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RAY TRACING IN INHOMOGENEOUS MEDIA

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5.1 Ray path equations

In this section I will derive the equations which describe the path of a wave packet in an inhomogeneous medium such as the ionosphere. More precisely, I will give the equations which have to be used for the simplified case of a horizontally stratified medium. If this refers to the ionosphere, then, for example, earth's curvature is neglected. More general ray equations are derived, for instance, in Chapt 14 of Budden (1961a).

The equations for ray paths in a horizontally stratified ionosphere make use of a variable, denoted with q and first introduced by Booker in his radio studies of the ionosphere: q is simply the vertical projection of the refractive index vector (i.e. a vector of magnitude equal to the refractive index and directed along the wave normal direction). Thus, if θ is the angle which the wave normal makes with respect to the vertical (z axis), which is taken as the direction of stratification, it is

$$q = n \cos \theta \qquad (5.1)$$

It can be derived (see, for example, Budden, 1961a) that q obeys a quartic equation

$$F(q) \equiv \alpha q^4 + \beta q^3 + \gamma q^2 + \delta q + \epsilon \qquad (5.2)$$

its four solutions representing therefore four rays in the ionosphere. If we further call S_1, S_2 the x and y components of the refractive index vector, then

$$S^2 \equiv S_1^2 + S_2^2 = n^2 \sin^2 \theta \qquad (5.3)$$

and Snell's law of refraction (see Appendix A) tells us that

$$S = \text{const} \qquad (5.4)$$

Furthermore it is, obviously

$$q^2 = n^2 - S^2 \qquad (5.5)$$

and the three direction cosines of the wave normal are given by

$$S_1/(S_1^2+S_2^2+q^2)^{1/2}\ ,\ S_2/(S_1^2+S_2^2+q^2)^{1/2}\ ,\ q/(S_1^2+S_2^2+q^2)^{1/2} \qquad (5.6)$$

The coefficients $\alpha,\ldots,\epsilon$ in (5.2) are function of S_1, S_2 and the frequency and will not be given here.

In terms of the above quantities any field component F in an electromagnetic wave moving in the medium, which is supposed to be slowly varying, can be written in the WKB form

$$F(z)=F_o(z)\exp(ik(ct-S_1x-S_2y - \int_o^z q\, dz)) \qquad (5.7)$$

where $F_o(z)$ represents an amplitude variation on a scale slower than that contained in the exponential factor. More precisely, (5.7) represents a single wave which extends indefinitely in the x and y directions and has a single frequency. A real electromagnetic signal, which is limited in lateral extent and in time, i.e. a wave packet, is formed, as we know, by adding together an infinite number of waves like (5.7) with various values of S_1, S_2, k (q which is a root of eq. (5.2) is a function of k, S_1, S_2). Such wave packet can therefore be generally written as

$$F(t,x,y,z) =$$

$$\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} F_o(k,S_1,S_2)\exp\{ik(ct-S_1x-S_2y-\int_o^z q\,dz)\}dk\,dS_1dS_2 \tag{5.8}$$

It is the path of this wave packet, or ray path, which we want to determine. Now, the contribution to the integral in (5.8) will only be appreciable near the values of S_1, S_2 for which the phase ϕ in the function under integration is stationary. The phase is given by

$$\phi = k(ct - S_1x - S_2y - \int_o^z q\,dz) \tag{5.9}$$

and, for this to be stationary, we must have

$$-\frac{\partial\phi}{\partial S_1} \equiv x + \int_0^z \frac{\partial q}{\partial S_1}dz = 0 \tag{5.10}$$

$$-\frac{\partial\phi}{\partial S_2} \equiv y + \int_0^z \frac{\partial q}{\partial S_2}dz = 0 \tag{5.11}$$

These equations must be solved for S_1, S_2 at any given position. If the solutions are S_{10}, S_{20}, the amplitude in (5.8) can be replaced approximately by $F(k,S_{10},S_{20})$. If we suppose, for example, that the

wave packet leaves, say, a transmitter in the x-z plane with $S_1=S_o$, $S_2=0$, the amplitude F_o will have a maximum for these two values of S_1, S_2. Corresponding to the equations (5.10), (5.11) we then find, for an element of the path of the wave packet, the differential equations

$$\frac{dx}{dz} = -\left(\frac{\partial q}{\partial S_1}\right)_{S_1=S_o} \qquad \frac{dy}{dz} = -\left(\frac{\partial q}{\partial S_2}\right)_{S_2=0} \tag{5.12}$$

Now, the quartic equation (5.2) for the variable q must be satisfied by each of the component plane waves of the wave packet considered. Therefore, at any level, it must be valid for any values of S_1, S_2 and we can write, correspondingly,

$$\frac{dF(q)}{dS_1} = \frac{dF(q)}{dS_2} = 0 \tag{5.13}$$

Thus

$$\frac{dF(q)}{dS_1} = \frac{\partial F}{\partial q}\frac{\partial q}{\partial S_1} + \frac{\partial\alpha}{\partial S_1}q^4 + \frac{\partial\beta}{\partial S_1}q^3 + \frac{\partial\gamma}{\partial S_1}q^2 + \frac{\partial\delta}{\partial S_1}q + \frac{\partial\epsilon}{\partial S_1} = 0 \tag{5.14}$$

and there is a similar equation with S_2 replaced by S_1. These two equations can be used to obtain explicit expressions for the right hand sides of eqs. (5.12) in terms of the Booker quartic and its coefficients. For the case considered $S_1=S_o$, $S_2=0$, we obtain

$$\frac{dx}{dz} = \left[\frac{\partial\alpha}{\partial S_1}q^4 + \frac{\partial\beta}{\partial S_1}q^3 + \frac{\partial\gamma}{\partial S_1}q^2 + \frac{\partial\delta}{\partial S_1}q + \frac{\partial\epsilon}{\partial S_1}\right] \Big/ \frac{\partial F}{\partial q}$$

(5.15)

$$\frac{dy}{dz} = \left[\frac{\partial\alpha}{\partial S_2}q^4 + \frac{\partial\beta}{\partial S_2}q^3 + \frac{\partial\gamma}{\partial S_2}q^2 + \frac{\partial\delta}{\partial S_2}q + \frac{\partial\epsilon}{\partial S_2}\right] \Big/ \frac{\partial F}{\partial q}$$

where derivates with respect to S_1, S_2 are calculated for $S_1=S_o$, $S_2=0$ respectively. These are the ray equations which must be integrated to find the ray path in an horizontally stratified ionosphere.

It is of interest to note what these equations tell us about reflection of a wave packet in such a stratified medium. At reflection, the ray path must become horizontal and, hence, dx/dz and dz/dz must become infinite. The numerators, on the right hand sides of eqs. (5.15) cannot become ∞ unless q is ∞. Hence the denominators must be zero which means that, at reflection, one must have

$$\frac{\partial F(q)}{\partial q} = 0 \qquad (5.16)$$

provided that neither numerator in (5.15) is zero at the same level. This is also the condition that implies that two roots of the quartic are equal. Notice that this double root needs not be zero. If it is different from zero, when the ray path is horizontal (at reflection) the wave normal is not. Similarly, at a level where one root for q is zero, the wave normal is horizontal but the wave packet is in general not so. We see therefore that the level where q=0 is not in general a level of reflection of the wave packet unless q=0 is a double root of the quartic.

5.2 Ray tracing using refractive index surfaces

We will now outline an alternative method, beyond that based on the differential equations (5.15) and the use of the Booker's quartic, of finding ray directions in the ionosphere. The method, which uses the refractive index surfaces together with Snell's law, has been introduced by Poeverlein (1949) and it is actually the one we will follow in outlining the phenomenon of trapping of whistler waves in ducts of

enhanced ionization in a later section.

We recall that refractive index surfaces are constructed in spherical coordinates. The polar angle, with respect to the magnetic field direction, represents the wave normal direction and the length of the radius vector to the surface represents the magnitude of the corresponding refractive index. This radius vector is called the refractive index vector. For each wave normal direction, as we saw in Sect. 3, the ray direction is given by the perpendicular to the refractive index surface at the point where the refractive index vector intersects that surface.

To illustrate the use of refractive index surfaces for ray tracing in non homogeneous media, let us first consider the case of a sharp plane boundary between two different homogeneous media. Consider a ray R_1 in direction ψ_1 (with respect to $\underline{H}_o$), with corresponding wave normal angle θ_1 travelling in medium I towards the boundary (see Fig. 28a). The direction of transmitted and reflected rays can be determined geometrically as indicated in Fig. 28b. This figure contains the refractive index surfaces of both media I and II superimposed on the same center O. We also assume that the magnetic field direction has not changed in going from I to II. Here a line is drawn from the center O in the direction θ_1 of the first wave normal. This intersects the refractive index surface of medium I at P_1 (μ_1 denotes in Fig. 28b the refractive index vector of the incident wave). The corresponding ray direction is, again, indicated with R_1. Now, by Snell's law of refraction (5.4) the incident, transmitted and reflected waves, must have refractive index vectors whose projections on the boundary are equal. Therefore, a line drawn at P_1 normal to the boundary, must also pass through all other possible terminations of refractive index vectors. This line in Fig. 30b intersects the refractive index surface of medium I at P_3 where therefore wave normal and ray direction of the reflected wave are determined. The same line intersects the refractive index surface of medium II at points P_2 and P_4. The construction at P_2 gives the transmitted ray R_2. The ray R_4 constructed at the other intersection P_4 would represent a wave in medium II with energy travelling towards medium I. This ray is actually not existing as we supposed no source of rays in medium II (but only in medium I).

The same type of construction, outlined in Figs. 28a and b for the case of a sharp boundary, can be applied to find the change in ray direction for non homogeneous slowly varying media. By slowly varying we mean that the refractive index varies on a scale larger than the wavelengths of the waves we are considering. Under this condition we can neglect the reflected components of the wave (see Appendix A). The graphycal procedure for ray tracing is indicated, for the bidimensional case, in Figs. 29a and b. The initial condition that we need, at some starting point P_1, is the wave normal angle θ_1 with respect to the direction of the local magnetic field H_{o1} at P_1. From this and the refractive index surface relative to P_1, we determine the direction of the ray at the initial point P_1. The ray is assumed to progress a certain distance, up to a point P_2, the distance being small enough to consider the index of refraction as essentially constant along this path element. At this point we apply the previous considerations by considering as the two media those corresponding to the local parameters at P_1 and P_2 respectively. In other words, we invent a plane boundary which is perpendicular to the local (in the segment P_1P_2) spatial variations of the refractive index. In Fig. 29b we have drawn the refractive index surfaces $\mu_1(\theta)$, $\mu_2(\theta)$ corresponding to points P_1, P_2 respectively (and having the two magnetic fields H_{o1}, H_{o2} as reference axes). Then a normal to the boundary is drawn at $\theta=\theta_1$ on the surface $\mu_1(\theta)$ and intersects $\mu_2(\theta)$ at $\theta=\theta_2$. This intersection gives then both magnitude and direction of the new wave normal at P_2. Correspondingly

we also determine a new ray direction (R_2), which is now supposed to hold for another small segment. The process is hence repeated until one obtains the desired path. It is clear that in such a construction to determine differential refraction in a slowly varying medium, the normals to the local "boundaries" have the meaning of local directions of the spatial gradient of the refractive index.

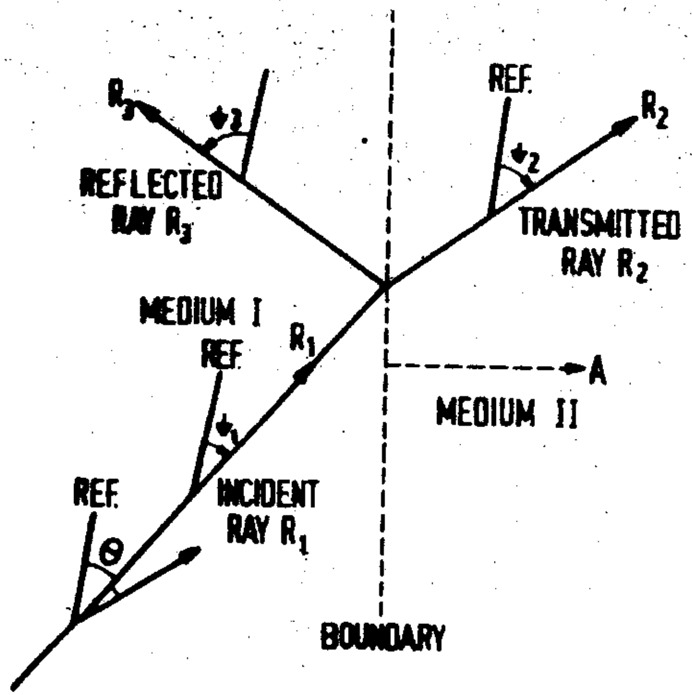


FIG 28a

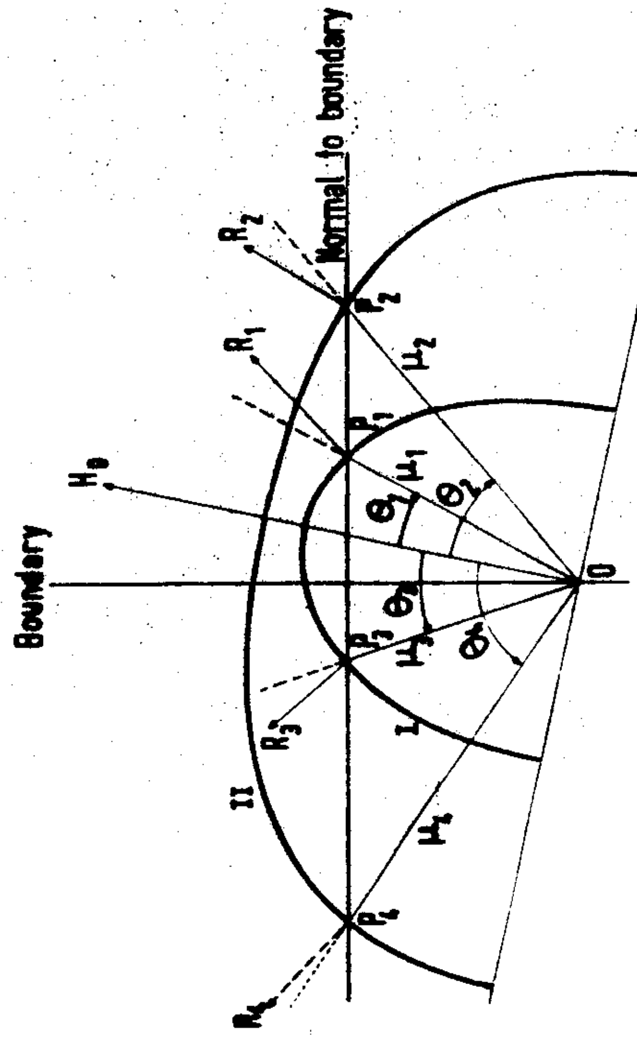
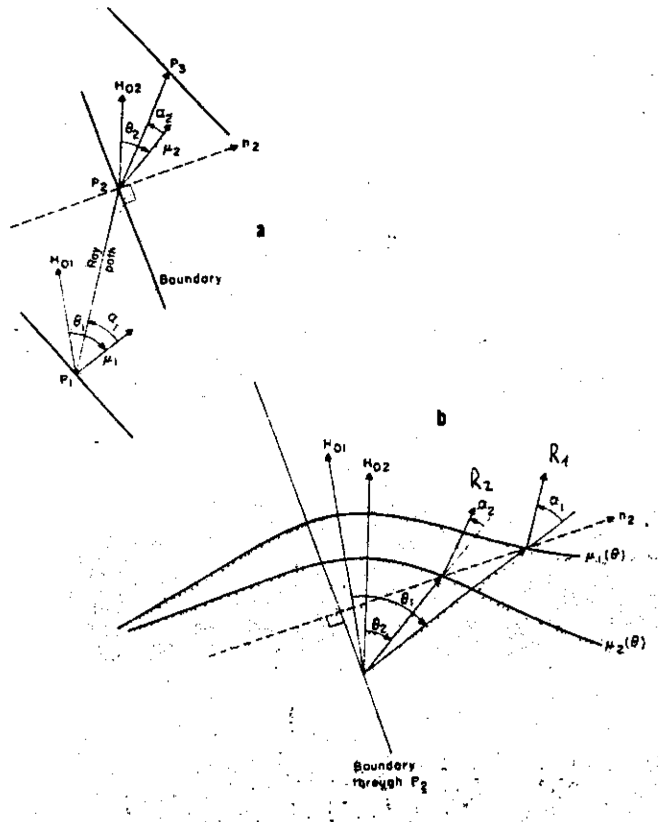


FIG 28b



a
P_3
H_{02}
θ_2
α_2
μ_2
n_2
P_2
Ray path
Boundary
H_{01}
θ_1
α_1
μ_1
P_1
b
H_{01}
H_{02}
R_2
R_1
α_2
α_1
n_2
$\mu_1(\theta)$
θ_1
θ_2
$\mu_2(\theta)$
Boundary through P_2

FIG. 29