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AUTUMN COURSE ON GEOMAGNETISM, THE IONOSPHERE
AND MAGNETOSPHERE

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RADIO PROPAGATION IN THE IONOSPHERE

Part 2

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Notes for lectures on

Radio Propagation in the Ionosphere

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III - PROPAGATION IN STRATIFIED MAGNETOIONIC MEDIUM.

These notes use the notation and some of the equations from the notes I Magnetoionic Theory. But here we use Θ (instead of θ) for the angle between the wave normal and the vector $\mathbf{Y}$.

1. Refractive index surface. This is a surface such that the line from a fixed origin to each point on it has length equal to the refractive index n, and the direction of the wave normal. It is formed by drawing a polar plot with n as radius and Θ as polar angle, and then rotating the curve about the direction of $\mathbf{Y}$ as axis. In a loss free medium the direction of the time averaged Poynting vector

$$\overline{\mathbf{\Pi}} = \tfrac{1}{2}\,\mathrm{Re}(\mathbf{E}\wedge\mathbf{H}^*). \tag{1}$$

is normal to the refractive index surface.

Proof Think of the refractive index as a vector $\mathbf{n}$ with Cartesian components n_x, n_y, n_z.

$$\text{Maxwell 3:}\quad \mathbf{n}\wedge\mathbf{E} = c\mu_0\mathbf{H}. \tag{2}$$

$$\text{Maxwell 4:}\quad \mathbf{n}\wedge\mathbf{H} = -c\mathbf{D}. \tag{3}$$

The plasma is a loss free dielectric so its permittivity is Hermitean:

$$\varepsilon_{ij} = \varepsilon^*_{ji} \tag{4}$$

where

$$D_i = \varepsilon_0\,\varepsilon_{ij}\,E_j \tag{5}$$

with summation over repeated suffixes.

Now

$$\delta\mathbf{E}^*.\mathbf{D} = \delta E_i^*.D_i = \varepsilon_0\,\delta E_i^*\,\varepsilon_{ij}\,E_j$$

$$= \varepsilon_0\,\delta E_i^*\,\varepsilon^*_{ji}\,E_j = \delta D_j^*.E_j = \delta\mathbf{D}^*.\mathbf{E}. \tag{6}$$

Differentiate (2) and scalar multiply by $\mathbf{H}^*$:

$$\mathbf{H}^*.(\mathbf{n}\wedge\delta\mathbf{E}) + \mathbf{H}^*.(\delta\mathbf{n}\wedge\mathbf{E}) = c\mu_0\,\mathbf{H}^*.\delta\mathbf{H}. \tag{7}$$

Permute vectors and use complex conjugate of (3):

$$c\,\delta\mathbf{E}.\mathbf{D}^* + \delta\mathbf{n}.(\mathbf{E}\wedge\mathbf{H}^*) = c\mu_0\,\mathbf{H}^*.\delta\mathbf{H}. \tag{8}$$

Differentiate (3) and scalar multiply by $\mathbf{E}^*$:

$$\mathbf{E}^*.(\mathbf{n}\wedge\delta\mathbf{H}) + \mathbf{E}^*.(\delta\mathbf{n}\wedge\mathbf{H}) = -c\,\mathbf{E}^*.\delta\mathbf{D}.$$

Permute vectors and use complex conjugate of (2), and reverse signs:

$$-c\mu_0\,\delta\mathbf{H}.\mathbf{H}^* + \delta\mathbf{n}.(\mathbf{E}^*\wedge\mathbf{H}) = c\,\mathbf{E}^*.\delta\mathbf{D}. \tag{9}$$

Add (8) and (9):

$$c(\mathbf{D}^*.\delta\mathbf{E} - \mathbf{E}^*.\delta\mathbf{D}) + \delta\mathbf{n}.(\mathbf{E}\wedge\mathbf{H}^* + \mathbf{E}^*\wedge\mathbf{H}) = 0. \tag{10}$$

But (6) shows that first term is zero, so from (1)

$$\delta\mathbf{n}.\overline{\mathbf{\Pi}} = 0. \tag{11}$$

Every possible $\delta \underset{\sim}{n}$ must lie in the refractive index surface. This completes the proof.

2. The Ray. Stationary phase. Consider a homogeneous medium with point source at the origin. The emitted wave system may be expressed as an angular spectrum of plane waves

$$F = \iint A(n_x, n_y) \exp\left\{ - i \frac{\omega}{c} (x n_x + y n_y + z n_z) \right\} dn_x \, dn_y . \qquad (12)$$

The main contribution to the integral comes from where the phase

$$\phi = \frac{\omega}{c} (x n_x + y n_y + z n_z) \qquad (13)$$

is stationary for small variations of $\underset{\sim}{n}$, so

$$\delta \phi = 0 \;\; ; \;\; x \delta n_x + y \delta n_y + z \delta n_z = 0. \qquad (14)$$

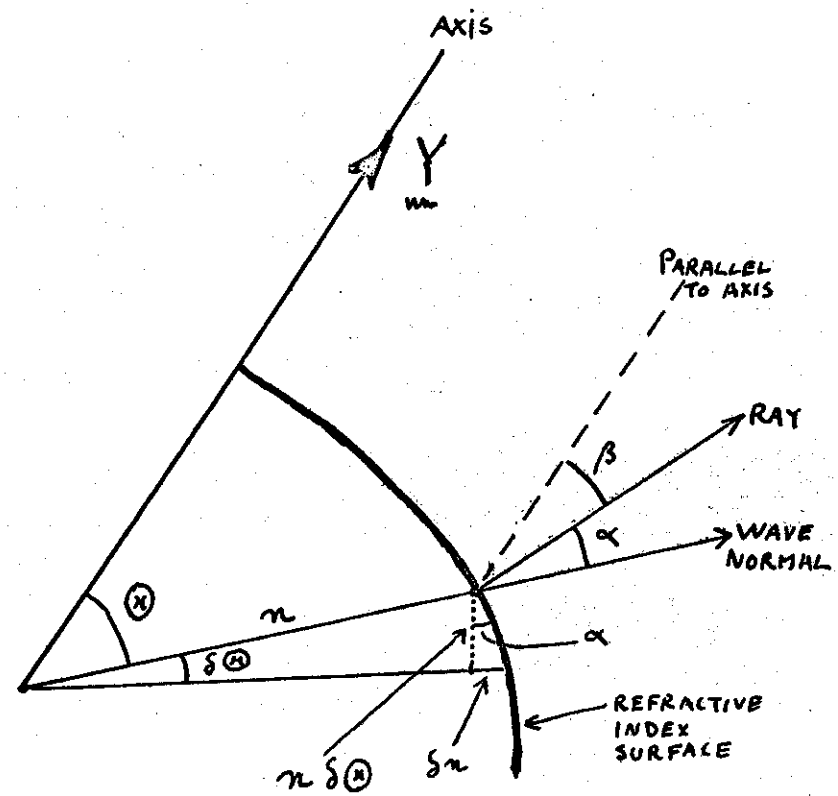
But any $\delta \underset{\sim}{n}$ lies in the refractive index surface. Thus the ray $\underset{\sim}{r} = (x, y, z)$, from the origin, is normal to the refractive index surface. The angle α between the ray and the wave normal is given by

$$\tan \alpha = \frac{1}{n} \frac{dn}{d\Theta} . \qquad (15)$$

The speed with which a wave front travels along the ray direction is

$$V_R = \frac{c}{n} \sec \alpha . \qquad (16)$$

The vector $\underset{\sim}{V}_R$ of magnitude V_R in the direction of the ray is called the



ray velocity. Then

$$\mathbf{V}_R \cdot \mathbf{n} = c. \qquad (17)$$

The locus of a point at V_R from the origin, in the direction of the ray, is called the "ray surface".

The Group Velocity is

$$\mathbf{U} = \partial \omega / \partial \mathbf{k}. \qquad (18)$$

It is normal to the surface $\omega(\mathbf{k})$ = constant. But this is just the refractive index surface because $\mathbf{k} = \omega \mathbf{n}/c$. Hence $\mathbf{U}$ has the direction of the ray.

The component of $\mathbf{U}$ in the direction of the wave normal is

$$U \cos\alpha = \partial\omega / \partial k = c\,\partial\omega/\partial(\omega n) = c/n'. \qquad (19)$$

(in these differentiations θ is assumed to be held constant.)

$$n' = \partial(\omega n)/\partial\omega \qquad (20)$$

is called the group refractive index. Hence

$$U = c/(n' \cos\alpha). \qquad (21)$$

(Note: $c/n' = U \cos\alpha$ is often wrongly called the "group velocity".)

3. Stratified Medium. Coordinate z is vertically upwards. The composition of the ionosphere is a function of z only. There is free space below it. Consider upgoing plane wave incident from below. Then the space variation of any of its field components is

$$F = \exp\left\{-i\frac{\omega}{c}(S_1 x + S_2 y + Cz)\right\}. \qquad (22)$$

The wave normal has direction cosines S_1, S_2, C and

$$S_1^2 + S_2^2 + C^2 = 1. \qquad (23)$$

Imagine the ionosphere to be divided into discrete thin horizontal strata. Then, at each boundary, Snell's law applies so that the x and y dependence of all field components is the same in all strata. Thus

$$n_x = S_1, \quad n_y = S_2. \qquad (24)$$

It is usual to use the symbol q for n_z. Hence, in any stratum

$$S_1^2 + S_2^2 + q^2 = n^2, \qquad (25)$$

where n is one of the two refractive indices. The results all apply in the limit when the strata are infinitesimally thin.

To find q we proceed as follows. The direction cosines of the wave normal are

$$S_1/n, \quad S_2/n, \quad q/n. \qquad (26)$$

Let the direction cosines of the vector $\mathbf{Y}$ be

$$\ell_x , \quad \ell_y , \quad \ell_z \quad . \tag{27}$$

Then the angle $\textcircled{H}$ Between $\underset{\sim}{Y}$ and the wave normal is given by

$$\cos \textcircled{H} = (S_1 \ell_x + S_2 \ell_y + q \ell_z)/n. \tag{28}$$

The dispersion relation is given by notes I, equations (33) to (36), with θ replaced by $\textcircled{H}$. Here use (28) to substitute for $\cos^2 \textcircled{H}$, $\sin^2 \textcircled{H}$ and then use (25) for n^2. This gives a quartic equation for q, the <u>Booker quartic:</u>

$$F(q) \equiv \alpha q^4 + \beta q^3 + \gamma q^2 + \delta q + \varepsilon = 0, \tag{29}$$

where

$$\alpha = U(U^2 - Y^2) + X(Y^2 \ell_z^2 - U^2)$$

$$\beta = 2XY^2 \ell_z (S_1 \ell_x + S_2 \ell_y)$$

$$\gamma = -2U(U-X)(C^2U-X) + 2Y^2(C^2U-X) + XY^2\left\{1 - C^2\ell_z^2 + (S_1\ell_x + S_2\ell_y)^2\right\}$$

$$\delta = -2C^2 XY^2 \ell_z (S_1 \ell_x + S_2 \ell_y)$$

$$\varepsilon = (U-X)(C^2U-X)^2 - C^2Y^2(C^2U - X) - C^2XY^2(S_1 \ell_x + S_2 \ell_y)^2. \tag{30}$$

(Booker, H.G.. 1939. Phil. Trans. R.Soc. Lond. A. <u>237</u>, 411.)

4. <u>Ray Tracing</u>. Consider again the thin strata. If one stratum, of thickness δz, is crossed in the vertical direction, the change of phase ϕ, (13), is $\frac{\omega}{c} q \delta z$. If many strata are crossed, going from the ground z = 0 to height z, the change of phase is $\frac{\omega}{c}\int_0^z q dz$.

The incident plane wave (22) therefore gives, at height z, a field

$$F = A(z) \exp\left\{-i \frac{\omega}{c} (S_1 x + S_2 y + \int_0^z q dz)\right\}. \tag{30}$$

This is called the "phase memory" concept. At the boundaries between strata the amplitude changes very slightly. Thus the amplitude factor A(z) is a slowly varying function of z. If the signal comes from a point source at the origin, there is an angular spectrum of plane waves like (12), (30) must be replaced by

$$F = \iint A(S_1, S_2, z) \exp\left\{- i \frac{\omega}{c} (S_1 x + S_2 y + \int_0^z q dz)\right\} dS_1 dS_2. \tag{31}$$

At any point (x, y, z) the dominant contribution to the integral comes from those values of S_1, S_2 where the phase is stationary, that is where $\partial / \partial S_1$ and $\partial / \partial S_2$ of the exponent in (31) are zero. This gives

$$x + \int_0^z \frac{\partial q}{\partial S_1} dz = 0 ; \qquad y + \int_0^z \frac{\partial q}{\partial S_2} dz = 0. \tag{32}$$

These can be written in differential form:-

$$\frac{dx}{dz} = -\frac{\partial q}{\partial S_1} \quad ; \qquad \frac{dy}{dz} = -\frac{\partial q}{\partial S_2} \quad . \tag{33}$$

These are Booker's form of the ray tracing equations for a stratified medium.

Reflection occurs where the ray is horizontal, that is $\delta z = 0$ where δx and δy are not both zero. Thus at least one of

$\frac{1}{\partial q/\partial S_1}$ and $\frac{1}{\partial q/\partial S_2}$ is zero. Now the quartic (29)

is $F(q) \equiv 0$. Hence

$$\frac{dF}{dS_1} = 0 = \frac{\partial F}{\partial S_1} + \frac{\partial F}{\partial q} \cdot \frac{\partial q}{\partial S_1} \quad . \tag{34}$$

$$\frac{1}{\partial q/\partial S_1} = -\frac{\partial F/\partial q}{\partial F/\partial S_1} \quad . \tag{35}$$

A similar result applies with S_2 for S_1. Here $\partial F/\partial S_1$ means

$$\frac{\partial \alpha}{\partial S_1} q^4 + \frac{\partial \beta}{\partial S_1} q^3 + \frac{\partial \gamma}{\partial S_1} q^2 + \frac{\partial \delta}{\partial S_1} q + \frac{\partial \varepsilon}{\partial S_1} \quad . \tag{36}$$

Now α, β, etc. in (30) are polynomials in S_1, S_2, so $\partial F/\partial S_1$ and $\partial F/\partial S_2$ are never infinite. Therefore, from (35), the reflection condition is

$$\partial F/\partial q = 0. \tag{37}$$

This is also the condition that the quartic (29) shall have two coincident roots.

5. <u>W.K.B. solutions.</u> Ray theory is an approximation. To justify its use we need to ask "how accurate is it?" and "when does it fail?" This requires a more detailed study of the differential equations for wave propagation. We shall do it here only for an isotropic ionosphere. (The treatment for an anisotropic ionosphere will come in the lectures that deal with "full wave solutions".)

For an isotropic medium the electric permittivity ε is a scalar given by

$$\varepsilon = n^2 = 1 - X/U \, . \tag{38}$$

Thus n is a function of z but independent of ⊛ and of S_1, S_2. We also use $k = \omega/c$. (NOTE: <u>not</u> the more usual meaning $\omega n/c$.)

A ray going out from the origin behaves in the same way for all azimuths so we can take $S_2 = 0$ without loss of generality. In S_1 we now omit the subscript 1. Snell's law therefore gives at all heights

$$\partial/\partial x \equiv -ikS; \quad \partial/\partial y \equiv 0. \tag{39}$$

Hence Maxwell's third equation gives

$$-\partial E_y/\partial z = -i\omega\mu_o H_x. \tag{40}$$

$$\partial E_x/\partial z + ikSE_z = -i\omega\mu_o H_y \, . \tag{41}$$

$$-ikSE_y = -i\omega\mu_o H_z. \tag{42}$$

and Maxwell's fourth equation gives

$$-\partial H_y/\partial z = i\omega\varepsilon_o\varepsilon E_x. \quad (43)$$

$$\partial H_x/\partial z + ikSH_z = i\omega\varepsilon_o\varepsilon E_y. \quad (44)$$

$$-ikSH_y = i\omega\varepsilon_o\varepsilon E_z. \quad (45)$$

These separate into two independent sets. We use the set (40), (42), (44) which has only E_y, H_x, H_z. The waves are linearly polarized with the electric field horizontal.

From (42)

$$H_z = S E_y/\mu_o c. \quad (46)$$

Substitution in (44) gives, with (38):

$$\frac{\partial H_x}{\partial z} = i\omega\varepsilon_o(n^2-S^2)E_y = i\omega\varepsilon_o q^2 E_y. \quad (47)$$

Differentiate (40) with respect to z and substitute (47). This gives

$$\frac{d^2E_y}{dz^2} + k^2q^2E_y = 0. \quad (48)$$

(We assume that the factor $\exp\{i(\omega t - kSx)\}$ is always omitted and we use total derivative signs.)

To obtain an approximate solution of (48) we use the W.K.B. method. Change the dependent variable.

Let

$$E_y = e^{-i\phi}. \quad (49)$$

so that

$$\frac{dE_y}{dz} = -i\frac{d\phi}{dz}e^{-i\phi}; \quad \frac{d^2E_y}{dz^2} = \left\{-i\frac{d^2\phi}{dz^2} - \left(\frac{d\phi}{dz}\right)^2\right\}e^{-i\phi}. \quad (50)$$

Then

$$-i\frac{d^2\phi}{dz^2} - \left(\frac{d\phi}{dz}\right)^2 + k^2q^2 = 0. \quad (51)$$

Solve by successive approximation. In a homogeneous medium q is constant and $\frac{d^2\phi}{dz^2} = 0$. In a slowly varying medium expect that $\frac{d^2\phi}{dz^2}$ is small. Therefore:

First approximation.

Neglect $\frac{d^2\phi}{dz^2}$ then:

$$\frac{d\phi}{dz} = \mp kq \quad (52)$$

Second approximation.

$$\frac{d^2\phi}{dz^2} = \mp k\frac{dq}{dz}. \quad (53)$$

so (51) gives

$$\left(\frac{d\phi}{dz}\right)^2 = k^2q^2 \mp ik\frac{dq}{dz}. \quad (54)$$

Take square root. Expand right hand side by binomial theorem (not allowed when q is very small.)

$$\frac{d\phi}{dz} = \mp kq - \frac{i}{2q} \frac{dq}{dz} . \qquad (55)$$

Same choice of sign must be used in (52) and (55). So

$$\phi = \mp k \int q dz - \tfrac{1}{2} i \ln q . \qquad (56)$$

and (49) gives

$$E_y = q^{-\frac{1}{2}} \exp (\mp ik \int q dz) . \qquad (57)$$

These are the two W.K.B. solutions. They are approximations and we expect them to be inaccurate when q is very small. It can be shown that they are accurate to better than about 5 % if

$$\frac{2}{k} \frac{d}{dz} \left(\frac{1}{q} \right) \leqslant 1. \qquad (58)$$

A further discussion of W.K.B. solutions and related topics is contained in "Notes on Asymptotic Approximations." Copies may be consulted in the library.