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DIELECTRIC TENSOR AND WAVE BRANCHES OF A COLD MAGNETOPLASMA

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1. DIELECTRIC TENSOR AND WAVE BRANCHES OF A COLD MAGNETOPLASMA

1.1 Dielectric tensor and wave dispersion equation

The dielectric permittivity tensor ϵ_{ij} is defined by the equation

$$D_i = \sum_j \epsilon_{ij} E_j \qquad (1.1)$$

where $\underline{D} = \underline{E} + \frac{4\pi i}{\omega}\underline{j}$ is the electrical induction. In the case of a cold magnetoplasma one obtains

$$\epsilon_{ij} = \begin{pmatrix} \epsilon_1 & i\epsilon_2 & 0 \\ -i\epsilon_2 & \epsilon_1 & 0 \\ 0 & 0 & \epsilon_3 \end{pmatrix} \qquad (1.2)$$

with

$$\epsilon_1 = 1 - \sum_\alpha \frac{\omega_{p\alpha}^2}{\omega^2 - \omega_{B\alpha}^2}, \quad \epsilon_2 = \frac{\omega_{p\alpha}^2\,\omega_{B\alpha}}{\omega(\omega^2 - \omega_{B\alpha}^2)}$$

$$\epsilon_3 = 1 - \sum_\alpha \frac{\omega_{p\alpha}^2}{\omega^2} \qquad (1.3)$$

Here the index α refers to particle species, $\omega_{p\alpha} = \frac{4\pi e^2 n_\alpha}{m_\alpha}$ is the plasma frequency and $\omega_{B\alpha} = \frac{q_\alpha B}{m_\alpha c}$ the cyclotron frequency of species α particles.

When there is no magnetic field, $\underline{B}_o=0$, the tensor ϵ_{ij} becomes proportional to the unit tensor

$$\epsilon_{ij} = \epsilon_3 \delta_{ij} \qquad (1.4)$$

and the plasma is isotropic.

The expressions above refer to a loss-free medium (no collisions) in which case ϵ_{ij} is a Hermitean tensor. The fact that there are off-diagonal elements of ϵ_{ij} (proportional to ϵ_2) when $\underline{B}_o \neq 0$, implies that electromagnetic waves in a cold magnetoplasma cannot be linearly polarized (with the exception of propagation across the field) and are, in general, elliptically polarized.

Notice that the elements ϵ_1 and ϵ_2 have singularities for $|\omega| = |\omega_{B\alpha}|$ corresponding to cyclotron resonance. In actual fact the cold plasma approximation becomes inapplicable near these resonances and pressure and kinetic effects must be taken into account to have a proper description of the plasma dielectric properties.

The dispersion equation for electromagnetic waves in an anisotropic medium with a dielectric permittivity tensor ϵ_{ij} has the form (Stix, 1962)

$$\Lambda = \det|\Lambda_{ij}| = 0 \qquad (1.5)$$

where

$$\Lambda_{ij} = n^2\left(\frac{k_i k_j}{k^2} - \delta_{ij}\right) + \epsilon_{ij} \qquad (1.6)$$

with $\underline{k}$ wave vector and

$$n = \frac{kc}{\omega} \qquad (1.7)$$

the index of refraction. Substituting in (1.5) eqs. (1.3) for the ϵ_{ij}'s, we obtain the dispersion relation for a cold magnetoplasma in the well known form

$$\Lambda \equiv An^4 + Bn^2 + C = 0 \qquad (1.8)$$

where

$$A = \epsilon_1 \sin^2\theta + \epsilon_3 \cos^2\theta$$

$$B = -\epsilon_1\epsilon_3\left(1+\cos^2\theta\right) - \left(\epsilon_1^2 - \epsilon_2^2\right)\sin^2\theta \qquad (1.9)$$

$$C = \epsilon_3\left(\epsilon_1^2 - \epsilon_2^2\right)$$

and θ is the angle between the wave vector $\underline{k}$ and the magnetic field $\underline{B}_o$. It follows from (1.8) that it is possible that two waves can propagate in a plasma with a given frequency but with different values of the refractive index

$$n^2 = \frac{-B \pm \sqrt{B^2 - 4AC}}{2A} \qquad (1.10)$$

as it will appear from the plots of $n(\omega)$ to be given later.

1.2 Wave branches in a cold magnetoplasma and their classification

Eq. (1.8), taking the wave vector $\underline{k}$ as given, is an equation for the eigen-frequencies $\omega^{(\nu)}(k,\theta)$ of the plasma oscillations and it is easily seen to be of fifth order in ω^2 so that it defines ten eigenfrequencies. As, corresponding to each eigenfrequency ω^ν, there is an eigenfrequency $-\omega^\nu$, we shall assume for simplicity that all eigenfrequencies are positive and, correspondingly, distinguish five branches of oscillations of a cold magnetoplasma.

In Fig. 1 (which is not drawn to scale), we show the five branches in a plot of n^2 versus ω referring to propagation oblique with respect to the magnetic field ($\theta \neq 0$, $\theta \neq \pi/2$). We will name the five branches following the classification of Shafranov (1966). As indicated in the figure, the branches are called: the Alfvén branch (A), the fast magnetosound branch (FMS), the slow extraordinary branch (SE); the ordinary branch (O); the fast extraordinary branch (FE). The relation of these branches to other names which one finds in the literature (ordinary O and extraordinary X, L and R waves) will be clarified later.

Let us now discuss the remaining notation of Fig. 1 (which also follows Shafranov (1966)). There are three cut-off frequencies, denoted with $\omega_o^{(1)}$, $\omega_o^{(2)}$, $\omega_o^{(3)}$, which separate regions with $n^2>0$, i.e. regions of transparency, from regions ($n^2<0$) where the waves cannot propagate. It is clear that $n^2=0$ when $C=\epsilon_3(\epsilon_1^2-\epsilon_2^2)=0$. From this we find, neglecting small ionic contributions to ϵ_1, ϵ_2, ϵ_3 the following expressions for the cut-off frequencies

$$\omega_o^{(1)} = \sqrt{\omega_{pe}^2 + \frac{1}{4}\omega_{Be}^2} + \frac{1}{2}|\omega_{Be}| \,, \qquad \omega_o^{(2)} = \omega_{pe} \,,$$

$$\omega_o^{(3)} = \sqrt{\omega_{pe}^2 + \frac{1}{4}\omega_{Be}^2} - \frac{1}{2}|\omega_{Be}| \qquad (1.11)$$

The meaning of these values is that

$$\omega_o^{(i)} = \lim_{k\to 0} \omega^{(i)}(k,\theta)$$

The corresponding behaviour of $\omega^{(\nu)}$ as a function of k in the transparency regions is indicated in Fig. 2 (always referring to oblique propagation).

The frequencies labelled ∞ in both Figs. 1 and 2 correspond, on the other hand, to infinities of the refractive index which means wave phase velocity going to zero. From the dispersion relation (1.10) we see that one of the solutions for the refractive index tends to ∞ when

$$A = 0 \tag{1.12}$$

The solution behaves then as

$$n^2 = -\frac{B}{A} \tag{1.13}$$

while the refractive index of the other wave (the second solution for the refractive index) stays finite and is given by

$$n^2 = -\frac{C}{B} \tag{1.14}$$

Using the expressions (1.3) for the dielectric tensor components, eq. (1.12) for the resonant frequencies becomes

$$1 - \frac{\omega_{pe}^2}{\omega^2}\cos^2\theta - \frac{\omega_{pe}^2}{\omega^2-\omega_{Be}^2}\sin^2\theta - \frac{\omega_{pi}^2}{\omega^2}\cos^2\theta - \frac{\omega_{pi}^2}{\omega^2-\omega_{Bi}^2}\sin^2\theta = 0 \tag{1.15}$$

This is of third order in ω^2 and thus defines three resonant frequencies

$$\omega = \omega_\infty^{(j)}(\theta) \qquad j = 1,2,3$$

which are the ones corresponding to the vertical asymptotes in Fig. 1 ($n^2\to\infty$) and to the horizontal asymptotes in Fig. 2 ($\omega/K\to 0$).

The behaviour of the resonant frequencies as a function of θ is depicted schematically in Fig. 3 where we find also the limiting values of these frequencies for parallel and perpendicular propagation. Thus we see that, going from perpendicular to parallel propagation,

$$0 < \omega_\infty^{(3)}(\theta) < \omega_{Bi}$$

$$\sqrt{\frac{\omega_{pi}^2+\omega_{Bi}^2}{1+\eta}} < \omega_\infty^{(2)}(\theta) < \mathrm{Min}(\omega_{pe}, \omega_{Be})$$

whereas $\omega_\infty^{(1)}$ decreases from $\sqrt{\omega_{pe}^2+\omega_{Be}^2}$ at perpendicular propagation to Max $(\omega_{pe}, \omega_{Be})$ for parallel propagation. In the above expressions it is $\eta=\omega_{pe}^2/\omega_{Be}^2$.

The resonance frequencies $\omega_\infty^{(j)}(\theta)$ have considerable physical meaning in that, as one approaches one of these frequencies, the corresponding electromagnetic wave becomes longitudinal, i.e. the component of the electrical field strength parallel to the wavevector, $\underline{E}_1=\underline{k}(\underline{k}\cdot\underline{E})/k^2$ becomes appreciably larger than the components at right angles to $\underline{k}$.

To see this, we multiply the equations $\Sigma\, \Lambda_{ij}E_j=0$ by k_i and sum over i from 1 to 3. As a result we get

$$\sum_{i,j} \epsilon_{ij} k_i E_j = 0 \tag{1.16}$$

Putting $\underline{E}=\underline{E}_1+\underline{E}_t$, we find from this equation

$$E_l = -\frac{1}{A}\sum_{i,j} \epsilon_{ij} k_i E_{t_j} \tag{1.17}$$

where

$$A = \frac{1}{k^2}\sum_{i,j}\epsilon_{ij}k_ik_j = \epsilon_1\sin^2\theta + \epsilon_3\cos^2\theta \qquad (1.18)$$

so that, indeed $|\underline{E}_l|/|\underline{E}_t|\to\infty$ as $A\to 0$. As one can always put $\underline{E}_l=-\nabla\phi=-i\underline{k}\phi$, $\underline{E}_t=\nabla\times A=i(\underline{k}\times\underline{A})$, with ϕ, $\underline{A}$ scalar and vector potentials respectively, one can also say that the plasma oscillations become irrotational (i.e. quasi-electrostatic) near the frequencies $\omega_\infty^{(j)}$.
In the literature the frequencies $\omega_\infty^{(j)}$ are often called hybrid resonances. Near the hybrid resonances, as the wave phase velocity tends to zero (see Fig. 2), one cannot use any more the cold plasma approximation. Temperature and kinetic effects must be taken into account. I will not however enter here into the new complexities introduced by a kinetic description.

Summarizing the results shown, referring to oblique propagation (Figs. 1 and 2), we have that a cold plasma in an external magnetic field is an anisotropic medium with temporal dispersion in which there are five branches of oscillations: the ordinary, the fast and slow extraordinary, the Alfvén and the fast magnetosonic mode. The frequency dependence of the refractive index of these branches is shown in Fig. 1. All waves have normal dispersion in the sense that their frequencies increase with increasing wave number, as indicated in Fig. 2, and are elliptically polarized. We must add that, as it is easy to show, the dispersion of the high frequency branches (O waves, FE waves and also SE waves with θ not too close to $\pi/2$) is determined by the electrons alone, while the dispersion of the low frequency branches (FMS waves A waves and also FE waves with $\theta \sim \pi/2$) is determined in general by both electrons and ions.

1.3 Parallel propagation of electromagnetic waves in a cold magnetoplasma

When $\theta=0$, we obtain from (1.9)

$$A = \epsilon_3 \quad , \quad B = -2\epsilon_1\epsilon_3 \quad , \quad C = \epsilon_3(\epsilon_1^2-\epsilon_2^2)$$

and, consequently, the dispersion relation (1.8) splits into three factors

$$\Lambda \equiv \epsilon_3(n^2-\epsilon_1-\epsilon_2)(n^2-\epsilon_1+\epsilon_2) = 0 \qquad (1.19)$$

The equation $\epsilon_3=0$ has solution

$$\omega(k) = \sqrt{\omega_{pe}^2+\omega_{pi}^2} \qquad (1.20)$$

corresponding to Langmuir oscillations unaffected by the magnetic field. The remaining two factors define two electromagnetic waves with circular polarization. More explicitly the respective indices of refraction are given by

$$n_+^2 = 1 - \frac{\omega_{pe}^2}{\omega(\omega-|\omega_{Be}|)} - \frac{\omega_{pi}^2}{\omega(\omega+\omega_{Bi})} \qquad (1.21)$$

$$n_-^2 = 1 - \frac{\omega_{pe}^2}{\omega(\omega+|\omega_{Be}|)} - \frac{\omega_{pi}^2}{\omega(\omega-\omega_{Bi})} \qquad (1.22)$$

The electric field vector of one wave rotates around the magnetic field $\underline{B}_o$ in the direction of the electron rotation in the magnetic field. Correspondingly (see eq. (1.21) its refractive index has a singularity at $\omega = |\omega_{Be}|$. This wave which is circularly polarized around $\underline{B}_o$ with a right-hand sense, is also called the R wave (where R stands for right hand polarization). For the second case (eq. (1.22)) the electric vector rotates in the direction of the ion rotation in the magnetic field $\underline{B}_o$ and the index of refraction has, correspondingly, a singularity at $\omega = |\omega_{Bi}|$. This is also commonly quoted as the L wave (L stands for left handed polarization). These waves are also commonly denoted, especially at radio frequencies, as the extraordinary (R) and ordinary (L) wave respectively.

Fig. 4 shows the frequency dependence of n^2 for parallel propagation and shows, contrary to the general case of oblique propagation only 4 branches of oscillations, which are the ones represented by eqs. (1.21) (1.22). The branch which has vanished is given by the Langmuir oscillations (1.20) (and corresponds to the O branch of Fig. 1). The notation in the Figure continues to be the Shafranov notation. What is called L wave, as introduced before, corresponds therefore to the two branches FMS and FE (with resonances at $|\omega_{Be}|$)(*) while the so called R wave corresponds to the two branches A and SE (with resonance at ω_{Bi}). Thus for parallel propagation Fast magnetosound waves and Fast extraordinary waves exhibit right hand polarization. Finally Fig. 5 gives the wave number dependences of eigenfrequencies for $\theta=0$. In this plot the dashed straight line with label O (ordinary wave) corresponds to the Langmuir oscillations.

1.4 Transverse propagation of electromagnetic waves in a cold magnetoplasma

When $\theta=\pi/2$ the dispersion equation (1.8) splits into two equations

(*) when $\omega_{Bi} \ll \omega \ll \omega_{Be}$, this R wave becomes the electron whistler, for parallel propagation.

$$n^2 = \epsilon_3 \tag{1.23}$$

$$n^2 = \frac{\epsilon_1^2 - \epsilon_2^2}{\epsilon_1} \tag{1.24}$$

The first equation determines the refractive index of a linearly polarized wave whose electrical field vector is parallel to the external magnetic field $\underline{B}_o$. The magnetic field does not affect the propagation of this wave (which is the $\theta=\pi/2$ limit of the ordinary wave branch O). Figs. 6 and 7 give the frequency dependence of the squared refractive index and the wave number dependencies of the corresponding eigenfrequencies. We have now three different branches, corresponding to eq. (1.24) and, in addition, the O branch which is dashed in both Figures. Like in the case $\theta=0$, also for $\theta=\pi/2$ one branch has disappeared with respect to the general case of oblique propagation. The branch which has disappeared for $\theta=\pi/2$ is the Alfvén wave branch. We realize that this must be so by looking at Fig. 1 (oblique propagation) where the A branch had a resonance at the frequency $\omega_\infty^{(3)}$. This frequency, as already shown in Fig. 3, goes however to 0 when $\theta=\pi/2$. As for the remaining three branches, described by eq. (1.24) two of them have resonances, in the region of transparency and precisely, a resonance at $\omega_\infty^{(2)} \sim \left(\frac{\omega_{pi}^2 + \omega_{Bi}^2}{1+ \eta}\right)^{1/2}$ (for the FMS branch) and a second at $\omega_\infty^{(1)} = \left(\omega_{pe}^2 + \omega_{Be}^2\right)^{1/2}$ (for the SE branch).

In relation to other terminology which one frequently finds in the literature, we must add that, for the case of transverse propagation the two factors (1.23) and (1.24) of the dispersion equation are denoted respectively by ordinary wave (O) (which is also our present terminology) and extraordinary wave (X).

The ordinary wave, as already mentioned, is not affected by the magnetic field while the X wave is. The polarization of the X wave is elliptical in a plane perpendicular to $\underline{B}_o$ and, hence, containing the wave phase speed. This notation of extraordinary (X) wave thus includes the three branches (FMS, SE, FE) of Figs. 6 and 7. We recall also that this same distinction into an O and an X wave is sometimes used also for oblique propagation. When this is done the term ordinary includes both the A branch and our O branch, while the term extraordinary refers conplexively to the remaining three branches (FMS, SE, FE).

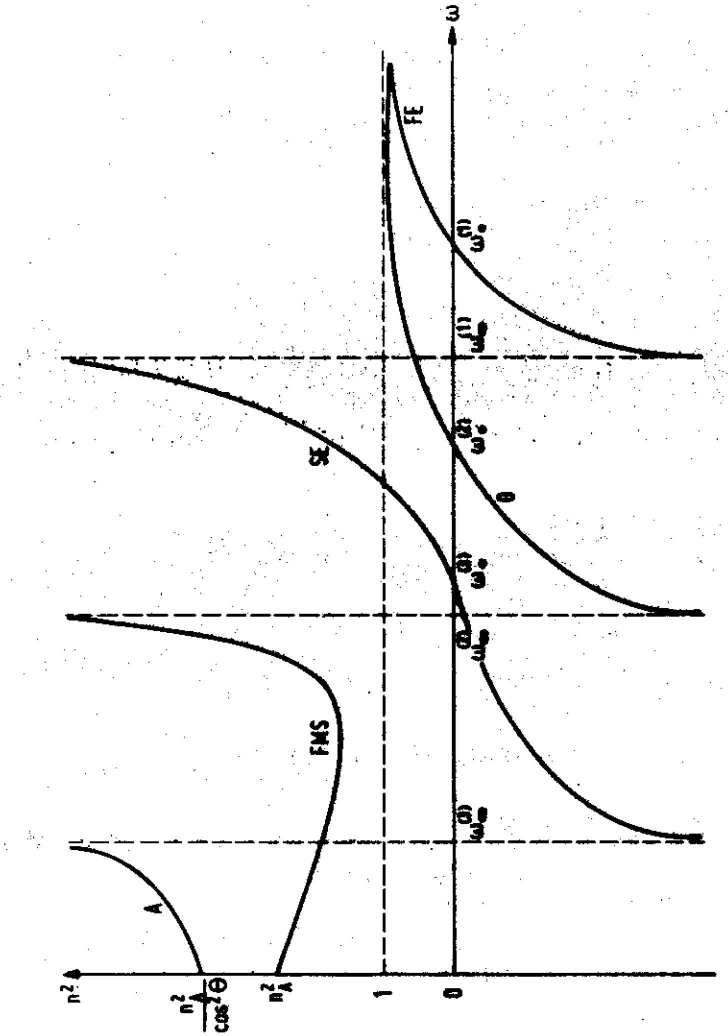


FIG.1

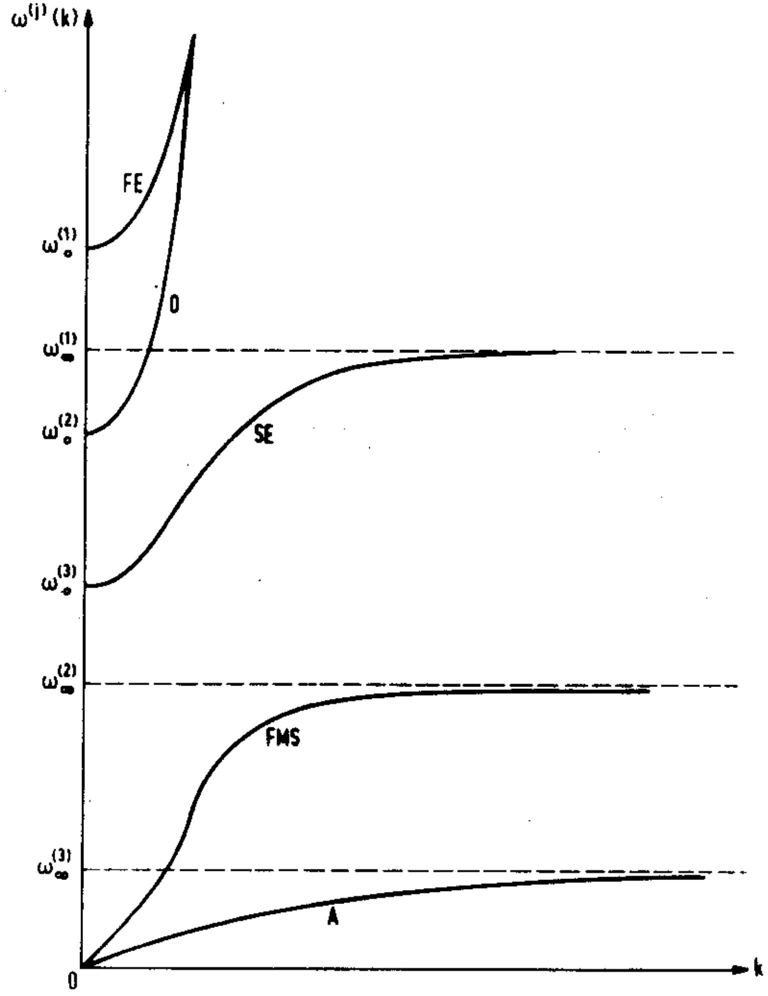

$\omega^{(j)}(k)$
FE
$\omega_o^{(1)}$
O
$\omega_\infty^{(1)}$
$\omega_o^{(2)}$
SE
$\omega_o^{(3)}$
$\omega_\infty^{(2)}$
FMS
$\omega_\infty^{(3)}$
A
0
k

FIG. 2

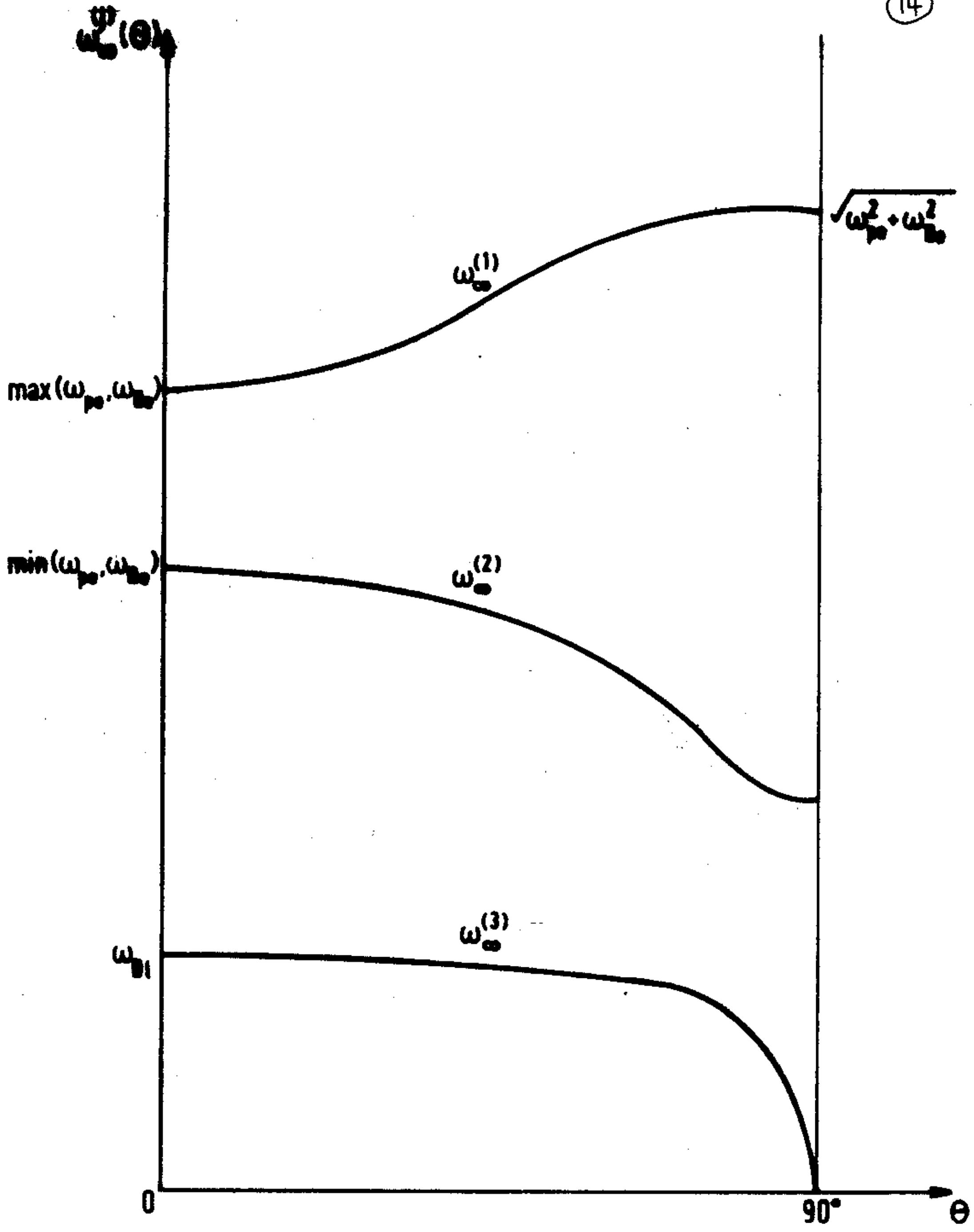
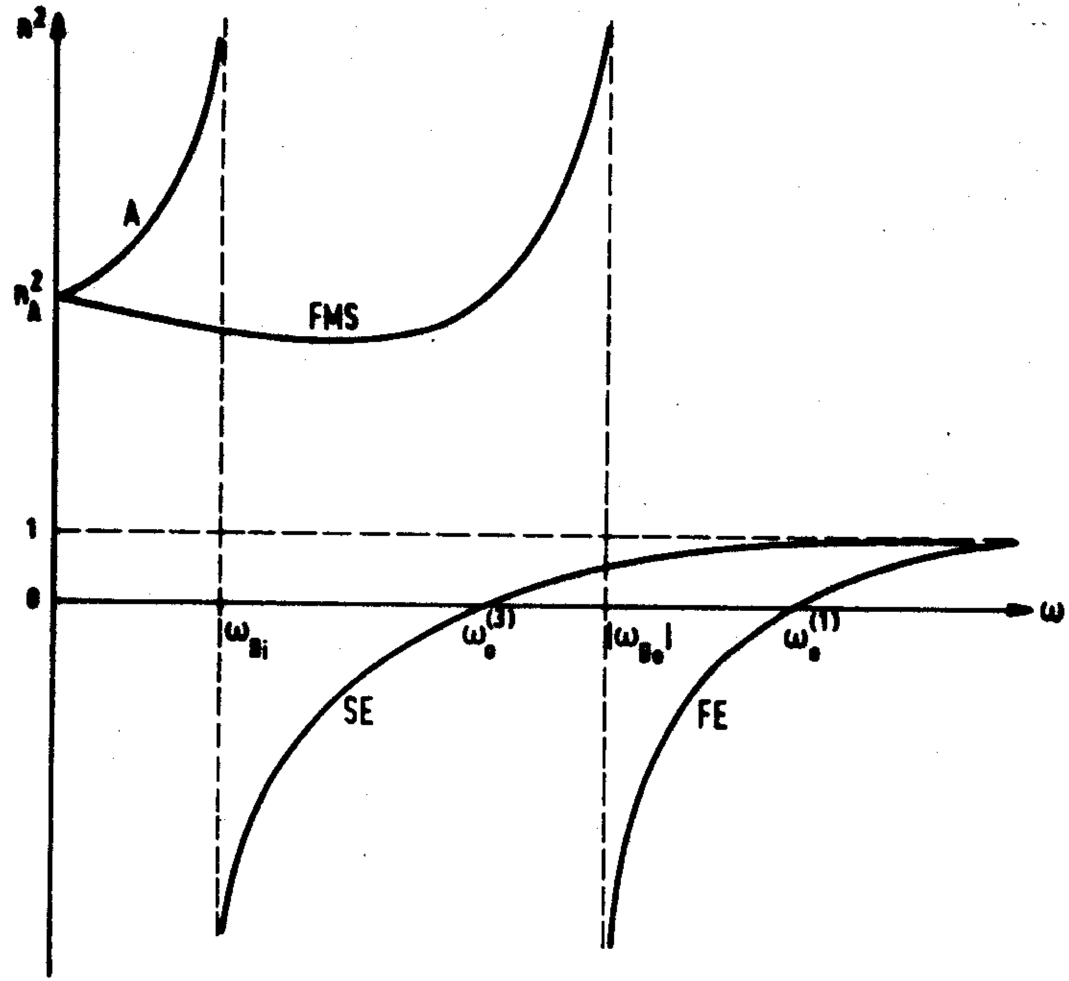
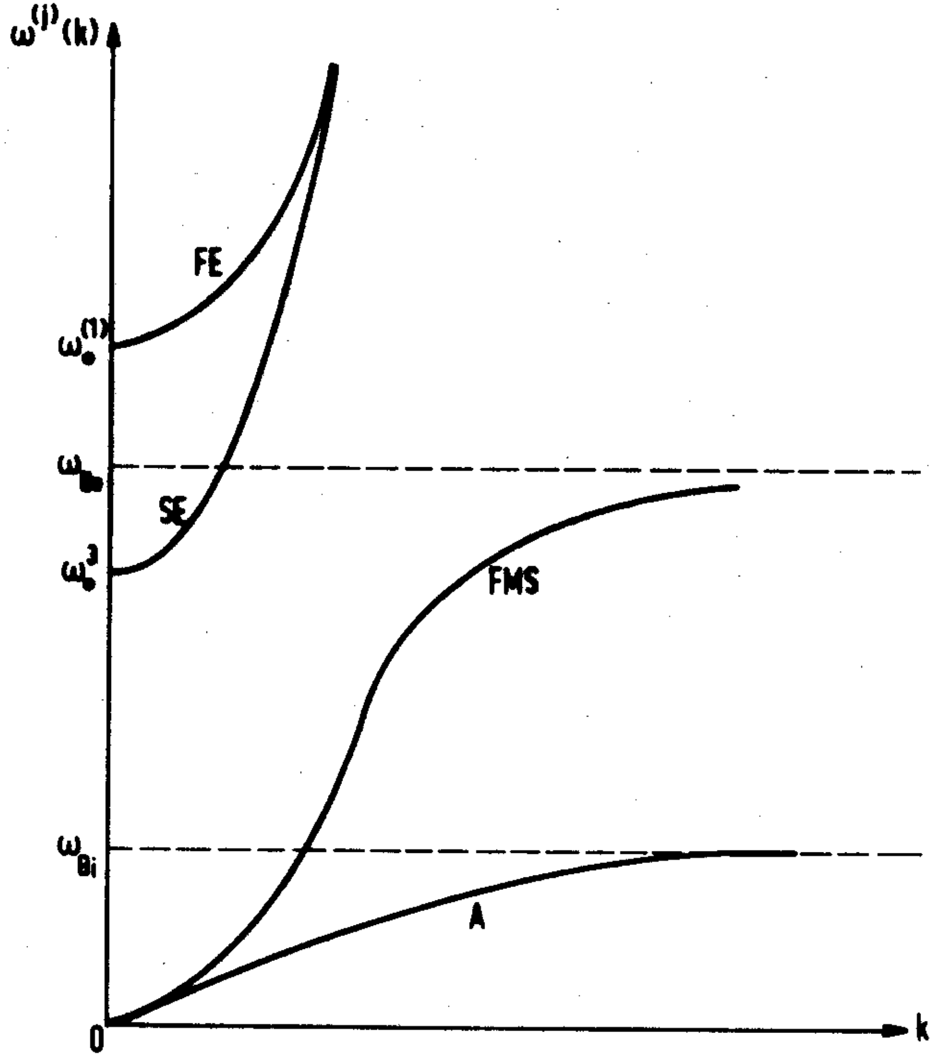

$\omega_\infty^{(j)}(\theta)$
$\sqrt{\omega_{pe}^2+\omega_{Be}^2}$
$\omega_\infty^{(1)}$
max$(\omega_{pe},\omega_{Be})$
min$(\omega_{pe},\omega_{Be})$
$\omega_\infty^{(2)}$
$\omega_\infty^{(3)}$
ω_{Bi}
0
90°
θ

FIG. 3



n^2
A
n_A^2
FMS
1
0
ω
ω_{Bi}
$\omega_e^{(3)}$
$|\omega_{Be}|$
$\omega_e^{(1)}$
SE
FE

FIG. 4



$\omega^{(j)}(k)$
FE
$\omega_e^{(1)}$
ω_{Be}
SE
ω_e^3
FMS
ω_{Bi}
A
0
k

FIG. 5

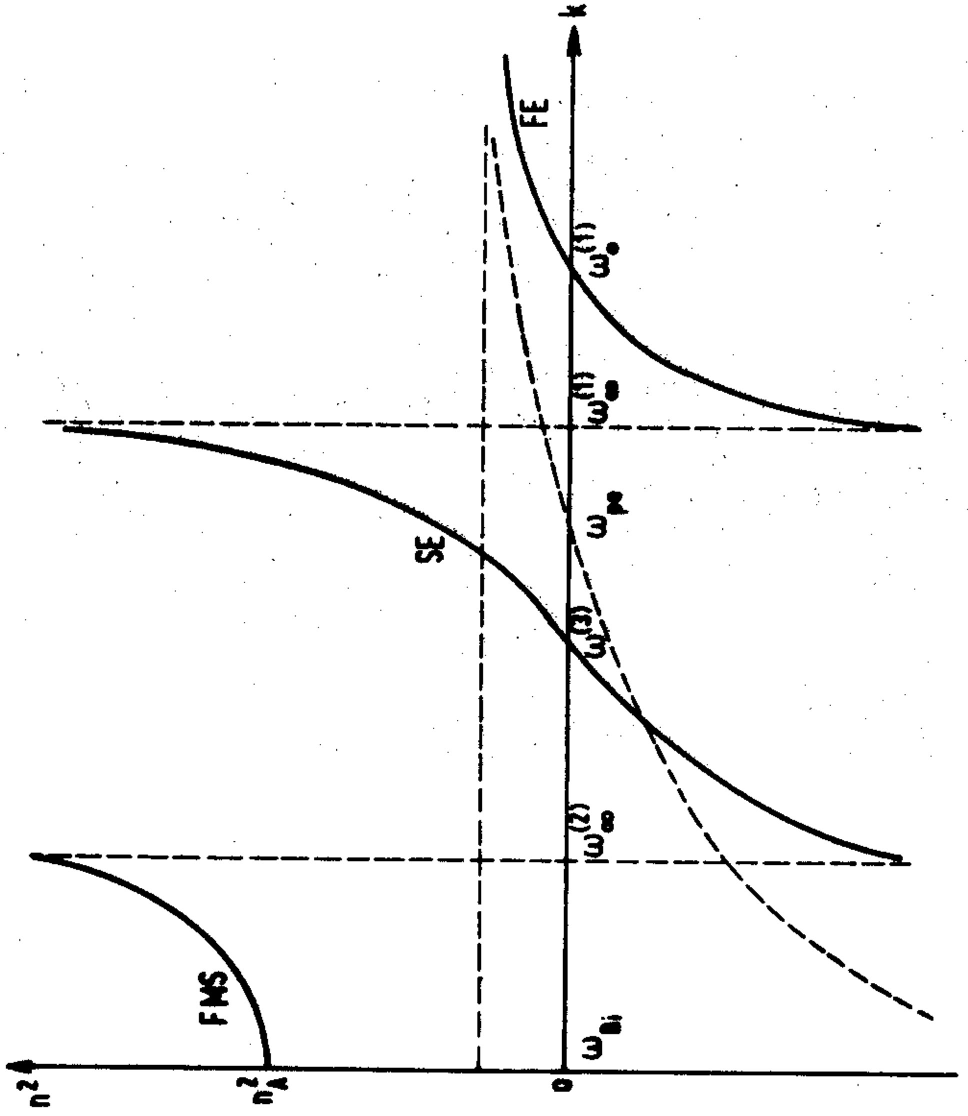


FIG. 6

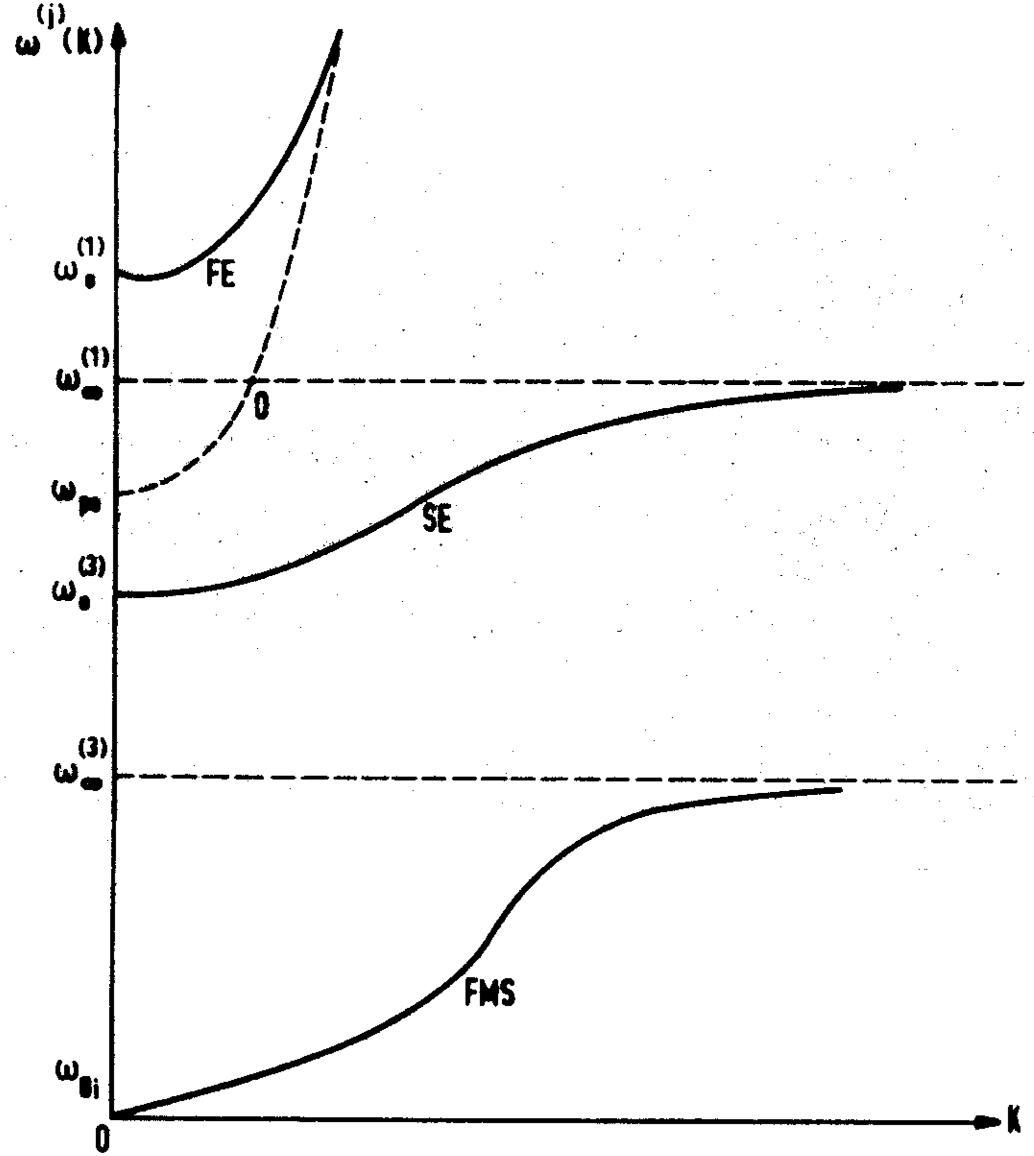


FIG. 7