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AUTUMN COURSE ON GEOMAGNETISM, THE IONOSPHERE
AND MAGNETOSPHERE

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RADIO PROPAGATION IN THE IONOSPHERE

PROBLEMS

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Lectures on

Radio Propagation in the Ionosphere

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PROBLEMS

(Harder problems are marked *).

* 1. A cold plasma contains free electrons. There is no superimposed magnetic field and collisions are negligible. A plane, linearly polarised, progressive or evanescent electromagnetic wave is present whose electric field at a given point is $E_o \sin \omega t$. Prove that the total stored energy per unit volume at that point (both electric and magnetic) is

$$\tfrac{1}{4} \varepsilon_o E_o^2 \left\{ 1 + (X - 1) \cos 2\omega t \right\} \quad \text{when } \omega > \omega_N$$

$$\tfrac{1}{4} \varepsilon_o E_o^2 \left\{ X + (X - 1) \cos 2\omega t \right\} \quad \text{when } \omega < \omega_N$$

where $X = \omega_N^2 / \omega^2$.

(see Budden: Radio Waves in the Ionosphere sections 2.11 and 3.10 Clemmow and Dougherty p.193).

2. A spherical cloud of uniform cold plasma contains free electrons and is surrounded by free space. There is no superimposed magnetic field and collisions are negligible. The electrons are all displaced by the same small distance parallel to a fixed direction, and are then released from rest. Show that they oscillate with angular frequency $\omega_N / \sqrt{3}$.

(Assume that all the electrons, including those near the boundary, experience the same electric field as an electron near the centre of the cloud).

Try the same problem for a cylindrical cloud if the electrons are initially displaced perpendicular to the axis. (Frequency $\omega_N / \sqrt{2}$).

3. For some purposes the ionosphere can be assumed to be a cold isotropic plasma containing free electrons with negligible collision damping. Prove that the group velocity of an electromagnetic wave of angular frequency ω is $c(1 - X)^{\frac{1}{2}}$ where $X = \omega_N^2/\omega^2$ and ω_N is the angular plasma frequency. Use these assumptions to solve the following problems.

A radio pulse or "wave packet" of central frequency ω is normally incident from below on a stratified ionosphere in which ω_N^2 is a function only of the height z. If ω is small enough the pulse is reflected at height z_o and returns to its starting level at z = 0. If the total time taken is T, then $h'(\omega) = \tfrac{1}{2}Tc$ is called the equivalent height of reflection. Find $h'(\omega)$ and the true height of reflection z_o in the following cases:

(a) $\omega_N^2 = 0$ for $z < a$. $\omega_N^2 = c(z-a)$ for $z > a$.

(b) $\omega_N^2 = \omega_p^2 \left\{ 1 - (z-b)^2/a^2 \right\}$ for $b - a < z < b + a$

$\omega_N^2 = 0$ otherwise.

(c) $\omega_N^2 = \omega_0^2 \exp(\alpha z)$.

4. The same assumptions are to be made as in the preceding question except that the pulse is now obliquely incident on the ionosphere at angle θ to the vertical. It leaves the transmitter at $x = 0$, $z = 0$ and, after reflection, returns to the ground at $x = D$, $z = 0$ (ignore the earth's curvature). If the time of travel to a given point (x,z) is τ then $p' = \tau c$ is called the equivalent path to that point. For the end point (D,0) it is denoted by P' and called the total equivalent path. Prove that $p' = x/\sin\theta$ and, in particular, that $P' = D/\sin\theta$. Note that this is the length of the two oblique sides of a triangle formed by producing the incident and emergent rays upwards to their point of intersection. $D/\sin\theta$ is called the 'triangulated path' This result is sometimes called Breit and Tuve's theorem. It is true for <u>any</u> electron distribution function N(z).* Find D and P' for cases (a) (b) and (c) of question 3.

*It is <u>not</u> true when the effect of the earth's magnetic field is allowed for.

*5. For the conditions of the two preceding questions $P'(\omega)$ is the total equivalent path for a pulse of frequency ω incident at an angle θ to the vertical. Let $h'(\omega\cos\theta)$ be the equivalent height of reflection of a vertically incident pulse of frequency $\omega\cos\theta$. Prove that

$$P'(\omega)\cos\theta = 2h'(\omega\cos\theta).$$

(this is sometimes called Martyn's theorem for equivalent path).

6. An elliptically polarised plane electromagnetic wave has its wave normal parallel to the z-axis. Its wave polarisation is $E_y/E_x = R$ (complex). (a) What are the axis ratio and the square of the length of the semi major axis of the polarisation ellipse (in terms of $|E_x|$ and R)? (b) The wave is incident normally on a cold magnetoplasma in which the characteristic polarisations are ρ and $1/\rho$. Find the ratio of the major axes of the polarisation ellipses of the two resulting waves in the plasma (and check that the answer is right for the special cases $R = \rho$ or $1/\rho$).

7. Two elliptically polarised plane electromagnetic waves of the same frequency, with their wave normals parallel to the z-axis, are combined. Their wave polarisations E_y/E_x are respectively ia and $-i/a$ where a is real, positve and less than

unity. The major axes of their polarisation ellipses are equal. What are the smallest and greatest possible axis ratios (minor axis/major axis) of the polarisation ellipse of the resultant wave and in each case what is the angle between the major axis and the x-axis?

8. An electromagnetic wave travels through a cold magnetoplasma with its wave normal always parallel to the superimposed magnetic field. The medium attenuates the waves and its thickness is such that the amplitude of the left handed circularly polarised characteristic wave is reduced by a factor F and that of the right handed wave is reduced by a factor $\frac{1}{2}$F. What is the state of polarisation of the emergent wave, if the incident wave is (a) linearly polarised, (b) unpolarised? (Try to give quantitative answers).

*9. Unpolarised radiation at radio frequencies comes into the earth's ionosphere from the galaxy (at great heights it is reasonable to ignore collisions). Show that within the ionosphere half the energy flux is in the ordinary and half in the extraordinary wave. (see Budden and Hugill 1964, Proc. Roy. Soc. A., 277, 365 especially § 4).

10. A homogeneous plasma contains free electrons, and collision damping is negligible. There is a superimposed steady

magnetic field which is so strong that it prevents the electrons from moving at right angles to it. The electrons can therefore move only in the direction of the field. A wave of angular frequency ω travels with its wave normal at an angle θ to the field. It is linearly polarized with its electric vector in the plane containing the wave normal and the direction of the field. Write down the equation of motion of an electron. Hence find the relation between the electric polarization $\underset{\sim}{P}$ and the electric intensity $\underset{\sim}{E}$. Show that the refractive index μ is given by:

$$\mu^2 = \frac{1 - X}{1 - X\cos^2\theta}$$

where $X = (\omega_N/\omega)^2$ and $\omega_N/2\pi$ is the plasma frequency.

11. For Cartesian axes, with the z axis anti-parallel to the superimposed magnetic field, the inverse of the electric susceptibility matrix of a cold magnetoplasma (electrons only) is

$$-\frac{1}{X}\begin{pmatrix} U & iY & 0 \\ -iY & U & 0 \\ 0 & 0 & U \end{pmatrix}$$

Find its eigen values and hence show that the three eigen values of the dielectric constant are $1 - X/(U \pm Y)$, $1 - X/U$. Find the corresponding eigen vectors and try to interpret them physically.

(In a homogeneous plasma of this kind, is there any solution of Maxwell's equations which uses only one of the three eigen vectors? If so, what kind of wave is it?).

Note. The directions of the eigen vectors are the "principal axes" They are often used in wave propagation and plasma probmems. They are not orthogonal but they are "Hermitean orthogonal". Why is this?

12. A neutral molecule in a plasma can be thought of as an infinitely massive, perfectly conducting small sphere, at rest.

(a) Show that an electron at distance r away experiences an attractive force proportional to $1/r^5$.

* (b) If the electron approaches the molecule with velocity v, show that the transport area of cross section A for the encounter is proportional to 1/v. (Hint: The differential equation of the orbit is $\frac{d^2u}{d\theta^2} + u = \frac{Ku^3}{b^2v^2}$ where $u = 1/r$. Show that the angle of deflection χ is given by

$$\pi + \chi = \int_0^{x_0} 2\left\{1 - x^2 + K x^4/(2 v^2 b^4)\right\}^{-\frac{1}{2}} dx \text{ where}$$

x_o is the greatest value of b/r. Hence show that χ and

therefore cos χ depend on b and v, only through the combination b^2v. Use this in the integral for A. It is not necessary to evaluate the integrals).

(c) Hence show that the effective collision frequency is independent of v.

13. A warm isotropic plasma contains electrons and one kind of positive ion (masses m_e, m_i, effective temperatures T_e, T_i, plasma frequencies ω_{Ne}, ω_{Ni}, respectively). Use the "compressible plasma" (acoustic) method and neglect collision damping.

(a) Show that, for longitudinal (plasma) waves, the dispersion relation is

$$\frac{\omega_{Ne}^{\ 2}}{\omega^2 - a_e^2 k^2} + \frac{\omega_{Ni}^{\ 2}}{\omega^2 - a_i^2 k^2} = 1$$

where $a_e^2 = \gamma KT_e/m_e$, $a_i^2 = \gamma KT_i/m_i$.

(b) Sketch curves of k^2 versus ω^2.

(c) Show that when ω^2 is very large, $k^2 \sim \omega^2/a_i^2$ or $k^2 \sim \omega^2/a_e^2$ (the asymptotes).

(d) Find the dispersion relation for transverse (electromagnetic) waves. (Read: Clemmow and Dougherty, section 5.5.3).

14. A uniform cold isotropic electron plasma is at rest and has plasma frequency ω_N. Moving through it is a weak stream of electrons with velocity $V (\ll c)$ and plasma frequency $\omega_1 (\ll \omega_N)$. Show that for longitudinal waves whose wave normal is parallel to V, the dispersion relation is

$$(\omega^2 - \omega_N^2)(\omega - kV)^2 = \omega_1^2 \omega^2.$$

Sketch the curve of k versus ω and indicate for what range of k instability might be expected.

ANSWERS TO PROBLEMS.

3. (a) $a + 2\omega^2/\alpha, \quad a + \omega^2/\alpha$

(b) If $\omega < \omega_p$: $b - a(1 - \omega^2/\omega_p^2)^{\frac{1}{2}}$,

$$b - a + \tfrac{1}{2} a \frac{\omega}{\omega_p} \ln \frac{\omega_p + \omega}{\omega_p - \omega}$$

($\omega_p / 2\pi$ is called the penetration frequency).

(c) $\frac{2}{\alpha} \ln\left(\frac{\omega}{\omega_o}\right)$, $\frac{2}{c}$ arc cosh (ω/ω_o)

4. $D =$ (a) $2a \tan \Theta + (\omega^2 \sin 2\Theta)/\alpha$

(b) $2a \tan \Theta + a \frac{\omega}{\omega_p} \sin \Theta \ln \frac{\omega_p + \omega \cos \Theta}{\omega_p - \omega \cos \Theta}$

(c) $\frac{4 \tan \Theta}{\alpha}$ arc cosh $\left(\frac{\omega \cos \Theta}{2 \omega_o}\right)$

$p' = D/\sin \Theta$ in all three cases.

6. (a) $\left(\frac{1 + |R^2| - |1 + R^2|}{1 + |R^2| + |1 + R^2|}\right)^{\frac{1}{2}}$, $\frac{1}{2} |E_x^2| (1 + |R^2| + |1 + R^2|)$

(b) $|(\rho - R)/(1 - R\rho)|$.

7. Smallest 0, (45°)

Greatest (1 - a)/(1 + a), (90°).

8. (a) Elliptical. Axis ratio 1/3.

(b) Partially circularly polarised. Energy flux ratio (circular component/unpolarised component) = 3/2.

10. If z-axis is parallel to magnetic field,

$P_x = P_y = 0, \; P_z = - \varepsilon_o X E_z.$

11. Eigen vectors (1, ± i, 0), (0, 0, 1).

First two: circularly polarised waves, wave normal parallel to magnetic field.

Third: linearly polarised (ordinary) wave, wave normal perpendicular to magnetic field.

13. (d) $\omega_{Ne}^2 + \omega_{Ni}^2 = \omega^2 - k^2 c^2.$