dependence of the current on the electric field present (eg Ohm's law). We shall assume in this article that Φ is a known function of the space coordinates, thus excluding processes such as Jeans instability leading to gravitational condensation (but see Chandrasekhar 1954a, 1961 and Lynden-Bell 1966).

We can illustrate the thermodynamic relations required for a typical compressible fluid by considering a perfect gas for which

$$p/\rho T = \text{constant} \tag{2.5}$$

where T denotes temperature. Two extreme cases will be cited to indicate the range of possibilities as regards entropy variations, without having to write down the field equations governing irreversible transport processes. The first is the isentropic case, in which changes of state are so rapid that transport processes can be ignored so that the entropy per unit mass

$$c_v \ln(p\rho^{-\gamma}) \tag{2.6}$$

(where c_v is the specific heat at constant volume, c_p is the specific heat at constant pressure, and $\gamma = c_p/c_v$) of a fluid element remains constant. At the other extreme, the isothermal case, transport processes are so efficient that the temperature of a fluid element remains constant.

It is convenient to consider two types of incompressible fluid, namely those which are 'barotropic' and those which are 'baroclinic'. Barotropic incompressible fluids have uniform density and in consequence the gravitational contribution to $\boldsymbol{F}$ cannot exert a torque. In the absence of hydromagnetic effects, hydrodynamic flow of a barotropic fluid has to be generated by applying forces at the bounding surfaces of the fluid. When ρ is kept constant in equations (2.1) and (2.2), the thermodynamic relations are redundant.

Baroclinicity is associated with density variations, the action of gravity on which gives rise to buoyancy forces (see the first term on the right-hand side of equation (2.4)). Owing to the torque exerted by these forces, it is not generally possible to maintain hydrostatic equilibrium in their presence and hydrodynamic flow must ensue. Baroclinicity arises in a variety of ways; it may be due to variations in temperature, chemical composition, or both.

Differential heating produces temperature variations from place to place in the fluid and owing to thermal expansion they give rise to density variations (see § 6). If the heated incompressible fluid has a volume coefficient of thermal expansion ϑ the equation of state is

$$\rho = \rho_0(1 - \vartheta T) \tag{2.7}$$

where ρ_0 is the density at $T = 0$. Entropy changes are taken into account by including the equation of heat transfer

$$\rho\frac{\mathrm{D}}{\mathrm{D}t}(c_v T) \equiv \rho\left(\frac{\partial}{\partial t}(c_v T) + (\boldsymbol{u}.\nabla)(c_v T)\right)$$
$$= \nabla.(K\rho c_v \nabla T) + Q \tag{2.8}$$

where K is the effective thermal diffusivity, equal to the coefficient of thermal conductivity divided by ρc_v, and Q is the rate of internal heating per unit volume, including radiative effects which could be negative (see Chandrasekhar 1961).

2.3. *Electrodynamic equations*

Now we must write down equations relating current density $\boldsymbol{j}$, magnetic field $\boldsymbol{B}$, electric field $\boldsymbol{E}$ and charge density φ. If

$$\boldsymbol{H} \equiv \boldsymbol{B}/\mu \tag{2.9a}$$

and

$$\boldsymbol{D} \equiv \epsilon\boldsymbol{E} \tag{2.9b}$$

where μ and ϵ denote, respectively, magnetic permeability and dielectric constant, then the equations are

$$\nabla \times \boldsymbol{H} = \boldsymbol{j} \tag{2.10}$$

$$\nabla \times \boldsymbol{E} = -\partial \boldsymbol{B}/\partial t \tag{2.11}$$

$$\nabla . \boldsymbol{B} = 0 \tag{2.12}$$

$$\nabla . \boldsymbol{D} = \varphi . \tag{2.13}$$

Equation (2.10) is Ampère's circuital law relating the magnetic field to its basic source, the electric current, Maxwell's displacement current being neglected (cf equation (2.17) below) and equation (2.12) expresses the fact that the magnetic field is solenoidal. Equation (2.11) is Faraday's law of induction which, in its differential form, contains many subtle difficulties of interpretation brought out clearly in relatively few standard texts (see Alfvén and Fälthammar 1963). Equations (2.13) and (2.9*b*) relate the electric field to the volume density of electric charge φ.

A unit electric charge moving at velocity $\boldsymbol{u}$ relative to a magnetic field $\boldsymbol{B}$ experiences a force $\boldsymbol{E}+\boldsymbol{u}\times\boldsymbol{B}$. Thus, if the conducting fluid satisfies Ohm's law with conductivity σ, then

$$\boldsymbol{j} = \sigma(\boldsymbol{E}+\boldsymbol{u}\times\boldsymbol{B}). \tag{2.14}$$

This completes our set of equations governing seven unknown vector quantities, $\boldsymbol{u}$, $\boldsymbol{F}$, $\boldsymbol{E}$, $\boldsymbol{D}$, $\boldsymbol{H}$, $\boldsymbol{B}$, $\boldsymbol{j}$ and four unknown scalar quantities p, ρ, T and φ, or twenty-five scalars in all.

2.4. *Range of validity*

Now we must consider the range of validity of our equations, and for this purpose we introduce a typical absolute flow speed W^* (relative to an inertial frame, cf equation (1.4)), the ordinary sound speed a_0 and the speed of electromagnetic waves $c_0 = (\mu\epsilon)^{-1/2}$. The ratios $(W^*/c_0)^2$, $(a_0/c_0)^2$ and $(V^*/c_0)^2$ (see equation (1.2)) are respective measures of the ordered kinetic energy, thermal energy and magnetic energy in terms of the rest energy of the fluid.

When $W^*/c_0 \ll 1$ the D/Dt terms in equations (2.1) and (2.2) are nonrelativistic. To the same approximation, although the effective electric field $\boldsymbol{E}+\boldsymbol{u}\times\boldsymbol{B}$ depends on the local frame of reference in which it is measured, the magnetic field is frame independent. When $a_0/c_0 \ll 1$, that is to say when the root mean square speed of thermal motion is much less than the speed of light, the relativistic correction p/c_0^2 to the density ρ is negligible.

If τ^* is a typical time scale associated with the hydromagnetic flow, L^* being a typical length, then by equations (2.9), (2.10), (2.11) and (2.13)

$$\left.\begin{aligned} \left|\frac{\partial \boldsymbol{D}}{\partial t}\right| &\simeq \left(\frac{L^*}{\tau^* c_0}\right)^2 |\boldsymbol{j}| \\ |\varphi \boldsymbol{E}| &\simeq \left(\frac{L^*}{\tau^* c_0}\right)^2 |\boldsymbol{j}\times\boldsymbol{B}| \\ |\varphi \boldsymbol{u}| &\simeq \left(\frac{L^*}{\tau^* c_0}\right)^2 |\boldsymbol{j}|. \end{aligned}\right\} \tag{2.15}$$

Thus, when τ^* is much greater than the time taken for electromagnetic waves to traverse a distance L^* the neglect of the displacement current $\partial\boldsymbol{D}/\partial t$ in equation (2.10), of the electrostatic body force $\varphi\boldsymbol{E}$ in the expression for the body force (equation (2.4)), and of the advective contribution $\varphi\boldsymbol{u}$ to $\boldsymbol{j}$ in equation (2.14) is justified. However, when our equations are nonrelativistic (ie $W^*/c_0 \ll 1$ and $a_0/c_0 \ll 1$) but V^* is not much less than c_0, then it may be shown that equations (2.4), (2.10) and (2.14) become

$$\boldsymbol{F} = -\rho\nabla\Phi + \boldsymbol{j}\times\boldsymbol{B} + \varphi\boldsymbol{E} \tag{2.16}$$

$$\nabla\times\boldsymbol{H} = \boldsymbol{j} + \frac{\partial \boldsymbol{D}}{\partial t} \tag{2.17}$$

$$\boldsymbol{j} = \sigma(\boldsymbol{E}+\boldsymbol{u}\times\boldsymbol{B}) + \varphi\boldsymbol{u}. \tag{2.18}$$

We see that the neglect of terms of order $(V^*/c)^2$ in equations (2.4), (2.10) and (2.14) filters out electromagnetic waves. Plasma oscillations are automatically excluded by taking Ohm's law (equation (2.14), cf Spitzer 1956) as the relationship between the current and the effective electric field.

In addition to supposing that relativistic effects can be neglected we have assumed implicitly that the fluid can be regarded as a continuum with isotropic transport coefficients ν, K and σ. This procedure is valid for sufficiently dense media, but in the case of a tenuous fluid continuum theory breaks down (see Spitzer 1956). It would, however, take us too far away from our main topic even to summarize how criteria for the validity of continuum theory can be deduced from the equations governing the motions of individual ions, electrons and neutral particles.

2.5. *Equations referred to a rotating frame*

When referred to a frame that rotates with instantaneous angular velocity $\boldsymbol{\Omega}$ relative to an inertial frame, the equations of motion and continuity for an incompressible fluid of uniform coefficient of viscosity $\nu\rho$ become

$$\begin{aligned}\rho\{\partial\boldsymbol{u}/\partial t + (\boldsymbol{u}.\nabla)\boldsymbol{u} + 2\boldsymbol{\Omega}\times\boldsymbol{u}\} \\ = -\nabla p + \boldsymbol{j}\times\boldsymbol{B} + \nu\rho\nabla^2\boldsymbol{u} - \rho\nabla\Phi + \rho\boldsymbol{r}\times \mathrm{d}\boldsymbol{\Omega}/\mathrm{d}t\end{aligned} \tag{2.19}$$

and

$$\nabla . \boldsymbol{u} = 0. \tag{2.20}$$

Here Φ now includes the centrifugal term $-\frac{1}{2}|\boldsymbol{\Omega}\times\boldsymbol{r}|^2$ as well as gravitational effects, $\boldsymbol{r}$ being a vector from the origin to a general point P at which the eulerian fluid velocity vector, now measured relative to the rotating frame (see Kibble 1966), is $\boldsymbol{u}$, and the corresponding pressure, magnetic field and electric current density are p, $\boldsymbol{B}$ and $\boldsymbol{j}$ respectively. The last term in equation (2.19) vanishes when $\boldsymbol{\Omega}$ is steady.

If the density of the fluid depends on temperature only then equation (2.7) holds, and if variations in $K\rho c_v$ and c_v are negligible, equation (2.8) reduces to

$$\frac{\partial T}{\partial t} + (\boldsymbol{u}.\nabla)T = K\nabla^2 T + q \tag{2.21}$$

where $q = Q/\rho c_v$.

Equations (2.19) and (2.20) hold when $|\boldsymbol{\Omega}\times\boldsymbol{r}+\boldsymbol{u}|$, the speed of absolute motion, is everywhere much less than the speed of sound. Provided that $|\boldsymbol{\Omega}\times\boldsymbol{r}+\boldsymbol{u}|$ is much less than the speed of light (see § 2.4) the electrodynamic equations (2.10) to (2.14) are the same in the rotating frame as in the nonrotating frame (Trocheris 1949).

When $\boldsymbol{j}$ and $\boldsymbol{E}$ are eliminated from equations (2.9*a*), (2.10), (2.11) and (2.14) we have

$$\frac{\partial \boldsymbol{B}}{\partial t} = \nabla\times(\boldsymbol{u}\times\boldsymbol{B})+\eta\nabla^2\boldsymbol{B}. \tag{2.22}$$

Here

$$\eta = (\mu\sigma)^{-1} \tag{2.23}$$

which has the same dimensions as ν and K, namely (length)^2(time)$^{-1}$. According to equation (2.22) the rate of change of $\boldsymbol{B}$ depends on two agencies, namely motional induction, represented by the term $\nabla\times(\boldsymbol{u}\times\boldsymbol{B})$, and ohmic dissipation due to electrical resistance, represented by the term $\eta\nabla^2\boldsymbol{B}$. The magnetic Reynolds number

$$\mathscr{S} \equiv U^*L^*\mu\sigma = U^*L^*/\eta \tag{2.24}$$

is a measure of the relative importance of these two agencies. When $\mathscr{S}\ll 1$, equation (2.22) reduces to the diffusion equation, with solutions corresponding to magnetic fields decaying on the timescale L^{*2}/η, which is typically $\sim 3\times 10^{11}$ s ($\sim 10^4$ yr) for the liquid core of the Earth. When, on the other hand, $\mathscr{S}\gg 1$, diffusion effects are negligible and the magnetic lines of force move with the fluid (Alfvén's theorem, see § 3.3 below). The quantitative basis for the dynamo theory of the Earth's magnetism (Bullard 1949, Elsasser 1946a,b, 1947, Parker 1955), investigations of which have been largely though not entirely concerned with the so-called 'kinematic dynamo problem' in which $\boldsymbol{u}$ is specified *a priori* when solving equation (2.22) (for up to date references see Moffatt (1972, 1973) and reviews by Roberts (1971), Roberts and Stix (1971) and Weiss (1971)), is the recognition that motions as slow as 10^{-3} m s^{-1} give $\mathscr{S}\sim 100$ in the core of the Earth. In these circumstances the magnetic energy increases at the expense of the kinetic energy of motion, the final balance being attained when the rate at which work is done on the system by the forces that drive the motion is offset by dissipation due to electrical resistivity and viscosity. For the Earth this dissipation rate is $\sim 3\times 10^{10}$ J s^{-1}, some two or three powers of ten less than the energy of thermal sources within the Earth (but evidently comparable, incidentally, with energy released by earthquakes and with the rate of working of the forces that move the continental plates). The source of energy for core motions has not yet been identified with certainty, but mechanical stirring associated with the Earth's precessional motion (cf the $\boldsymbol{r}\times \mathrm{d}\boldsymbol{\Omega}/\mathrm{d}t$ term in equation (2.19)) or thermal stirring due to radioactive heating or to the release of heat of crystallization might suffice (see Jacobs *et al* 1972, Malkus 1968, 1971 and Stacey 1969 for recent discussions of this question and references to earlier work).

2.6. *Energy and vorticity equations*

It is instructive to consider the energetics of hydromagnetic flows (see Chandrasekhar 1961, Hide 1956, Hide and Roberts 1962, Roberts 1967). Multiplying equation (2.22) scalarly by $\mu^{-1}\boldsymbol{B}$ we find, after some manipulation making

use of equations (2.10) and (2.14), that

$$\frac{\partial}{\partial t}\left(\frac{B^2}{2\mu}\right) = -\frac{j^2}{\sigma}-\frac{1}{\mu}\nabla.(\boldsymbol{E}\times\boldsymbol{B})-(\boldsymbol{j}\times\boldsymbol{B}).\boldsymbol{u}. \tag{2.25}$$

The left-hand side is the time rate of change of magnetic energy density, while on the right-hand side the first term represents decay of magnetic energy due to ohmic heating, the second term is the Poynting effect and represents the flux of electromagnetic energy across the boundaries of the (elementary) region considered, and the third term is a measure of effects due to motional induction. We observe that $\mathscr{S}$, the magnetic Reynolds number, is a measure of the third term divided by the first term.

We can deduce an expression for $(\boldsymbol{j}\times\boldsymbol{B}).\boldsymbol{u}$ by multiplying equation (2.19) scalarly by $\boldsymbol{u}$. On combining the resulting equation with equation (2.25) we find for the rate of change of total energy density in the system, magnetic plus kinetic, the following expression:

$$\begin{aligned}\frac{\partial}{\partial t}\left(\frac{B^2}{2\mu}+\frac{1}{2}\rho u^2\right) = &-\left\{\frac{j^2}{\sigma}+\frac{1}{\mu}\nabla.(\boldsymbol{E}\times\boldsymbol{B})\right\}\\ &-\left\{\nu\rho|\nabla\times\boldsymbol{u}|^2+\nu\rho\nabla.(\nabla\times\boldsymbol{u})\times\boldsymbol{u}\right\}-\nabla.\left\{\left(p+\frac{1}{2}\rho u^2+\rho\Phi\right)\boldsymbol{u}\right\}\\ &+\left\{\Phi\boldsymbol{u}.\nabla\rho+\frac{1}{2}u^2\frac{\mathrm{D}\rho}{\mathrm{D}t}\right\}+\rho(\boldsymbol{u}\times\boldsymbol{r}).\frac{\mathrm{d}\boldsymbol{\Omega}}{\mathrm{d}t}.\end{aligned} \tag{2.26}$$

The first group of terms on the right-hand side comprise the electromagnetic contributions already discussed. Combined in the second group are terms representing viscous dissipation and work done by tangential viscous forces. Terms representing the rate of working of normal pressure forces and the advection of kinetic and gravitational energy into the elementary fluid volume make up the third group. The first contribution to the fourth group of terms represents the internal release of gravitational potential energy, which vanishes when ρ is uniform. This contribution is much greater than $\frac{1}{2}u^2\,\mathrm{D}\rho/\mathrm{D}t$ when the gravitational field is so large that density variations are negligible in all terms of the equation of motion (equation (2.19)) save the buoyancy term ('Boussinesq' approximation, see Spiegel and Veronis 1960). Finally, we have the only term directly involving $\boldsymbol{\Omega}$ (for $\boldsymbol{u}.2\boldsymbol{\Omega}\times\boldsymbol{u}=0$), and this vanishes when $\mathrm{d}\boldsymbol{\Omega}/\mathrm{d}t=0$. It represents the rate at which work is done on the system by the forces that produce time variations in $\boldsymbol{\Omega}$, and in the case of the Earth's core is a measure of the contribution to the hydromagnetic energy of the Earth made by the gravitational torques that cause the Earth's rotation axis to precess about an axis fixed in space.

The vorticity equation, obtained by taking the curl of equation (2.19) making use of the full continuity equation (2.1), may usefully be written for future reference in §§ 6.1–6.5 (where nonhomogeneous fluids are considered) as follows:

$$\begin{aligned}&(2\boldsymbol{\Omega}.\nabla)(\rho\boldsymbol{u})+\nabla\times(\boldsymbol{j}\times\boldsymbol{B})+(\nabla\Phi)\times(\nabla\rho)\\ &\quad= -2\boldsymbol{\Omega}\,\partial\rho/\partial t+\nabla\times\{\rho\,\partial\boldsymbol{u}/\partial t+\rho(\boldsymbol{u}.\nabla)\,\boldsymbol{u}-\nu\rho\nabla^2\boldsymbol{u}-\rho\boldsymbol{r}\times\mathrm{d}\boldsymbol{\Omega}/\mathrm{d}t\}.\end{aligned} \tag{2.27}$$

However, for the problems treated in §§ 3–5, *where attention is confined to fluids of constant density* (ie $\nabla\rho=\partial\rho/\partial t=0$) *for which the angular velocity of basic rotation is* *steady* (ie $d\boldsymbol{\Omega}/dt = 0$), it follows from the last equation that the relative vorticity $\boldsymbol{\xi} = \nabla \times \boldsymbol{u}$ satisfies

$$\partial\boldsymbol{\xi}/\partial t + (\boldsymbol{u}.\nabla)\boldsymbol{\xi} - \{(\boldsymbol{\xi}+2\boldsymbol{\Omega}).\nabla\boldsymbol{u}\} = \nabla\times\{(\boldsymbol{j}\times\boldsymbol{B})/\rho\} + \nu\nabla^2\boldsymbol{\xi}. \qquad (2.28)$$

As we shall see in what follows next in §§ 3.1–3.5, equations (2.28) and (2.22) lead to several useful general theorems and concepts for rapidly rotating nonhydromagnetic systems (Proudman–Taylor theorem, Ertel's potential vorticity theorem, 'geostrophic flow') and for corresponding hydromagnetic systems (Alfven's theorem, 'aligned field' flows, a hydromagnetic potential vorticity theorem, 'magnetostrophic flow').

3. Some theorems

3.1. *Proudman–Taylor theorem and geostrophy*

If, in addition to making the foregoing simplifications, we can suppose that the motion (relative to the rotating frame) is so steady and slow and the hydromagnetic and viscous forces so tiny that all the following dimensionless parameters tend to zero

$$\mathscr{A} = \overline{\rho\,\partial\boldsymbol{u}/\partial t}\big/\overline{2\rho\,\boldsymbol{\Omega}\times\boldsymbol{u}} \qquad (3.1)$$

$$\mathscr{R} = \overline{\rho(\boldsymbol{u}.\nabla)\,\boldsymbol{u}}\big/\overline{2\rho\,\boldsymbol{\Omega}\times\boldsymbol{u}} \qquad (3.2)$$

(see equation (1.4))

$$\mathscr{M} = \overline{\boldsymbol{j}\times\boldsymbol{B}}\big/\overline{2\rho\,\boldsymbol{\Omega}\times\boldsymbol{u}} \qquad (3.3)$$

and

$$\mathscr{E} = \overline{\rho\nu\nabla^2\boldsymbol{u}}\big/\overline{2\rho\,\boldsymbol{\Omega}\times\boldsymbol{u}} \qquad (3.4)$$

(where the overbar denotes the root mean square value), then equation (2.19) takes the very simple form

$$2\rho\,\boldsymbol{\Omega}\times\boldsymbol{u} = -\nabla p - \rho\nabla\Phi. \qquad (3.5)$$

This expresses a 'geostrophic' balance between the Coriolis force (per unit volume) and the nonhydrostatic part of the pressure gradient. The corresponding vorticity equation (cf equation (6.8)) is, by equation (2.28),

$$2(\boldsymbol{\Omega}.\nabla)\,\boldsymbol{u} = 0. \qquad (3.6)$$

This is the celebrated 'two-dimensional' theorem due originally to Proudman (1916) and Taylor (1921) (see Greenspan 1968), which shows that all components of $\boldsymbol{u}$ are independent of the coordinate parallel to $\boldsymbol{\Omega}$ (see § 6.2 for the extension of this result to the case $\nabla\rho \neq 0$). The Proudman–Taylor theorem together with certain compatibility conditions to be satisfied where an effectively inviscid 'interior' flow meets a viscous Ekman layer on a bounding surface (see equation (5.31)) are concepts of central importance which greatly simplify the theoretical analysis of complex rotating *non*hydromagnetic systems. (There is an analogy here with certain theorems satisfied by slow motions of a nonrotating fluid pervaded by a nearly uniform magnetic field (Hide 1956, Hide and Roberts 1962, Hunt and Ludford 1968, Kulikovsky 1968, Lundquist 1952; see also § 6.4) and compatibility conditions imposed by the presence of viscous Hartmann layers on bounding surfaces (see equation (5.25)).

The gyroscopic forces due to the rapid basic rotation thus have the effect of aligning elementary vortices so that their axes are parallel to the rotation axis. Displacement of these axes would produce restoring forces resulting in 'inertial' oscillations. In circumstances when these oscillations can be described in terms of a superposition of plane waves of small amplitude proportional to $\exp i(\boldsymbol{\kappa}.\boldsymbol{r} - \omega t)$, the corresponding dispersion relationship connecting the angular frequency ω to the wavenumber vector $\boldsymbol{\kappa}$ is

$$\omega^2 = (2\boldsymbol{\Omega}.\boldsymbol{\kappa})^2/\kappa^2 \qquad (3.7)$$

(see eg Batchelor 1967, Lighthill 1966, 1967 and § 4.1). Particle orbits in these inertial oscillations are circular and lie in the plane of the wavefront (see figure 2).

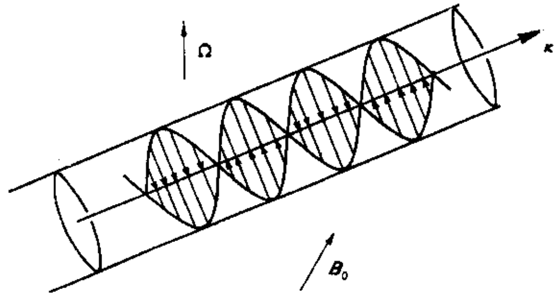

Figure 2. Illustrating the velocity field in a single plane wave of wavenumber $\boldsymbol{\kappa}$ in an unbounded rotating fluid with or without a corotating uniform magnetic field $\boldsymbol{B}_0$. Particle orbits are circular and lie in the plane of the wavefront. After Moffatt (1970).

Observe that when $\omega = 0$ (steady motion) the wavenumber vector $\boldsymbol{\kappa}$ is perpendicular to $\boldsymbol{\Omega}$, in keeping with the 'two-dimensional' theorem expressed by equation (3.6).

The foregoing results have proved useful in the discussion of effects of irregularly shaped boundaries on flows of rapidly rotating fluids. We conclude this section with a basic result concerning the effect of rotation on flows which, in virtue of the boundary conditions and method of generation, are strictly two-dimensional in planes perpendicular to the axis of rotation.

Consider the vorticity equation (2.28) when $(2\boldsymbol{\Omega}.\nabla)\,\boldsymbol{u} \equiv 0$. Thus we have (when $\boldsymbol{B} = 0$)

$$\partial\boldsymbol{\xi}/\partial t + (\boldsymbol{u}.\nabla)\boldsymbol{\xi} - (\boldsymbol{\xi}.\nabla)\,\boldsymbol{u} - \nu\nabla^2\boldsymbol{\xi} = 0 \qquad (3.8)$$

which shows that *if the boundary conditions are independent of* $\boldsymbol{\Omega}$ then so is the two-dimensional field of relative motion. This result, which was first enunciated by Taylor (1917), and its extension to cases when it *cannot* be supposed that the boundary conditions are independent of $\boldsymbol{\Omega}$, have been the subject of several theoretical and experimental studies (Hide 1968, see also Greenspan 1968).

Since $\boldsymbol{\Omega}$ does not appear in the electromagnetic equations (see § 2.4), Taylor's result carries over trivially to the hydromagnetic case provided, of course, that the electromagnetic as well as the mechanical boundary conditions are now independent of $\boldsymbol{\Omega}$. Thus, problems in which the Coriolis torque $(2\boldsymbol{\Omega}.\nabla)\,\boldsymbol{u}$ vanishes *by hypothesis* fall into a special category, for then there is no stretching or bending of fluid filaments that would couple the vorticity of the basic rotation to the vorticity of the relative flow and the novel phenomena with which this article is largely concerned do not then arise (see eg Causse and Poirier 1960, Poirier and Robert 1960, cf Loper and Benton 1970), although other interesting phenomena are by no means excluded (see eg Moffatt 1965).

3.2. *Ertel's potential vorticity theorem*

We define the 'potential vorticity' as

$$\mathscr{Z} \equiv \rho^{-1}(\boldsymbol{\xi}+2\boldsymbol{\Omega}).\nabla\Lambda \tag{3.9}$$

where Λ is any scalar quantity that is conserved by individual fluid elements throughout their motion (ie $\mathrm{D}\Lambda/\mathrm{D}t \equiv \partial\Lambda/\partial t + \boldsymbol{u}.\nabla\Lambda = 0$). A theorem, due originally to Ertel (1942, see also Batchelor 1967, Ertel and Rossby 1949, Greenspan 1968, Lighthill 1966, Pedlosky 1971), can be obtained by multiplying the vorticity equation scalarly by $\nabla\Lambda$. When $\nabla\rho = 0$ and hydromagnetic effects vanish, this gives

$$\frac{\mathrm{D}}{\mathrm{D}t}\{(\boldsymbol{\xi}+2\boldsymbol{\Omega}).\nabla\Lambda\} = \nu\nabla\Lambda.\nabla^2\boldsymbol{\xi}. \tag{3.10}$$

When $\nu = 0$ the quantity $\mathscr{Z}$ is conserved by individual fluid elements throughout their motion; an application of this result and of a hydromagnetic extension (see § 3.4) is given in § 4.5 below, where oscillations of spherically bounded rotating systems are considered.

3.3. *Alfvén's theorem*

We turn now to the hydromagnetic situation, but before examining the equation of motion it is instructive to consider the equation (2.22) for $\boldsymbol{B}$. In the limit when the magnetic Reynolds number $\mathscr{S} \to \infty$ (see equation (2.24)), equation (2.22) reduces to

$$\frac{\partial \boldsymbol{B}}{\partial t} + (\boldsymbol{u}.\nabla)\boldsymbol{B} - (\boldsymbol{B}.\nabla)\boldsymbol{u} = 0 \tag{3.11}$$

(since $\nabla.\boldsymbol{u} = \nabla.\boldsymbol{B} = 0$, see equations (2.20) and (2.12)). The line integral of equation (2.22) around a material circuit expresses the impossibility when $\sigma \to \infty$ of changing the flux linkage of such a circuit. The lines of force are then effectively 'frozen' into the fluid (Alfvén 1942a,b). So far as their mechanical effects are concerned (see below) the lines of force behave like elastic strings and permit wave motions which, in their simplest form (ie no Coriolis forces, etc), are plane 'Alfvén waves' satisfying the dispersion relationship

$$\omega^2 = (\boldsymbol{V}.\boldsymbol{\kappa})^2 \tag{3.12}$$

(cf equation (3.7) and § 4.1). Here

$$\boldsymbol{V} = \frac{\boldsymbol{B}_0}{(\mu\rho)^{1/2}} \tag{3.13}$$

is the 'Alfvén velocity' based on the undisturbed magnetic field (cf equation (1.2)). Particle orbits in these Alfvén waves are linear and lie in planes parallel to the wavefronts. In contrast to inertial waves (see equation (3.7)), Alfvén waves are non-dispersive. Their group velocity $\partial\omega/\partial\boldsymbol{\kappa}$ is equal to $\pm\boldsymbol{V}$.

3.4. *'Aligned field' flows*

We now consider the *dynamical* aspects of hydromagnetic rotating fluid flows, and it is of some interest to examine first the restrictive but important class of motions for which the velocity and magnetic fields *as measured relative to the rotating frame* are everywhere parallel (Acheson 1971, cf Hasimoto 1959). Thus we write

$$\boldsymbol{B} = C\boldsymbol{u}(\mu\rho)^{1/2} \tag{3.14}$$

where C is a dimensionless scalar. Such flows are necessarily steady when C is uniform and the fluid is a perfect conductor of electricity, for then by equations (3.11) and (3.14) $\partial\boldsymbol{B}/\partial t = 0$. The equation of motion (2.19) then reduces to

$$(1-C^2)(\boldsymbol{u}.\nabla)\boldsymbol{u} + 2\boldsymbol{\Omega}\times\boldsymbol{u} = -\rho^{-1}\nabla(p+\tfrac{1}{2}\mu^{-1}B^2) - \nabla\Phi + \nu\nabla^2\boldsymbol{u}. \tag{3.15}$$

Further, in view of equation (3.14), equations (2.3) and (2.12), $\nabla.\boldsymbol{u} = 0$ and $\nabla.\boldsymbol{B} = 0$ respectively, become identical. For such 'aligned field' flows the governing equations therefore reduce to a set *identical with the corresponding nonhydromagnetic equations* (consisting of equations (2.3) and (3.15) with $\boldsymbol{B} = C = 0$), provided only that we make the transformations

$$(\boldsymbol{\Omega}_*, p_*, \Phi_*, \nu_*) \equiv (\boldsymbol{\Omega}, p+\tfrac{1}{2}\mu^{-1}B^2, \Phi, \nu)/(1-C^2), \tag{3.16}$$

for equation (3.15) may then be written

$$(\boldsymbol{u}.\nabla)\boldsymbol{u} + 2\boldsymbol{\Omega}_*\times\boldsymbol{u} = -\rho^{-1}\nabla p_* - \nabla\Phi_* + \nu_*\nabla^2\boldsymbol{u}. \tag{3.17}$$

For the Proudman–Taylor theorem to hold for such flows it is evidently necessary for $|\mathscr{R}(1-C^2)|$ and $\mathscr{E}$ to be much less than unity, a condition which may be much more restrictive than $\mathscr{R} \ll 1$ and $\mathscr{E} \ll 1$ (see equations (3.2) and (3.4)), especially when C, the ratio of the local Alfvén speed to the local fluid speed, is much greater than unity. The corresponding potential vorticity theorem takes the form

$$(\boldsymbol{u}.\nabla)\{(\boldsymbol{\xi}+2\boldsymbol{\Omega}_*).\nabla\Lambda\} = \nu_*\nabla\Lambda.\nabla^2\boldsymbol{\xi} \tag{3.18}$$

(cf equation (3.10)).

We have already noted that these 'aligned field' results do not in general apply to unsteady motions. Nevertheless, in cases when the unsteadiness can be removed by a simple transformation of coordinates (eg simple waves—see §§ 4.1 and 4.5) the above concepts can be both useful and illuminating.

3.5. *'Magnetostrophic' flows*

By analogy with geostrophic motion (see equation (3.5)) *magneto*strophic motion is defined as satisfying the equation

$$2\rho\boldsymbol{\Omega}\times\boldsymbol{u} - \boldsymbol{j}\times\boldsymbol{B} = -\nabla p - \rho\nabla\Phi. \tag{3.19}$$

In this case the dynamical pressure gradient (ie the actual pressure gradient plus $\rho\nabla\Phi$) is balanced by the difference between the Coriolis and Lorentz forces. The vorticity equation then becomes

$$2(\boldsymbol{\Omega}.\nabla)\boldsymbol{u} = -\frac{1}{\mu\rho}\nabla\times\{(\nabla\times\boldsymbol{B})\times\boldsymbol{B}\}, \tag{3.20}$$

expressing a balance between the gyroscopic and hydromagnetic torques acting on an individual fluid element.

In order to specify the conditions under which magnetostrophic flows can be expected to occur, we must consider the order of magnitude of the various terms in equation (2.22) for $\boldsymbol{B}$. Now

$$\left|\frac{\partial \boldsymbol{B}}{\partial t}\right| \sim \frac{B^*}{\tau^*},$$

$$|\nabla \times (\boldsymbol{u} \times \boldsymbol{B})| \sim \frac{U^* B^*}{L^*}$$

and

$$|\eta \nabla^2 \boldsymbol{B}| \sim \frac{\eta B^*}{L^{*2}}$$

if B^* and U^* are typical values of $|\boldsymbol{B}|$ and $|\boldsymbol{u}|$ respectively and L^* and τ^* are corresponding length and time scales. Hence

$$U^* \sim \left(\frac{L^*}{\tau^*} + \frac{\eta}{L^*}\right)$$

so that the Coriolis acceleration is of order

$$2\Omega\left(\frac{L^*}{\tau^*} + \frac{\eta}{L^*}\right)$$

in magnitude. Thus when the flow is 'magnetostrophic' we infer from equation (3.20) that

$$2\Omega\left(\frac{1}{\tau^*} + \frac{\eta}{L^{*2}}\right) \sim \frac{V^{*2}}{L^{*2}} \tag{3.21}$$

where V^* is given by equation (1.2). Hence

$$\frac{1}{\tau^*} \sim \frac{V^{*2}}{2\Omega L^{*2}} + \frac{\eta}{L^{*2}}$$

so that the neglect of $\partial \boldsymbol{u}/\partial t$, $(\boldsymbol{u}.\nabla)\boldsymbol{u}$ and $\nu \nabla^2 \boldsymbol{u}$ in deriving equation (3.19) from (2.19) is valid provided that

$$\left(\frac{V^{*2}}{2\Omega^2 L^{*2}} + \frac{\eta}{\Omega L^{*2}}\right) \ll 1$$

$$\frac{U^*}{\Omega L^*} \ll 1$$

and

$$\nu \ll \Omega L^{*2}.$$

In terms of the dimensionless parameters already introduced, these criteria are satisfied when

$$\left.\begin{aligned} &\mathscr{L}^2(1+\mathscr{C}^{-1}) \ll 1 \\ &\mathscr{R} \ll 1 \\ \text{and} \quad &\mathscr{E} \ll 1 \end{aligned}\right\} \tag{3.22}$$

(see equations (1.1)–(1.4) and equation (3.4)).

We see immediately from equation (3.22) that $\mathscr{C}^{-1}$ is the appropriate measure of dissipative effects due to ohmic heating. When $\mathscr{C} \gg 1$ (ie $\tau^* \ll L^{*2}/\eta$ as in the case of the geomagnetic secular variation; see § 1.2) the criteria for magnetostrophic flow become

$$\left.\begin{aligned} &\mathscr{L}^2 \ll 1 \\ &\mathscr{R} \ll 1 \\ \text{and} \quad &\mathscr{E} \ll 1., \end{aligned}\right\} \tag{3.23}$$

When on the other hand $\mathscr{C} \not\gg 1$ (ie $\tau^* \not\ll L^{*2}/\eta$, as in the case of the geomagnetic dynamo problem, see eg ~~Bullard and Gellman 1954,~~ Elsasser 1956), the criteria reduce to

$$\left.\begin{aligned} &\mathscr{L}^2 \ll \mathscr{C} \\ &\mathscr{R} \ll 1 \\ \text{and} \quad &\mathscr{E} \ll 1. \end{aligned}\right\} \tag{3.24}$$

We must note here that the satisfaction of these criteria is not sufficient to ensure that *all* modes of motion will be magnetostrophic, as evinced by the examples discussed below.

When dealing with wave motions we shall frequently suppose that the electrical resistivity of the fluid is so small that η may be taken to be zero. In order to appreciate the interesting (and sometimes apparently paradoxical) results that emerge in this limit it is helpful to consider equation (3.21). If $\eta = 0$ it is clear that *no matter how small the magnetic field may be* magnetostrophic flow may always occur (inasmuch as viscous forces can be neglected; see § 5.2 especially equation (5.30) and cf § 4.3, especially equation (4.35)) *if the timescale of the fluid motions is sufficiently long*. More precisely, as $V^* \to 0$ hydromagnetic and Coriolis forces still remain equally important if $\tau^* \sim 2\Omega L^{*2}/V^{*2}$. This persistence of hydromagnetic effects in the dynamics is a direct consequence of the assumption of perfect conductivity, for Alfvén's theorem (see equation (3.11)) follows directly from that assumption and *the permanency of attachment of the lines of force to the fluid is then in no way dependent on the strength of the magnetic field.* As soon as the fluid is permitted a nonzero electrical resistivity, however small, it is clear from equation (3.21) that no matter how long the timescale of the fluctuations in the fluid motion they cannot be characterized by a magnetostrophic balance in the limit $V^* \to 0$.

Finally, we note (for reference when meeting phenomena described in §§ 5.1 and 6.4) that *steady* magnetostrophic motions are characterized by values of the parameter $\mathscr{C}$, the Chandrasekhar number, of order unity.

The magnetic field in the liquid core of the Earth consists of two parts, a poloidal field $\boldsymbol{B}_{\text{pol}}$ with lines of force that penetrate the upper reaches of the solid Earth and pass through the Earth's surface into space, and a toroidal field $\boldsymbol{B}_{\text{tor}}$ which is largely confined to the core (as it would be completely if the surrounding mantle were a perfect insulator). Near the Earth's surface, where magnetic measurements are made, $B_{\text{pol}} \sim 5 \times 10^{-5}$ Wb m^{-2} (0·5 gauss) in magnitude and largely dipolar in character, so that $B_{\text{pol}} \sim 5 \times 10^{-4}$ Wb m^{-2} in the core. The toroidal field $\boldsymbol{B}_{\text{tor}}$ cannot be determined directly, but various lines of indirect evidence indicate that

$$B_{\text{tor}}/B_{\text{pol}} \sim 10^{1\cdot5\pm0\cdot5}.$$

Taking $\rho \sim 10^4\ \mathrm{kg\,m^{-3}}$ and $B \sim 10^{-2}\ \mathrm{Wb\,m^{-2}}$ we find that $V^* \sim 10^{-1}\ \mathrm{m\,s^{-1}}$ and

$$\mathscr{L} \sim 10^{-3} \tag{3.25}$$

for motions on the scale of the core (see table 1 and equation (1.1)) and $\mathscr{L} > 10^{-3}$ for any smaller scale motions that might be present. For comparison we note that for $U^* \sim 10^{-3}\ \mathrm{m\,s^{-1}}$, a crude estimate of the speed of core motions based on the displacement of the magnetic field pattern at the Earth's surface, we have

$$\mathscr{R} \gtrsim 10^{-5} \tag{3.26}$$

(see equation (1.4)). Finally we note that if the electrical conductivity of the core is $10^{5\cdot5\pm0\cdot5}\ \mathrm{ohm^{-1}\,m^{-1}}$ (see Gardiner and Stacey 1971) then $\eta \sim 10^{0\cdot5\pm0\cdot5}\ \mathrm{m^2\,s^{-1}}$ and

$$\mathscr{C} \sim 10^{1\cdot5\pm0\cdot5} \tag{3.27}$$

(see equation (1.3)).

Acknowledgments

We thank the Director-General of the Meteorological Office for permission to publish this review. One of us (DJA) is indebted to the Natural Environment Research Council for a Research Fellowship during the tenure of which this review was written.

References

Acheson D J 1971 *PhD Thesis* University of East Anglia
—— 1972a *J. Fluid Mech.* **52** 529–41
—— 1972b to be published
—— 1972c *J. Fluid Mech.* **53** 401–15
Akasofu S-I and Chapman S 1972 *Solar-terrestrial Physics* (Oxford: Clarendon Press)
Alfvén H 1942a *Ark. Mat. Astr. Fys.* **29** 1–7
—— 1942b *Nature, Lond.* **150** 405–6
Alfvén H and Fälthammar C G 1963 *Cosmical Electrodynamics* (London: Oxford University Press)
Ariel P D 1970 *Proc. Indian Nat. Acad. Sci.* **2** 230–45
Backus G 1968 *Phil. Trans. R. Soc.* A **263** 239–66
—— 1971 *Lect. Appl. Math.* **14** 1–106 (American Mathematical Society)
Baldwin P and Roberts P H 1972 *Phil. Trans. R. Soc.* A **272** 303–30
Barcilon V 1970 *Phys. Fluids* **13** 537–44
Batchelor G K 1967 *An Introduction to Fluid Dynamics* (London: Cambridge University Press)
Benton E R and Loper D E 1969 *J. Fluid Mech.* **39** 561–86

Birch F 1972 *Geophys. J. R. Astr. Soc.* **29** 373–87
Booker J R and Bretherton F P 1967 *J. Fluid Mech.* **27** 513–39
Braginsky S I 1964 *Geomag. Aeronomy* **4** 698–712
—— 1967a *Geomag. Aeronomy* **7** 323–9
—— 1967b *Geomag. Aeronomy* **6** 851–9
—— 1970a *Geomag. Aeronomy* **10** 1–8
—— 1970b *Geomag. Aeronomy* **10** 172–81
Bullard E C 1949 *Proc. R. Soc.* A **197** 433–53
Bullard E C and Gellman H 1954 *Phil. Trans. R. Soc.* A **247** 213–78
Busse F H 1968a *J. Fluid Mech.* **33** 577–89
—— 1968b *J. Fluid Mech.* **33** 739–51
—— 1972 *J. Geophys. Res.* **77** 1589–90
Calkin M G 1963 *Can. J. Phys.* **41** 2241–51
Carstoiu J 1964 *C. R. Acad. Sci., Paris* **258** 2745–7
Causse R and Poirier Y 1960 *C. R. Acad. Sci., Paris* **251** 1056–8
Chandrasekhar S 1953 *Proc. R. Soc.* A **216** 293–309
—— 1954a *Astrophys. J.* **119** 7–9
—— 1954b *Proc. R. Soc.* A **225** 173–84
—— 1956 *Proc. R. Soc.* A **237** 476–84
—— 1960 *Proc. Nat. Acad. Sci., Wash.* **46** 253–7
—— 1961 *Hydrodynamic and Hydromagnetic Stability* (London: Oxford University Press)
Chandrasekhara B C and Rudraiah N 1971 *Appl. Sci. Res.* **25** 179–92
Chang T S and Sartory W K 1967 *Proc. R. Soc.* A **301** 451–71
Chawla S S 1972 *J. Fluid Mech.* **53** 545–55
Cowling T G 1955 *Proc. R. Soc.* A **233** 319–22
—— 1957 *Magnetohydrodynamics* (New York: Interscience)
—— 1962 *Rep. Prog. Phys.* **25** 244–86
Cox A 1968 *J. Geophys. Res.* **73** 3247–60
Cox A and Cain J C 1972 *Trans. Am. Geophys. Un.* **53** 591–623
Datta N 1964 *J. Phys. Soc. Japan* **19** 2219–23
Datta N and Biswas K K 1971 *J. Phys. Soc. Japan* **31** 1259–69
Donnelly R J and Ozima M 1960 *Phys. Rev. Lett.* **4** 497–8
Edmonds F N 1958 *Phys. Fluids* **1** 30–41
Ekman V W 1905 *Ark. Mat. Astr. Fys.* **2** 1–52
Elsasser W M 1946a *Phys. Rev.* **69** 106–16
—— 1946b *Phys. Rev.* **70** 202–12
—— 1947 *Phys. Rev.* **72** 821–33
—— 1956 *Rev. Mod. Phys.* **28** 135–63
Eltayeb I A 1972 *Proc. R. Soc.* A **326** 229–54
Eltayeb I A and Roberts P H 1970 *Astrophys. J.* **162** 699–701
Ertel H 1942 *Met. Z.* **59** 277–81
Ertel H and Rossby C G 1949 *Geofis. Pura Appl.* **14** 189–93
Ferraro V C A and Plumpton C 1966 *An Introduction to Magneto-fluid Mechanics* (Oxford: Clarendon Press)
Gans R F 1970 *J. Fluid Mech.* **45** 111–30
—— 1971 *J. Fluid Mech.* **50** 449–68
—— 1972 *J. Geophys. Res.* **77** 360–6
Gardiner R B and Stacey F D 1971 *Phys. Earth Planet. Interiors* **4** 406–10
Gilman P A 1967a *J. Atmos Sci.* **24** 101–43
—— 1967b *J. Atmos Sci.* **24** 333–6
—— 1968 *Science* **160** 760–3
—— 1969 *J. Atmos. Sci.* **26** 1003–9
—— 1971 *Phys. Fluids* **14** 7–12
Gilman P A and Benton E R 1968 *Phys. Fluids* **11** 2397–401
Govindarajulu T 1971 *Def. Sci. J. (India)* **21** 171–4
Greenspan H P 1968 *The Theory of Rotating Fluids* (London: Cambridge University Press)
Greenspan H P and Howard L N 1963 *J. Fluid Mech.* **17** 385–404
Gupta A S 1968 *J. Sci. Engng Res.* **12** 103–9

Hartmann J 1937 *K. Danske Vidensk. Selsk. Math.-fys. Medd.* **15** 6

Hasimoto H 1959 *Phys. Fluids* **2** 337–8

Hide R 1956 *Phys. Chem. Earth* **1** 94–137

—— 1966a *Phil. Trans. R. Soc.* A **259** 615–47

—— 1966b *Planet. Space Sci.* **14** 579–86

—— 1967 *Science* **157** 55–6

—— 1968 *J. Fluid Mech.* **32** 737–64

—— 1969a *J. Atmos Sci.* **26** 841–53

—— 1969b *Nature, Lond.* **222** 1055

—— 1969c *J. Fluid Mech.* **39** 283–8

—— 1969d *The Global Circulation of the Atmosphere* ed G A Corby (London: Royal Meteorological Society) pp196–221

—— 1970 *Q. J. R. Met. Soc.* **96** 579–90

—— 1971a *Nature, Lond.* **229** 114–5

—— 1971b *Nature Phys. Sci.* **233** 100–1

—— 1971c *J. Fluid Mech.* **49** 745–51

Hide R and Jones M V 1972 *Scientific Papers, Meteorological Office, London* no33

Hide R and Malin S R C 1970 *Nature, Lond.* **225** 605–9

—— 1971a *Nature Phys. Sci.* **230** 63

—— 1971b *Nature Phys. Sci.* **232** 31–3

—— 1971c *Comments Earth Sci.: Geophys.* **2** 1–13

Hide R and Roberts P H 1960 *Rev. Mod. Phys.* **32** 799–806

—— 1962 *Adv. Appl. Mech.* ed G Kuerti (New York: Academic Press) **7** pp215–316

Hide R and Stewartson K 1972 *Rev. Geophys. Space Phys.* **10** 579–98

Hide R and Titman C W 1967 *J. Fluid Mech.* **29** 39–60

Higgins G and Kennedy G C 1971 *J. Geophys. Res.* **76** 1870–8

Howard L N and Gupta A S 1962 *J. Fluid Mech.* **14** 463–76

Hunt J C R and Ludford G S S 1968 *J. Fluid Mech.* **33** 693–714

Ingham D B 1969 *Phys. Fluids* **12** 389–96

Irving E 1964 *Palaeomagnetism* (New York: Wiley)

Jacobs J A 1971 *Comments Earth Sci.: Geophys.* **2** 61–8

Jacobs J A, Chan T and Frazer M 1972 *Nature Phys. Sci.* **236** 24–5

Kakutani T 1962 *J. Phys. Soc. Japan* **17** 1496–506

Kalra G L 1969 *Publ. Astron. Soc. Japan* **21** 263–6

Kalra G L and Talwar S P 1967 *Mon. Not. R. Astr. Soc.* **135** 391–400

Kendall P C 1960 *Q. J. Mech. Appl. Math.* **13** 285–99

Khan M A A and Srivastava R B 1970 *Indian J. Phys.* **44** 647–56

Kibble T W B 1966 *Classical Mechanics* (New York: McGraw-Hill)

Kulikovsky A G 1968 *Izv. Akad. Nauk SSSR Mekh. Zhidk. I Gaza* no2 p3

Kurzweg U H 1963 *J. Fluid Mech.* **17** 52–60

Lai W 1962 *Phys. Fluids* **5** 560–6

Lehnert B 1954 *Astrophys. J.* **119** 647–54

—— 1955a *Astrophys J.* **121** 481–90

—— 1955b *Q. Appl. Math.* **12** 321–41

Lighthill M J 1966 *J. Fluid Mech.* **26** 411–31

—— 1967 *J. Fluid Mech.* **27** 725–52

Longuet-higgins M S 1968 *Phil. Trans. R. Soc.* A **262** 511–607

Loper D E 1970 *Phys. Fluids* **13** 2999–3002

—— 1971 *J. Fluid Mech.* **50** 609–23

Loper D E and Benton E R 1970 *J. Fluid Mech.* **43** 785–99

Lorenz E N 1967 *World Meteorological Organization Publication* no218

Ludwieg H 1962 *Z. Flugwiss.* **10** 242–9

Lundquist S 1952 *Ark. Fys.* **5** 297–347

Lynden-Bell D 1966 *Observatory* **86** 57–60

Malkus W V R 1959 *Astrophys. J.* **130** 259–75

—— 1967a *J. Fluid Mech.* **28** 793–802

—— 1967b *Proc. Symp. Appl. Math.* **18** 64–77

—— 1968 *Science* **160** 259–64

Malkus W V R 1971 *Lect. Appl. Math.* **14** 207–28 (American Mathematical Society)

Michael D H 1954 *Mathematika* **1** 45–50

Moffatt H K 1965 *J. Fluid Mech.* **22** 521–8

—— 1970 *J. Fluid Mech.* **44** 705–19

—— 1972 *J. Fluid Mech.* **53** 385–99

—— 1973 *J. Fluid Mech.* in press

Murphy J O and van de Borght R 1972 *Proc. Aust. Sci. Ass.* **2** 215–31

Murthy S N, Rudraiah N and Mariyappa B V 1972 *Appl. Sci. Res.* **26** 285–306

Nakagawa Y 1957 *Proc. R. Soc.* A **242** 81–8

—— 1959 *Proc. R. Soc.* A **249** 138–45

—— 1962 *Bull. Soc. R. Sci. Liége* no11–12 685–95

Namikawa T 1955 *J. Geomag. Geoelec.* **7** 97–104

—— 1961 *J. Inst. Polytechnics, Osaka City Univ.* ser G **4** 125–32

Nanda R S and Mohanty H K 1971 *Appl. Sci. Res.* **24** 65–78

Nigam P D 1965 *Appl. Sci. Res.* B **12** 139–50

Nigam S D and Nigam P D 1963 *Proc. R. Soc.* A **272**, 529–41

Pao H-P 1966 *Phys. Fluids* **9** 1254–5

Parker E N 1955 *Astrophys. J.* **122** 293–314

Pathak R S and Pathak L 1971 *Indian J. Phys.* **45** 49–58

Pedley T J 1968 *J. Fluid Mech.* **31** 603–7

Pedlosky J 1964 *J. Atmos. Sci.* **21** 201–19

—— 1971 *Lect. Appl. Math.* **13** 1–60 (American Mathematical Society)

Plumpton C and Ferraro V C A 1955 *Astrophys. J.* **121** 168–74

Poirier Y and Robert J 1960 *C. R. Acad. Sci., Paris* **251** 2880–2 (in French)

Pop I 1969 *Rev. Math. Pures Appl., Buc.* **14** 1173–86

Prandtl L 1952 *Essentials of Fluid Dynamics* (London: Blackie)

Proudman J 1916 *Proc. R. Soc.* A **92** 408–24

Rao A R 1971 *J. Fluid Mech.* **46** 357–63

Rayleigh Lord 1920 *Scientific Papers* **6** 447–53 (London: Cambridge University Press)

Rickard J A 1970 *PhD Thesis* University of London

Rikitake T 1956 *Bull. Earthq. Res. Inst. Tokyo Univ.* **34** 139–56

—— 1966 *Electromagnetism and the Earth's Interior* (Amsterdam: Elsevier)

Roberts B and Sozou C 1971 *J. Plasma Phys.* **6** 249–55

Roberts P H 1964 *Proc. Camb. Phil. Soc.* **60** 635–51

—— 1967 *An Introduction to Magnetohydrodynamics* (London: Longmans)

—— 1971 *Lect. Appl. Math.* **13** 129–206 (American Mathematical Society)

Roberts P H and Soward A M 1972 *Ann. Rev. Fluid Mech.* **4** 117–54

Roberts P H and Stix M 1971 *The Turbulent Dynamo* (a translation of a series of papers by F Krause, K H Rädler and M Steenbeck) *National Center for Atmospheric Research, Boulder, Colorado, Tech. Note* IA–60

Rochester M G 1970 *Earthquake Displacement Fields and the Rotation of the Earth* ed L Mansinha *et al* (Dordrecht: Reidel) pp136–48

Rudraiah N 1964 *Appl. Sci. Res.* B **11** 105–17

Rudraiah N and Murthy S N 1971 *Current Sci. (India)* **40** 343

Rudraiah N and Venkatachalappa M 1972 *J. Fluid Mech.* **52** 193–206

Sakurai K 1967 *Report of Ionosphere and Space Research in Japan* **21** 107–12

Schubert G 1968 *Astrophys. J.* **151** 1099–110

Shercliff J A 1965 *A Textbook of Magnetohydrodynamics* (Oxford: Pergamon Press)

Soundalgekar V M and Pop I 1970 *Bull. Math. Soc. Sci. Math. Phys. Répub. Pop. Roum.* **14** 375–81

Spiegel E A and Veronis G 1960 *Astrophys. J.* **131** 442–7

Spitzer L 1956 *Physics of Fully Ionized Gases* (New York: Interscience)

Squire H B 1933 *Proc. R. Soc.* A **142** 621–8

Stacey F D 1969 *Physics of the Earth* (London: Wiley)

Stephenson G 1961 *Mon. Not. R. Astr. Soc.* **122** 455–9

Stewartson K 1957 *Z. Angew. Math. Phys.* **8** 290–7

—— 1960 *J. Fluid Mech.* **8** 82–96

—— 1967 *Proc. R. Soc.* A **299** 173–87

Stewartson K 1971 *Lect. Appl. Math.* **14** 229–37 (American Mathematical Society)
Suffolk G C J and Allan D W 1969 *The Application of Modern Physics to the Earth and Planetary Interiors* ed S K Runcorn (London: Wiley-Interscience) pp653–6
Talwar S P 1960 *J. Fluid Mech.* **9** 581–92
—— 1962 *Indian J. Phys.* **36** 351–8
Tassoul J L 1967 *Mon. Not. R. Astr. Soc.* **137** 1–13
Taylor G I 1917 *Proc. R. Soc.* A **93** 99–113
—— 1921 *Proc. Camb. Phil. Soc.* **20** 326–9
Taylor J B 1963 *Proc. R. Soc.* A **274** 274–83
Trehan S K and Nakagawa Y 1969 *Astrophys. J.* **158** 1075–80
Trocheris M G 1949 *Phil. Mag.* **40** 1143–54
Velikhov E P 1959 *J. Exp. Theor. Phys. (USSR)* **36** 995–8
Veronis G 1970 *Ann. Rev. Fluid Mech.* **2** 37–67
Vidyanidhi V 1969 *J. Math. Phys. Sci. (India)* **3** 193–9
Vidyanidhi V and Rao V 1968 *J. Phys. Soc. Japan* **25** 1694–710
Weiss N O 1971 *Q. J. R. Ast. Soc.* **12** 432–46
Yih C-S 1958 *Q. Appl. Math.* **16** 409–15
—— 1959 *J. Fluid Mech.* **5** 436–44
—— 1965 *Dynamics of Non-homogeneous Fluids* (New York: Macmillan)
Yukutake T 1970 *Comments Earth Sci.: Geophys.* **1** 55–64