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ON THE MAGNETIC FLUX LINKAGE OF AN ELECTRICALLY-CONDUCTING FLUID

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On the Magnetic Flux Linkage of an Electrically-Conducting Fluid

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A method has recently been proposed for finding the radius r_c of the electrically-conducting fluid core of a planet of outer radius r_s from observations of the magnetic field **B** in the accessible region near or well above the surface of the planet (Hide, 1978). The method is based on the supposition that when the magnetic field is produced by hydromagnetic dynamo action in the core, implying that the magnetic Reynolds number R there is large, (a) fluctuations in **B** can occur everywhere on the comparatively short advective time-scale τ_A associated with fluid motions in the core and so can fluctuations in the quantity N, defined for any closed surface S as the total number of intersection of magnetic lines of force with S, provided that S lies well outside the core, but (b) at the surface of the core, where lines of magnetic force emerge from their region of origin, concomitant fluctuations in N are negligibly small, of the order of τ_A/τ_O, where $\tau_O(=R\tau_A)$ is the Ohmic decay time of the core.

A proof of this supposition follows directly from the general expression derived in the present paper showing that when S is a material surface the time rate of change of N is equal to minus twice the line integral of the current density divided by the electrical conductivity around all the lines on S where the magnetic field is tangential to S. This expression (which Palmer in an accompanying paper rederives and extends to the relativistic case using the mathematical formalism of Cartan's exterior calculus) also provides a direct demonstration of the well-known result that although high electrical conductivity, sufficient to make $R\gg 1$, is a necessary condition for hydromagnetic dynamo action, such action is impossible in a perfect conductor, when $R\to\infty$

It is now generally accepted that the main magnetic fields of the Earth, Jupiter and other planets are due to ordinary electric currents produced by "hydromagnetic dynamo" action, involving fluid motions in their electrically-conducting fluid cores. I have recently proposed a method (Hide, 1978, hereafter cited as H) for determining the radius r_c of the fluid core of such a planet from magnetic observations external to the planet or at its surface near $r=r_s$ (where r_s is the mean radius of the planet and

$r=|\mathbf{r}|$ if $\mathbf{r}$ is the vector distance of a general point from the centre of the planet). When applied to the planet Earth (H: Hide and Malin, 1979), for which the value of r_s is 6371.2 km, making use of a recently-published determination of the main geomagnetic field and its secular changes (Barraclough, Harwood, Leaton and Malin, 1978) based on all available geomagnetic data for the period 1955 to 1975, the new method gives for r_c a value in agreement, within the errors involved, with the accepted value of 3786 ± 5 km based on studies by seismologists of the propagation of elastic waves generated by earthquakes. Seismological data are not available in the case of other planets and it will therefore be particularly important to make detailed measurements of the magnetic fields of these planets which can be exploited to the full in the study of their internal structure, by methods that are theoretically sound and adequately tested against terrestrial observations.

The method proposed in H considers the quantity N defined as the number of intersections of lines of force of the magnetic field $\mathbf{B}(\mathbf{r},t)$ (where t denotes time) with any closed (and not necessarily spherical) surface S:

$$N\equiv\iint_S|\mathbf{B}\cdot d\mathbf{S}|=N(t;S), \tag{1}$$

if $d\mathbf{S}$ denotes the vector element of area of S over the whole of which the area integral is taken. We denote by $S(+)$ those regions of S where the magnetic lines of force emerge from the volume enclosed by S, i.e. where $\mathbf{B}\cdot d\mathbf{S}>0$, by $S(-)$ those regions where lines of force enter the enclosed volume, i.e. where $\mathbf{B}\cdot d\mathbf{S}<0$, and by C the one or more closed "null flux" lines on S where $\mathbf{B}\cdot d\mathbf{S}=0$, and which are thus the boundaries separating the $S(+)$ and $S(-)$ regions. According to its definition [see equation (1)]

$$N=\iint_{S(+)}\mathbf{B}\cdot d\mathbf{S}-\iint_{S(-)}\mathbf{B}\cdot d\mathbf{S}. \tag{2}$$

Moreover, since **B** satisfies Gauss's law

$$\nabla\cdot\mathbf{B}=0, \tag{3}$$

we have

$$N=2\iint_{S(+)}\mathbf{B}\cdot d\mathbf{S}=-2\iint_{S(-)}\mathbf{B}\cdot d\mathbf{S}. \tag{4}$$

The magnetic field **B** also satisfies Ampère's law

$$\nabla\times(\mathbf{B}\mu^{-1})=\mathbf{j} \tag{5}$$

where μ is the magnetic permeability and $\mathbf{j}$ is the electric current density. Outside the core, in the region $r>r_c$ where, in the idealized model used in H, the electrical conductivity is taken as zero, $\mathbf{j}$ must vanish and equations (3) and (5) can in consequence be satisfied when $\nabla\mu=0$ by writing

$$\mathbf{B}=-\nabla V \text{ (say), where } \nabla^2 V=0. \tag{6}$$

Within the core $(r<r_c)$, where the electrical conductivity σ does not vanish, $\mathbf{j}$ satisfies Ohm's law as it applies to a conductor moving with the (Eulerian) fluid velocity $\mathbf{u}$; thus

$$\mathbf{j}=\sigma[\mathbf{E}+\mathbf{u}\times\mathbf{B}], \tag{7}$$

where $\mathbf{E}$ is the electric field, which is related to $\mathbf{B}$ by Faraday's law

$$\nabla\times\mathbf{E}=-\partial\mathbf{B}/\partial t. \tag{8}$$

By taking the *curl* of equation (7) and using equations (8) and (5), thus eliminating $\mathbf{E}$ and $\mathbf{j}$, we find the well-known equation for the time-rate of change of the magnetic field in a moving conductor

$$\partial\mathbf{B}/\partial t=\nabla\times(\mathbf{u}\times\mathbf{B})-\nabla\times[\sigma^{-1}\nabla\times(\mathbf{B}\mu^{-1})] \tag{9}$$

(see e.g. Roberts, 1967; Moffatt, 1978).

An advective time-scale $\tau_A=L/U$ can be associated with the first term on the right-hand side of equation (9) (where L is a typical length-scale and U a typical value of the speed of fluid motion) and an "Ohmic decay" time-scale $\tau_O=L^2\mu\sigma$ with the second-term. The "magnetic Reynolds number" R is given by

$$R\equiv UL\mu\sigma=\tau_O/\tau_A, \tag{10}$$

and when this is large one expects that fluctuations $\mathbf{B}$ on time-scales that are much less than τ_O can be treated on the basis of Alfvén's "frozen field" theorem, to which equation (9) reduces when the second term on the right-hand side can be neglected. Under these conditions magnetic lines of force move with fluid particles in the core and fluctuations that occur in $\mathbf{B}$ on these time-scales at points outside the core (in $r>r_c$) – which are known as the "geomagnetic secular variation" in the case of the Earth are then to be regarded (see e.g. Roberts and Scott, 1965; Backus, 1968) as being a consequence of the redistribution (rather than the creation and destruction) of magnetic lines of force by fluid motions in the upper

reaches of the core, just below a thin boundary layer on $r=r_c$. Concomitant fluctuations in N [see equation (1)] do not in general vanish in $r>r_c$ and well above the level $r=r_c$ they will be pronounced and associated with the same time-scale as fluctuations in $\mathbf{B}$, namely the advective time-scale τ_A. However, at the surface of the electrically-conducting core, where $r=r_c$, temporal changes in N cannot occur on the advective time-scale τ_A; such changes can only occur on the Ohmic time-scale τ_O, since the total flux linkage of a perfect conductor cannot change. Whence, we can suppose when $R\gg 1$ that on time-scales comparable with τ_A the quantity $\mathrm{d}N/\mathrm{d}t=\mathrm{O}(\tau_A/\tau_O)$ when $r=r_c$, so that to a good approximation we can take

$$\mathrm{d}N/\mathrm{d}t=0 \text{ for the surface } r=r_c, \tag{11}$$

implying that all higher time derivatives of N also vanish on the time scale τ_A. The basis of the method introduced in H for finding r_c is to determine, from observations of $\mathbf{B}(\mathbf{r},t)$ near $r=r_c$ and with the aid of equation (6), that value of r at which N effectively satisfies equation (11).

Colleagues have criticized the supposition made in H that equation (11) can be taken as an obvious consequence of equation (9) in the case of a perfect conductor, urging that a formal mathematical demonstration is desirable. Thus, we shall show in what follows that when S is a material surface, N satisfies

$$\frac{\mathrm{d}N}{\mathrm{d}t}\equiv\frac{\mathrm{d}}{\mathrm{d}t}\iint_S|\mathbf{B}\cdot\mathbf{dS}|=-2\oint_C\frac{\mathbf{j}\cdot\mathbf{dC}}{\sigma}, \tag{12}$$

which reduces to equation (11) when applied to surface of the core in $r=r_c$ in the limit when σ tends to infinity. In evaluating the line integrals along the "null flux" curves C on S the vector element of length $\mathbf{dC}$ is taken as positive when an $S(+)$ region is on the left to an observer facing in the direction of $\mathbf{dC}$, implying that the direction of a vector joining a point near C just inside an $S(+)$ region to a neighbouring point in the $S(-)$ region on the other side of C is the same is that of the vector $\mathbf{dC}\times\mathbf{dS}$.

To obtain equation (12) we first observe that

$$\frac{\mathrm{d}}{\mathrm{d}t}\iint_{S(+)}\mathbf{B}\cdot\mathbf{dS}=\iint_{S(+)}\frac{\partial\mathbf{B}}{\partial t}\cdot\mathbf{dS}+\oint_C\mathbf{B}\cdot\mathbf{u}\times\mathbf{dC}, \tag{13}$$

the second term on the right-hand side of which represents the contribution to $\frac{1}{2}\mathrm{d}N/\mathrm{d}t$ [see equation (4)] due to the movement of the "null-flux" lines C. This equation, when combined with equation (4) making use of Stokes's theorem and a well-known vector identity, gives

$$\frac{dN}{dt} = 2\iint_{S(+)} \left(\frac{\partial \mathbf{B}}{\partial t} - \nabla \times (\mathbf{u} \times \mathbf{B})\right) \cdot \mathbf{dS}. \qquad (14)$$

Finally, we note that the integrand of the right-hand side of equation (14) is equal to the second term on the right-hand side of the equation (9): whence

$$\frac{dN}{dt} \equiv \frac{d}{dt}\iint_S |\mathbf{B} \cdot \mathbf{dS}| = -2\oint_C \frac{(\nabla \times (\mathbf{B}\mu^{-1})) \cdot \mathbf{dC}}{\sigma}, \qquad (15)$$

which, by virtue of Ampère's law (see equation (5)), is equivalent to, and for some purposes more convenient than, the result we set out to prove, equation (12). Palmer (1979), in the companion paper, extends the foregoing analysis to the relativistic case, using the powerful and elegant method of Cartan's exterior calculus. His equations (14) and (17) correspond to equations (12) and (15) above, recognizing that when S is not a material surface it is necessary to add the term,

$$2\oint_C \mathbf{B} \cdot (\mathbf{v} - \mathbf{u}) \times \mathbf{dC},$$

to the right-hand sides of equations (12) and (15), where $\mathbf{v}$ denotes the motion of S in the frame of reference with respect to which the Eulerian flow velocity is $\mathbf{u}$.

A systematic discussion of the implications of these useful general results lies beyond the scope of the present note. It is sufficient to remark here that equation (15) [or (12)] leads directly to the supposition expressed by equation (11) and stated without proof in H. By equation (15) we can associate a time-scale

$$\tau_N \simeq 2r_c^2 \mu\sigma \hat{B}/\gamma\bar{B}, \qquad (16)$$

with fluctuations in N near $r = r_c$, where $\hat{B}$ is a typical value of $N/4\pi r_c^2$, $2\pi\gamma r_c$ is the total length of the "null flux" lines C and $\bar{B}/\mu r_c$ is the average value of the component of $\mathbf{j}$ along C. The value of τ_N would fail to go to infinity with σ if and only if the quantity $\bar{B}\gamma/\hat{B}$ increases with σ at least as rapidly as σ raised to the first power, which is unlikely. Such behaviour would imply in the limit when $\sigma \to \infty$ the existence of current sheets where $|\mathbf{j}|$ tends to infinity, and in such circumstances, except in the unlikely event that *all* components of $\mathbf{B}$ vanish everywhere on C, the Lorentz force per unit volume $\mathbf{j} \times \mathbf{B}$ would be infinite and therefore accelerate a fluid of finite density at an infinite rate, which is an impossible situation. Hydromagnetic dynamo action can be said to occur if and only if $dN/dt > 0$, so that a necessary condition for such action is that the term on the right-hand side of equation (15) should be a positive definite quantity. As we have shown, this term vanishes in the limit of infinite conductivity, which nicely demonstrates the well-known result (see e.g. Bondi and Gold, 1950; Moffatt, 1978) that although high conductivity [more precisely $R \gg 1$, see equation (10)] is a necessary condition for dynamo action to occur, such action is impossible in a perfect conductor, for which $R \to \infty$.

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